Dynamic analysis of non-proportionally damped stochastic MDOF systems

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What this talk is about

Impulse response of stochastic oscillators: Ensemble damping and phase decoherence

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Central theme: the **characteristic phase decoherence time**

Outline

- The equation of motion of a damped multi-degree-of-freedom (MDOF) system is
$$\mathbf{M}\ddot{\mathbf{u}}(\tau) + \mathbf{C}\dot{\mathbf{u}}(\tau) + \mathbf{K}\mathbf{u}(\tau) = \mathbf{f}(\tau) \tag{1}$$
with mass **M**, damping **C** and stiffness **K** matrices.
- In real structures, **M**, **C** and **K** carry inherent scatter: manufacturing tolerances, material variability, geometric imperfections and modelling idealisation.
- The natural frequencies, damping ratios and modal coefficients therefore become random quantities.
- When **C** is non-proportional, the system has no classical normal modes, and the response is governed by complex modes.
- Central question: how does parameter uncertainty alter the time-domain response, and can we describe it in closed form?

The impulse response is the building block

- For any linear system, the response to arbitrary forcing follows from the impulse response function $h(\tau)$ through the convolution integral

$$u(\tau) = \int_0^{\tau} h(\tau - s)\, f(s)\, \mathrm{d}s \tag{2}$$

- $h(\tau)$ is the Green's function of the equation of motion: it encodes the complete dynamic character of the system in a single time-domain quantity.
- Modal projection of Eq. (1) reduces the dynamics to a set of single-degree-of-freedom (SDOF) oscillators, one per mode, each with its own natural frequency and damping.
- Understanding the stochastic SDOF oscillator is therefore the essential first step. Its impulse response is the kernel from which the MDOF response statistics are assembled.
- This talk derives the time-domain statistics of that kernel in exact closed form, then returns to the MDOF case at the end.

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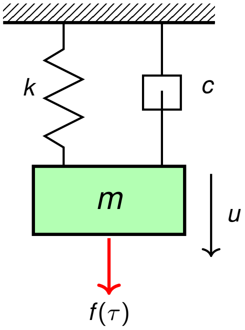
The analytical gap

- Classical random vibration theory treats the system as deterministic and the forcing as a stochastic process.
- Stochastic finite elements, polynomial chaos and stochastic collocation propagate parameter randomness, but predominantly in the frequency domain.
- Recent surrogate and machine-learning methods give accurate pointwise statistics but do not reveal the structure of the time-domain mean response, its decay mechanism, or its intrinsic timescales.
- For the SDOF oscillator, frequency-domain analysis has shown that frequency uncertainty produces an apparent increase in effective damping of the mean-square frequency response function.
- No closed-form expression for the ensemble mean impulse response of a stochastic oscillator has been reported. The complementary time-domain problem has had little analytical attention.

What this talk delivers

1. An exact closed-form ensemble mean impulse response of a stochastic oscillator with Gaussian frequency uncertainty, through a sinc-kernel identity and the complex error function.
2. Identification of ensemble damping: phase dispersion across the ensemble attenuates the mean response independently of viscous dissipation.
3. The characteristic phase decoherence time t_{d}, the central quantity of the talk, together with the ensemble damping ratio ζ_{en}.
4. Distributional robustness across uniform, normal, log-normal and chi-square frequency models.
5. The same analytical route extends to MDOF systems through modal projection.



- Equation of motion

$$m\ddot{u} + c\dot{u} + ku = f(\tau) \tag{3}$$

- In standard form

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = \frac{f(\tau)}{m} \tag{4}$$

with $\omega_n = \sqrt{k/m}$, $\zeta = c/(2\sqrt{km})$.

- Unit impulse $f(\tau) = \delta(\tau)$ gives $u(0^+) = 0$, $\dot{u}(0^+) = 1/m$. The displacement u is measured downward.

Deterministic impulse response

$$h(\tau) = \frac{1}{m\omega_d} e^{-\zeta\omega_n\tau}\sin(\omega_d\tau), \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} \tag{5}$$

- Let the natural frequency be uncertain

$$\omega_n = \omega_{n_0} \, z \tag{6}$$

where ω_{n_0} is the deterministic value and z is a random variable with mean 1 and standard deviation σ.

