Structural Dynamic Analysis with Generalised Damping Models: Identification

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Contents

Preface

Among the various ingredients of structural dynamics, damping remains one of the least understood topics. The main reason is that unlike the stiffness and inertia forces, the damping forces cannot be always obtained from 'first principles'. The past two decades have seen significant developments in the modelling and analysis of damping in the context of engineering dynamic systems. Developments in composite materials including nanocomposites and their applications in advanced structures, such as new generation of aircrafts and large wind-turbines, have lead to the need for understanding damping in a superior manner. Additionally, the rise of vibration energy harvesting technology using piezoelectric and electromagnetic principles further enhanced the importance of looking at damping more rigorously. The aim of this book is to systematically present the latest developments in the area modelling and analysis of damping in the context of general linear dynamic systems with multiple degrees of freedom. The focus has been on the mathematical and computational aspects. This book will be relevant to aerospace, mechanical and civil engineering disciplines and various sub-disciplines within them. The intended readers of this book include senior undergraduate students and graduate students doing projects or doctoral research in the filed of damped vibration. Researchers, Professors and and practicing engineers working in the field of advanced vibration will find this book useful. This book will also be useful for researchers working in the fields of aeroelasticity and hydroelasticity, where complex eigenvalue problems routinely arise due to fluid-structure interactions.

There are some excellent books which already exist in the filed of damped vibration. The book by Nashif *et al.* [NAS 85] covers various material damping models and their applications in the design and analysis of dynamic systems. A valuable reference on dynamic analysis of damped structures is the book by Sun and Lu [SUN 95]. The book by Beards [BEA 96] takes a pedagogical approach towards structural vibration of damped systems. The handbook by Jones [JON 01] focuses on viscoelastic damping and analysis of structures with such damping models. These books represent the sate-of-the art at the time of their publications. Since these publications significant research works have gone into the dynamics of damped systems. The aim of this book is to cover some of these latest developments. The attention is mainly limited to theoretical and computational aspects, although some reference to experimental works are given.

One of the key feature of this book is the consideration of general nonviscous damping and how such general models can be seamlessly integrated into the framework of conventional structural dynamic analysis. New results are illustrated by numerical examples and wherever possible connections were made to well-known concepts of viscously damped systems. The book is divided into two volumes. The first volume deals with analysis of linear systems with general damping models. The second volume deals with identification and quantification of damping. There are ten chapters and one appendix in the book - covering analysis and identification of dynamic systems with viscous and nonviscous damping. Chapter 1 gives an introduction to the various damping models. Dynamics of viscously damped systems are discussed in chapter 2. Chapter 3 considers dynamics of nonviscously damped single-degree-of-freedom systems in details. Chapter 4 discusses nonviscously damped multiple-degree-of-freedom systems. Linear systems with general nonviscous damping are studied in Chapter 5. Chapter 6 proposes reduced computational methods for damped systems. Chapter 7 describes parametric sensitivity of damped systems. Chapter 8 takes up the problem of identification of viscous damping. The identification of nonviscous damping is detailed in Chapter 9. Chapter 10 gives some tools for the quantification of damping. A method to deal with general asymmetric systems is described in the appendix.

This book is a result of last 15 years of research and teaching in the area of damped vibration problems. Initial chapters started taking shape when I offered a course on advanced vibration at the University of Bristol. The later chapters originated from research works with numerous colleagues, students, collaborators and mentors. I am deeply indebted to all of them for numerous stimulating scientific discussions, exchanges of ideas and in many occasions direct contributions towards the intellectual content of the book. I am grateful to my teachers Professor C. S. Manohar (Indian Institute of Science, Bangalore), Professor R. S. Langley (University of Cambridge) and in particular Professor J. Woodhouse (University of Cambridge), who was heavily involved with the works reported in chapters 8-10. I am very thankful to my colleague Professor M. I. Friswell with whom I have a long-standing collaboration. Some joint works are directly related to the content of this book (chapter 7 in particular). I would also like to thank Professor D. J. Inman (University of Michigan) for various scientific discussions during his visits to Swansea. I am thankful to Professor A. Sarkar (Carleton University) and his doctoral student M. Khalil for joint research works. I am deeply grateful to Dr A. S. Phani (University of British Columbia) for various discussions related to damping identification and contributions towards chapters 2, 5 and 8. Particular thanks to Dr N. Wagner (Intes GmbH, Stuttgart) for joint works on nonviscously damped systems and contributions in chapter 4. I am also grateful to Professor F. Papai for involving me on research works on damping identification. My former PhD students B Pascual (contributed in chapter 6), J. L. du Bois, F. A. Diaz De la O deserves particular thanks for various contributions throughout their time with me ad putting up with my busy schedules. I am grateful to Dr Y. Lei (University of Defence Technology, Changsha) for carrying out joint research with me on nonviscously damped continuous systems. I am grateful to Professor A. W. Lees (Swansea University), Professor N. Lieven, Professor F. Scarpa (University of Bristol), Professor D. J. Wagg (University of Sheffield), Professor S. Narayanan (IIT Madras), Professor G. Litak (Lublin University), E. Jacquelin (Université Lyon), Dr A. Palmeri (Loughborough University), Professor S. Bhattacharya (University of Surrey), Dr S. F. Ali (IIT Madras), Dr R. Chowdhury (IIT Roorkee), Dr P. Duffour (University College London) and Dr P. Higino, Dr G. Caprio & Dr A. Prado (Embraer Aircraft) for their intellectual contributions and discussions at different times. Beside the names taken here, I am thankful to many colleagues, fellow researchers and students working in this field of research around the world, whose name cannot be listed here for page limitations. The lack of explicit mentions by no means implies that their contributions are any less. The opinions presented in the book are entirely mine, and none of my colleagues, students, collaborators and mentors has any responsibility for any shortcomings.

I have been fortunate to receive grants from various companies, charities and government organisations including an Advanced Research Fellowship from UK Engineering and Physical Sciences Research Council (EPSRC), the Wolfson research merit award from The Royal Society and the Philip Leverhulme Prize from The Leverhulme Trust. Without these finding it would be impossible to conduct the works leading to this book. Finally I want thank my colleagues at the College of Engineering in Swansea University. Their support proved to be a key factor in materialising the idea of writing this book.

Last, but by no means least, I wish to thank my wife Sonia and my parents for their constant support, encouragements and putting up with my ever so increasing long periods of ‘non-engagement’ with them.

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Nomenclature

C'_{jj}	diagonal element of the modal damping matrix
$\alpha_k^{(j)}$	terms in the expansion of approximate complex modes
α_1, α_2	proportional damping constants
α_j	coefficients in Caughey series, $j = 0, 1, 2 \cdots$
$\mathbf{0}_j$	a vector of j zeros
$\mathbf{A}$	state-space system matrix
$\mathbf{a}_j$	a coefficient vector for the expansion of j-th complex mode
$\boldsymbol{\alpha}$	a vector containing the constants in Caughey series
$\bar{h}(\mathrm{i}\omega)$	frequency response function of a SDOF system
$\mathbf{B}$	state-space system matrix
$\mathbf{b}_j$	a vector for the expansion of j-th complex mode
$\bar{\mathbf{f}}(s)$	forcing vector in the Laplace domain
$\bar{\mathbf{f}}'(s)$	modal forcing function in the Laplace domain
$\bar{\mathbf{p}}(s)$	effective forcing vector in the Laplace domain
$\bar{\mathbf{q}}(s)$	response vector in the Laplace domain
$\bar{\mathbf{u}}(s)$	Laplace transform of the state-vector of the first-order system
$\bar{\mathbf{y}}(s)$	modal coordinates in the Laplace domain
$\bar{\mathbf{y}}_k$	Laplace transform of the internal variable $\mathbf{y}_k(t)$
$\mathbb{R}^+$	positive real line
$\mathbf{C}$	viscous damping matrix
$\mathbf{C}'$	modal damping matrix
$\mathbf{C}_0$	viscous damping matrix (with a nonviscous model)
$\mathbf{C}_k$	coefficient matrices in the exponential model for $k = 0, ..., n$, where n is the number of kernels
$\boldsymbol{\mathcal{G}}(t)$	nonviscous damping function matrix in the time domain
$\boldsymbol{\Delta}\mathbf{K}$	error in the stiffness matrix
$\boldsymbol{\Delta}\mathbf{M}$	error in the mass matrix
β	nonviscous damping factor
β_c	critical value of β for oscillatory motion, $\beta_c = \frac{1}{3\sqrt{3}}$
$\beta_i(\bullet)$	proportional damping functions (of a matrix)
$\beta_k(s)$	Coefficients in the state-space modal expansion
β_{mU}	the value of β above which the frequency response function always has a maximum
$\mathbf{F}$	linear matrix pencil with time step in state-space, $\mathbf{F} = \mathbf{B} - \frac{h}{2}\mathbf{A}$
$\mathbf{F}_1, \mathbf{F}_2$	linear matrix pencils with time step in the configuration space
$\mathbf{F}_j$	Regular linear matrix pencil for the j-th mode
$\mathbf{f}'(t)$	forcing function in the modal coordinates
$\mathbf{f}(t)$	forcing function

$\mathbf{G}(s)$	nonviscous damping function matrix in the Laplace domain
$\mathbf{G}_0$	the matrix $\mathbf{G}(s)$ at $s \to 0$
$\mathbf{G}_\infty$	the matrix $\mathbf{G}(s)$ at $s \to \infty$
$\mathbf{H}(s)$	frequency response function matrix
$\hat{\mathbf{u}}_j$	Real part of $\hat{\mathbf{z}}_j$
$\hat{\mathbf{v}}_j$	Imaginary part of $\hat{\mathbf{z}}_j$
$\hat{\mathbf{z}}_j$	j-th measured complex mode
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{O}_{ij}$	a null matrix of dimension $i \times j$
$\mathbf{\Omega}$	diagonal matrix containing the natural frequencies
$\mathbf{p}$	parameter vector (in chapter 1)
$\mathbf{P}_j$	a diagonal matrix for the expansion of j-th complex mode
$\boldsymbol{\phi}_j$	eigenvectors in the state-space
$\boldsymbol{\psi}_j$	left eigenvectors in the state-space
$\mathbf{q}(t)$	displacement response in the time-domain
$\mathbf{q}_0$	vector of initial displacements
$\mathbf{Q}_j$	an off-diagonal matrix for the expansion of j-th complex mode
$\mathbf{r}(t)$	forcing function in the state-space
$\mathbf{R}_k$	rectangular transformation matrices (in **??**)
$\mathbf{R}_k$	residue matrix associated with pole s_k
$\mathbf{S}$	a diagonal matrix containing eigenvalues s_j
$\mathbf{T}$	a temporary matrix, $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$ (chapter 2)
$\mathbf{T}_k$	Moore-Penrose generalised inverse of $\mathbf{R}_k$
$\mathbf{T}_k$	a transformation matrix for the optimal normalisation of the k-th complex mode
$\mathbf{\Theta}$	Normalisation matrix
$\mathbf{u}(t)$	the state-vector of the first-order system
$\mathbf{u}_0$	vector of initial conditions in the state-space
$\mathbf{u}_j$	displacement at the time step j
$\mathbf{v}(t)$	velocity vector $\mathbf{v}(t) = \dot{\mathbf{q}}(t)$
$\mathbf{v}_j$	a vector of the j-modal derivative in Nelson's methods (in chapter 1)
$\mathbf{v}_j$	velocity at the time step j
$\boldsymbol{\varepsilon}_j$	Error vector associated with j-th complex mode
$\boldsymbol{\varphi}_k(s)$	eigenvectors of the dynamic stiffness matrix
$\mathbf{W}$	coefficient matrix associated with the constants in Caughey series
$\mathbf{X}$	matrix containing the undamped normal modes $\mathbf{x}_j$
$\mathbf{x}_j$	undamped eigenvectors, $j = 1, 2, \cdots, N$
$\mathbf{y}(t)$	modal coordinate vector (in **??**)
$\mathbf{y}_k(t)$	vector of internal variables, $k = 1, 2, \cdots n$
$\mathbf{y}_{k,j}$	internal variable $\mathbf{y}_k$ at the time step j
$\mathbf{Z}$	matrix containing the complex eigenvectors $\mathbf{z}_j$
$\mathbf{z}_j$	complex eigenvectors in the configuration space
$\boldsymbol{\zeta}$	diagonal matrix containing the modal damping factors
$\boldsymbol{\zeta}_v$	a vector containing the modal damping factors
χ	merit function of a complex mode for optimal normalisation
χ_R, χ_I	merit functions for real and imaginary parts of a complex mode
Δ	perturbation in the real eigenvalues
δ	perturbation in complex conjugate eigenvalues

$\dot{q}_0$	initial velocity (SDOF systems)
ϵ	small error
η	ratio between the real and imaginary parts of a complex mode
$\mathcal{F}$	dissipation function
γ	Non-dimensional characteristic time constant
γ_j	complex mode normalisation constant
γ_R, γ_I	weights for the normalisation of the real and imaginary parts of a complex mode
$\hat{\theta}(\omega)$	Frequency dependent estimated characteristic time constant
$\hat{\theta}_j$	Estimated characteristic time constant for j-th mode
$\hat{t}$	an arbitrary independent time variable
κ_j	real part of the complex optimal normalisation constant for the j-th mode
λ	complex eigenvalue corresponding to the oscillating mode (in **??**)
λ_j	complex frequencies MDOF systems
$\mathcal{M}_r$	moment of the damping function
$\mathcal{D}$	dissipation energy
$\mathcal{G}(t)$	nonviscous damping kernel function in a SDOF system
$\mathcal{T}$	kinetic energy
$\mathcal{U}$	potential energy
μ	relaxation parameter
μ_k	relaxation parameters associated with coefficient matrix $\mathbf{C}_k$ in the exponential nonviscous damping model
ν	real eigenvalue corresponding to the overdamped mode
$\nu_k(s)$	eigenvalues of the dynamic stiffness matrix
ω	driving frequency
ω_d	damped natural frequency of SDOF systems
ω_j	undamped natural frequencies of MDOF systems, $j = 1, 2, \cdots, N$
ω_n	undamped natural frequency of SDOF systems
$\omega_{\max}$	frequency corresponding to the maximum amplitude of the response function
ω_{d_j}	damped natural frequency of MDOF systems
ρ	mass density
i	unit imaginary number, $\mathrm{i} = \sqrt{-1}$
τ	dummy time variable
θ_j	characteristic time constant for j-th nonviscous model
$\tilde{\mathbf{f}}(t)$	forcing function in the modal domain
$\tilde{\omega}$	normalised frequency ω/ω_n
ς_j	imaginary part of the complex optimal normalisation constant for the j-th mode
ϑ	phase angle of the response of SDOF systems
ϑ_j	phase angle of the modal response
ψ	a trail complex eigenvector (in **??**)
$\widehat{\mathbf{A}}$	asymmetric state-space system matrix
$\widehat{\mathbf{C}}$	fitted damping matrix
$\widehat{f}(\omega_j)$	fitted generalised proportional damping function (in chapter 2)
$\widetilde{\mathbf{A}}$	state-space system matrix for rank deficient systems
$\widetilde{\mathbf{B}}$	state-space system matrix for rank deficient systems
$\widetilde{\mathbf{i}}_r$	integration of the forcing function in the state-space for rank deficient systems
$\widetilde{\mathbf{i}}_r$	integration of the forcing function in the state-space
$\widetilde{\mathbf{\Phi}}$	matrix containing the state-space eigenvectors for rank deficient systems
$\widetilde{\boldsymbol{\phi}}_j$	eigenvectors in the state-space for rank deficient systems
$\widetilde{\mathbf{r}}(t)$	forcing function in the state-space for rank deficient systems

