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Data-Driven Non-Central Wishart Framework for Hybrid Uncertainty Quantification of Aerostructures

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Outline

Introduction and Motivation

How can we quantify uncertainty in FEM-based dynamic predictions?

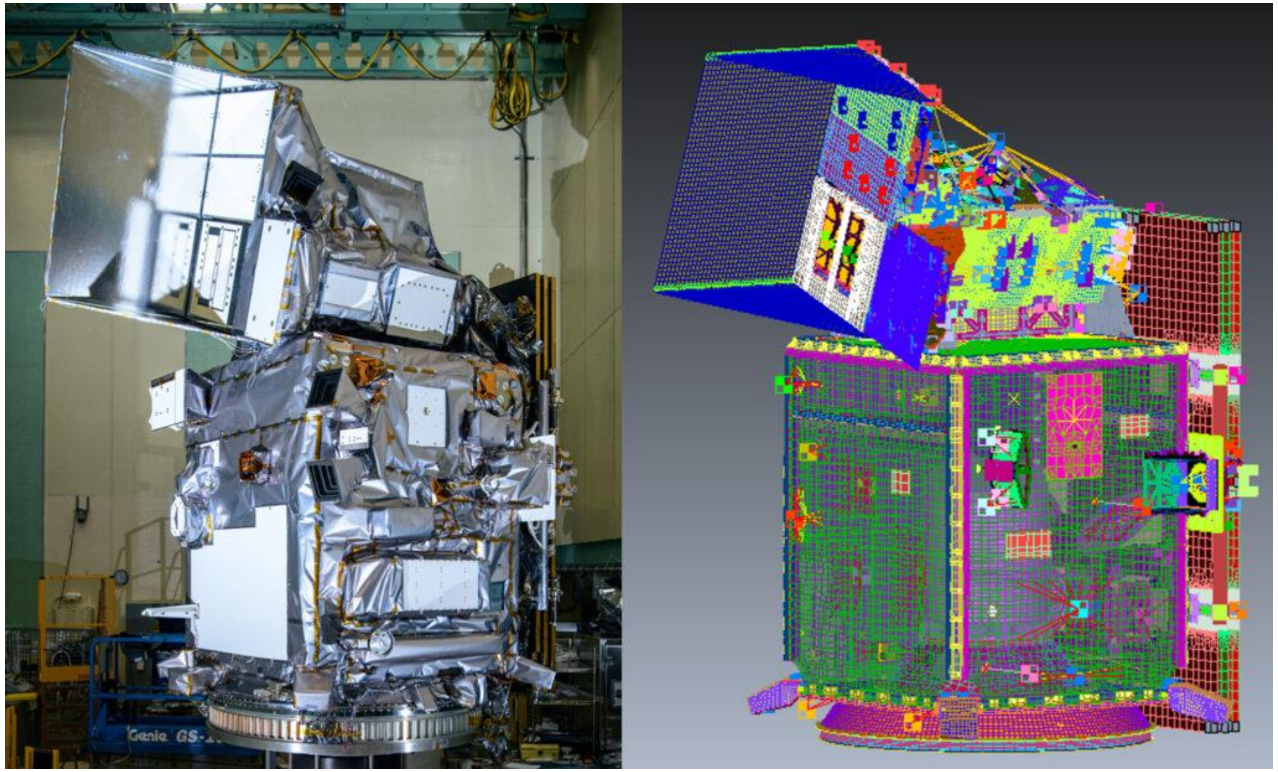


Figure 1: Actual vs. Finite Element Model (FEM) of NASA PACE Observatory.

Image Source: NASA Goddard Space Flight Center. (n.d.). *Mechanical Systems Analysis & Simulation – Finite Element Model and Physical Hardware of a Satellite.* Retrieved from https://etd.gsfc.nasa.gov/capabilities/capabilities-listing/mechanical-systems-analysis-simulation/

Why does uncertainty arise?

Model parameters:
Young's modulus, density, Poisson's ratio, thickness

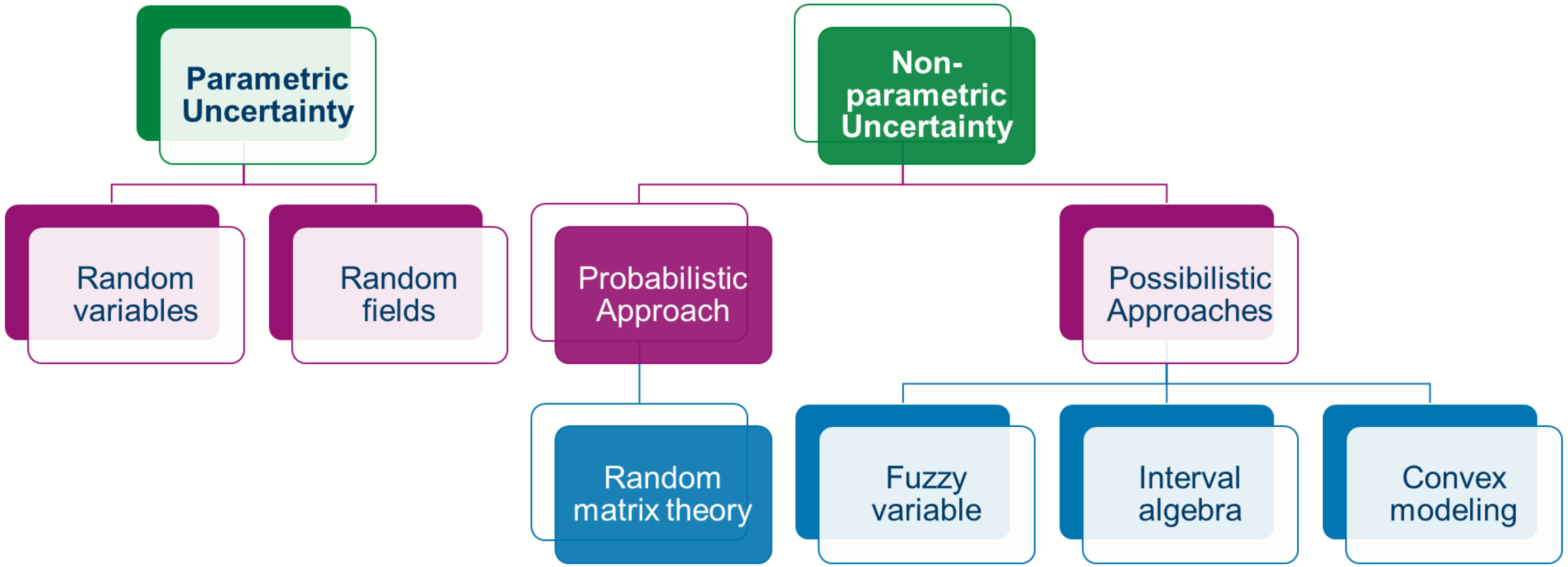
Model form:
Boundary conditions, damping models, subsystem coupling

Numerical modelling:
Discretisation, truncation, solver tolerances

Measurements:
Noise, limited sensors, excitation uncertainty

Introduction and Motivation

Uncertainty Quantification Taxonomy



Parametric Uncertainty (Aleatoric)
Variability in material and geometric properties, often spatially correlated, typically modelled using random variables or random fields (e.g. KLE).

Model-Form Uncertainty (Epistemic)
Uncertainty arising from modelling assumptions such as boundary conditions, damping representations, subsystem coupling, and numerical discretisation.