- Introduce a standardised random variable x with zero mean and unit variance

$$z = 1 + \epsilon x \tag{7}$$

For most distributions ϵ is the coefficient of variation of the frequency.

- The Gaussian model $x \sim N(0, 1)$, $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, is used for the analytical derivation because expectations become Gaussian integrals.
- Four distributions are considered overall: uniform, normal, log-normal, chi-square, spanning the models common in engineering uncertainty quantification.

Impulse response as a random process

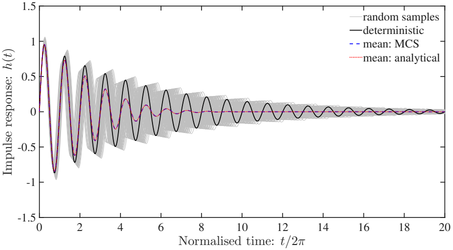
- With the uncertain frequency, the impulse response inherits the randomness through z

$$h(\tau, z) = \frac{e^{-\zeta\omega_n\tau}\sin(\omega_d\tau)}{\omega_d} \tag{8}$$

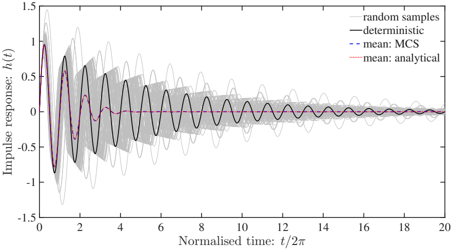
- Using non-dimensional time $t = \omega_{n_0}\tau$ and $\omega_n = \omega_{n_0}(1+\epsilon x)$,

$$h(t, x) = \frac{\exp[-\zeta(1+\epsilon x)t]\,\sin\big((1+\epsilon x)\beta t\big)}{(1+\epsilon x)\beta}, \qquad \beta = \sqrt{1-\zeta^2} \tag{9}$$

- The same random variable x governs the response at all times, so $h(t, x)$ is not a random variable but a random process indexed by time.
- Each realisation is a deterministic oscillator with its own natural frequency; the ensemble represents the statistical variability of the system.



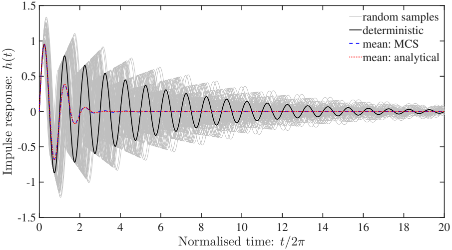
(a) $\epsilon = 0.05,\ \zeta = 0.03$



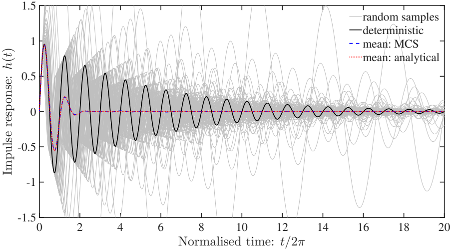
(b) $\epsilon = 0.10,\ \zeta = 0.03$

Grey: individual sample realisations. Blue: Monte Carlo ensemble mean (10,000 samples). The smooth line through the mean is the analytical result, derived later in the talk. The mean decays faster than each realisation.

Ensemble impulse response: higher randomness



(c) $\epsilon = 0.15,\ \zeta = 0.03$



(d) $\epsilon = 0.20,\ \zeta = 0.03$

As ϵ grows, the ensemble mean is suppressed after fewer cycles: tracking the deterministic response for several cycles in (c), but extinguished within roughly two cycles in (d), while individual realisations still oscillate at full amplitude.

1. Ensemble damping. The mean response decays substantially faster than the deterministic impulse response, even though the physical damping ratio $\zeta = 0.03$ is unchanged. The attenuation is not extra dissipation.
2. Uncertainty controls the decay. The rate of attenuation grows systematically with the randomness ϵ: from tracking the deterministic response for about eight cycles at $\epsilon = 0.05$, to near-immediate suppression at $\epsilon = 0.20$.
3. A characteristic timescale appears. In every subfigure there is a time beyond which the mean collapses toward zero, separating a coherent early regime from an incoherent late regime.