$\widetilde{\mathbf{u}}(t)$	the state vector for rank deficient systems
$\widetilde{\mathbf{y}}_k(t)$	vector of internal variables for rank deficient systems, $k = 1, 2, \cdots n$
$\widetilde{\mathbf{y}}_{k,j}$	internal variable $\mathbf{y}_k$ at the time step j for rank deficient systems
$\widetilde{\boldsymbol{y}}_{k_j}$	jth eigenvector corresponding to the k-th the internal variable for rank deficient systems
ξ	a function of ζ defined in equation [**??**]
ζ	viscous damping factor
ζ_c	critical value of ζ for oscillatory motion, $\zeta_c = \frac{4}{3\sqrt{3}}$
ζ_j	modal damping factors
ζ_L	lower critical damping factor
ζ_n	equivalent viscous damping factor
ζ_U	upper critical damping factor
ζ_{mL}	the value of ζ below which the frequency response function always has a maximum
a_k, b_k	nonviscous damping parameters in the exponential model
B	response amplitude of SDOF systems
B_j	modal response amplitude
c	viscous damping constant of a SDOF system
c_k	coefficients of exponential damping in a SDOF system
c_{cr}	critical damping factor
d_j	a constant of the j-modal derivative in Nelson's methods
E	Young's modulus
$f(t)$	forcing function (SDOF systems)
$f_d(t)$	nonviscous damping force
$G(\mathrm{i}\omega)$	non-dimensional frequency response function
$G(s)$	nonviscous damping kernel function in the Laplace domain (SDOF systems)
$g_{(i)}$	scalar damping functions, $i = 1, 2, \cdots$
h	constant time step
$h(t)$	impulse response function of SDOF systems
$h(t)$	impulse response function
I_k	non-proportionally indices, $k1 = 1, 2, 3, 4$
k	spring stiffness of a SDOF system
L	length of the rod
l_e	length of an element
m	dimension of the state-space for nonviscously damped MDOF systems
m	mass of a SDOF system
N	number of degrees of freedom
n	number of exponential kernels
n_d	number of divisions in the time axis
p	any element in the parameter vector **p** (in chapter 1)
$q(t)$	displacement in the time domain
q_0	initial displacement (SDOF systems)
Q_{nc_k}	non-conservative forces
$R(\mathbf{x})$	Rayleigh quotient for a trail vector **x**
R_1, R_2, R_3	three new Rayleigh quotients
r_j	normalised eigenvalues of nonviscously damped SDOF systems (in **??**)
r_k	rank of $\mathbf{C}_k$ matrices
s	Laplace domain parameter
s_j	eigenvalues of dynamic systems
t	time
T_n	natural time period of an undamped SDOF system

T_{min}	Minimum time period for the system
$varrho_j$	complex optimal normalisation constant for the j-th mode
x	normalised frequency-squared, $x = \omega^2/\omega_n^2$ (in **??**)
y_j	modal coordinates (in **??**)
$\bar{f}(s)$	forcing function in the Laplace domain
$\bar{q}(s)$	displacement in the Laplace domain
$\hat{\mathbf{U}}$	Matrix containing $\hat{\mathbf{u}}_j$
$\hat{\mathbf{V}}$	Matrix containing $\hat{\mathbf{v}}_j$
$\mathbf{\Phi}$	matrix containing the eigenvectors $\boldsymbol{\phi}_j$
$\dot{\mathbf{q}}_0$	vector of initial velocities
$\mathcal{F}_i(\bullet,\bullet)$	nonviscous proportional damping functions (of a matrix)
$\boldsymbol{Y}_k$	a matrix of internal eigenvectors
$\boldsymbol{y}_{kj}$	jth eigenvector corresponding to the k-th the internal variable
PSD	Power spectral density
$\mathbf{0}$	a vector of zeros
$\mathcal{L}$	Lagrangian (in **??**)
$\delta(t)$	Dirac-delta function
δ_{jk}	Kroneker-delta function
$\Gamma(\bullet)$	Gamma function
γ	Lagrange multiplier (in **??**)
$(\bullet)^*$	complex conjugate of $(\bullet)$
$(\bullet)^T$	matrix transpose
$(\bullet)^{-1}$	matrix inverse
$(\bullet)^{-T}$	matrix inverse transpose
$(\bullet)^H$	Hermitian transpose of $(\bullet)$
$(\bullet)_e$	elastic modes
$(\bullet)_{nv}$	nonviscous modes
$(\dot{\bullet})$	derivative with respect to time
$\mathbb{C}$	space of complex numbers
$\mathbb{R}$	space of real numbers
$\perp$	orthogonal to
$\mathcal{L}(\bullet)$	Laplace transform operator
$\mathcal{L}^{-1}(\bullet)$	inverse Laplace transform operator
$\det(\bullet)$	determinant of $(\bullet)$
$\mathrm{diag}[\bullet]$	a diagonal matrix
$\forall$	for all
$\Im(\bullet)$	imaginary part of $(\bullet)$
$\in$	belongs to
$\notin$	does not belong to
$\otimes$	Kronecker product
$\overline{(\bullet)}$	Laplace transform of $(\bullet)$
$\Re(\bullet)$	real part of $(\bullet)$
vec	vector operation of a matrix
$O(\bullet)$	in the order of
ADF	Anelastic Displacement Field model
$\mathrm{adj}(\bullet)$	adjoint matrix of $(\bullet)$
GHM	Golla, McTavish and Hughes model
MDOF	multiple-degree-of-freedom
SDOF	single-degree-of-freedom

Chapter 1

Parametric Sensitivity of Damped Systems

Changes of the eigenvalues and eigenvectors of a linear vibrating system due to changes in system parameters are of wide practical interest. Motivation for this kind of study arises, on one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualise the changes in the dynamic response with respect to system parameters. Besides, this kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods. Sensitivity of eigenvalues and eigenvectors are useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities due to this slight alteration. Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems. Possibly, the earliest work on the sensitivity of the eigenvalues was carried out by Rayleigh [RAY 77]. In his classic monograph he derived the changes in natural frequencies due to small changes in system parameters. Fox and Kapoor [FOX 68] have given exact expressions for the sensitivity of eigenvalues and eigenvectors with respect to any design variables. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure in its current state, and have been used extensively in a wide range of application areas of structural dynamics. Nelson [NEL 76] proposed an efficient method to calculate eigenvector derivative which requires only the eigenvalue and eigenvector under consideration. A comprehensive review of research on this kind of sensitivity analysis can be obtained in Adelman and Haftka [ADE 86]. A brief review of some of the existing methods for calculating sensitivity of the eigenvalues and eigenvectors are given in **??**.

The purpose of this chapter is to consider parametric sensitivity of the eigensolutions of damped systems. We first start with undamped systems in section 1.1. Parametric sensitivity of viscously damped systems is discussed in section 1.2. In section 1.3 we discuss the sensitivity of eigensolutions of general nonviscously damped systems. In section 1.4 a summary of the techniques introduced in this chapter is provided.

1.1. Parametric sensitivity of undamped systems

The eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{p})\mathbf{x}_j = \lambda_j \mathbf{M}(\mathbf{p})\mathbf{x}_j \qquad [1.1]$$

Here λ_j and $\mathbf{x}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{p}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$ and $\mathbf{K}(\mathbf{p}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$, the mass and stiffness matrices, are assumed to be smooth, continuous and differentiable functions of a parameter vector $\mathbf{p} \in \mathbb{R}^m$. Note that $\lambda_j = \omega_j^2$ where ω_j is the j-th undamped natural frequency. The vector $\mathbf{p}$ may consist of material properties, e.g., mass density, Poisson's ratio, Young's modulus; geometric properties, e.g., length, thickness, and boundary conditions. The eigenvalues and eigenvectors are smooth differentiable functions of the parameter vector $\mathbf{p}$.

1.1.1. *Sensitivity of the eigenvalues*

We rewrite the eigenvalue equation as

$$[\mathbf{K} - \lambda_j \mathbf{M}]\, \mathbf{x}_j = \mathbf{0} \qquad [1.2]$$

$$\text{or} \quad \mathbf{x}_j^T \, [\mathbf{K} - \lambda_j \mathbf{M}] \qquad [1.3]$$

The functional dependence of $\mathbf{p}$ is removed for notational convenience. Differentiating the eigenvalue equation [1.2] with respect to the element p of the parameter vector we have

$$\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{x}_j + [\mathbf{K} - \lambda_j \mathbf{M}] \, \frac{\partial \mathbf{x}_j}{\partial p} = \mathbf{0} \qquad [1.4]$$

Premultiplying by $\mathbf{x}_j^T$ we have

$$\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{x}_j + \mathbf{x}_j^T \, [\mathbf{K} - \lambda_j \mathbf{M}] \, \frac{\partial \mathbf{x}_j}{\partial p} = \mathbf{0} \qquad [1.5]$$

Using the identity in [1.3] we have

$$\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{x}_j = \mathbf{0} \qquad [1.6]$$

$$\text{or} \quad \frac{\partial \lambda_j}{\partial p} = \frac{\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{x}_j}{\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j} \qquad [1.7]$$

Note that when the modes are mass normalised $\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j = 1$. Equation [1.7] shows that the derivative of a given eigenvalue depends only eigensolutions corresponding to that particular eigensolutions only. Next we show that this fact is not true when we consider the derivative of the eigenvectors.

1.1.2. *Sensitivity of the eigenvectors*

Different methods have been developed to calculate the derivatives of the eigenvectors. One way to express the derivative of an eigenvector is by a linear combination of all the eigenvectors

$$\frac{\partial \mathbf{x}_j}{\partial p} = \sum_{r=1}^{N} \alpha_{jr} \mathbf{x}_r \qquad [1.8]$$

This can be always done as $\mathbf{x}_r, r = 1, 2, \cdots, N$ forms a complete basis. It is necessary to find expressions for the constant α_{jr} for all $r = 1, 2, \cdots N$. Substituting this in equation [1.4] we have

$$\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{x}_j + \sum_{r=1}^{N} [\mathbf{K} - \lambda_j \mathbf{M}] \, \alpha_{jr} \mathbf{x}_r = \mathbf{0} \qquad [1.9]$$

Premultiplying by $\mathbf{x}_k^T$ we have

$$\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{x}_j + \sum_{r=1}^{N} \mathbf{x}_k^T \, [\mathbf{K} - \lambda_j \mathbf{M}] \, \alpha_{jr} \mathbf{x}_r = 0 \qquad [1.10]$$

We consider $r = k$ and the orthogonality of the eigenvectors

$$\mathbf{x}_k^T \mathbf{K} \mathbf{x}_r = \lambda_k \delta_{kr} \quad \text{and} \quad \mathbf{x}_k^T \mathbf{M} \mathbf{x}_r = \delta_{kr} \qquad [1.11]$$

Using these we have

$$\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + (\lambda_k - \lambda_j)\, \alpha_{jik} = 0 \qquad [1.12]$$

From this we obtain

$$\alpha_{jik} = -\frac{\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\lambda_k - \lambda_j}, \quad \forall\, k \neq j \qquad [1.13]$$

To obtain the j-th term α_{jj} we differentiate the mass orthogonality relationship in [1.11] as

$$\frac{\partial (\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j)}{\partial p} = 0 \quad \text{or} \quad \frac{\partial \mathbf{x}_j^T}{\partial p} \mathbf{M} \mathbf{x}_j + \mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j + \mathbf{x}_j^T \mathbf{M} \frac{\partial \mathbf{x}_j}{\partial p} = 0 \qquad [1.14]$$

Considering the symmetry of the mass matrix and using the expansion of the eigenvector derivative we have

$$\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j + 2\mathbf{x}_j^T \mathbf{M} \frac{\partial \mathbf{x}_j}{\partial p} = 0 \quad \text{or} \quad \sum_{r=1}^{N} 2\mathbf{x}_j^T \mathbf{M} \alpha_{jr} \mathbf{x}_r = -\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j \qquad [1.15]$$

Utilising the othonormality of the mode shapes we have

$$\alpha_{jj} = -\frac{1}{2} \mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j \qquad [1.16]$$

The complete eigenvector derivative is therefore given by

$$\frac{\partial \mathbf{x}_j}{\partial p} = -\frac{1}{2} \left(\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j \right) \mathbf{x}_j + \sum_{k=1 \neq j}^{N} \frac{\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\lambda_j - \lambda_k} \mathbf{x}_k \qquad [1.17]$$

From equation [1.17] it can be observed that when two eigenvalues are close, the modal sensitivity will be higher as the denominator of the right hand term will be very small. Unlike the derivative of the eigenvalues given in [1.7], the derivative of an eigenvector requires all the other eigensolutions. This can be computationally demanding for large systems. The method proposed by Nelson [NEL 76] can address this problem. We will discuss Nelson's method in the context of damped system in the next sections.