Introduction and Motivation

Limitations of Existing UQ Approaches

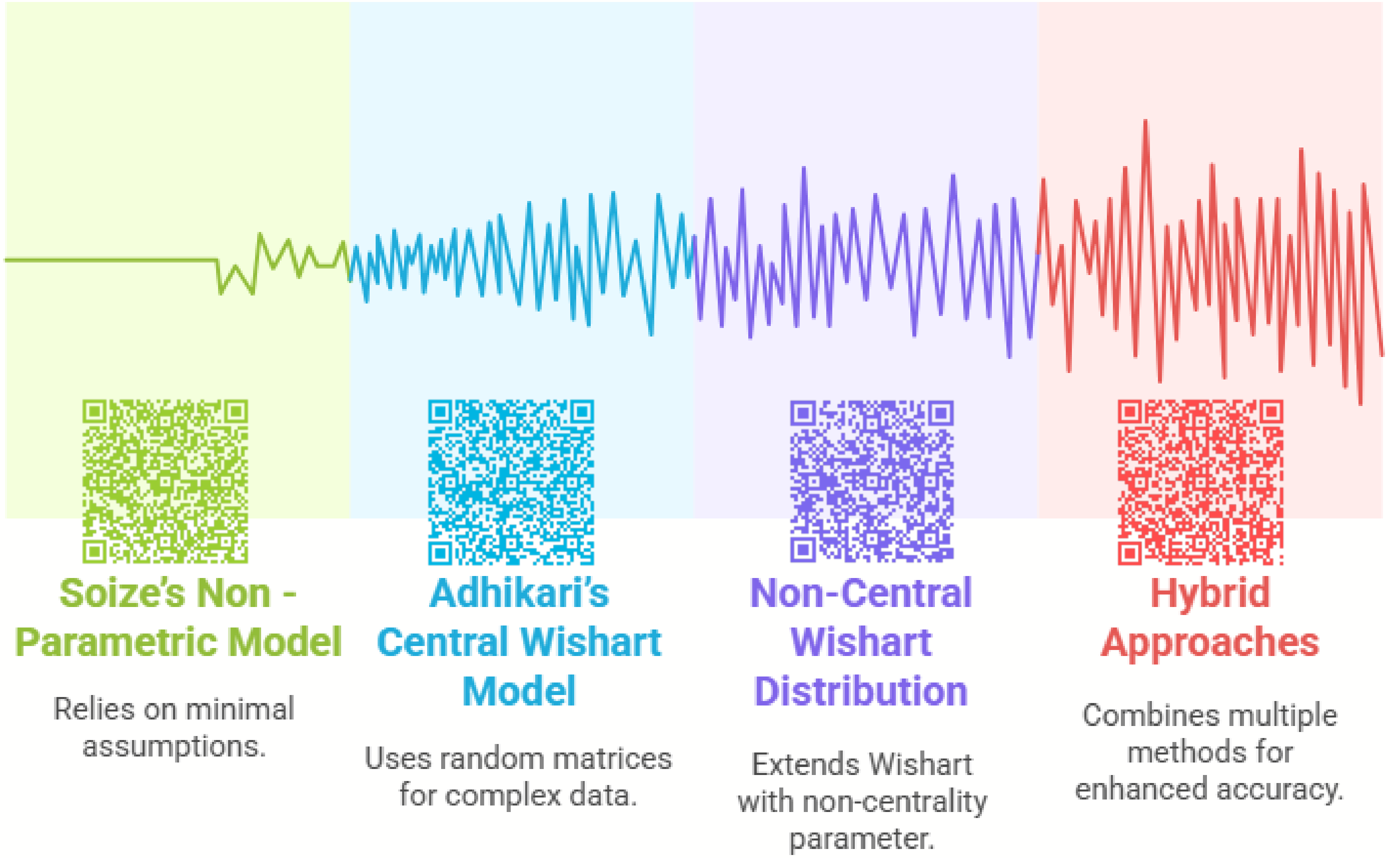
- **Parametric Approaches**
- Stochastic FEM, random fields
- Designed to model **aleatoric uncertainty**
- Require detailed knowledge of spatial variability
- Computationally expensive for large-scale models
- Primarily effective at low frequencies

Non-Parametric Approaches

- Random matrix theory (central Wishart)
- Designed to **model model-form uncertainty**
- Preserve symmetry and positive definiteness
- Usually use a **single dispersion parameter**
- Cannot reproduce structured covariance information

Background and Literature

Evolution of Random Matrix UQ Methods



1. C. Soize. A nonparametric model of random uncertainties for reduced matrix models in structural dynamics. Probabilistic Engineering Mechanics, 15(3):277– 294, 2000.
2. Adhikari, S.: "Wishart Random Matrices in Probabilistic Structural Mechanics," J. Eng. Mech., vol. 134, no. 12, pp. 1029-1044, 2008.
3. S. Adhikari. Uncertainty quantification in structural dynamics using non-central Wishart distribution. Technical report, 2010.
4. Dexter Johnson, / Nesc, Joel Sills, Daniel Kammer, and Paul Blelloch. Uncertainty models for the hybrid parametric variation method of uncertainty quantification; analysis. NASA Technical report, 2022.

Theoretical Framework

Proposed reduced unified approach

- The objective : To develop a hybrid approach which takes both parametric and nonparametric uncertainties into account and reduce the computational cost by implementing an easy reduced order model.

- The rationale : No matter what the nature of uncertainty is (parametric/nonparametric or both), at the end it will result in random **M, C** and **K** matrices.

- The methodology : Derive the matrix variate probability density functions of **M, C** and **K** based on parametric information (e.g. mean and covariance of the elements) and overall, physically realistic mathematical constraints (such as the symmetry and positive definiteness)

Introduction and Motivation

Towards Unified Hybrid UQ Framework

In practical engineering systems, it is **difficult or impossible to distinguish** parametric and model-form uncertainties. Regardless of their origin, all sources of uncertainty ultimately result in **random mass and stiffness matrices**.

$$\mathbf{M}\ddot{q}(t) + \mathbf{C}\dot{q}(t) + \mathbf{K}q(t) = f(t)$$

Methodology:

Treat the system matrices as **symmetric positive-definite random matrices**.

Use **parametric uncertainty simulations** to obtain:

- Mean matrices
- Covariance information

Construct a **non-central Wishart distribution** consistent with:

- Known first- and second-order statistics
- Physical constraints (symmetry, positive definiteness)

Perform identification and sampling in a **reduced modal subspace** for computational tractability.

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Theoretical Framework

Structural Dynamics and Moment Conditions

For a linear structural system, the governing equation in the frequency domain is written as

$$\mathbf{D}(\omega)\mathbf{q}(\omega) = \mathbf{f}(\omega), \qquad \boldsymbol{D}(\omega) = -\omega^2\boldsymbol{M} + i\omega\boldsymbol{C} + \boldsymbol{K}.$$

- In the presence of uncertainty, the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$are modelled as **random matrices**.
- These matrices are assumed to be symmetric, with $\mathbf{M}$and $\mathbf{K}$ positive-definite and $\mathbf{C}$non-negative definite.
- In order for the response statistics to be well-defined, the moments of $\mathbf{D}(\omega)$must exist for all frequencies of interest.

The frequency response function (FRF) matrix is defined as

$$\boldsymbol{H}(\omega) = \boldsymbol{D}^{-\mathbf{1}}(\omega).$$

To capture the stochastic behaviour of the system response, the associated probability distributions of the random matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ must ensure that the inverse moments of the dynamic stiffness matrix exist for all excitation frequencies ω.
Accordingly, the following moment condition must be satisfied:

$$\mathbb{E}\{\|\mathbf{H}(\omega)\|^{\mu}\} < \infty, \qquad \forall\, \omega,\ \mu > 0.$$

Central Wishart Random Matrix Formulation

A $n \times n$ symmetric positive definite random matrix $\boldsymbol{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$\mathrm{p}_S(\mathbf{S}) = \left\{ 2^{\left(\frac{1}{2}\mathrm{np}\right)} \Gamma_\mathrm{n}\left(\frac{\mathrm{p}}{2}\right) |\mathbf{\Sigma}|^{\frac{1}{2}\mathrm{p}} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{e}^{\mathrm{Tr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\}}$$

The distribution is denoted as $\boldsymbol{S} \sim W_n(p, \boldsymbol{\Sigma})$

Assumption:

Here for each system matrix, the mean of the inverse of the random matrix is equal to the inverse of the deterministic matrix and the normalized standard deviation is same as the measured normalized standard deviation. Suppose $\boxed{\mathbf{G} = \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}}$ & $\boxed{\mathbf{G} \sim W_n(p, \boldsymbol{\Sigma})}$

The parameters are estimated mathematically, and

$$p = \lfloor n + 1 + \theta \rfloor \text{ (Degrees of Freedom) and } \boldsymbol{\Sigma} = \frac{\bar{\mathbf{G}}}{\theta} \text{ (Scale Matrix)}$$

Where, $\boxed{\theta = \frac{1}{\tilde{\sigma}_G^2}\left\{1 + \frac{\{\mathrm{Trace}(\bar{\mathbf{G}})\}^2}{\mathrm{Trace}(\bar{\mathbf{G}}^2)}\right\} - (n+1)}$ & $\boxed{\widetilde{\sigma_G^2} = \delta_G^2 = \frac{E\left[\|\mathbf{G} - E[\mathbf{G}]\|_F^2\right]}{\|E[\mathbf{G}]\|_F^2} = \frac{\mathrm{Tr}(\mathrm{Cov}(\mathrm{vec}(\mathbf{G})))}{\mathrm{Tr}(\overline{\mathbf{G}^2})}}$

(obtained from experiment, experience, or using the stochastic finite-element method)

Ref: Adhikari, S.: "Wishart Random Matrices in Probabilistic Structural Mechanics," J. Eng. Mech., vol. 134, no. 12, pp. 1029-1044, 2008.

Central Wishart Random Matrix Formulation

Limitation:

- The main limitation: $cov(G_{ij}, G_{kl}) = \frac{1}{\theta}\left(\overline{G_{ik}}\,\overline{G_{jl}} + \overline{G_{il}}\,\overline{G_{jk}}\right)$
- Only one parameter controls the uncertainty - an oversimplification.
- The covariance matrix of $\mathbf{G}$ can have $n(n\ +\ 1) \times \frac{n(n+1)+2}{8}$ number of independent parameters
- To account for parametric uncertainties, we need a matrix variate distribution which not only satisfy the mathematical constrains, but also must offer more parameters to fit the 'known' covariance tensor of $\mathbf{G}$.

Matrix Factorization Approach

- Because $\mathbf{G}$ is a symmetric and positive-definite random matrix, it can be always factorized as

$$\mathbf{G} = \mathbf{X}\mathbf{X}^{\mathrm{T}}$$

where $\mathbf{X} \in \mathbb{R}^{n\times n}$, $p \geq n$ is in general a rectangular matrix.

- Extending the standard maximum entropy argument to the matrix case we can say that the pdf of $\mathbf{X}$ is given by the matrix variate Gaussian distribution, that is,

$$\mathbf{X} \sim N_{n,p}\left(\boldsymbol{\mathcal{M}}, \Sigma \otimes \mathbf{I_p}\right)$$

- This shows that $\mathbf{G}$ has non central Wishart distribution and unified parametric-nonparametric probability density function a random system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ follows the noncentral Wishart distribution, that is

$\boxed{\mathbf{G} \sim W_n(p, \boldsymbol{\Sigma}, \boldsymbol{\Theta})}$ where $p > n$ is a real scalar, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Theta}$ are symmetric positive-definite $n \times n$ real matrices.