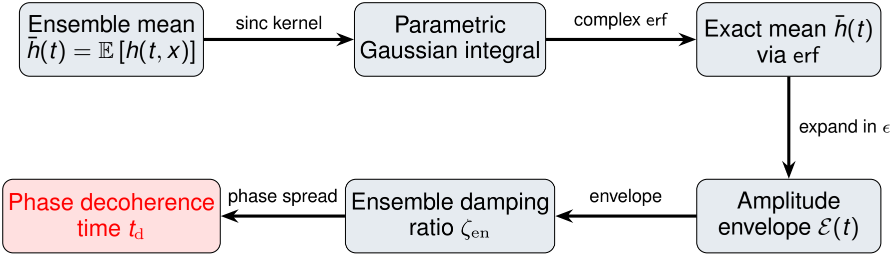
- These observations set up the talk. The analytical line through the mean is the exact closed-form expression developed in the following sections.

The central quantity: a characteristic decoherence time

- The most important feature of the plots is the time at which the ensemble mean stops following the deterministic response and rapidly decays.
- This time shortens as ϵ increases, and it is present even though each individual oscillator keeps vibrating.
- It is a property of the ensemble, not of any single realisation: the realisations drift out of phase, and their average cancels.
- The goal of this talk is to pin this timescale down in closed form and to show that it is
  - intrinsic to the stochastic system,
  - absent from the deterministic theory,
  - and independent of viscous damping to leading order.
- We call it the characteristic phase decoherence time t_{d}. Everything that follows builds toward its closed-form expression.

Outline of the analytical route

- The path from the ensemble mean to the two descriptors ζ_{en} and t_{d} proceeds in four steps.



- The red box is the destination: the closed-form characteristic phase decoherence time.
- The next sections fill in each step. They are detailed and may be skimmed; the results are what matter.

- The ensemble mean impulse response is the expectation over x

$$\bar{h}(t) = \mathbb{E}\left[h(t,x)\right] = \int_{-\infty}^{\infty} h(t,x)\, p_X(x)\, \mathrm{d}x \tag{10}$$

- Substituting Eq. (9), the Gaussian model gives the integral

$$\bar{h}(t) = \frac{1}{\beta\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp[-\zeta(1+\epsilon x)t]\, \sin\big((1+\epsilon x)\beta t\big)}{(1+\epsilon x)}\, e^{-x^2/2}\, \mathrm{d}x \tag{11}$$

- The factor $1/(1+\epsilon x)$ obstructs a direct evaluation. The key is to remove it with an integral identity.

- Write $a = \zeta t$, $b = t\sqrt{1-\zeta^2}$ and $u = 1 + \epsilon x$, with $\phi(x) = p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ the standard Gaussian pdf. The mean reduces to

$$I = \int_{-\infty}^{\infty} \phi(x)\, e^{-au}\, \frac{\sin(bu)}{u}\, \mathrm{d}x, \qquad \bar{h}(t) = \frac{I}{\beta} \tag{12}$$

- The factor $\sin(bu)/u$ is regular ($\to b$ as $u \to 0$) and admits the integral representation

Sinc-kernel identity

$$\frac{\sin(bu)}{u} = \int_0^b \cos(su)\, \mathrm{d}s \tag{13}$$

- This replaces the awkward $1/u$ by an ordinary integral over s, leaving only $e^{-au}\cos(su)$, linear in $u = 1 + \epsilon x$ in both exponent and argument.
- The expectation over x then becomes a standard Gaussian integral.

- Insert Eq. (13) into Eq. (12) and interchange the order of integration

$$I = \int_0^b \underbrace{\left[\int_{-\infty}^{\infty} \phi(x)\, e^{-a(1+\epsilon x)} \cos\big(s(1+\epsilon x)\big)\,\mathrm{d}x\right]}_{J(s)} \mathrm{d}s \tag{14}$$

- Write the cosine as $\mathrm{Re}\ e^{is(1+\epsilon x)}$ and collect the x-dependence

$$J(s) = \mathrm{Re}\left[e^{-(a-is)} \int_{-\infty}^{\infty} \phi(x)\, e^{-(a-is)\epsilon x}\,\mathrm{d}x\right] \tag{15}$$

- The Gaussian identity $\int_{-\infty}^{\infty} \phi(x)\, e^{cx}\mathrm{d}x = e^{c^2/2}$ with $c = -(a-is)\epsilon$ gives

$$J(s) = e^{-a+\frac{\epsilon^2}{2}(a^2-s^2)} \cos\big((1-\epsilon^2 a)s\big) \tag{16}$$