1.2. Parametric sensitivity of viscously damped systems

The analytical method in the preceding section is for undamped systems. For damped systems, unless the system is proportionally damped (see **??**), the mode shapes of the system will not coincide with the undamped mode shapes. In the presence of general non-proportional viscous damping, the equation of motion in the modal coordinates will be coupled through the off-diagonal terms of the modal damping matrix, and the mode shapes and natural frequencies of the structure will in general be complex. The solution procedures for such non-proportionally damped systems follow mainly two routes: the state space method and approximate methods in the configuration space, as discussed in **??** and **??**. The state-space method, see [NEW 89, GÉR 97] for example, although exact in nature requires significant numerical effort for obtaining the eigensolutions as the size of the problem doubles. Moreover, this method also lacks some of the intuitive simplicity of traditional modal analysis. For these reasons there has been considerable research effort to analyse non-proportionally damped structures in the configuration space. Most of these methods either seek an optimal decoupling of the equation of motion or simply neglect the off-diagonal terms of the modal damping matrix. It may be noted that following such methodologies the mode shapes of the structure will still be real. The accuracy of these methods, other than the light damping assumption, depends upon various factors, for example, frequency separation between the modes and driving frequency [PAR 92a, GAW 97] and the references therein for discussions on these topics. A convenient way to avoid the problems which arise due to the use of real normal modes is to incorporate complex modes in the analysis. Apart from the mathematical consistency, conducting experimental modal analysis also one often identifies complex modes: as Sestieri and Ibrahim [SES 94] have put it ' ... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality!' But surprisingly in most of the current application areas of structural dynamics which utilise the eigensolution derivatives, e.g. modal updating, damage detection, design optimisation and stochastic finite element methods, do not use complex modes in the analysis but rely on the real undamped modes only. This is partly because of the problem of considering appropriate damping model in the structure and partly because of the unavailability of complex eigensolution sensitivities. Although, there has been considerable research efforts towards damping models, sensitivity of complex eigenvalues and eigenvectors with respect to system parameters appear to have received less attention.

In this section we determine the sensitivity of complex natural frequencies and mode shapes with respect to some set of design variables in non-proportionally damped discrete linear systems. It is assumed that the system does not posses repeated eigenvalues. In **??**, mathematical background on linear multiple-degree-of-freedom discrete systems needed for further derivations was already discussed. Sensitivity of complex eigenvalues is derived in subsection 1.2.1 in terms of complex modes, natural frequencies and changes in the system property matrices. The approach taken here avoids the use of state-space formulation. In subsection 1.2.2, sensitivity of complex eigenvectors is derived. The derivation method uses state-space representation of equation of motion for intermediate calculations and then relates the eigenvector sensitivities to the complex eigenvectors of the second-order system and to the changes in the system property matrices. In subsubsection 1.2.2.3, a two-degree-of-freedom system which shows the 'curve-veering' phenomenon has been considered to illustrate the application of the expression for rates of changes of complex eigenvalues and eigenvectors. The results are carefully analysed and compared with presently available sensitivity expressions of undamped real modes.

1.2.1. *Sensitivity of the eigenvalues*

The equation of motion for free vibration of a linear damped discrete system with N degrees of freedom can be written as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \tag{1.18}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are mass, damping and stiffness matrices, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of the generalised coordinates and $t \in \mathbb{R}^+$ denotes time. The eigenvalue problem associated with equation [1.18] is given by

$$s_j^2\mathbf{M}\mathbf{z}_j + s_j\mathbf{C}\mathbf{z}_j + \mathbf{K}\mathbf{z}_j = \mathbf{0}, \quad \forall\, j = 1, 2, \cdots 2N. \tag{1.19}$$

Here $\mathbf{z}_j$ are the mode shapes and the natural frequencies s_j are defined by $s_j = \mathrm{i}\lambda_j$. Unless system [1.18] is proportionally damped, i.e. $\mathbf{C}$ is simultaneously diagonalisable with $\mathbf{M}$ and $\mathbf{K}$ (conditions were derived by Caughey and O'Kelly [CAU 65], in general s_j and $\mathbf{z}_j$ will be complex in nature. The calculation of complex modes and natural frequencies are discussed in details in **??** and **??**.

Complex modes and frequencies can be exactly obtained by the state space (first-order) formalisms. Transforming equation [1.18] into state space form we obtain

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) \tag{1.20}$$

where $\mathbf{A} \in \mathbb{R}^{2N\times 2N}$ is the system matrix and $\mathbf{u}(t) \in \mathbb{R}^{2N}$ response vector in the state-space given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \qquad \mathbf{u}(t) = \left\{ \begin{array}{c} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{array} \right\}. \tag{1.21}$$

In the above equation $\mathbf{O} \in \mathbb{R}^{N \times N}$ is the null matrix and $\mathbf{I} \in \mathbb{R}^{N \times N}$ is the identity matrix. The eigenvalue problem associated with the above equation is now in term of an asymmetric matrix and can be expressed as

$$\mathbf{A}\boldsymbol{\phi}_j = s_j \boldsymbol{\phi}_j, \quad \forall j = 1, \cdots, 2N \qquad [1.22]$$

where s_j is the $j-$th eigenvalue and $\boldsymbol{\phi}_j \in \mathbb{C}^{2N}$ is the $j-$th *right* eigenvector which is related to the eigenvector of the second-order system as

$$\boldsymbol{\phi}_j = \left\{ \begin{array}{c} \mathbf{z}_j \\ s_j \mathbf{z}_j \end{array} \right\}. \qquad [1.23]$$

The *left* eigenvector $\boldsymbol{\psi}_j \in \mathbb{C}^{2N}$ associated with s_j is defined by the equation

$$\boldsymbol{\psi}_j^T \mathbf{A} = s_j \boldsymbol{\psi}_j^T \qquad [1.24]$$

where $(\bullet)^T$ denotes matrix transpose. For distinct eigenvalues it is easy to show that the right and left eigenvectors satisfy an orthogonality relationship, that is

$$\boldsymbol{\psi}_j^T \boldsymbol{\phi}_k = 0; \quad \forall j \neq k \qquad [1.25]$$

and we may also normalise the eigenvectors so that

$$\boldsymbol{\psi}_j^T \boldsymbol{\phi}_j = 1. \qquad [1.26]$$

The above two equations imply that the dynamic system defined by equation [1.20] posses a set of biorthonormal eigenvectors. As a special case, when all eigenvalues are distinct, this set forms a *complete* set. Henceforth in our discussion it will be assumed that all the system eigenvalues are distinct.

Suppose the structural system matrices appearing in [1.18] is a function of a parameter p. This parameter can be an element of a larger parameter vector. This can denote a material property (such as Young's modulus) or a geometric parameter (such as thickness). We wish to find the sensitivity of the eigenvalues and eigenvectors with respect to this general parameter. We aim to derive expressions of derivative of eigenvalues and eigenvectors with respect to p without going into the state space.

For the j-th set, equation [1.19] can be rewritten as

$$\mathbf{F}_j \mathbf{z}_j = 0 \qquad [1.27]$$

where the regular matrix pencil

$$\mathbf{F}_j = s_j^2 \mathbf{M} + s_j \mathbf{C} + \mathbf{K}. \qquad [1.28]$$

Note that complex frequencies can be obtained by substituting $s_j = \mathrm{i}\lambda_j$. Premultiplication of equation [1.27] by $\mathbf{z}_j^T$ yields

$$\mathbf{z}_j^T \mathbf{F}_j \mathbf{z}_j = 0. \qquad [1.29]$$

Differentiating the above equation with respect to p_j one obtains

$$\frac{\partial \mathbf{z}_j}{\partial p}^T \mathbf{F}_j \mathbf{z}_j + \mathbf{z}_j^T \frac{\partial \mathbf{F}_j}{\partial p} \mathbf{z}_j + \mathbf{z}_j^T \mathbf{F}_j \frac{\partial \mathbf{z}_j}{\partial p} = 0 \qquad [1.30]$$

where $\frac{\partial \mathbf{F}_j}{\partial p}$ stands for $\frac{\partial \mathbf{F}_j}{\partial p_j}$, and can be obtained by differentiating equation [1.28] as

$$\frac{\partial \mathbf{F}_j}{\partial p} = \left[\frac{\partial s_j}{\partial p} (2 s_j \mathbf{M} + \mathbf{C}) + s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right]. \qquad [1.31]$$

Now taking the transpose of equation [1.27] and using the symmetry property of $\mathbf{F}_j$ it can shown that the first and third terms of the equation [1.30] are zero. Therefore we have

$$\mathbf{z}_j^T \frac{\partial \mathbf{F}_j}{\partial p} \mathbf{z}_j = 0 \qquad [1.32]$$

Substituting $\frac{\partial \mathbf{F}_j}{\partial p}$ from equation [1.31] into the above equation one writes

$$-\frac{\partial s_j}{\partial p} \mathbf{z}_j^T \left(2 s_j \mathbf{M} + \mathbf{C}\right) \mathbf{z}_j = \mathbf{z}_j^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right] \mathbf{z}_j \qquad [1.33]$$

From this we have

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right] \mathbf{z}_j}{\mathbf{z}_j^T \left(2 s_j \mathbf{M} + \mathbf{C}\right) \mathbf{z}_j} \qquad [1.34]$$

which is the derivative of the j-th complex eigenvalue. For the undamped case, when $\mathbf{C} = 0$, $s_j \to \mathrm{i}\omega_j$ and $\mathbf{z}_j \to \mathbf{x}_j$ (ω_j and $\mathbf{x}_j$ are undamped natural frequencies and modes satisfying $\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j$), with usual mass normalisation the denominator $\to 2\mathrm{i}\omega_j$, and we obtain

$$-\mathrm{i}\frac{\partial \omega_j}{\partial p} = \frac{\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{2\mathrm{i}\omega_j} \quad \text{or} \quad \frac{\partial \omega_j^2}{\partial p} = \mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j. \qquad [1.35]$$

This is exactly the well-known relationship derived by Fox and Kapoor [FOX 68] for the undamped eigenvalue problem. Thus, equation [1.34] can be viewed as a generalisation of the familiar expression of the sensitivity of undamped eigenvalues to the damped case. Following observations may be noted from this result

– The derivative of a given eigenvalue requires the knowledge of only the corresponding eigenvalue and eigenvector under consideration, and thus a complete solution of the eigenproblem, or from the experimental point of view, eigensolution determination for *all* the modes is not required.

– Changes in mass and/or stiffness introduce more change in the real part of the eigenvalues whereas changes in the damping introduce more change in the imaginary part.

Since $\frac{\partial s_j}{\partial p}$ is complex in equation [1.34], it can be effectively used to determine the sensitivity of the modal damping factors with respect to the system parameters. For small damping, the modal damping factor for the j-th mode can be expressed in terms of complex frequencies as $\zeta_j = \Im(\lambda_j)/\Re(\lambda_j)$, with $\Re(\bullet)$ and $\Im(\bullet)$ denoting real and imaginary parts respectively. Consequently the derivative can be evaluated from

$$\frac{\partial \zeta_j}{\partial p} = \frac{\partial \Im(\lambda_j)/\Re(\lambda_j)}{\partial p} = \left[\frac{\Im(\frac{\partial \lambda_j}{\partial p})\Re(\lambda_j) - \Im(\lambda_j)\Re(\frac{\partial \lambda_j}{\partial p})}{\Re(\lambda_j)^2} \right]. \qquad [1.36]$$

This expression may turn out to be useful since we often directly measure the damping factors from experiment.

1.2.2. *Sensitivity of the eigenvectors*

1.2.2.1. *Modal approach*

We use the state-space eigenvectors to calculate the derivative of the eigenvectors in the configuration space. Since $\mathbf{z}_j$ is the first N rows of $\boldsymbol{\phi}_j$ (see equation [1.23]) we first try to derive $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ and subsequently obtain $\frac{\partial \mathbf{z}_j}{\partial p}$ using their relationships.

Differentiating [1.22] with respect to p_j one obtains

$$(\mathbf{A} - s_j)\frac{\partial \boldsymbol{\phi}_j}{\partial p} = -\left(\frac{\partial \mathbf{A}}{\partial p} - \frac{\partial s_j}{\partial p} \right) \boldsymbol{\phi}_j. \qquad [1.37]$$

Since it has been assumed that $\mathbf{A}$ has distinct eigenvalues the right eigenvectors, $\boldsymbol{\phi}_j$, forms a complete set of vectors. Therefore we can expand $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ as

$$\frac{\partial \boldsymbol{\phi}_j}{\partial p} = \sum_{l=1}^{2N} a_{jl} \boldsymbol{\phi}_l \tag{1.38}$$

where $a_{jl}, \forall l = 1, \cdots 2N$ are set of complex constants to be determined. Substituting $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ in equation [1.37] and premultiplying by the left eigenvector $\boldsymbol{\psi}_k^T$ one obtains the scalar equation

$$\sum_{l=1}^{2N} (\boldsymbol{\psi}_k^T \mathbf{A} \boldsymbol{\phi}_l - s_j \boldsymbol{\psi}_k^T \boldsymbol{\phi}_l)\, a_{jl} = -\boldsymbol{\psi}_k^T \frac{\partial \mathbf{A}}{\partial p} \boldsymbol{\phi}_j + \frac{\partial s_j}{\partial p} \boldsymbol{\psi}_k^T \boldsymbol{\phi}_j. \tag{1.39}$$

Using the orthogonality relationship of left and right eigenvectors from the above equation we obtain

$$a_{jk} = \frac{\boldsymbol{\psi}_k^T \frac{\partial \mathbf{A}}{\partial p} \boldsymbol{\phi}_j}{s_j - s_k}; \quad \forall k = 1, \cdots, 2N; \neq j \tag{1.40}$$

The a_{jk} as expressed above is not very useful since it is in terms of the left and right eigenvectors of the first-order system. In order to obtain a relationship with the eigenvectors of second-order system we assume

$$\boldsymbol{\psi}_j = \left\{ \begin{array}{c} \boldsymbol{\psi}_{1j} \\ \boldsymbol{\psi}_{2j} \end{array} \right\} \tag{1.41}$$

where $\boldsymbol{\psi}_{1j}, \boldsymbol{\psi}_{2j} \in \mathbb{C}^N$. Substituting $\boldsymbol{\psi}_j$ in equation [1.24] and taking transpose one obtains

$$\begin{aligned} s_j \boldsymbol{\psi}_{1j} &= -\mathbf{K}\mathbf{M}^{-1} \boldsymbol{\psi}_{2j} \\ s_j \boldsymbol{\psi}_{2j} &= \boldsymbol{\psi}_{1j} - \mathbf{C}\mathbf{M}^{-1} \boldsymbol{\psi}_{2j} \\ \text{or} \quad \boldsymbol{\psi}_{1j} &= \left[s_j \mathbf{I} + \mathbf{C}\mathbf{M}^{-1} \right] \boldsymbol{\psi}_{2j}. \end{aligned} \tag{1.42}$$

Elimination of $\boldsymbol{\psi}_{1j}$ from the above two equation yields

$$\begin{aligned} & s_j \left(s_j \boldsymbol{\psi}_{2j} + \mathbf{C}\mathbf{M}^{-1} \boldsymbol{\psi}_{2j} \right) = -\mathbf{K}\mathbf{M}^{-1} \boldsymbol{\psi}_{2j} \\ \text{or} \quad & \left[s_j^2 \mathbf{M} + s_j \mathbf{C} + \mathbf{K} \right] \left(\mathbf{M}^{-1} \boldsymbol{\psi}_{2j} \right) = 0. \end{aligned} \tag{1.43}$$

By comparison of this equation with equation [1.19] it can be seen that the vector $\mathbf{M}^{-1} \boldsymbol{\psi}_{2j}$ is parallel to $\mathbf{z}_j$; that is, there exist a non-zero $\beta_j \in \mathbb{C}$ such that

$$\mathbf{M}^{-1} \boldsymbol{\psi}_{2j} = \beta_j \mathbf{z}_j \quad \text{or} \quad \boldsymbol{\psi}_{2j} = \beta_j \mathbf{M} \mathbf{z}_j. \tag{1.44}$$

Now substituting $\boldsymbol{\psi}_{1j}$, $\boldsymbol{\psi}_{2j}$ and using the definition of $\boldsymbol{\phi}_j$ from equation [1.23] into the normalisation condition [1.26] the scalar *constant* β_j can be obtained as

$$\beta_j = \frac{1}{\mathbf{z}_j^T \left[2 s_j \mathbf{M} + \mathbf{C} \right] \mathbf{z}_j}. \tag{1.45}$$

Using $\boldsymbol{\psi}_{2j}$ from equation [1.44] into the second equation of [1.42] we obtain

$$\boldsymbol{\psi}_j = \beta_j \mathbf{P}_j \boldsymbol{\phi}_j; \quad \text{where} \quad \mathbf{P}_j = \left[\begin{array}{cc} s_j \mathbf{M} + \mathbf{C} & \mathbf{O} \\ \mathbf{O} & \frac{\mathbf{M}}{s_j} \end{array} \right] \in \mathbb{C}^{2N \times 2N}. \tag{1.46}$$

The above equation along with the definition of $\boldsymbol{\phi}_j$ in [1.23] completely relates the left and right eigenvectors of the first-order system to the eigenvectors of the second-order system.