- We match the mean and covariance of the distribution of $\mathbf{G}$ with 'measured/known' quantities

$$\mathrm{E}[\mathbf{G}] = p\boldsymbol{\Sigma} + \boldsymbol{\Omega}$$

$$cov\big(vec(\mathbf{G})\big) = (\mathbf{I}_{n^2} + \mathbf{K}_{nn})(p\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma} + \boldsymbol{\Omega}\otimes\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\otimes\boldsymbol{\Omega})$$

- Mean is satisfied exactly while the covariance is satisfied in least-square sense

Non-Central Wishart Distribution

A $n \times n$ symmetric positive definite random matrix $\boldsymbol{S}$ is said to have a non-central Wishart distribution with parameters $p \geq n$, $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Theta} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_S(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\boldsymbol{\Sigma}|^{\frac{1}{2}p}\right\}^{-1} \exp\left(-\frac{1}{2}\mathrm{tr}(\boldsymbol{\Theta})\right)\exp\left(-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\right)|\mathbf{S}|^{\frac{1}{2}(p-n-1)}\ {}_0F_1\left(\frac{1}{2}p, \boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\mathbf{S}/4\right)$$

The distribution is denoted as $\mathbf{S} \sim W_n(p,\ \boldsymbol{\Sigma}, \boldsymbol{\Theta})$

Here, ${}_0F_1$ is the hypergeometric function of matrix argument (also known as the matrix Bessel function), and when $\boldsymbol{\Theta} = 0$,i.e. if the non-centrality parameter is a null matrix, this PDF reduces to the central Wishart distribution.

The function $\Gamma_n(a)$ is the multivariate gamma function, which can be expressed in terms of products of the univariate gamma functions as

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)}\prod_{k=1}^{n}\Gamma\left(a - \frac{1}{2}(k-1)\right), \quad \text{for } \Re(a) > \frac{1}{2}(n-1)$$

this ensures the pdf is properly normalised (i.e. The total integral over all positive-definite matrices S is 1). $\Re$- real part.

Ref: Adhikari, S., “Uncertainty quantification in structural dynamics using non-central Wishart distribution,” *International Journal of Engineering Under Uncertainty: Hazards, Assessment and Mitigation*, Vol. 2, No. 3-4, 2010, pp. 123–139.

Non-Central Wishart Method

1. Obtain the mean matrix (corresponding to the baseline system) $\bar{\mathbf{G}} \in \mathbb{R}_n^+$ from the standard FE analysis. Obtain the covariance matrix $\mathbf{C}_{\mathrm{G}} = \mathrm{Cov}\big(\mathrm{vec}(\mathbf{G})\big) \in \mathbb{R}_{n^2}^+$ corresponding to $\bar{\mathbf{G}}$ (e.g., using the Karhunen–Loève expansion of the parametric random fields).

2. Obtain the normalized standard deviation δ_G of $\mathbf{G}$ from $\delta_G^2 = \frac{E\left[\|\mathbf{G}-E[\mathbf{G}]\|_F^2\right]}{\|E[\mathbf{G}]\|_F^2} = \frac{\mathrm{Tr}(\mathbf{C}_{\mathrm{G}})}{\mathrm{Tr}(\bar{\mathbf{G}}^2)}$.

3. Begin with $\mathbf{\Omega} = 0_{n,n}$ (as in the case of a non-parametric distribution), and obtain the initial value of p from:

$$p = \frac{1}{\delta_G^2} \cdot \frac{\mathrm{Tr}\left(\overline{G^2}\right) + \left(\mathrm{Tr}(\bar{G})\right)^2}{\mathrm{Tr}\left(\overline{G^2}\right)}$$

4. Now, form the matrix

$$\mathcal{A} = \bar{\mathbf{G}} \otimes \bar{\mathbf{G}} - \frac{p}{2}\boldsymbol{C}_{\boldsymbol{G}} \in \mathbb{R}^{n^2 \times n^2}$$

and obtain $\mathbf{\Omega} \in \mathbb{R}^{n\times n}$ by least-square minimisation of Frobenius norm: $\|\mathcal{A} - \mathbf{\Omega} \otimes \mathbf{\Omega}\|_F$

5. Using this value of $\mathbf{\Omega} \in \mathbb{R}^{n\times n}$, calculate the new value of p as:

$$p_{new} = \frac{1}{\delta_G^2} \cdot \frac{\mathrm{Tr}\left(\overline{\mathbf{G}^2} - \mathbf{\Omega}^2\right) + \left(\mathrm{Tr}(\bar{\mathbf{G}})\right)^2 - \left(\mathrm{Tr}(\mathbf{\Omega})\right)^2}{\mathrm{Tr}\left(\overline{\mathbf{G}^2}\right)}$$

Non-Central Wishart Method

6. Calculate $\delta p = |p - p_{\text{new}}|$.

   (a) If $\delta p \leq \epsilon_p$ (where ϵ_p is a small number, say $\epsilon_p = 0.01$) then iteration stops and set $p = p_{\text{new}}$.

   (b) Else, if $\delta p > \epsilon_p$, then the iteration stops. Select $p = p_{\text{new}}$ and go back to step 4.

7. Finally, calculate $\Sigma = (\bar{G} - \Omega)/p, \quad \Theta = \Sigma^{-1}\boldsymbol{\Omega}$.

8. Perform the Cholesky factorizations of the positive definite matrices $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Omega} \in \mathbb{R}_n^+$ as:

   $$\boldsymbol{\Sigma} = \mathbf{D}\mathbf{D}^\top, \quad \mathbf{D} \in \mathbb{R}^{n\times n} \text{ and } \boldsymbol{\Omega} = \widehat{\boldsymbol{\mathcal{M}}}\widehat{\boldsymbol{\mathcal{M}}}^\top, \quad \widehat{\boldsymbol{\mathcal{M}}} \in \mathbb{R}^{n\times n}$$

9. Calculate the square mean matrix $\widetilde{\boldsymbol{\mathcal{M}}} = \boldsymbol{D}^{-1}\widehat{\boldsymbol{\mathcal{M}}}, \quad \widehat{\boldsymbol{\mathcal{M}}} \in \mathbb{R}^{n\times n}$

10. Construct the rectangular mean matrix of size $n \times p$ as: $\boldsymbol{\mathcal{M}} = \left[\widetilde{\boldsymbol{\mathcal{M}}},\ O_{n,(n-p)}\right] \in \mathbb{R}^{n\times p}$

11. Obtain the matrix $\mathbf{Y} \in \mathbb{R}^{n\times p}$ containing uncorrelated standard normal random variables, where the **mean** is $\boldsymbol{\mathcal{M}}$ and the standard deviation is unitary. Each row of $\boldsymbol{Y}$ can be understood as $y_i \sim \mathcal{N}(\boldsymbol{\mathcal{M}}_i, I)$

12. Generate the samples of the system matrix as: $\mathbf{G} = \mathbf{D}\mathbf{Y}\mathbf{Y}^\top\mathbf{D}^\top \in \mathbb{R}_+^{n\times n}$

Computational Challenges for Full NCW Method

- In the full non-central Wishart model, the matrix

$$\mathcal{A} = \bar{\mathbf{G}} \otimes \bar{\mathbf{G}} - \frac{\mathrm{p}}{2}\mathbf{C}_{\mathbf{G}} \qquad \in \mathbb{R}^{\mathrm{n}^2\times\mathrm{n}^2}$$

  and obtain $\mathbf{\Omega} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ by least-square minimisation of Frobenius norm:

$$\|\mathcal{A} - \mathbf{\Omega} \otimes \mathbf{\Omega}\|_{\mathrm{F}} \qquad \in \mathbb{R}^{\mathrm{n}^2\times\mathrm{n}^2}$$

  this step becomes computationally very heavy for large-scale FE models $(\mathrm{n} \approx 10^4)$, because it requires forming $\mathcal{A}$ of size $n^2 \times n^2$.
- Further the, steps for iteration of the parameter p, adds to the calculation of this $n^2 \times n^2$ matrix at every iteration.
- For a high-fidelity aircraft wing finite element model with $n \sim 10^5$ degrees of freedom, the full non-central Wishart formulation requires forming the matrix which would have dimension $10^{10} \times 10^{10}$, far exceeding practical memory and computational limits.
- Moreover, the least-squares identification of the non-centrality matrix $\mathbf{\Omega}$ and iterative updates of the parameter p require repeated construction of this matrix at each iteration.

Proposed Methodology

Reduced Non-Central Wishart Method

Let $\mathbf{G}(\omega) \in \mathbb{S}_+^n$ denote a symmetric positive (semi-)definite system matrix (e.g., stiffness $\mathbf{K}$ or mass $\mathbf{M}$) assembled on the **free DOF space** of dimension n after enforcing essential BCs. Let $\Phi_r \in \mathbb{R}^{n\times r}$ be the deterministic modal basis with $r \ll n$, extracted from the baseline FE model (e.g., first r eigenmodes), and assumed full column-rank.