- Substituting $J(s)$ back into the outer integral gives the exact representation

Exact mean response integral

$$I = e^{-a+\frac{\epsilon^2 a^2}{2}} \int_0^b e^{-\frac{\epsilon^2 s^2}{2}} \cos\big((1-\epsilon^2 a)s\big)\,\mathrm{d}s \tag{17}$$

- This already shows how frequency randomness ϵ modifies both amplitude and phase of the deterministic response.
- The remaining integral has the form $\int_0^b e^{-\alpha s^2}\cos(\gamma s)\,\mathrm{d}s$ with $\alpha = \epsilon^2/2$, $\gamma = 1-\epsilon^2 a$.
- Completing the square and writing $\cos(\gamma s) = \mathrm{Re}\, e^{i\gamma s}$ yields the complex error function erf of a complex argument.

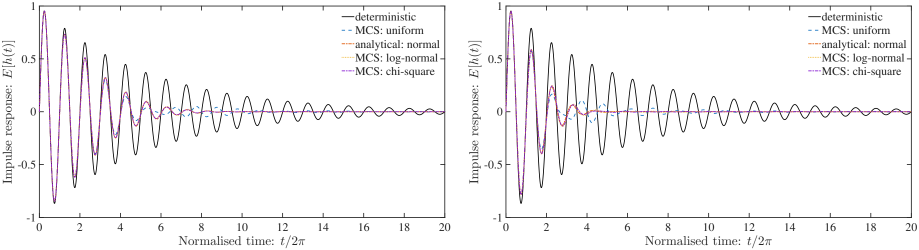
- Evaluating Eq. (17) in closed form gives the main analytical result.

Exact ensemble mean impulse response (Gaussian frequency uncertainty)

$$\bar{h}(t) = e^{-a+\frac{\epsilon^2 a^2}{2}} \frac{\sqrt{\pi}}{\sqrt{2}\,\epsilon\beta}\, e^{-\frac{(1-\epsilon^2 a)^2}{2\epsilon^2}}\ \mathrm{Re}\left[\mathrm{erf}\left(\frac{\epsilon b}{\sqrt{2}} - i\frac{1-\epsilon^2 a}{\sqrt{2}\,\epsilon}\right) - \mathrm{erf}\left(-i\frac{1-\epsilon^2 a}{\sqrt{2}\,\epsilon}\right)\right] \tag{18}$$

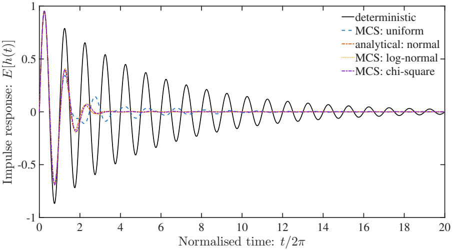
- Here $a = \zeta t$, $b = \beta t$, $\beta = \sqrt{1-\zeta^2}$.
- This is the analytical line shown through the Monte Carlo mean in the earlier figures; it matches to graphical precision.
- It is exact, requires no numerical averaging, and uses only the standard special function erf.

Four distributions: low to moderate randomness

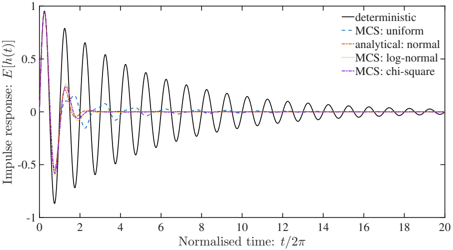


(a) $\epsilon = 0.05$ (b) $\epsilon = 0.10$

Mean response for the normal (analytical), uniform, log-normal and chi-square frequency models, $\zeta = 0.03$. The four curves nearly coincide; the uniform model differs only slightly.



(c) $\epsilon = 0.15$



(d) $\epsilon = 0.20$

Even at higher randomness the normal, log-normal and chi-square models remain nearly indistinguishable. The mean depends only on the first two moments of x, so the result is largely distribution-free.