The derivative of the system matrix $\mathbf{A}$ can be expressed as

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial p} &= \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \frac{\partial \mathbf{M}^{-1}\mathbf{K}}{\partial p} & \frac{\partial \mathbf{M}^{-1}\mathbf{C}}{\partial p} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ -\mathbf{M}^{-2}\frac{\partial \mathbf{M}}{\partial p}\mathbf{K} + \mathbf{M}^{-1}\frac{\partial \mathbf{K}}{\partial p} & -\mathbf{M}^{-2}\frac{\partial \mathbf{M}}{\partial p}\mathbf{C} + \mathbf{M}^{-1}\frac{\partial \mathbf{C}}{\partial p} \end{bmatrix}\end{aligned} \quad [1.47]$$

from which after some simplifications the numerator of the right hand side of equation [1.40] can be obtained as

$$\boldsymbol{\psi}_k^T \frac{\partial \mathbf{A}}{\partial p}\boldsymbol{\phi}_j = -\beta_k \mathbf{z}_k^T \left\{ -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial p}\left[\mathbf{K} + s_j\mathbf{C}\right] + \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right\} \mathbf{z}_j. \quad [1.48]$$

Since $\mathbf{I} = \mathbf{M}\mathbf{M}^{-1}$, $\frac{\partial \mathbf{I}}{\partial p} = \frac{\partial \mathbf{M}}{\partial p}\mathbf{M}^{-1} + \mathbf{M}\left[-\mathbf{M}^{-2}\frac{\partial \mathbf{M}}{\partial p}\right] = \mathbf{O}$ or $\frac{\partial \mathbf{M}}{\partial p}\mathbf{M}^{-1} = \mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial p}$, that is $\mathbf{M}^{-1}$ and $\frac{\partial \mathbf{M}}{\partial p}$ commute in product. Using this property and also from [1.19] noting that $s_j^2\mathbf{z}_j = -\mathbf{M}^{-1}\left[s_j\mathbf{C} + \mathbf{K}\right]\mathbf{z}_j$ we finally obtain

$$a_{jk} = -\beta_k \frac{\mathbf{z}_k \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{s_j - s_k}; \; \forall k = 1, \cdots, 2N; \neq j. \quad [1.49]$$

This equation relates the a_{jk} with the complex modes of the second-order system.

To obtain a_{jj} we begin with differentiation of the normalisation condition [1.26] with respect to p and obtain the relationship

$$\frac{\partial \boldsymbol{\psi}_j}{\partial p}^T \boldsymbol{\phi}_j + \boldsymbol{\psi}_j^T \frac{\partial \boldsymbol{\phi}_j}{\partial p} = 0. \quad [1.50]$$

Substitution of $\boldsymbol{\psi}_j$ from equation [1.46] further leads to

$$\beta_j \left\{ \frac{\partial \boldsymbol{\phi}_j}{\partial p}^T \mathbf{P}_j^T \boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p}^T \boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T \mathbf{P}_j^T \frac{\partial \boldsymbol{\phi}_j}{\partial p} \right\} = 0 \quad [1.51]$$

where $\frac{\partial \mathbf{P}_j}{\partial p}$ can be derived from equation [1.46] as

$$\frac{\partial \mathbf{P}_j}{\partial p} = \begin{bmatrix} \frac{\partial s_j}{\partial p}\mathbf{M} + s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p} & \mathbf{O} \\ \mathbf{O} & -\frac{\mathbf{M}}{s_j^2}\frac{\partial s_j}{\partial p} + \frac{\frac{\partial \mathbf{M}}{\partial p}}{s_j} \end{bmatrix}. \quad [1.52]$$

Since $\mathbf{P}_j$ is a symmetric matrix, equation [1.51] can be rearranged as

$$2\left(\beta_j \boldsymbol{\phi}_j^T \mathbf{P}_j\right) \frac{\partial \boldsymbol{\phi}_j}{\partial p} = -\beta_j \boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p}\boldsymbol{\phi}_j. \quad [1.53]$$

Note that the term within the bracket is $\boldsymbol{\psi}_j^T$ (see equation [1.46]). Using the assumed expansion of $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ from [1.40] this equation reads

$$2\boldsymbol{\psi}_j^T \sum_{l=1}^{2N} a_{jl}\boldsymbol{\phi}_l = -\beta_j \boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p}\boldsymbol{\phi}_j. \quad [1.54]$$

The left hand side of the above equation can be further simplified

$$\begin{aligned}\phi_j^T \frac{\partial \mathbf{P}_j}{\partial p}\phi_j =&\mathbf{z}_j^T\left[\frac{\partial s_j}{\partial p}\mathbf{M} + s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j + \\ &\mathbf{z}_j^T s_j \left[-\frac{\mathbf{M}}{s_j^2}\frac{\partial s_j}{\partial p} + \frac{\frac{\partial \mathbf{M}}{\partial p}}{s_j}\right] s_j \mathbf{z}_j = \mathbf{z}_j^T\left[2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j.\end{aligned} \qquad [1.55]$$

Finally using the orthogonality property of left and right eigenvectors, from equation [1.54] we obtain

$$a_{jj} = -\frac{1}{2}\frac{\mathbf{z}_j^T\left[2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{z}_j}. \qquad [1.56]$$

In the above equation a_{jj} is expressed in terms of the complex modes of the second-order system. Now recalling the definition of ϕ_j in [1.23], from the first N rows of equation [1.38] one can write

$$\begin{aligned}\frac{\partial \mathbf{z}_j}{\partial p} &= a_{jj}\mathbf{z}_j + \sum_{k\neq j}^{2N} a_{jk}\mathbf{z}_k = -\frac{1}{2}\frac{\mathbf{z}_j^T\left[2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{z}_j}\mathbf{z}_j \\ &- \sum_{k\neq j}^{2N}\beta_k\frac{\mathbf{z}_k\left[s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{s_j - s_k}\mathbf{z}_k.\end{aligned} \qquad [1.57]$$

We know that for any real symmetric system first-order eigenvalues and eigenvectors appear in complex conjugate pairs. Using usual definition of natural frequency, that is, $s_k = \mathrm{i}\lambda_k$ and consequently $s_k^* = -\mathrm{i}\lambda_k^*$, where $(\bullet)^*$ denotes complex conjugate, the above equation can be rewritten in a more convenient form as

$$\begin{aligned}\frac{\partial \mathbf{z}_j}{\partial p} &= -\frac{1}{2}\frac{\mathbf{z}_j^T\left[\frac{\partial \mathbf{M}}{\partial p} - \mathrm{i}\frac{\partial \mathbf{C}}{\partial p}/2\lambda_j\right]\mathbf{z}_j}{\mathbf{z}_j^T\left[\mathbf{M} - \mathrm{i}\mathbf{C}/2\lambda_j\right]\mathbf{z}_j}\mathbf{z}_j \\ &+ \sum_{k\neq j}^{N}\left[\frac{\alpha_k(\mathbf{z}_k^T\frac{\partial \tilde{\mathbf{F}}}{\partial p}\mathbf{z}_j)\mathbf{z}_k}{\lambda_j - \lambda_k} - \frac{\alpha_k^*(\mathbf{z}_k^{*T}\frac{\partial \tilde{\mathbf{F}}}{\partial p}^*\mathbf{z}_j^*)\mathbf{z}_k^*}{\lambda_j + \lambda_k^*}\right]\end{aligned} \qquad [1.58]$$

where

$$\frac{\partial \tilde{\mathbf{F}}}{\partial p} = \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j^2\frac{\partial \mathbf{M}}{\partial p} + \mathrm{i}\lambda_j\frac{\partial \mathbf{C}}{\partial p}\right]$$

$$\text{and} \quad \alpha_k = \mathrm{i}\beta_k = \frac{1}{\mathbf{z}_k^T\left[2\lambda_k\mathbf{M} - \mathrm{i}\mathbf{C}\right]\mathbf{z}_k}.$$

This result is a generalisation of the known expression of the sensitivity of real undamped eigenvectors to complex eigenvectors. The following observations can be made from this result

– Unlike the eigenvalue derivative, the derivative of a given complex eigenvector requires the knowledge of all the other complex eigenvalues and eigenvectors.

– The sensitivity depends very much on the modes whose frequency is close to that of the considered mode.

– Like eigenvalue derivative, changes in mass and/or stiffness introduce more changes in the real part of the eigenvector whereas changes in damping introduce more changes in the imaginary part.

From equation [1.58], it is easy to see that in the undamped limit $\mathbf{C} \rightarrow 0$, and consequently $\lambda_k, \lambda_k^* \rightarrow \omega_k$; $\mathbf{z}_k, \mathbf{z}_k^* \rightarrow \mathbf{x}_k$; $\frac{\partial \tilde{\mathbf{F}}}{\partial p}, \frac{\partial \tilde{\mathbf{F}}}{\partial p}^* \rightarrow \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2\frac{\partial \mathbf{M}}{\partial p}\right]$ and also with usual mass normalisation of the undamped modes $\alpha_k, \alpha_k^* \rightarrow \frac{1}{2\omega_k}$ reduces the above equation exactly to the corresponding well known expression derived by Fox and Kapoor [FOX 68] for derivative of undamped modes.

1.2.2.2. *Nelson's method*

The method outlined in the previous subsection obtained the eigenvector derivative as a linear combination of all the eigenvectors. For large scale structures, with many degrees of freedom, obtaining all the eigenvectors is a computationally expensive task. Nelson [NEL 76] introduced the approach, extended here, where only the eigenvector of interest was required. Lee et. al. [LEE 99a] calculated the eigenvector derivatives of self-adjoint systems using a similar approach to Nelson. This subsection extends Nelson's method to non-proportionally damped systems with complex modes. This method has the great advantage that only the eigenvector of interest is required.

Since the eigenvectors are not unique, in the sense that any scalar (complex) multiple of an eigenvector is also an eigenvector. As a result their derivatives are also not unique. It is necessary to normalise the eiegnvector for further mathematical derivations. There are numerous ways of introducing a normalisation to ensure uniqueness. For undamped systems mass normalisation is the most common. A useful normalisation for damped systems follows from equation [**??**] is

$$\mathbf{z}_j^T \left[s_j \mathbf{M} + \ (1/s_j) \, \mathbf{K} \right] \, \mathbf{z}_j = \mathbf{z}_j^T \left[2 s_j \mathbf{M} + \ \mathbf{C} \right] \, \mathbf{z}_j = 1. \qquad [1.59]$$

Differentiating the equation governing the eigenvalues [1.19] with respect to the parameter p, gives

$$\left[s_j^2 \, \frac{\partial \mathbf{M}}{\partial p} + s_j \, \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right] u_j + \left[2 s_j \mathbf{M} + \ \mathbf{C} \right] \, u_j \, \frac{\partial s_j}{\partial p} + \left[s_j^2 \, \mathbf{M} + s_j \, \mathbf{C} + \mathbf{K} \right] \, \frac{\partial u_j}{\partial p} = 0. \qquad [1.60]$$

Rewriting this, we see that the eigenvector derivative satisfies

$$\left[s_j^2 \, \mathbf{M} + s_j \, \mathbf{C} + \mathbf{K} \right] \, \frac{\partial \mathbf{u}_j}{\partial p} = \mathbf{h}_j \qquad [1.61]$$

where the vector $\mathbf{h}_j$ consists of the first two terms in equation [1.60], and all these quantities are now known. Equation [1.61] cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues this matrix has a null space of dimension 1. Following Nelson's approach the eigenvector derivative is written as

$$\frac{\partial \mathbf{u}_j}{\partial p} = \mathbf{v}_j + d_j \mathbf{u}_j \qquad [1.62]$$

where $\mathbf{v}_j$ and d_j have to be determined. These quantities are not unique since any multiple of the eigenvector may be added to $\mathbf{v}_j$. A convenient choice is to identify the element of maximum magnitude in $\mathbf{u}_j$ and make the corresponding element in $\mathbf{v}_j$ equal to zero. Although other elements of $\mathbf{v}_j$ could be set to zero, this choice is most likely to produce a numerically well-conditioned problem. Substituting equation [1.62] into equation [1.61], gives

$$\left[s_j^2 \, \mathbf{M} + s_j \, \mathbf{C} + \mathbf{K} \right] \, \mathbf{v}_j = \mathbf{F}_j \, \mathbf{v}_j = \mathbf{h}_j. \qquad [1.63]$$

This may be solved, including the constraint on the zero element of $\mathbf{v}_j$, by solving the equivalent problem,

$$\begin{bmatrix} \mathbf{F}_{j11} & \mathbf{0} & \mathbf{F}_{j13} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{F}_{j31} & \mathbf{0} & \mathbf{F}_{j33} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{v}_{j1} \\ x_{j2} \, (= 0) \\ \mathbf{v}_{j3} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{h}_{j1} \\ 0 \\ \mathbf{h}_{j3} \end{array} \right\}. \qquad [1.64]$$

where the $\mathbf{F}_j$ is defined in equation [1.63], and has the row and column corresponding to the zeroed element of $\mathbf{v}_j$ replaced with the corresponding row and column of the identity matrix. This approach maintains the banded nature of the structural matrices, and hence is computationally efficient.