Reduction operator (Galerkin projection):

$$\overline{\boldsymbol{G}_r} = \Phi_r^{\top}\boldsymbol{G}\Phi_r \in \mathbb{S}_+^r$$

- If $\mathbf{G} \in \mathbb{S}_{++}^n$ and Φ_r has full column rank, then $\mathbf{G}_\mathrm{r} \in \mathbb{S}_{++}^r$.
- In practice, $\mathbf{G}_\mathrm{r}$ is symmetrised numerically: $\mathbf{G}_\mathrm{r} \leftarrow \frac{1}{2}(\mathbf{G}_\mathrm{r} + \mathbf{G}_\mathrm{r}{}^\mathrm{T})$

Nature of the reduced matrix:

- $\mathbf{G}_\mathrm{r}$ is the Rayleigh–Ritz / Galerkin reduced operator of $\mathbf{G}$ on the subspace v_r (the set of all linear combinations of the columns of Φ_r , $r \ll n$)
- $\Phi_r \in \mathbb{R}^{n\times r}$ is constructed from the deterministic baseline FE model, consists of the first r structural modes which are obtained from the generalised eigenvalue problem. $\mathbf{K}\phi_i = \lambda_i\mathbf{M}\phi_i$
- The eigenvectors are mass-orthonormal: $\phi_r\mathbf{M}\phi_r = \mathbf{I}_r$
- Consequently, $\phi_r\mathbf{K}\phi_r = \boldsymbol{\Lambda}_r$, where $\boldsymbol{\Lambda}_r = diag(\lambda_1, \dots, \lambda_r)$

Proposed Methodology

Implicit reshuffled operator for rank-1 Kronecker identification

For a given p, define the reduced Kronecker operator implicitly as

$$\boldsymbol{\mathcal{A}}_r(p) = \overline{\mathbf{G}_r} \otimes \overline{\mathbf{G}_r} - \frac{p}{2}\mathbf{C}_{G_r}$$

where only matrix-vector products with $\boldsymbol{\mathcal{A}}_r(p)$ are evaluated. No $r^2 \times r^2$ matrix is explicitly formed.

Instead, we define an operator

$$\mathbf{B}(p) \in \mathbb{E}^{r^2 \times r^2}$$

implicitly through its action on $\mathbf{u} = \text{vec}(\mathbf{U})$, $\mathbf{U} \in \mathbb{R}^{r \times r}$:

$$\mathbf{B}(p)\,\text{vec}(\mathbf{U}) = \text{vec}(\overline{\mathbf{G}_r})\big(\text{vec}(\overline{\mathbf{G}_r})^\top vec(\mathbf{U})\big) - \frac{p}{2}\,\text{vec}(\mathbb{E}[\Delta\mathbf{G}_r\,\mathbf{U}\,\Delta\mathbf{G}_r^\top])$$

with the sample approximation

$$\mathbb{E}[\Delta\mathbf{G}_r\,\mathbf{U}\,\Delta\mathbf{G}_r^\top] = \frac{1}{N_s - 1}\sum_{s=1}^{N_s} \Delta\mathbf{G}_r^{(s)}\;\mathbf{U}\,\Delta\mathbf{G}_r^{(s)\top}.$$

Proposed Methodology

Reduced Non-Central Wishart Method

1. In our reduced-order framework, mean matrix (corresponding to the baseline system) $\overline{\mathbf{G}} \in \mathbb{R}_n^+$ from the standard FE analysis is obtained in the modal subspace (using $r \ll n$), with $\overline{\boldsymbol{G}_r} = \Phi_r^\top \boldsymbol{G} \Phi_r$ and a vech-compressed reduced covariance $C_{G_r}^{(\text{vech})}$ computed from random-field Monte Carlo i.e,

$$\overline{\mathbf{G}_\mathrm{r}} \in \mathbb{R}_\mathrm{r}^+ \qquad \& \qquad C_{G_r}^{(\mathrm{vech})} \in S_+^q, \quad q = \frac{r(r+1)}{2}$$

2. Define the reduced normalised dispersion

$$\delta_{G_r}^2 \frac{\mathrm{tr}\left(\boldsymbol{C}_{\boldsymbol{G}_r}^{(\mathbf{vc})}\right)}{\mathrm{tr}\left(\overline{G_r^2}\right).}$$

3. Initialise the degree-of-freedom parameter as

$$p^{(0)} = \max\left(2r,\ \min\left(p_{\mathrm{CW}}, \left\lceil \frac{c}{\delta_{G_r}^2} \right\rceil\right)\right), \qquad c \in [1,2],$$

where p_{CW} denotes the reduced central Wishart degree of freedom and the lower bound $p \geq 2r$ ensures admissibility of the Wishart distribution.

The lower bound $\mathrm{p} = 2r$ corresponds to the minimal degree for which the reduced Wishart model remains well-defined and numerically stable. When the observed variability is predominantly structured rather than diffuse, the identification procedure naturally drives p towards this bound.

Proposed Methodology

Reduced Non-Central Wishart Method

4. For a given p, define the reduced Kronecker operator implicitly as

$$\boldsymbol{\mathcal{A}}_r(p) = \overline{\mathbf{G}_r} \otimes \overline{\mathbf{G}_r} - \frac{p}{2}\mathbf{C}_{G_r}$$

where only matrix-vector products with $\boldsymbol{\mathcal{A}}_r(p)$ are evaluated. No $r^2 \times r^2$ matrix is explicitly formed.

5. The non-centrality matrix Ω_r is identified by a rank-one Kronecker approximation

$$\boldsymbol{\Omega}_r = \arg\min_{\Omega_r \geq 0} \quad ||\boldsymbol{\mathcal{A}}_r(p) - \boldsymbol{\Omega}_\mathbf{r} \otimes \boldsymbol{\Omega}_\mathbf{r}||_F,$$

which is solved via dominant eigenpair extraction of the reshaped operator followed by symmetric positive semidefinite projection.

6. The parameter p is updated using a reduced consistency condition

$$p^{(k+1)} = \frac{1}{\delta^2_{G_r}}\left(1 - \frac{\mathrm{tr}(\boldsymbol{\Omega}_r^2)}{\mathrm{tr}(\overline{\boldsymbol{G}_r^2})}\right) \quad, p^{(k+1)} \geq 2r,$$

and iteration continues until convergence of both $\boldsymbol{\Omega}_r$ and p.

Reduced Non-Central Wishart Method

7. Using the reduced NCW mean relation $\overline{G_r} = p\,\Sigma_r + \Omega_r$,

Compute $\Sigma_r = \frac{\overline{G_r} - \Omega_r}{p}$

8. Perform the Cholesky factorizations of the positive definite matrices $\boldsymbol{\Sigma} \in \mathbb{R}_{\mathrm{r}}^{+}$ and $\boldsymbol{\Omega} \in \mathbb{R}_{\mathrm{r}}^{+}$ as:

$$\boldsymbol{\Sigma}_{\mathrm{r}} = \mathbf{D}_{\mathbf{r}}\mathbf{D}_{\mathrm{r}}^{\mathrm{T}}, \quad \mathbf{D}_{\mathbf{r}} \in \mathbb{R}^{\mathrm{r}\times\mathrm{r}} \text{ and } \boldsymbol{\Omega}_{\mathbf{r}} = \widehat{\boldsymbol{\mathcal{M}}}\widehat{\boldsymbol{\mathcal{M}}}^{\top}, \quad \widehat{\boldsymbol{\mathcal{M}}} \in \mathbb{R}^{\mathrm{r}\times\mathrm{r}}$$

9. Calculate the square mean matrix $\widetilde{\boldsymbol{\mathcal{M}}} = \boldsymbol{D}^{-1}\widehat{\boldsymbol{\mathcal{M}}}, \quad \widehat{\boldsymbol{\mathcal{M}}} \in \mathbb{R}^{\mathrm{r}\times\mathrm{r}}$ and construct the rectangular mean matrix of size $n \times p$ as: $\boldsymbol{\mathcal{M}} = \left[\widetilde{\boldsymbol{\mathcal{M}}},\ O_{\mathrm{r},(\mathrm{r}-p)}\right] \in \mathbb{R}^{\mathrm{r}\times p}$

10. Obtain the matrix $\mathbf{Y} \in \mathbb{R}^{\mathrm{r}\times p}$ containing uncorrelated standard normal random variables, where the **mean** is $\boldsymbol{\mathcal{M}}$ and the standard deviation is unitary. Each row of $\boldsymbol{Y}$ can be understood as $y_{\mathrm{r},i} \sim \mathcal{N}(\boldsymbol{\mathcal{M}}_i, I)$

11. Generate the samples of the system matrix as: $\mathbf{G}_{\mathbf{r}} = \mathbf{D}_{\mathbf{r}}\mathbf{Y}\mathbf{Y}^{\top}\mathbf{D}_{\mathrm{r}}^{\mathrm{T}} \in \mathbb{R}_{+}^{\mathrm{r}\times\mathrm{r}}$

12. The resulting samples are now $\mathrm{G}_{\mathrm{r}}^{(\mathrm{s})}$ are lifted back to physical DOFs via $\mathbf{G}^{(\mathrm{s})} = \boldsymbol{\Phi}_{\mathbf{r}}\mathbf{G}_{\mathbf{r}}^{(\mathbf{s})}\boldsymbol{\Phi}_{\mathbf{r}}^{\top}$ before FRF computation.