- Expand the exact integrand of Eq. (17) in powers of ϵ, retaining terms to $O(\epsilon^2)$

$$e^{-\frac{\epsilon^2 s^2}{2}} \cos\big((1-\epsilon^2 a)s\big) = \cos s + \epsilon^2 \left(as \sin s - \tfrac{s^2}{2} \cos s \right) + O(\epsilon^4) \qquad (19)$$

- Using $\int_0^b \cos s \,\mathrm{d}s$, $\int_0^b s \sin s \,\mathrm{d}s$, $\int_0^b s^2 \cos s \,\mathrm{d}s$ and collecting terms, with $a = \zeta t$, $b = \beta t$

$$\bar{h}(t) \approx \frac{e^{-\zeta t}}{\beta} \Big[\sin(\beta t) + \epsilon^2 \Big(\Big(\zeta t + 1 + \tfrac{\zeta^2 t^2}{2} - \tfrac{\beta^2 t^2}{2} \Big) \sin(\beta t) - (\zeta t + 1)\beta t \cos(\beta t) \Big) \Big] \qquad (20)$$

- The leading correction is $O(\epsilon^2)$: the $O(\epsilon)$ term vanishes because x has zero mean. Frequency fluctuation is a second-order effect on the mean.

- Write Eq. (20) as $\bar{h}(t) \approx \frac{e^{-\zeta t}}{\beta}\left[A(t)\sin(\beta t) - B(t)\cos(\beta t)\right]$ with

$$A(t) = 1 + \epsilon^2\left(\zeta t + 1 - \tfrac{(1-2\zeta^2)t^2}{2}\right), \qquad B(t) = \epsilon^2(\zeta t + 1)\beta t \tag{21}$$

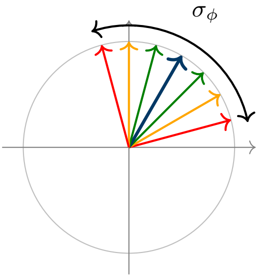
(using $\beta^2 - \zeta^2 = 1 - 2\zeta^2$).

- The envelope is $\mathcal{E}(t) = \frac{e^{-\zeta t}}{\beta}\sqrt{A^2 + B^2}$. Retaining $O(\epsilon^2)$ and resumming the polynomial into an exponential gives the compact form.

Mean-response amplitude envelope

$$\mathcal{E}(t) = \frac{1}{\beta}\exp\left[-\zeta t - \frac{(1-2\zeta^2)}{2}\,\epsilon^2 t^2\right] \tag{22}$$

- An additional Gaussian-in-time decay, absent from the deterministic theory, grows quadratically with ϵ and drives the collapse of the mean.



Phasors of individual realisations spread with time.

Each realisation is a unit phasor at angle $(1+\epsilon x)\beta t$. Early on they align and add constructively. As they rotate at different rates the bundle fans out with width $\sigma_\phi = \epsilon\beta t$, and the average cancels. A Gaussian spread gives a resultant $\propto e^{-\sigma_\phi^2/2}$, matching $\mathcal{E}(t)$.

- Each realisation oscillates at its own damped frequency $(1+\epsilon x)\beta$.
- At time t the relative phase of a realisation is $\epsilon x \beta t$. Since $x \sim N(0,1)$, the phase spread is

$$\sigma_\phi(t) = \epsilon\beta t \tag{23}$$

- For $\sigma_\phi \ll 1$ the realisations stay in phase and $\bar{h}$ tracks the deterministic response.
- As $\sigma_\phi \to 1$, phases spread over the unit circle and destructive interference suppresses the mean.
- The mean decays even when $\zeta = 0$: this is not dissipation.

- The transition from coherent to incoherent ensemble behaviour occurs when $\sigma_\phi \sim 1$, defining the central quantity of this talk.

Characteristic phase decoherence time

$$t_{\mathrm{d}} = \frac{1}{\epsilon\beta} = \frac{1}{\epsilon\sqrt{1-\zeta^2}} \qquad \Longrightarrow \qquad \tau_{\mathrm{d}} = \frac{2m}{\epsilon\sqrt{4km-c^2}} \tag{24}$$

- t_{d} is absent from deterministic theory. For $t \ll t_{\mathrm{d}}$ the mean tracks the deterministic response; for $t \gg t_{\mathrm{d}}$ it is extinguished.
- It is independent of ζ to leading order. In the undamped limit $\zeta^2 \to 0$

$$t_{\mathrm{d}} \to \frac{1}{\epsilon}, \qquad \tau_{\mathrm{d}} \to \frac{1}{\epsilon}\sqrt{\frac{m}{k}} \tag{25}$$

governed solely by mass, stiffness and randomness level. Phase decoherence persists with no energy dissipation.