It only remains to compute the scalar constant d_j to obtain the eigenvector derivative. For this the normalisation equation must be used. Differentiating equation [1.59], substituting equation [1.62] and rearranging produces

$$d_j = -\mathbf{u}_j^T \left[2 s_j \mathbf{M} + \ \mathbf{C} \right] \, \mathbf{v}_j - \frac{1}{2} \mathbf{u}_j^T \left[2\mathbf{M} \frac{\partial s_j}{\partial p} + 2 s_j \frac{\partial \mathbf{M}}{\partial p} + \ \frac{\partial \mathbf{C}}{\partial p} \right] \mathbf{u}_j. \qquad [1.65]$$

1.2.2.3. *Example: Two-degree-of-freedom system*

Sensitivity of complex frequencies

A two degree-of-freedom system has been considered to illustrate a possible use of the expressions developed so far. Figure 1.1 shows the example taken together with the numerical values. When eigenvalues are plotted

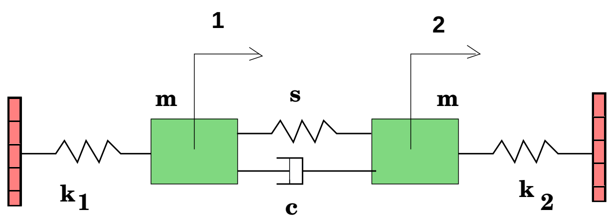

Figure 1.1: Two degree-of-system shows veering, $m = 1$ kg, $k_1 = 1000$ N/m, $c = 4.0$ Ns/m

versus a system parameter they create family of 'root loci'. When two loci approach together they may cross or rapidly diverge. The later case is called 'curve veering'. The veering of the real part of the complex frequencies for the system considered is shown in Figure 1.2. During veering, rapid changes take place in the eigensolutions,

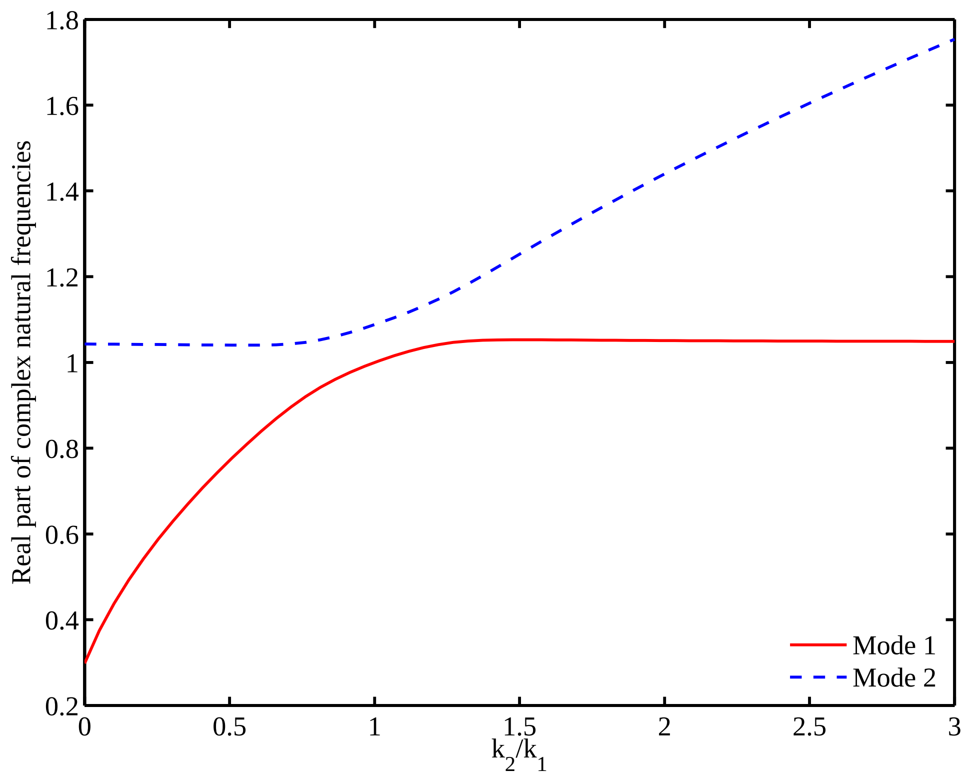

Figure 1.2: Real part of the complex frequencies of the two modes as a function of k_2 showing the veering phenomenon

as Leissa [LEI 74] pointed out ' ... the (eigenfunctions) must undergo violent change – figuratively speaking, a dragonfly one instant, a butterfly the next, and something indescribable in between'. Thus this is an interesting problem for applying the general results derived in this section.

Figure 1.3 shows the imaginary part (normalised by dividing with $\sqrt{k_1/m}$) of the derivative of first natural frequency with respect to the damping parameter 'c' over a parameter variation of k_2 and s. This plot was obtained

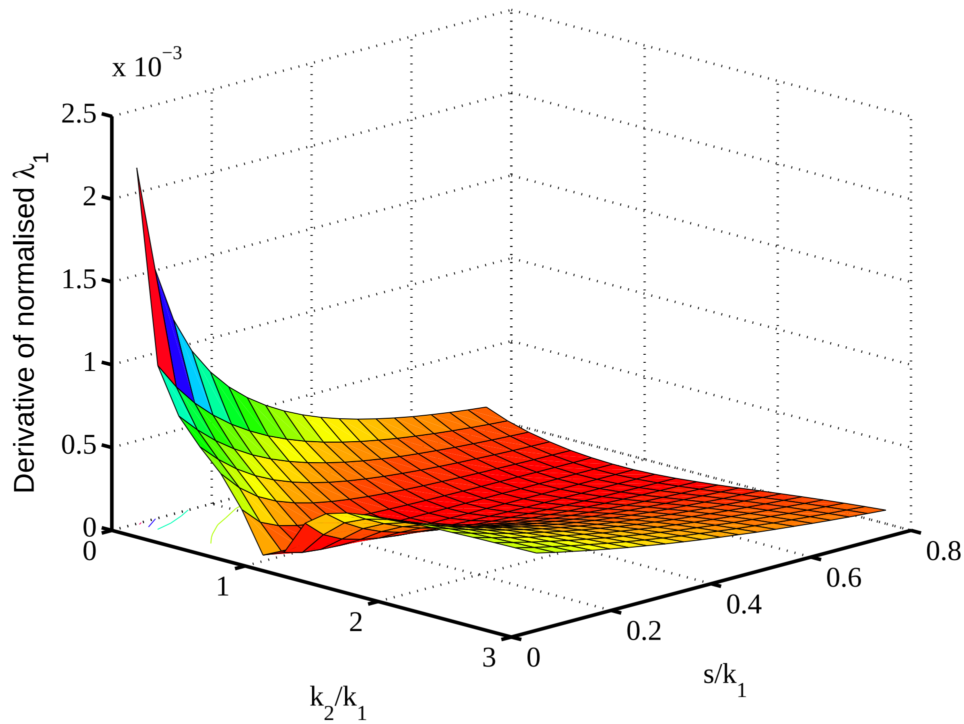

Figure 1.3: Imaginary part of the derivative of the first natural frequency, λ_1, with respect to the damping parameter, c

by programming of equation [1.34] in `Matlab`™ with substituting $s_j = \mathrm{i}\lambda_j$. The imaginary part has been chosen to be plotted here because a change in damping is expected to contribute a significant change in the imaginary part. The sharp rise of the rate in the low-value region of k_2 and s could be intuitively guessed because there the damper becomes the only 'connecting element' between the two masses and so any change made there is expected to have a strong effect. As we move near to the veering range ($k_2 \approx k_1$ and $s \approx 0$) the story becomes quite different. In the first mode, the two masses move in the same direction, in fact in the limit the motion approaches a 'rigid body mode'. Here, the change is no longer remains sensitive to the changes in connecting the element (i.e., only the damper since $s \approx 0$) as hardly any force transmission takes place between the two masses. For this reason we expect a sharp fall in the derivative as can be noticed along the $s \approx 0$ region of the figure. For the region when s is large, we also observe a lower value of derivative, but the reason there is different. The stiffness element 's' shares most of the force being transmitted between the two masses and hence does not depend much on the change of the value of the damper. A similar plot has been shown in Figure 1.4 for the second natural frequency. Unlike the previous case, here the derivative increases in the veering range. For the second mode the masses move in the opposite direction and in the veering range the difference between them becomes maximal. Since $s \approx 0$, only the damper is being stretched and as a result of this, a small change there produces a large effect. Thus, the use of equation [1.34] can provide good physical insight into the problem and can effectively be used in modal

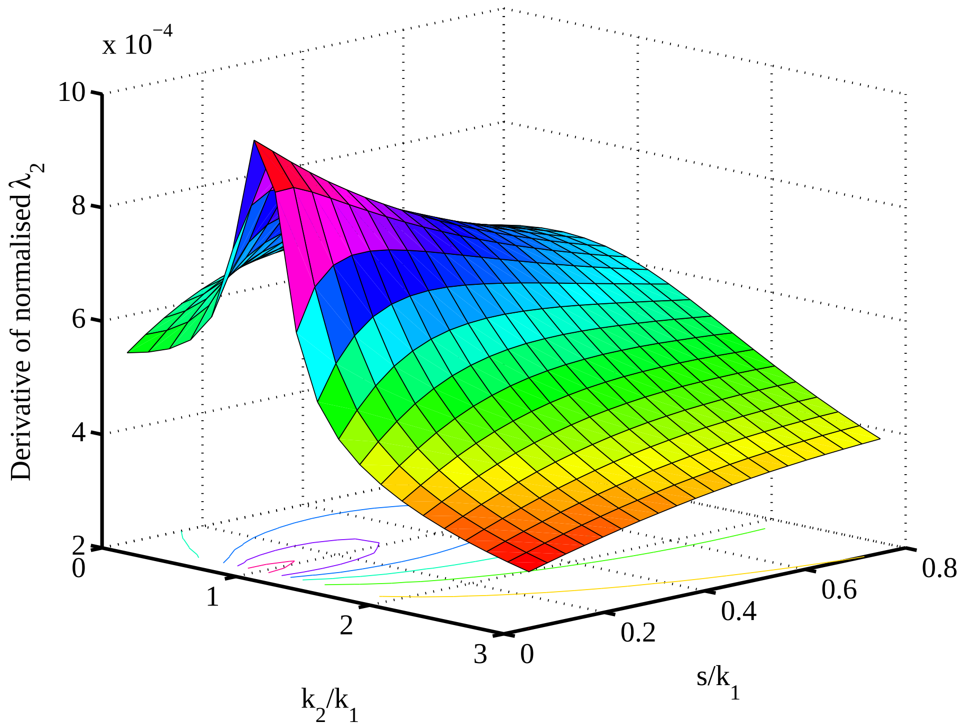

Figure 1.4: Imaginary part of the derivative of the second natural frequency, λ_2, with respect to the damping parameter, c

updating, damage detection and for design purposes by taking the damping matrix together with the mass and stiffness matrices improving the current practice of using the mass and stiffness matrices only.

Sensitivity of eigenvectors

Sensitivity of eigenvectors for the problem shown in Figure 1.1 can be directly obtained from equation [1.58]. Here we have focused our attention to calculate the sensitivity of eigenvectors with respect to the parameter k_2. Figure 1.5 shows the real part of the sensitivity of the first eigenvector normalised by its $\mathcal{L}^2$ norm (that is $\Re\left\{\frac{\mathrm{d}\mathbf{z}_1}{\mathrm{d}k_2}\right\} / \parallel \mathbf{z}_1 \parallel$) plotted over a variation of k_2/k_1 from 0 to 3 for both the coordinates. The value of the spring constant for the connecting spring is kept fixed at $s = 100$ N/m. The real part of the sensitivity of complex eigenvectors has been chosen mainly for two reasons:(a) any change in stiffness is expected to have made more changes in the real part; and (b) to compare it with the corresponding changes of the real undamped modes. Derivative of the first eigenvector (normalised by its $\mathcal{L}^2$ norm) with respect to k_2 corresponding to the undamped system (i.e., removing the damper) is also shown in the same figure (see the figure legend for details). This is calculated from the expression derived by Fox and Kapoor [FOX 68]. Similar plots for the second eigenvector are shown in Figure 1.6. Both of these figures reveal a common feature: around the veering range i.e. $0.5 < k_2/k_1 < 1.5$, the damped and the undamped sensitivities show considerable differences while outside this region they almost traces each other. A physical explanation of this phenomenon can be given. For the problem considered here the damper acts as an additional ‘connecting element’ between the two masses together with the spring ‘s’. As a result it ‘prevents’ the system to be close to show a ‘strong’ veering effect (i.e. when $k_2 = k_1$ and the force transmission between the masses is close to zero) and thus reduces the sensitivity of both the modes. However, for the first mode both masses move in the same direction and the damper has less effect compared to second mode where the masses move in the opposite directions and have much greater effect on the sensitivities.