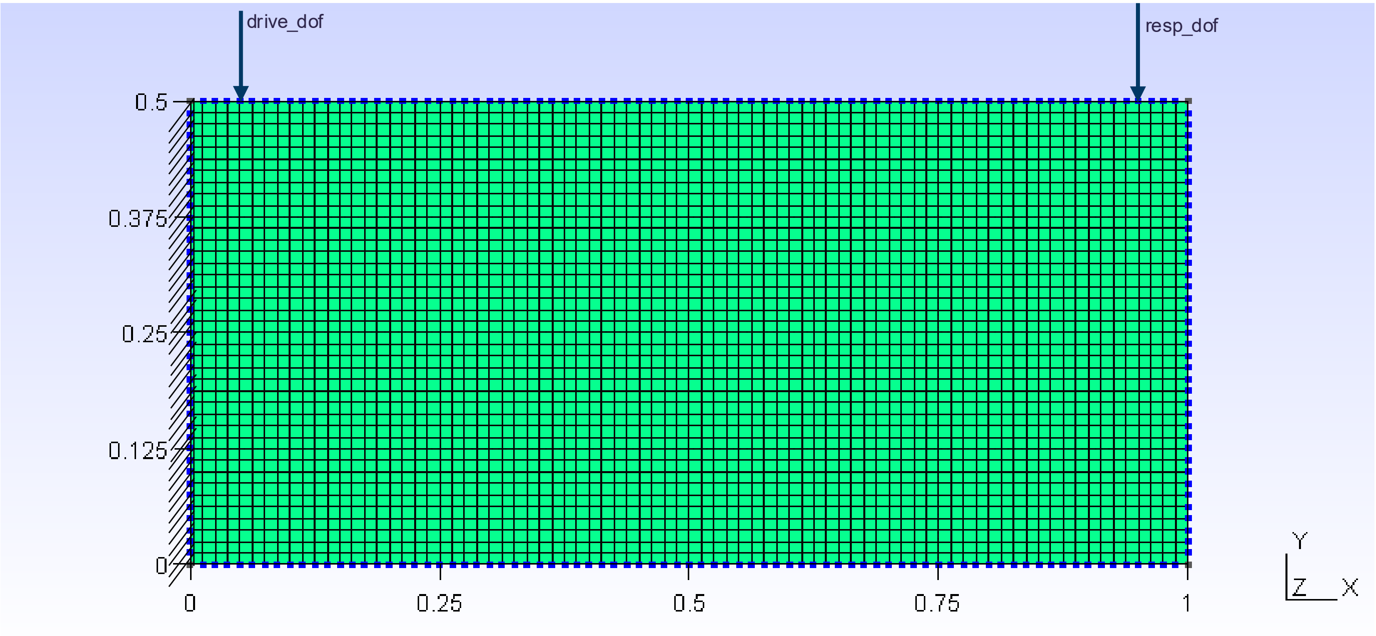
Proposed Methodology

Comparison of Full non-central Wishart identification vs reduced NCW formulation

Aspect	Full NCW	Reduced NCW
State space	Full matrix $\boldsymbol{G} \in \mathbb{R}^{n\times n}$	Reduced operator $\boldsymbol{G_r} \in \mathbb{R}^{r\times r}$ via $\boldsymbol{\Phi_r}$
Target statistics	$\overline{\boldsymbol{G}}$, $\boldsymbol{C_G}$ on $vec(\boldsymbol{G})$	$\overline{\boldsymbol{G}_r}$, $\boldsymbol{C}_{\boldsymbol{G_r}}^{(\boldsymbol{vech})}$ on reduced space
Covariance model	Kronecker form with $\boldsymbol{\Sigma}$, $\boldsymbol{\Omega}$ in Cov(vec(G)) $$\left\lVert \overline{\mathbf{G}} \otimes \overline{\mathbf{G}} - \frac{p}{2}\boldsymbol{C_G} - \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \right\rVert_F \in \mathbb{R}^{n^2\times n^2}$$	Direct covariance on $vech(\boldsymbol{G_r})$; no explicit Kronecker inversion
Scale matrix $\boldsymbol{\Sigma}$	Fitted from covariance via Least Squares (LS), often dense	Computed as $\Sigma_r = (\overline{G_r} - \Omega_r)/p$, PSD-projected
Non-central part $\boldsymbol{\Omega}$	Fitted jointly with $\boldsymbol{\Sigma}$ to match covariance	$\boldsymbol{\Omega_r}$ from Implicit reshuffled operator for rank-1 Kronecker identification
Degrees of freedom $\boldsymbol{p}$	Part of multi-parameter LS optimisation	Data-driven from reduced CW and RF-KLE, with iterative NCW refinement
Computational cost	$\mathcal{O}(n^4)$ storage/ops for explicit covariance	$\mathcal{O}(r^4)$ with $r \ll n$, based on Monte Carlo reduced statistics

Numerical Implementation

- thin rectangular plate
- dimensions $1.0\text{m} \times 0.5\text{m}$
- clamped-free boundary conditions, with one of its shorter edges (left) fixed ; cantilever configuration.
- discretised using quadrilateral elements based on the Kirchhoff–Love plate theory, implemented in the **FEniCSx** [1-2]



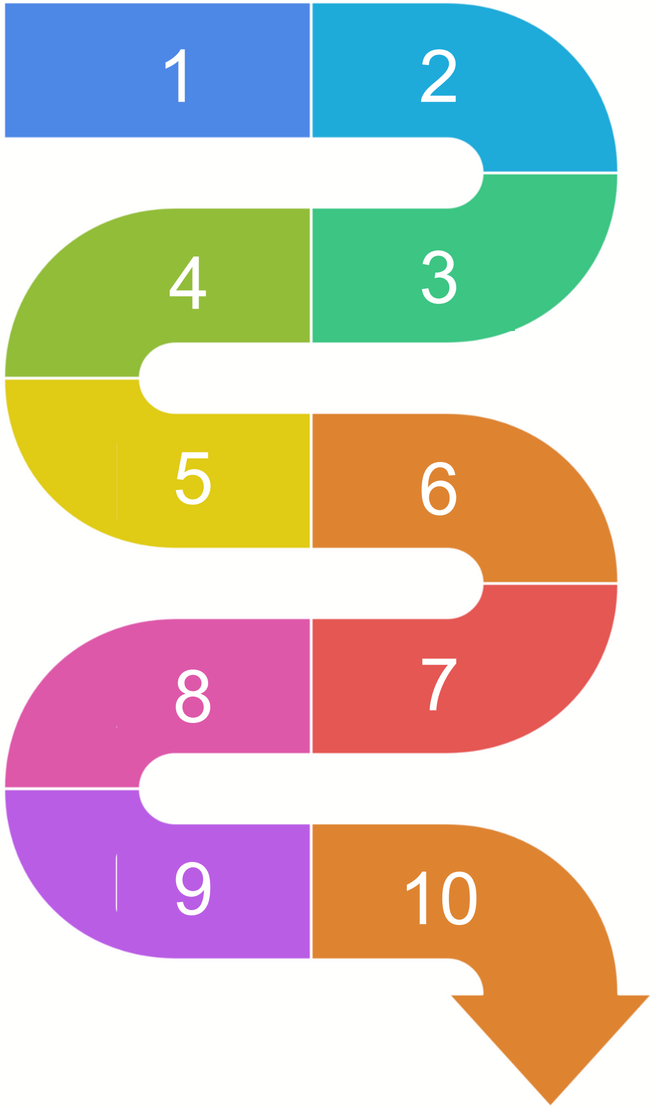
Material Properties

- Young's modulus: $(E = 68, \text{GPa})$
- Poisson's ratio: $(\nu = 0.22)$
- Density: $\left(\rho = 2500\ \text{kg/m}^3\right)$
- Thickness: $(t = 3.0\ \text{mm})$

1. Scroggs, M. W., Dokken, J. S., Richardson, C. N., and Wells, G. N., "Construction of arbitrary order finite element degree-of-freedom maps on polygonal and polyhedral cell meshes," ACM Transactions on Mathematical Software, Vol. 48, No. 2, 2022,pp. 18:1–18:23.
2. Scroggs, M. W., Baratta, I. A., Richardson, C. N., and Wells, G. N., "Basix: a runtime finite element basis evaluation library,"Journal of Open Source Software, Vol. 7, No. 73, 2022, pp. 3982.

Numerical Implementation

Hybrid Parametric–Non-Parametric UQ Approach



Define Random Field Inputs

Identify uncertain fields: E(x), v(x), ρ(x), t(x)

Generate KLE Representations

Perform Karhunen–Loève expansion for each random field

Run Parametric Monte Carlo Simulations

Sample random fields and perform NASTRAN-based FEM analysis

Extract System Matrices

Use PyNastran to retrieve Mi, Ki, Ci for each realization

Compute Ensemble Statistics

Evaluate sample mean G¯ and covariance C_G

Estimate Dispersion Parameter

Calculate δG^2 from empirical matrix statistics

Determine Effective Degrees of Freedom

Compute parameter pp for Wishart distribution

Compute Structured Mean Contribution

Estimate non-centrality: Ω=G¯−pΣ

Evaluate Non-Centrality Matrix

Compute Θ=Σ^(−1)Ω

Run Hybrid Monte Carlo Simulations

Follow with visualization

Algorithm 1: Reduced Non-Central Wishart (NCW) workflow for stochastic structural dynamics

Data : KLE-driven Monte Carlo samples of material fields $\{E^{(i)}(\mathbf{x}), \nu^{(i)}(\mathbf{x}), \rho^{(i)}(\mathbf{x}), t^{(i)}(\mathbf{x})\}_{i=1}^{N_{\mathrm{KLE}}}$; deterministic reduced basis $\mathbf{\Phi}_r \in \mathbb{R}^{n_f \times r}$ (mass-normalised deterministic modes); free-DOF operators $\mathbf{K}^{\mathrm{det}}, \mathbf{M}^{\mathrm{det}}$; Rayleigh damping (α, β); drive/response DOFs; frequency grid $\{\omega_\ell\}_{\ell=1}^{N_\omega}$.