- The instantaneous log-decay rate of the envelope Eq. (22) is

$$\gamma_{\text{eff}}(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathcal{E}(t) = \zeta + (1 - 2\zeta^2)\epsilon^2 t \tag{26}$$

- Averaging over one deterministic period $T_0 = 2\pi/\beta$ gives a single representative measure

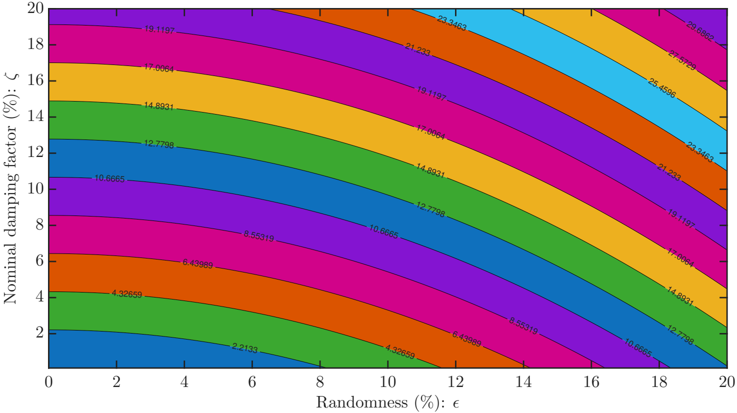
$$\zeta_{\text{en}} = \frac{1}{T_0} \int_0^{T_0} \gamma_{\text{eff}}(t)\,\mathrm{d}t = \zeta + \frac{1 - 2\zeta^2}{2} \epsilon^2 T_0 \tag{27}$$

Ensemble damping ratio

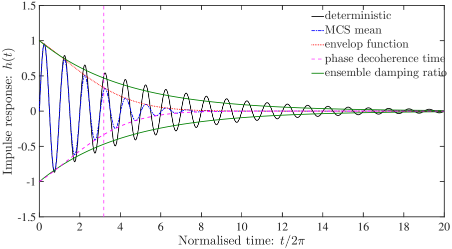
$$\zeta_{\text{en}} = \zeta + \frac{\pi(1 - 2\zeta^2)}{\sqrt{1 - \zeta^2}}\,\epsilon^2 \tag{28}$$

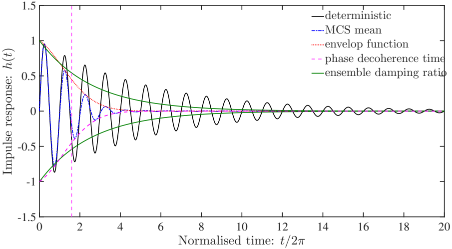
- Quadratic in ϵ, consistent with the $O(\epsilon^2)$ structure of the mean.
- Example: $\zeta = 2\%$ with $\epsilon = 10\%$ gives $\zeta_{\text{en}} \approx 6.4\%$, a threefold apparent increase, from statistics, not dissipation.

Ensemble damping ratio: design chart



Contours of ζ_{en} from Eq. (28) over randomness ϵ and nominal damping ζ (both in %). $\zeta_{\text{en}} > \zeta$ everywhere; even modest frequency scatter substantially raises the effective damping.



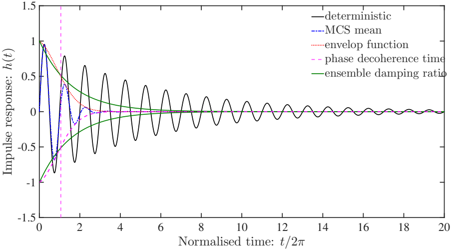


(a) $\epsilon = 0.05$, $t_{\mathrm{d}}/2\pi = 3.18$

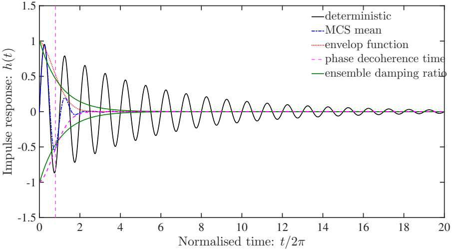
(b) $\epsilon = 0.10$, $t_{\mathrm{d}}/2\pi = 1.59$

Envelope $\mathcal{E}(t)$ (red), ensemble-damping exponential $\beta^{-1}e^{-\zeta_{\mathrm{en}}t}$ (green) and the decoherence time t_{d} (magenta), with the MCS mean and deterministic response, $\zeta = 0.03$.