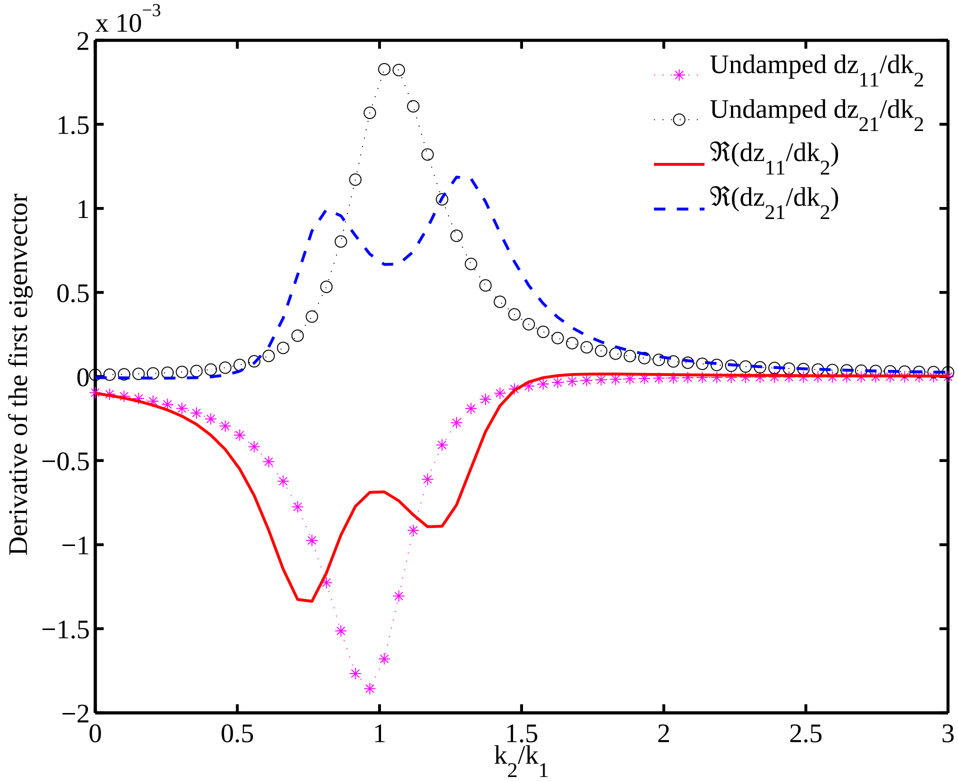

Figure 1.5: Real part of the derivative of the first eigenvector with respect to the stiffness parameter k_2

To analyse the results from quantitative point of view at this point it is interesting to look at the variation of the modal damping factors shown in Figure 1.7. For the first mode the damping factor is quite low (in the order of $\approx 10^{-4}$ near the veering range but still the sensitivities of the undamped mode and that of the real part of the complex mode for both coordinates are quite different. Again, away from the veering range, $k_2/k_1 > 2$, the damping factor is high but the sensitivities of the undamped mode and that of real part of the complex mode are quite similar. This is opposite to what we normally expect, as the common belief is that, when the damping factors are low, the undamped modes and the real part of complex modes should behave similarly and vice versa. For the second mode the damping factor does not change very much due to a variation of k_2 except becomes slightly higher in the vicinity of the veering range. But the difference between the sensitivities of the undamped mode and that of real part of the complex mode for both coordinates changes much more significantly than the damping factor. This demonstrates that even when the damping factors are similar, the sensitivity of the undamped modes and that of the real part of the complex modes can be significantly different. Thus, the use of the expression for the derivatives of undamped mode shapes can lead to a significant error even when the damping is very low. The expressions derived in this section should be used for any kind of study involving such a sensitivity analysis.

Since the expression in equation [1.34] and [1.58] has been derived exactly, the numerical results obtained here are also exact within the precision of the arithmetic used for the calculations. The only instance for arriving at an approximate result is when approximate complex frequencies and modes are used in the analysis. However, for this example it was verified that the use of approximate methods to obtain complex eigensolutions in the configuration space discussed in chapters **??** and **??** and the exact ones obtained from the state space method produce negligible discrepancy. Since in most engineering applications we normally do not encounter very high value of damping one can use approximate methods to obtain eigensolusions in the configuration space in conjunction with the sensitivity

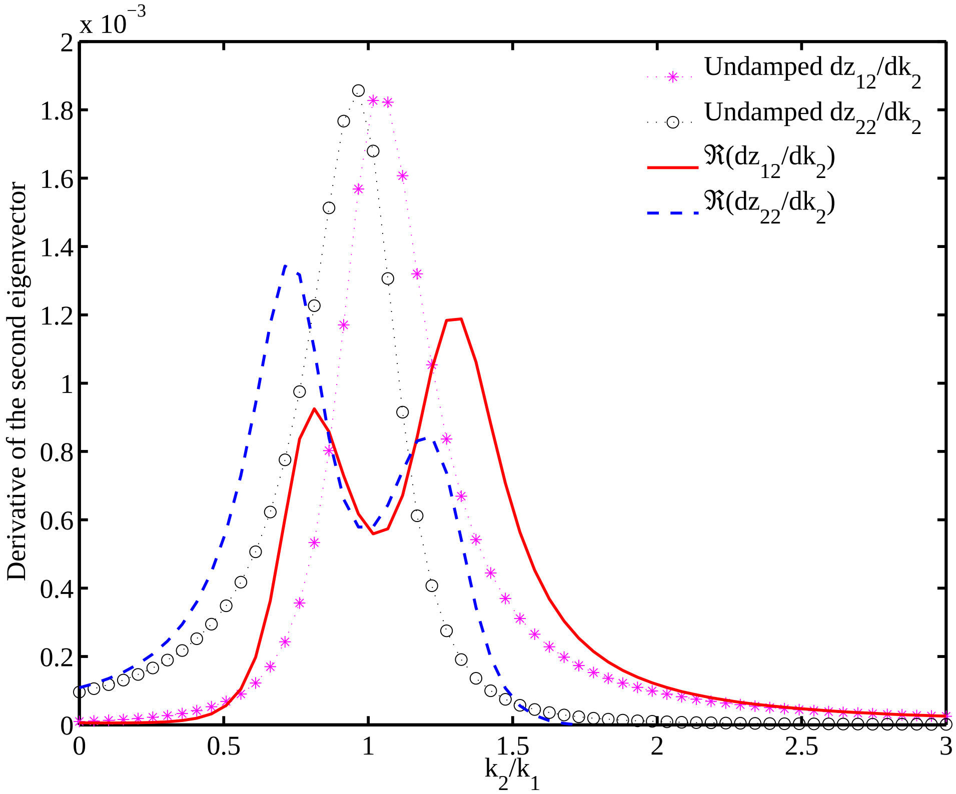

Figure 1.6: Real part of the derivative of the second eigenvector with respect to the stiffness parameter k_2

expressions derived here. This will allow the analyst to study the sensitivity of eigenvalues and eigenvectors of non-classically damped systems in a similar way to those of undamped systems.

1.3. Parametric sensitivity of nonviscously damped systems

The studies so far consider only viscous damping model. However, it is well known that the viscous damping is not the only damping model within the scope of linear analysis, examples are: damping in composite materials [BAB 94], energy dissipation in structural joints [EAR 66, BEA 77], damping mechanism in composite beams [BAN 91], to mention only a few. We consider a class of nonviscous damping models in which the damping forces depend on the past history of motion via convolution integrals over some kernel functions (see chapters **??** and **??**). The equation of motion describing free vibration of a N-degree-of-freedom linear system with such damping can be expressed by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{q}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{q}(t) = \mathbf{0}. \qquad [1.66]$$

Here $\mathbf{M}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrices, $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is the matrix of kernel functions and $\mathbf{0}$ is an $N \times 1$ vector of zeros. In the special case when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, equation [1.66] reduces to the case of viscously damped systems. The damping model of this kind is a further generalisation of the familiar viscous damping. The central aim of this section is to extend the eigensensitivity analysis to nonviscously damped systems of the form [1.66]. In the subsequent subsections the derivative of eigenvalues and eigenvectors are derived. Unlike viscously damped systems, the conversion of equation [1.66] into the state-space form may not be

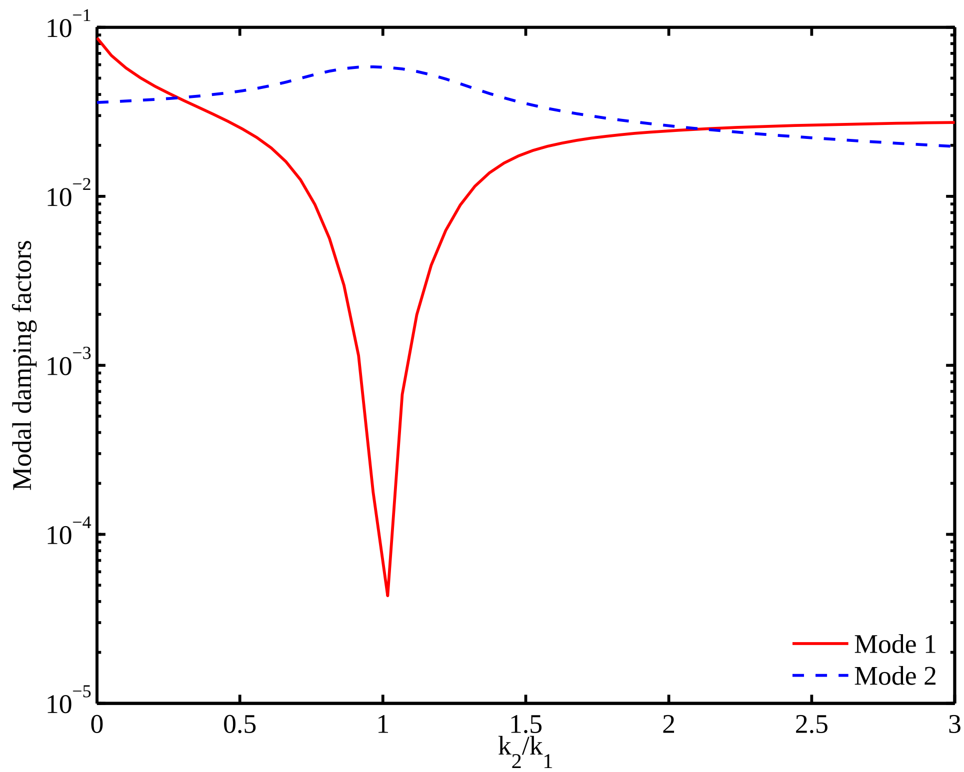

Figure 1.7: Modal damping factors for both the modes

advantageous because the eigenvalue problem in the state-space cannot be casted in the form of the conventional matrix eigenvalue problem involving *constant* matrices. For this reason the approach adopted here does not employ the state-space formulation of the equation of motion. An application of the derived expressions for the derivative of eigensolutions is illustrated by considering a two-degree-of-freedom system with nonviscous damping.

The determination of eigenvalues and eigenvectors of general nonviscously damped systems has been discussed in **??**. Taking the Laplace transform of equation [1.66] we have

$$s^2\mathbf{M}\bar{\mathbf{q}}(s) + s\,\mathbf{G}(s)\bar{\mathbf{q}}(s) + \mathbf{K}\bar{\mathbf{q}}(s) = \mathbf{0} \quad \text{or} \quad \mathbf{D}(s)\bar{\mathbf{q}}(s) = \mathbf{0} \qquad [1.67]$$

where the *dynamic stiffness matrix*

$$\mathbf{D}(s) = s^2\mathbf{M} + s\,\mathbf{G}(s) + \mathbf{K} \ \in \mathbb{C}^{N\times N}. \qquad [1.68]$$

Here $\bar{\mathbf{q}}(s) = \mathcal{L}[\mathbf{q}(t)] \in \mathbb{C}^N$, $\mathbf{G}(s) = \mathcal{L}[\boldsymbol{\mathcal{G}}(t)] \in \mathbb{C}^{N\times N}$ and $\mathcal{L}[\bullet]$ denotes the Laplace transform. In the context of structural dynamics, $s = \mathrm{i}\omega$, where $\omega \in \mathbb{R}^+$ denotes the frequency. We consider the damping to be 'non-proportional' (conditions for proportionality of nonviscous damping were derived in **??**), that is, the mass and stiffness matrices as well as the matrix of kernel functions cannot be simultaneously diagonalised by any linear transformation. However, it is assumed that $\mathbf{M}^{-1}$ exist and $\mathbf{G}(s)$ is such that the motion is dissipative. Conditions which $\mathbf{G}(s)$ must satisfy in order to produce dissipative motion were given by Golla and Hughes [GOL 85].

The eigenvalue problem associated with equation [1.66] can be defined from [1.67] as

$$\left[s_j^2\mathbf{M} + s_j\,\mathbf{G}(s_j) + \mathbf{K}\right]\mathbf{z}_j = \mathbf{0} \quad \text{or} \quad \mathbf{D}(s_j)\mathbf{z}_j = \mathbf{0}, \quad \forall j = 1,\cdots,m \qquad [1.69]$$

where $\mathbf{z}_j \in \mathbb{C}^N$ is the j-th eigenvector. The eigenvalues, s_j, are roots of the characteristic equation

$$\det\left[s^2\mathbf{M} + s\,\mathbf{G}(s) + \mathbf{K}\right] = 0. \qquad [1.70]$$

We consider that the order of the characteristic equation is m. Following **??** one may group the eigenvectors as (a) elastic modes (corresponding to N complex conjugate pairs of eigenvalues), and (b) nonviscous modes (corresponding to the 'additional' $m-2N$ eigenvalues). The elastic modes are related to the N modes of vibration of structural systems. We assume that *all* m eigenvalues are distinct. Following **??** the eigenvectors can be normalised as

$$\begin{aligned} &\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j} \mathbf{z}_j = \gamma_j \\ \text{or} \quad &\mathbf{z}_j^T \left[2s_j\mathbf{M} + \mathbf{G}(s_j) + s_j \frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j}\right] \mathbf{z}_j = \gamma_j, \quad \forall j = 1, \cdots, m \end{aligned} \qquad [1.71]$$

where $\gamma_j \in \mathbb{C}$ is some non-zero constant. Note that equation [1.71] reduces to the corresponding normalisation relationship for viscously damped systems (see [VIG 86, SES 94] for example) when $\mathbf{G}(s)$ is constant with respect to s. Numerical values of γ_j can be selected in various ways, see the discussions in **??**.