Result : NCW parameters $(p, \mathbf{\Sigma}_r, \mathbf{\Omega}_r)$; NCW Monte Carlo FRF envelopes.

Stage NCW-1: Reduced operator statistics from KLE Monte Carlo
for $i = 1, \ldots, N_{\mathrm{KLE}}$ **do**
- Assemble sample operators $\mathbf{K}_i, \mathbf{M}_i$ from $(E^{(i)}, \nu^{(i)}, \rho^{(i)}, t^{(i)})$
- $\mathbf{K}_{r,i} = \mathbf{\Phi}_r^\top \mathbf{K}_i \mathbf{\Phi}_r, \quad \mathbf{M}_{r,i} = \mathbf{\Phi}_r^\top \mathbf{M}_i \mathbf{\Phi}_r$
- $\mathbf{C}_K^{(\mathrm{vech})} = \mathrm{cov}(\{\mathrm{vech}(\mathbf{K}_{r,i})\}_i), \mathbf{C}_M^{(\mathrm{vech})} = \mathrm{cov}(\{\mathrm{vech}(\mathbf{M}_{r,i})\}_i)$

Stage NCW-2: Reduced NCW identification (independently for $G \in \{K, M\}$)
foreach $G \in \{K, M\}$ **do**
- Compute $\mathrm{tr}(\mathbf{C}_G^{(\mathrm{vech})}) = \sum_j \mathrm{var}(\mathrm{vech}(\mathbf{G}_{r,i})_j)$ and $\mathrm{tr}(\bar{\mathbf{G}}_r^2) = \mathrm{tr}(\bar{\mathbf{G}}_r \bar{\mathbf{G}}_r)$
- $\delta_G^2 \leftarrow \mathrm{tr}(\mathbf{C}_G^{(\mathrm{vech})}) / \max(\mathrm{tr}(\bar{\mathbf{G}}_r^2), \varepsilon)$
- Initialise $p \leftarrow \max(2r, \lceil c/\max(\delta_G^2, \varepsilon) \rceil)$ and cap $p \le p_{\mathrm{cw}}$
- Set $\mathbf{\Omega}^{(0)} \leftarrow \mathbf{0}$
- **for** $k = 0, \ldots, k_{\max}$ **do**
  - $p_{\mathrm{new}} \leftarrow \min\left(p_{\mathrm{cw}}, \max\left(2r, \left\lceil \frac{1}{\max(\delta_G^2, \varepsilon)} \frac{\mathrm{tr}(\bar{\mathbf{G}}_r^2) - \mathrm{tr}(\mathbf{\Omega}^2)}{\max(\mathrm{tr}(\bar{\mathbf{G}}_r^2), \varepsilon)} \right\rceil\right)\right)$
- $\mathbf{\Sigma}_{G,r} \leftarrow \Pi_{\mathbb{S}_+}((\bar{\mathbf{G}}_r - \mathbf{\Omega})/p)$
- Store $(p_G, \mathbf{\Sigma}_{G,r}, \mathbf{\Omega}_{G,r}) \leftarrow (p, \mathbf{\Sigma}_{G,r}, \mathbf{\Omega})$

Stage NCW-3: Reduced NCW sampling and FRF propagation
$\mathbf{\Sigma}_{G,r} = \mathbf{D}\mathbf{D}^\top, \quad \mathbf{\Omega}_{G,r} = \widehat{\mathbf{M}}\widehat{\mathbf{M}}^\top$
Form $\widetilde{\mathbf{M}} = \mathbf{\Sigma}_{G,r}^{-1/2}\widehat{\mathbf{M}}$ and the rectangular mean $\mathbf{M} = [\widetilde{\mathbf{M}} \ \ \mathbf{0}] \in \mathbb{R}^{r \times p_G}$
for $s = 1, \ldots, N_{\mathrm{NCW}}$ **do**
- Draw $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ in $\mathbb{R}^{r \times p_G}$ and set $\mathbf{Y} = \mathbf{M} + \mathbf{Z}$
- Draw reduced operator sample:
  - $\mathbf{G}_r^{(s)} = \mathbf{D}\mathbf{Y}\mathbf{Y}^\top\mathbf{D}^\top$

Stage NCW-4: Build random eigenpairs and compute FRFs
for $s = 1, \ldots, N_{\mathrm{NCW}}$ **do**
- Assemble reduced dynamics using $(\mathbf{K}_r^{(s)}, \mathbf{M}_r^{(s)})$
- Solve reduced generalised EVP: $\mathbf{K}_r^{(s)}\mathbf{w}_j = \lambda_j^{(s)}\mathbf{M}_r^{(s)}\mathbf{w}_j, \omega_j^{(s)} = \sqrt{\max(\lambda_j^{(s)}, 0)}$
- Lift eigenvectors to free DOFs: $\boldsymbol{\phi}_j^{(s)} = \mathbf{\Phi}_r \mathbf{w}_j$
- Compute modal-sum FRFs at $\{\omega_\ell\}$ with Rayleigh damping (α, β) to obtain $H_d^{(s)}(\omega_\ell)$ and $H_c^{(s)}(\omega_\ell)$

Post-process to envelopes (e.g., 5–95 percentile) and optionally apply peak-alignment/warping.

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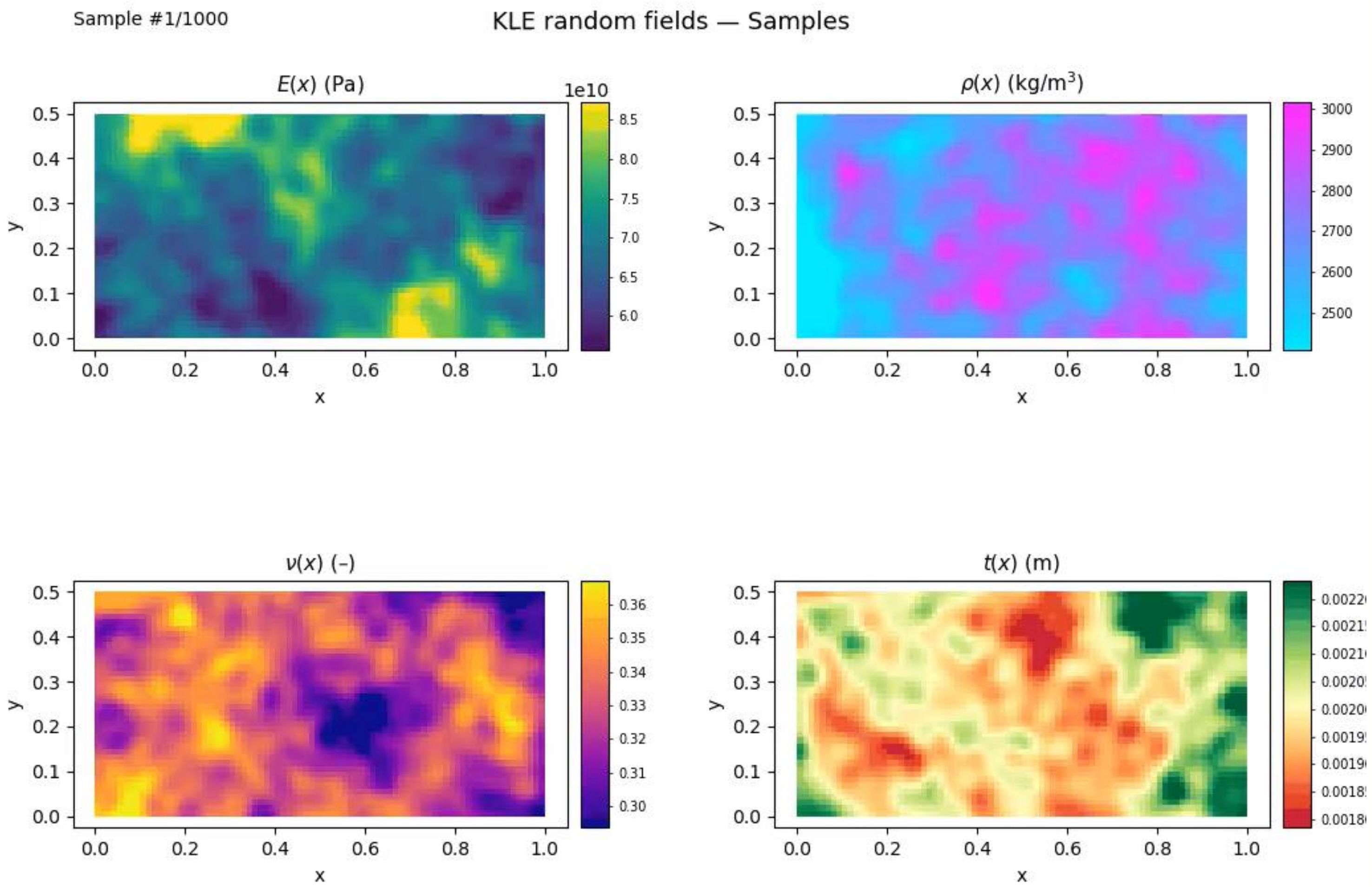