Descriptors validated: higher randomness



(c) $\epsilon = 0.15$, $t_{\rm d}/2\pi = 1.06$



(d) $\epsilon = 0.20$, $t_{\rm d}/2\pi = 0.80$

The magenta line $t_{\rm d}$ marks where the mean departs from the deterministic response in every subfigure, moving toward the origin as ϵ grows. The envelope $\mathcal{E}(t)$ bounds the mean throughout.

- For the MDOF system in Eq. (1), the impulse response matrix in state space is a sum over the complex modes

$$\mathbf{h}(\tau) = \sum_j \left(\mathbf{r}_j \, e^{\lambda_j \tau} + \mathbf{r}_j^* \, e^{\lambda_j^* \tau} \right) \tag{29}$$

  with complex eigenvalues λ_j and residues $\mathbf{r}_j$. Non-proportional damping is handled directly through the complex modes.
- Each modal contribution is the impulse response of a stochastic SDOF oscillator of exactly the type analysed here, with $\lambda_j = -\zeta_j \omega_j \pm i\omega_j \sqrt{1 - \zeta_j^2}$ and uncertain ω_j.
- The same sinc-kernel and Gaussian-integral route carries over mode by mode: the ensemble mean modal response, its envelope, an ensemble damping ratio and a decoherence time follow for each mode.
- Phase decoherence acts per mode; modes with larger frequency scatter decohere first.

Ensemble damping is already visible in MDOF data

- The mechanism is not an artefact of the SDOF idealisation chosen here for tractability.
- The ensemble-mean deflection of a randomly parametered cantilever beam under impulse loading, computed by Monte Carlo simulation in earlier works, shows precisely the accelerated mean decay relative to the deterministic response that is the signature of ensemble damping, across all uncertainty levels.
- That multi-degree-of-freedom result was not identified as a distinct mechanism at the time; the closed-form SDOF theory here explains it.
- This is why closed-form analysis matters: it names and characterises phenomena that remain invisible to purely computational approaches, however accurate their pointwise predictions.
- Natural continuation: covariance and forced-response statistics, and fully distributed (continuous) systems.

1. Exact mean impulse response for Gaussian frequency uncertainty, Eq. (18), through the sinc-kernel identity and the complex error function.
2. Amplitude envelope with an additional Gaussian-in-time decay, $\mathcal{E}(t) = \beta^{-1}\exp[-\zeta t - \frac{(1-2\zeta^2)}{2}\epsilon^2 t^2]$.
3. Ensemble damping: the mean decays faster than the deterministic response by phase dispersion, not dissipation.
4. Ensemble damping ratio $\zeta_{\rm en} = \zeta + \pi(1 - 2\zeta^2)\epsilon^2/\sqrt{1-\zeta^2}$ and the central phase decoherence time $t_{\rm d}$.
5. Distributional robustness across uniform, normal, log-normal and chi-square models.

- The stochastic SDOF oscillator admits an exact closed-form time-domain mean response, providing physical insights into stochastic structural dynamics.
- The central result is the characteristic phase decoherence time, in non-dimensional and dimensional form:

Characteristic phase decoherence time

$$t_{\mathrm{d}} = \frac{1}{\epsilon\sqrt{1-\zeta^2}} \qquad \tau_{\mathrm{d}} = \frac{1}{\epsilon\,\omega_{n_0}\sqrt{1-\zeta^2}} = \frac{2m}{\epsilon\sqrt{4km-c^2}}$$

- It is intrinsic to the stochastic system, absent from the deterministic theory, and independent of viscous damping to leading order; in the undamped limit $\boxed{\tau_{\mathrm{d}} \to \frac{1}{\epsilon}\sqrt{\frac{m}{k}} = \frac{T_0}{2\pi\epsilon}}$, about $1/(2\pi\epsilon)$ undamped periods.
- An ensemble of lightly damped structures with scattered natural frequencies appears, when averaged, more heavily damped than any single member. This must not be misattributed to physical dissipation.
- The same analytical route extends mode by mode to damped stochastic MDOF systems.