1.3.1. *Sensitivity of the eigenvalues*

Suppose the system matrices in equation [1.66] are functions of some design parameter p. In this subsection we intend to obtain an expression of the derivative of the j-th eigenvalue with respect to the design parameter p. Differentiating equation [1.69] with respect to p one obtains

$$\begin{aligned} &\left[2s_j \frac{\partial s_j}{\partial p}\mathbf{M} + s_j^2 \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial s_j}{\partial p}\mathbf{G}(s_j) + s_j \frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j \\ &\qquad + \left[s_j^2\mathbf{M} + s_j\,\mathbf{G}(s_j) + \mathbf{K}\right]\frac{\partial \mathbf{z}_j}{\partial p} = \mathbf{0}. \end{aligned} \qquad [1.72]$$

The term $\frac{\partial[\mathbf{G}(s_j)]}{\partial p}$ appearing in the above equation can be expressed as

$$\frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} = \frac{\partial s_j}{\partial p}\frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j} + \frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j}. \qquad [1.73]$$

Premultiplying equation [1.72] by $\mathbf{z}_j^T$ and using the symmetry property of the system matrices it may be observed that the second term of the equation vanishes due to [1.69]. Substituting [1.73] into equation [1.72] we obtain

$$\begin{aligned} &\mathbf{z}_j^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j + \\ &\mathbf{z}_j^T \left[2s_j \frac{\partial s_j}{\partial p}\mathbf{M} + + \frac{\partial s_j}{\partial p}\mathbf{G}(s_j) + s_j \frac{\partial s_j}{\partial p}\frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j}\right]\mathbf{z}_j = 0. \end{aligned} \qquad [1.74]$$

Rearranging the preceding equation, the derivative of eigenvalues can be obtained as

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T \left[2s_j\mathbf{M} + \mathbf{G}(s_j) + s_j \frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j}\right]\mathbf{z}_j}. \qquad [1.75]$$

Note that the denominator of equation [1.75] is exactly the normalisation relationship given by equation [1.71]. In view of this, equation [1.75] can be expressed in a concise form as

$$\begin{aligned} &\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j}\mathbf{z}_j}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j}\mathbf{z}_j} \\ \text{or} \quad &\frac{\partial s_j}{\partial p} = -\frac{1}{\gamma_j}\left(\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j}\mathbf{z}_j\right). \end{aligned} \qquad [1.76]$$

This is the most general expression for the derivative of eigenvalues of linear dynamic systems. Equation [1.76] can be used to derive the derivative of eigenvalues for various special cases:

1) *Undamped systems* (section 1.1): In this case $\mathbf{G}(s) = 0$ results

$$\begin{aligned}&\mathbf{D}(s) = s^2\mathbf{M} + \mathbf{K}\\ \text{and}\quad &\gamma_j = 2s_j\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j.\end{aligned} \qquad [1.77]$$

Assuming $s_j = \mathrm{i}\omega_j$ where $\omega_j \in \mathbb{R}$ is the j-th undamped natural frequency from equation [1.76] one obtains

$$-2\mathrm{i}\omega_j\mathrm{i}\frac{\partial\omega_j}{\partial p} = \frac{\partial\omega_j^2}{\partial p} = \frac{\mathbf{z}_j^T\left[\frac{\partial\mathbf{K}}{\partial p} - \omega_j^2\frac{\partial\mathbf{M}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j}, \qquad [1.78]$$

which is a well known result.

2) *Viscously damped systems* (section 1.2): In this case $\mathbf{G}(s) = \mathbf{C}$, a constant matrix with respect to s. This results

$$\begin{aligned}&\mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\\ \text{and}\quad &\gamma_j = \mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{z}_j.\end{aligned} \qquad [1.79]$$

Using these, from equation [1.76] one obtains

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T\left[s_j^2\frac{\partial\mathbf{M}}{\partial p} + s_j\frac{\partial\mathbf{C}}{\partial p} + \frac{\partial\mathbf{K}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{z}_j}. \qquad [1.80]$$

Thus, the result obtained in equation [1.76] generalise earlier expressions of the derivative of eigenvalues. The derivative of associated eigenvectors are considered in the next subsection.

1.3.2. *Sensitivity of the eigenvectors*

1.3.2.1. *Modal approach*

The various methods of calculating the derivative of eigenvectors can be divided into three main categories [MUR 88]: (a) adjoint method or modal method, (b) direct method, and (c) iterative method. We adopt the modal method where the derivative of each eigenvector is expanded in the space of the complete set of eigenvectors. The main difficulty in applying available methodologies for the modal method to nonviscously damped systems is that the eigenvectors do not satisfy any familiar orthogonality relationship. We propose an approach to calculate the derivative of eigenvector without using the orthogonality relationship.

It turns out that the eigenvalue problem of the dynamic stiffness matrix (given by equation [1.68]) plays an important role. For any given $s \in \mathbb{C}$, the eigenvalue problem associated with the dynamic stiffness matrix can be expressed by equation [**??**] given in **??**. The eigenvalues and eigenvectors of the dynamic stiffness matrix are given by $\nu_k(s)$ and $\boldsymbol{\varphi}_k(s)$ respectively. It is assumed that all the eigenvalues are distinct for any fixed value of s. The symbols $\nu_k(s)$ and $\boldsymbol{\varphi}_k(s)$ indicate functional dependence of these quantities on the complex parameter s. Such a continuous dependence is expected whenever $\mathbf{D}(s)$ is a sufficiently smooth matrix function of s. It should be noted that because $\mathbf{D}(s)$ is a $N \times N$ complex matrix for a fixed s, the number of eigenvalues (and consequently the eigenvectors) must be N. Further, it can be shown that, for distinct eigenvalues, $\boldsymbol{\varphi}_k(s)$ also satisfy an orthogonality relationship although $\mathbf{z}_k$ do not enjoy any such simple relationship. We normalise $\boldsymbol{\varphi}_k(s)$ as in equation [**??**] .

It is possible to establish the relationships between the original eigenvalue problem of the system defined by equation [1.69] and that by equation [**??**]. Consider the case when the parameter s approaches any one of the system eigenvalues, say s_j. Since *all* the $\nu_k(s)$ are assumed to be distinct, for nontrivial eigenvectors, comparing equations [1.69] and [**??**] we can conclude that one and only one of the $\nu_k(s)$ must be zero when $s \to s_j$. Further discussion is given in **??**. Considering the r-th set, equation [**??**] can be rewritten as

$$\mathbf{Z}_r(s)\boldsymbol{\varphi}_r(s) = \mathbf{0} \tag{1.81}$$

where

$$\mathbf{Z}_r(s) = \mathbf{D}(s) - \nu_r(s)\mathbf{I} \in \mathbb{C}^{N\times N}. \tag{1.82}$$

In view of [**??**], from the preceding equation it is clear that

$$\lim_{s\to s_j} \mathbf{Z}_r(s) = \mathbf{D}(s)|_{s=s_j}. \tag{1.83}$$

From this equation together with [**??**] we conclude that in the limit $s \to s_j$ the eigenvalue problem given by equation [1.81] approaches to the original eigenvalue problem given by [1.69].

Differentiating [1.81] with respect to the design parameter p one has

$$\begin{aligned} &\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s) + \mathbf{Z}_r(s)\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \mathbf{0} \\ \text{or} \quad &\mathbf{Z}_r(s)\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = -\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s). \end{aligned} \tag{1.84}$$

Premultiplying the preceding equation by $\mathbf{D}^{-1}(s)$ and using [1.82] we have

$$\left[\mathbf{I} - \mathbf{D}^{-1}(s)\nu_r(s)\right]\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = -\mathbf{D}^{-1}(s)\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s). \tag{1.85}$$

The derivative of eigenvector of the original system with respect to the design parameter p, that is $\frac{\partial \mathbf{z}_j}{\partial p}$ should be obtained from equation [1.85] by taking the limit $s \to s_j$. Because $\lim_{s\to s_j} \mathbf{D}(s)$ is at the most of rank $(N-1)$, it is not possible to obtain $\frac{\partial \mathbf{z}_j}{\partial p}$ directly from equation [1.85]. We avoid this difficulty by expanding $\mathbf{D}^{-1}(s)$ in terms of the poles and their associated residues. From equation [**??**] the inverse of the dynamic stiffness matrix can be expressed as

$$\mathbf{D}^{-1}(s) = \sum_{j=1}^{m} \frac{\mathbf{R}_j}{s - s_j} \tag{1.86}$$

where

$$\mathbf{R}_j = \frac{\mathbf{z}_j\mathbf{z}_j^T}{\gamma_j}. \tag{1.87}$$

Substituting $\mathbf{D}^{-1}(s)$ from equation [1.86] into equation [1.85], using [**??**] and [1.87], and taking the limit as $s \to s_j$ one obtains

$$\begin{aligned} \frac{\partial \mathbf{z}_j}{\partial p} &= -\lim_{s\to s_j} \sum_{k=1}^{m} \frac{\mathbf{z}_k\mathbf{z}_k^T}{\gamma_j(s-s_k)}\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s) \\ &= a_{jj}\mathbf{z}_j - \sum_{\substack{k=1 \\ k\neq j}}^{m} \frac{\mathbf{z}_k^T \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j}\mathbf{z}_j}{\gamma_k(s_j - s_k)}\mathbf{z}_k \end{aligned} \tag{1.88}$$

where

$$a_{jj} = - \lim_{s \to s_j} \frac{\mathbf{z}_j^T \frac{\partial \mathbf{Z}_{r(s)}}{\partial p} \boldsymbol{\varphi}_r(s)}{\gamma_j (s - s_j)}. \tag{1.89}$$

In deriving equation [1.88] we have also made use of the relationships [**??**] and [1.83]. Note that the limiting value of a_{jj}, the coefficient associated with $\mathbf{z}_j$, cannot be obtained from [1.89] because the denominator approaches to zero in the limit. A different approach is presented below to bypass this difficulty.

For a fixed value of s, $\boldsymbol{\varphi}_k(s), \forall\, k = 1, \cdots, N$ form a complete basis. For this reason $\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} \in \mathbb{C}^N$ can be expanded uniquely in terms of all $\boldsymbol{\varphi}_k(s)$, that is, one can write

$$\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \sum_{k=1}^{N} \alpha_k^{(r)}(s) \boldsymbol{\varphi}_k(s), \tag{1.90}$$

where $\alpha_k^{(r)}(s) \in \mathbb{C}$ are non-zero constants. The normalisation relationship for the r-th mode can be expressed from equation [**??**] as

$$\boldsymbol{\varphi}_r^T(s) \mathbf{D}(s) \boldsymbol{\varphi}_r(s) = \nu_r(s). \tag{1.91}$$

Differentiating this equation with respect to the design parameter p we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{\varphi}_r^T(s)}{\partial p} \mathbf{D}(s) \boldsymbol{\varphi}_r(s) + \boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{D}(s)}{\partial p} \boldsymbol{\varphi}_r(s) \\ + \boldsymbol{\varphi}_r^T(s) \mathbf{D}(s) \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \frac{\partial \nu_r(s)}{\partial p}. \end{aligned} \tag{1.92}$$

Using the symmetry property of $\mathbf{D}(s)$ and [1.82] the above equation can be rearranged as

$$2 \boldsymbol{\varphi}_r^T(s) \mathbf{D}(s) \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = - \boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s). \tag{1.93}$$

Substituting $\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p}$ from equation [1.90] and using the orthogonality relationship given by [**??**] from the above equation one obtains

$$\alpha_r^{(r)}(s) = - \frac{\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s)}{2 \nu_r(s)}. \tag{1.94}$$

Now, taking the limit $s \to s_j$ on equation [1.90] and using [**??**] we have

$$\begin{aligned} \lim_{s \to s_j} \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} &= \lim_{s \to s_j} \sum_{k=1}^{N} \alpha_k^{(r)}(s) \boldsymbol{\varphi}_k(s) \\ \text{or} \quad \frac{\partial \mathbf{z}_j}{\partial p} &= \left(\lim_{s \to s_j} \alpha_r^{(r)}(s) \right) \mathbf{z}_j + \lim_{s \to s_j} \sum_{\substack{k=1 \\ k \neq r}}^{N} \alpha_k^{(r)}(s) \boldsymbol{\varphi}_k(s). \end{aligned} \tag{1.95}$$

Because it is assumed that all the eigenvalues are distinct, the associated eigenvectors are also distinct. Thus, $\lim_{s \to s_j} \boldsymbol{\varphi}_k(s) \neq \mathbf{z}_j, \forall k = 1, \cdots, N; \neq r$. So, comparing the coefficient of $\mathbf{z}_j$ in equations [1.88] and [1.95] it is clear that

$$\begin{aligned} a_{jj} &= \lim_{s \to s_j} \alpha_r^{(r)}(s) \\ &= - \lim_{s \to s_j} \frac{\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s)}{2 \nu_r(s)} \qquad \text{(from [1.94])}. \end{aligned} \tag{1.96}$$

The above limit cannot be evaluated directly because from [**??**] $\lim_{s \to s_j} \nu_r(s) = 0$. Now, differentiate equation [1.81] with respect to p to obtain

$$\frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s) + \mathbf{Z}_r(s) \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \mathbf{0}. \qquad [1.97]$$

Premultiplying the above equation by $\boldsymbol{\varphi}_r^T(s)$ one obtains

$$\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s) + \boldsymbol{\varphi}_r^T(s) \mathbf{Z}_r(s) \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = 0. \qquad [1.98]$$

Taking transpose of equation [1.81] and considering the symmetry property of $\mathbf{Z}_r(s)$ it follows that the second term of the left-hand side of the above equation is zero. Thus, equation [1.98] reduces to

$$\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s) = 0. \qquad [1.99]$$

The above equation shows that in the limit the left hand side of equation [1.96] has a '0 by 0' form. So, applying l'Hôspital's rule, using [**??**], [1.83] and [**??**] in **??**, from equation [1.96] one obtains

$$a_{jj} = -\frac{\mathbf{z}_j^T \frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j} \mathbf{z}_j}{2 \frac{\partial \nu_r(s)}{\partial s}|_{s=s_j}} = -\frac{\mathbf{z}_j^T \frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j} \mathbf{z}_j}{2 \left(\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j} \mathbf{z}_j \right)}. \qquad [1.100]$$

This expression can now be used to obtain the derivative of $\mathbf{z}_j$ in equation [1.88].

The denominator in the above equation can be related to the normalisation constant γ_j given by equation [1.71]. The term $\frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j}$ appearing in the numerator may be obtained by differentiating equation [1.68] as

$$\frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j} = 2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j} + s_j \frac{\partial^2 [\mathbf{G}(s)]}{\partial s\, \partial p}|_{s=s_j} \qquad [1.101]$$

From equations [1.88] and [1.101] the derivative of $\mathbf{z}_j$ is obtained as

$$\frac{\partial \mathbf{z}_j}{\partial p} = -\frac{1}{2\gamma_j} \left(\mathbf{z}_j^T \frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j} \mathbf{z}_j \right) \mathbf{z}_j - \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{\mathbf{z}_k^T \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j} \mathbf{z}_j}{\gamma_k (s_j - s_k)} \mathbf{z}_k. \qquad [1.102]$$

This is the most general expression for the derivative of eigenvectors of linear dynamic systems. Equation [1.102] can be applied directly to derive the derivative of eigenvectors for various special cases:

1) *Undamped systems* (section 1.1): In this case $\mathbf{G}(s) = 0$ results the order of the characteristic polynomial $m = 2N$; s_j is purely imaginary so that $s_j = \mathrm{i}\omega_j$. Using [1.77], equation [1.101] results

$$\frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j} = 2 s_j \frac{\partial \mathbf{M}}{\partial p}. \qquad [1.103]$$

Recalling that the eigenvalues appear in complex conjugate pairs and all $\mathbf{z}_j$ are real, from [1.102] one obtains

$$\begin{aligned} \frac{\partial \mathbf{z}_j}{\partial p} = -\frac{1}{2} \frac{\left(2\mathrm{i}\omega_j \mathbf{z}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{z}_j \right)}{2\mathrm{i}\omega_j \left(\mathbf{z}_j^T \mathbf{M} \mathbf{z}_j \right)} \mathbf{z}_j - \\ \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\mathbf{z}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{z}_j}{2\mathrm{i}\omega_k \left(\mathbf{z}_k^T \mathbf{M} \mathbf{z}_k \right)} \left[\frac{1}{\mathrm{i}\omega_j - \mathrm{i}\omega_k} - \frac{1}{\mathrm{i}\omega_j + \mathrm{i}\omega_k} \right] \mathbf{z}_k \end{aligned} \qquad [1.104]$$

Considering the unity mass normalisation, that is, $\mathbf{z}_k^T \mathbf{M} \mathbf{z}_k = 1, \forall k = 1, \cdots, N$ the preceding equation can be rewritten as

$$\frac{\partial \mathbf{z}_j}{\partial p} = -\frac{1}{2}\left(\mathbf{z}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{z}_j\right) \mathbf{z}_j + \sum_{\substack{k=1\\k\neq j}}^{N} \frac{\mathbf{z}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{z}_j}{(\omega_j^2 - \omega_k^2)} \mathbf{z}_k \qquad [1.105]$$

which is a well known result.