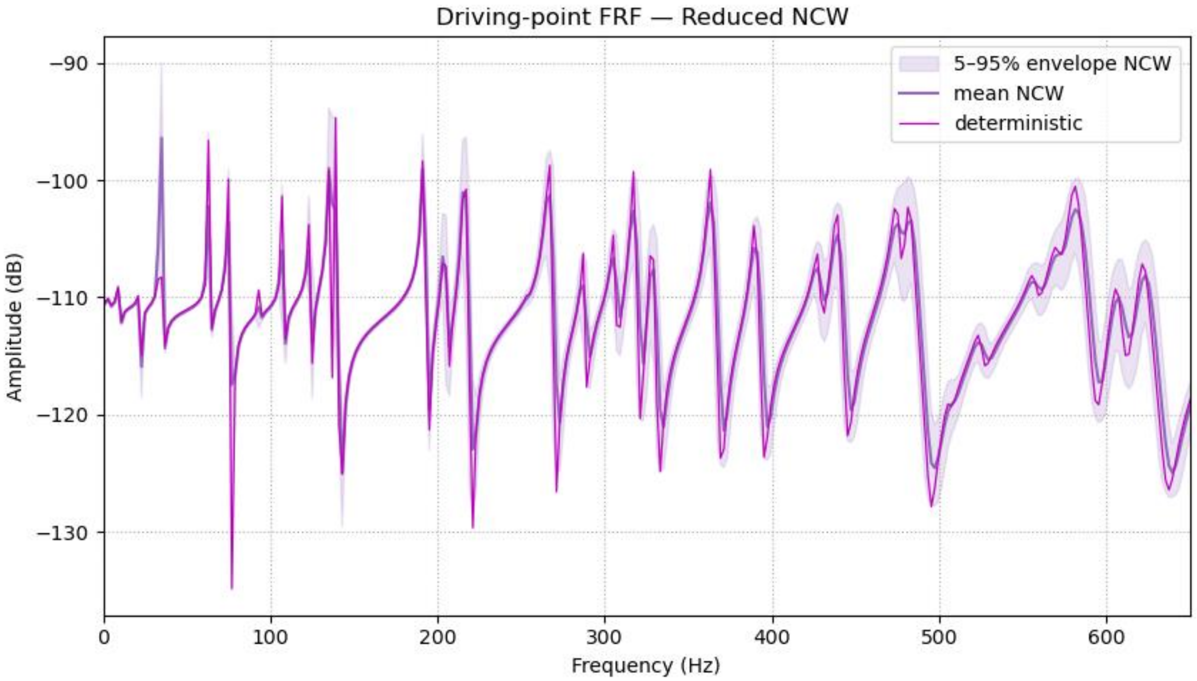
Figure: spatially correlated random fields used in the parametric Monte Carlo simulations: Young's modulus $E(x)$, density $\rho(x)$, Poisson's ratio $\nu(x)$, and thickness $t(x)$. All fields are generated from the same set of KLE coefficients to preserve cross-field correlation.

Each of these fields is represented using a truncated Karhunen- Loève (KL) expansion:

$$P(x, \omega) = \bar{P}(x) + \sum_{j=1}^{M} \sqrt{\lambda_j}\, \phi_j(x) \xi_j(\omega),$$

where $\bar{P}(x)$ is the mean field, λ_j and $\phi_j(x)$ are the eigenvalues and eigenfunctions of the covariance kernel, and $\xi_j \sim \mathcal{N}(0,1)$ are independent Gaussian variables.

Results



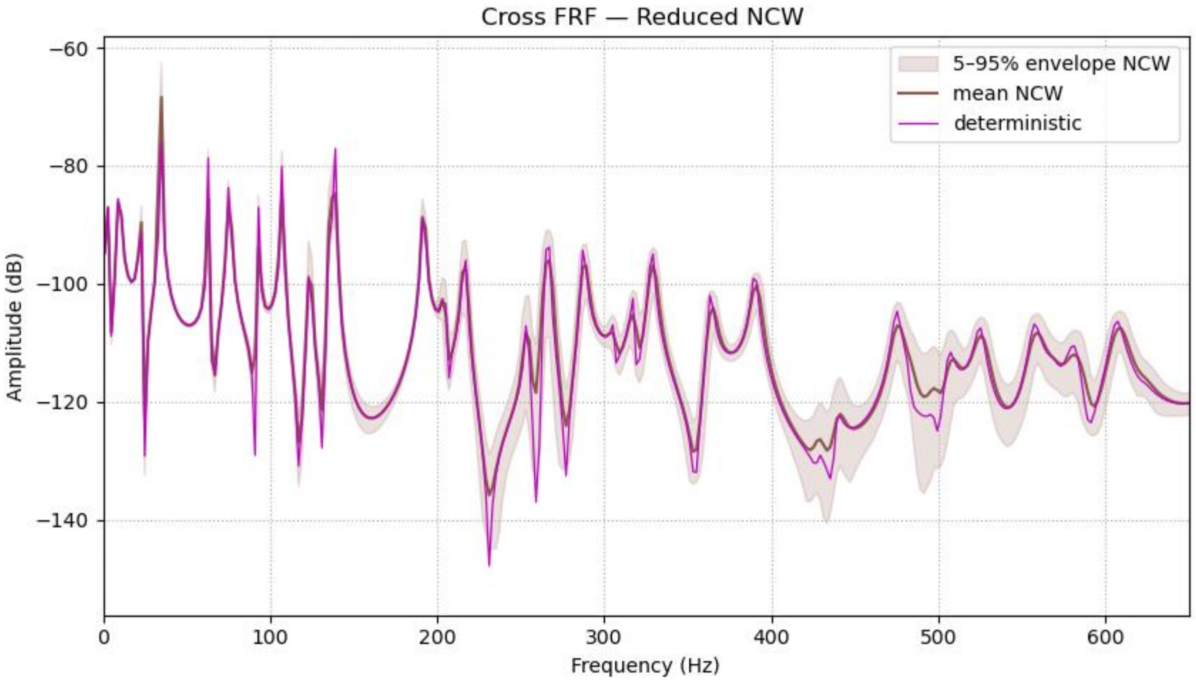


Figure : Comparison of deterministic and reduced NCW mean FRFs over 0–650 Hz for the driving-point (left) and cross (right) responses.

Results

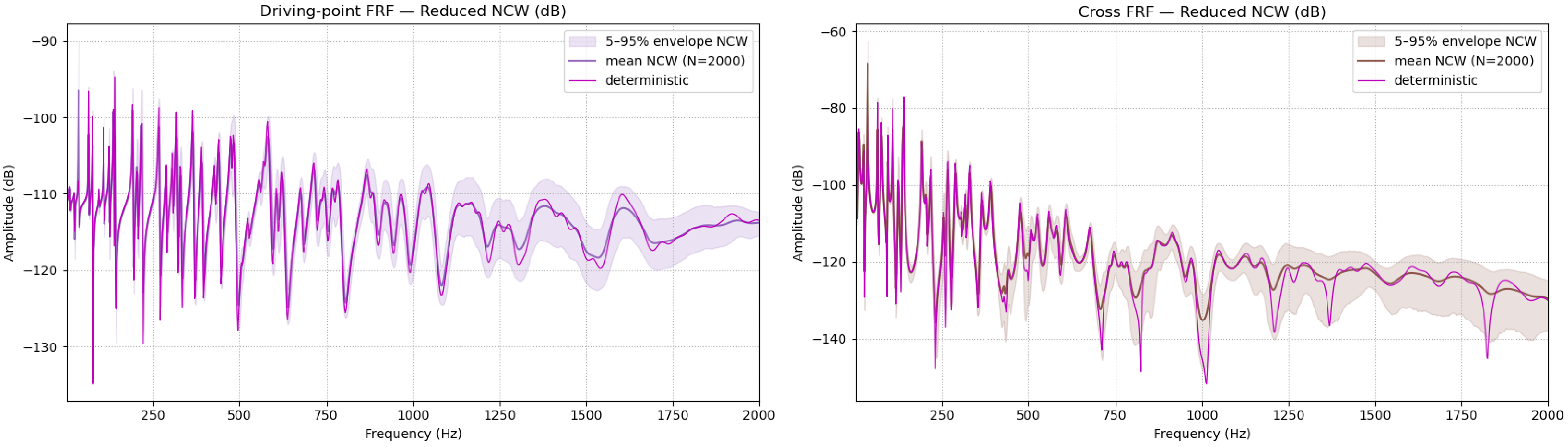


Figure : Comparison of deterministic and reduced NCW mean FRFs over 0–2000 Hz for the driving-point (left) and cross (right) responses.

Results

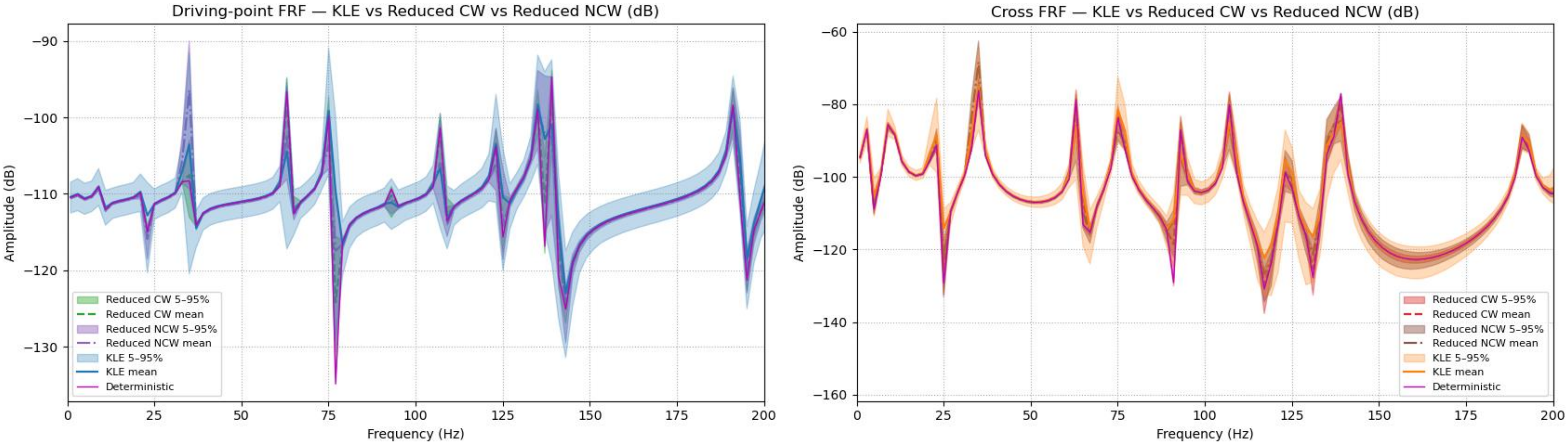


Figure : Comparison of deterministic, KLE, CW, and reduced NCW mean FRFs over 0–200 Hz for the driving point (left) and cross (right) responses.

Results

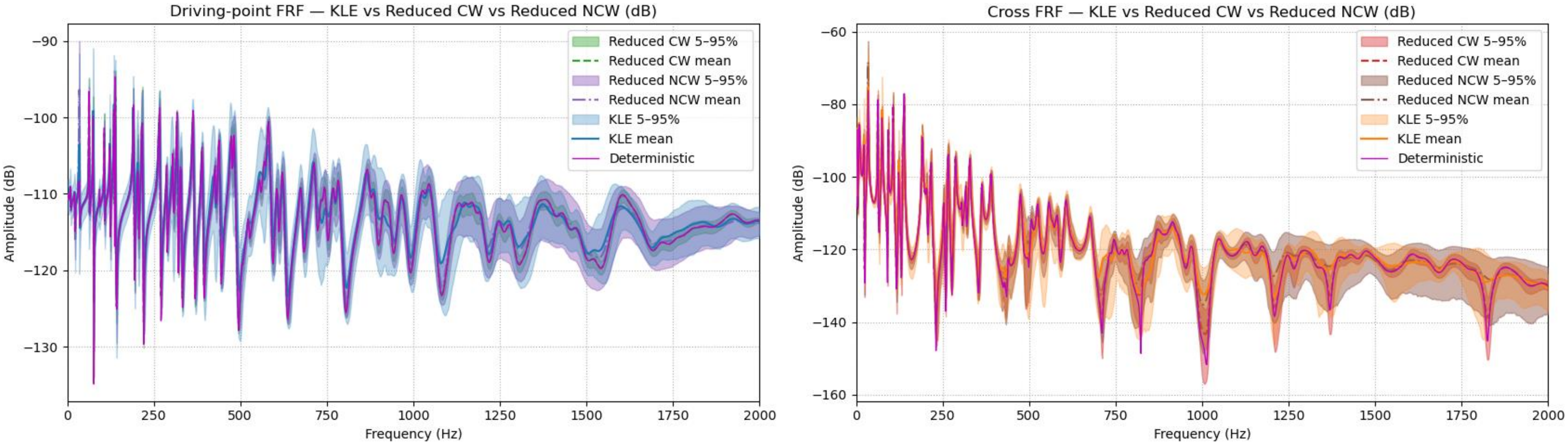


Figure : Comparison of deterministic, KLE, CW, and reduced NCW mean FRFs over 0–2000 Hz for the driving point (left) and cross (right) responses.

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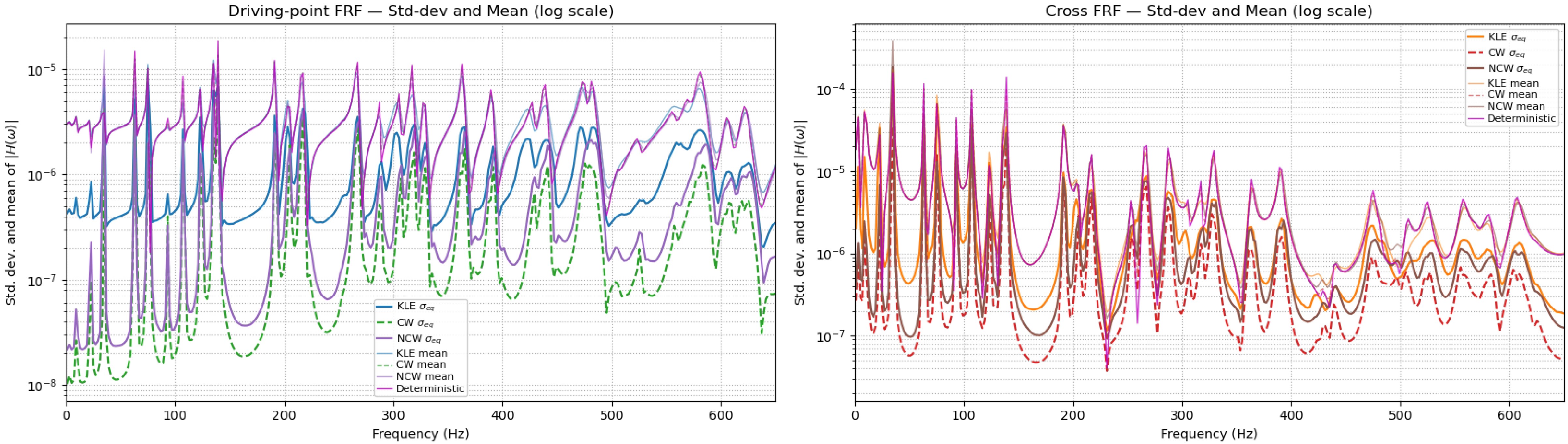
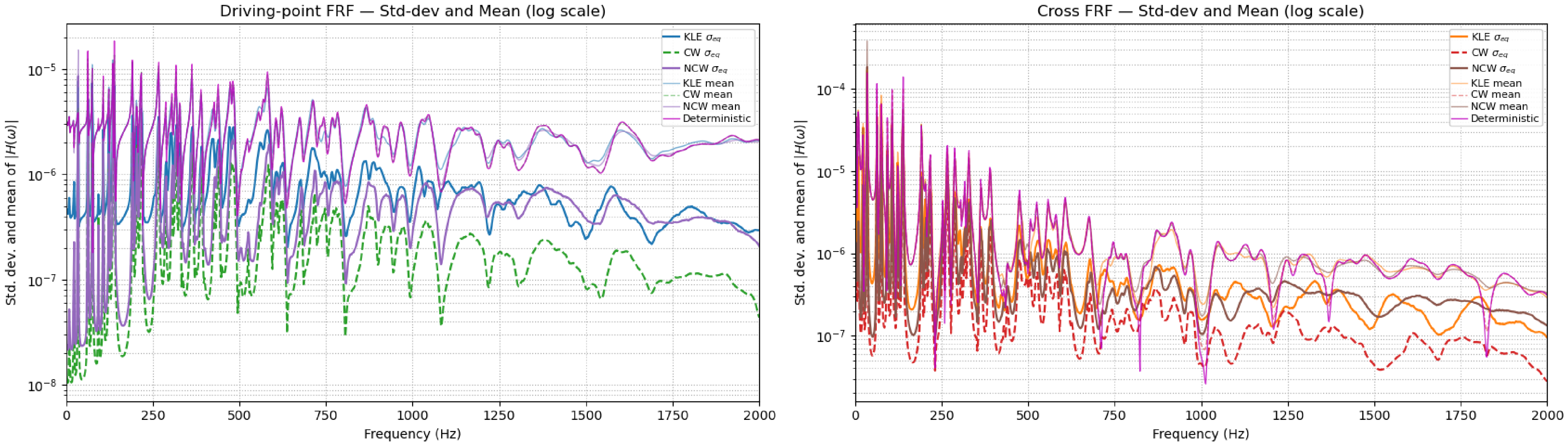


Figure : Standard deviation of the FRF amplitude over 0–650 Hz for deterministic baseline, KLE, CW, and reduced NCW ensembles, shown for the driving-point (left) and cross (right) responses.

Results



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Conclusion

- A **data-informed non-central Wishart framework** has been developed for uncertainty quantification in structural dynamics.
- The proposed reduced NCW formulation enables **identification and sampling in a reduced modal subspace** while preserving the statistical structure of the full NCW model.
- This **hybrid framework** provides a systematic link between parametric uncertainty simulations and non-parametric random matrix modelling.
- Reduced-space identification yields **substantial computational savings** relative to full NCW approaches.
- The approach is suitable for **easy scaling** on large-scale finite element models and frequency-domain response analysis.
- The reduced NCW ensemble captures FRF variability more accurately than both KL-expansion based Monte Carlo and central Wishart models, especially at high modal densities.

References

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