2) *Viscously damped systems* (section 1.2): In this case $\mathbf{G}(s) = \mathbf{C}$, a constant matrix with respect to s and $m = 2N$. Using [1.79], equation [1.101] results

$$\frac{\partial^2 [\mathbf{D}(s)]}{\partial s\, \partial p}|_{s=s_j} = 2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}. \qquad [1.106]$$

Recalling that the eigenvalues and eigenvectors appear in complex conjugate pairs, from [1.102] one obtains

$$\begin{aligned} \frac{\partial \mathbf{z}_j}{\partial p} = &-\frac{1}{2\gamma_j}\left(\mathbf{z}_j^T \left[2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right] \mathbf{z}_j\right) \mathbf{z}_j - \\ &\frac{1}{\gamma_j^* 2\mathrm{i}\Im(s_j)}\left(\mathbf{z}_j^{*T} \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j} \mathbf{z}_j\right) \mathbf{z}_j^* - \\ &\sum_{\substack{k=1\\k\neq j}}^{N} \left[\frac{\mathbf{z}_k^T \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j} \mathbf{z}_j}{\gamma_k (s_j - s_k)} \mathbf{z}_k + \frac{\mathbf{z}_k^{*T} \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j} \mathbf{z}_j}{\gamma_k^* (s_j - s_k^*)} \mathbf{z}_k^*\right]. \end{aligned} \qquad [1.107]$$

Thus, the result obtained in equation [1.102] generalise earlier expressions of the derivative of eigenvectors.

1.3.2.2. *Numerical example: A two-degree-of-freedom system*

We consider a two-degree-of-freedom system shown in Figure 1.8 to illustrate a possible use of the expressions derived so far. The system considered here is similar to the one used in subsubsection 1.2.2.3 except that the

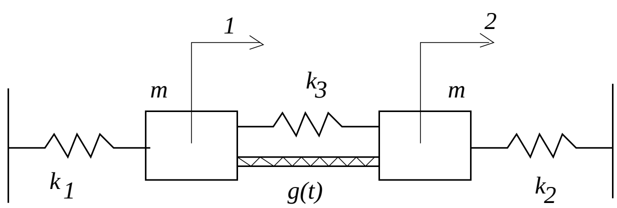

Figure 1.8: A two degrees-of-freedom spring-mass system with nonviscous damping, $m = 1$ Kg, $k_1 = 1000$ N/m, $k_3 = 100$ N/m, $g(t) = c\,(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t})$, $c = 4.0$ Ns/m, $\mu_1 = 10.0$ s^{-1}, $\mu_2 = 2.0$ s^{-1}

dissipative element connected between the two masses is not a simple viscous dashpot but a nonviscous damper. The equation of motion describing the free vibration of the system can be expressed by [1.66], with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \qquad [1.108]$$

and

$$\boldsymbol{\mathcal{G}}(t) = g(t)\hat{\mathbf{I}}, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \qquad [1.109]$$

The damping function $g(t)$ is assumed to be the GHM model [GOL 85, MCT 93] so that

$$g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right); \quad c, \mu_1, \mu_2 \geq 0, \tag{1.110}$$

where c is a constant and μ_1 and μ_2 are known as the relaxation parameters. In equation [1.110] if the function associated with c was a delta-function, c would serve the purpose of the familiar viscous damping constant. Taking the Laplace transform of [1.109] one obtains

$$\begin{aligned} &\mathbf{G}(s) = G(s)\hat{\mathbf{I}}, \\ &\quad \text{where} \quad G(s) = \mathcal{L}\left[g(t)\right] = c\left(\frac{\mu_1}{s+\mu_1} + \frac{\mu_2}{s+\mu_2}\right). \end{aligned} \tag{1.111}$$

Substituting [1.108] and [1.111] in equation [1.70] it may be shown that the system has six eigenvalues – four of which correspond to the two elastic modes (together with corresponding complex conjugate pairs) and the rest two correspond to two nonviscous modes. For convenience arrange the eigenvalues as

$$s_{e_1}, s_{e_2}, s^*_{e_1}, s^*_{e_2}, s_{nv_1}, s_{nv_1} \tag{1.112}$$

where $(\bullet)_e$ denotes elastic modes and $(\bullet)_{nv}$ denotes nonviscous modes.

We consider the derivative of eigenvalues with respect to the relaxation parameter μ_1. The derivative of the system matrices with respect to this parameter may be obtained as

$$\frac{\partial \mathbf{M}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial \mu_1} = \hat{\mathbf{I}}\frac{c\,s}{(s+\mu_1)^2} \quad \text{and} \quad \frac{\partial \mathbf{K}}{\partial \mu_1} = \mathbf{O}. \tag{1.113}$$

Further, from equation [1.111] one also obtains

$$\begin{aligned} \frac{\partial \mathbf{G}(s)}{\partial s} &= -\hat{\mathbf{I}}c\left\{\frac{\mu_1}{(s+\mu_1)^2} + \frac{\mu_2}{(s+\mu_2)^2}\right\} \\ \frac{\partial^2[\mathbf{G}(s)]}{\partial s\,\partial \mu_1} &= -\hat{\mathbf{I}}c\frac{s-\mu_1}{(s+\mu_1)^3}. \end{aligned} \tag{1.114}$$

Using equations [1.113] and [1.114] the terms γ_j, $\frac{\partial \mathbf{D}(s)}{\partial p}$ and $\frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial \mu_1}$ appearing in equations [1.76] and [1.102] can be evaluated.

Figures 1.9 and 1.10 show the real part of the derivative of first and second eigenvalue with respect to μ_1 over a parameter variation of k_2 and k_3. These results are obtained by direct application of equation [1.76]. The system considered here shows the so called 'veering' [BOI 11, BOI 09] when the eigenvalues are plotted versus a system parameter. In the veering range (that is when $k_2 \approx k_1$ and $k_3 \approx 0$) rapid changes take place in the eigensolutions. From Figures 1.9 and 1.10 it may be noted that around the veering range the first eigenvalue is not very sensitive to μ_1 while the second eigenvalue is very sensitive in this region. In the first mode both the blocks move in the same direction and consequently the damper is not stretched, resulting insensitiveness to the relaxation parameter μ_1. In the second mode the blocks move away from each other. This results stretching of the damping block and increases sensitiveness to the relaxation parameter μ_1.

It is useful to understand the effect of different parameters on the eigenvalues. Figures 1.11 and 1.12 show the imaginary part of the derivative of first and second eigenvalue with respect to the damping parameters c, μ_1 and μ_2 over a parameter variation of k_2. The value of k_3 is fixed at $k_3 = 100$. These plots show that the damping parameters not only affect the real part of the eigenvalues but also affect the imaginary part. Again, observe that in the veering range the first eigenvalue is insensitive to the damping parameters while the second eigenvalue is sensitive to them.

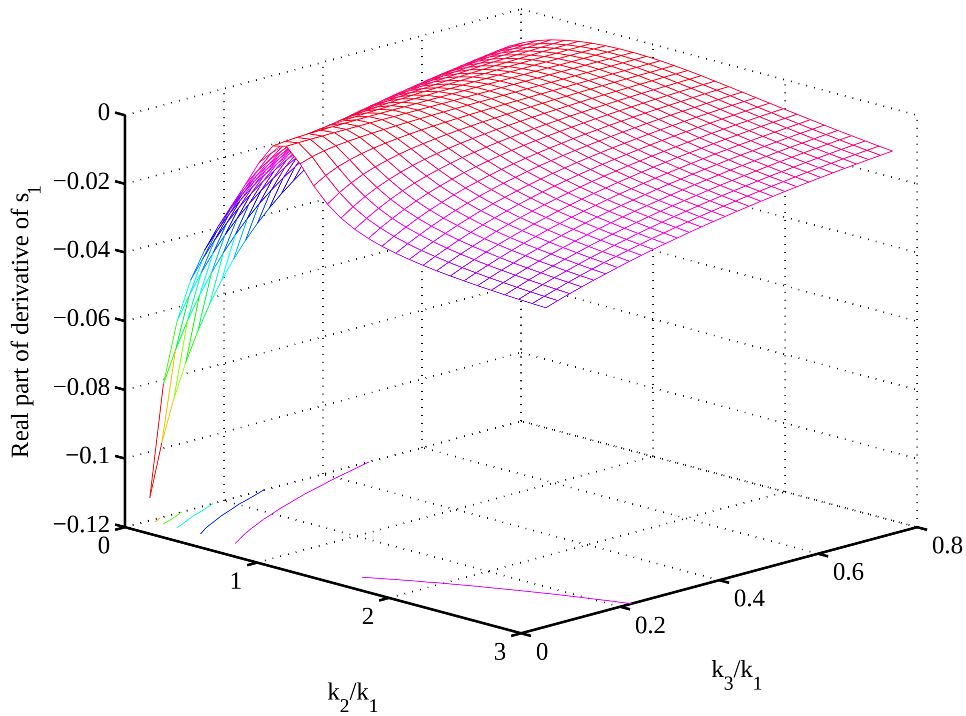

Figure 1.9: Real part of the derivative of the first eigenvalue with respect to the relaxation parameter μ_1

Now we turn our attention to the derivative of eigenvectors. Figures 1.13 and 1.14 show the real part of the derivative of first and second eigenvector with respect to k_2 over a parameter variation of k_2. It is useful to compare these results with the corresponding results by considering the damping mechanism to be viscous, that is, when $g(t)$ given by equation [1.110] has the form $g(t) = c\,\delta(t)$. In Figures 1.13 and 1.14 the derivative of both eigenvectors for the corresponding viscously damped system is also plotted. Observe that around the veering range the derivatives of both eigenvectors are different for viscously and nonviscously damped systems. This illustrates that the nature of damping affects the parameter sensitivity of the real part of complex modes.

1.3.2.3. *Nelson's method*

For large scale structures with nonviscous damping, obtaining all of the eigenvectors is a computationally expensive task because the number of eigenvectors of a nonviscously damped system is much larger, in general, than the number for a viscously damped system. This motivates the extension of Nelson's method to calculate the derivatives of eigenvectors of nonviscously damped systems.

Differentiating equation [1.69] with respect to the design parameter p we have

$$\mathbf{D}(s_j)\frac{\partial \mathbf{z}_j}{\partial p} = \mathbf{h}_j \qquad [1.115]$$

where

$$\mathbf{h}_j = -\frac{\partial \mathbf{D}(s_j)}{\partial p}\mathbf{z}_j = -\left[2s_j\frac{\partial s_j}{\partial p}\mathbf{M} + s_j^2\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial s_j}{\partial p}\mathbf{G}(s_j) + s_j\frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j \qquad [1.116]$$

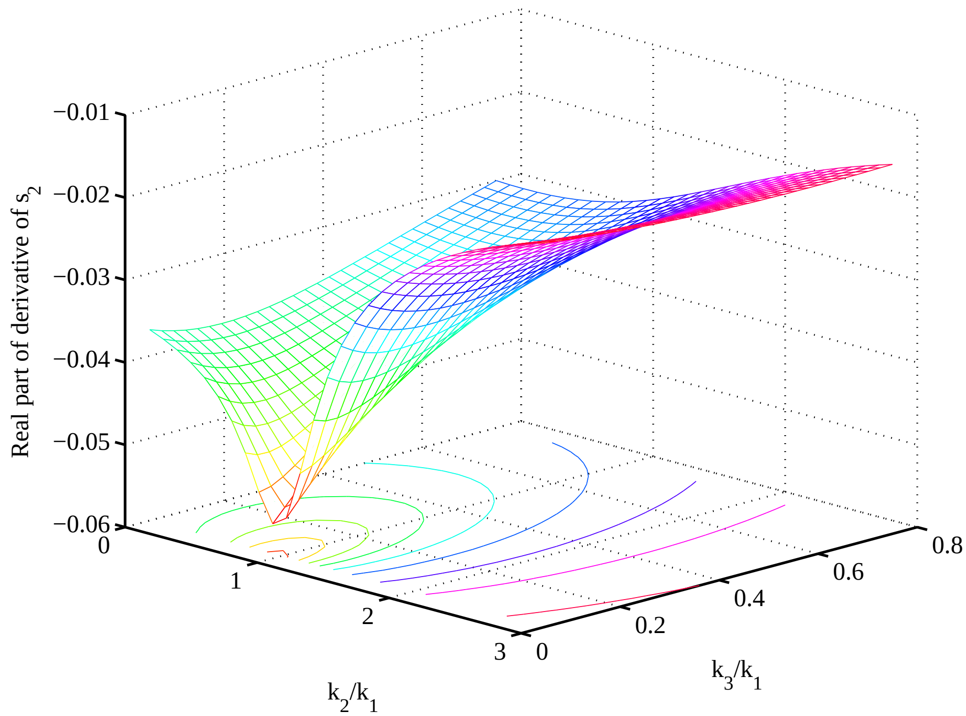

Figure 1.10: Real part of the derivative of the second eigenvalue with respect to the relaxation parameter μ_1

is known. For unique results we need to normalise the eigenvectors. There are many approaches to the normalisation of the eigenvectors. A convenient approach (see **??**) is to normalise $\mathbf{z}_j$ such that

$$\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j} \mathbf{z}_j = \gamma_j \tag{1.117}$$

$$\text{or} \quad \mathbf{z}_j^T \mathbf{D}'(s_j) \mathbf{z}_j = \gamma_j, \quad \forall j = 1, \cdots, m \tag{1.118}$$

where

$$\mathbf{D}'(s) = \frac{\partial \mathbf{D}(s)}{\partial s} = [2s\mathbf{M} + \mathbf{G}(s) + s\mathbf{G}'(s)] \ \in \mathbb{C}^{N \times N} \tag{1.119}$$

and $\gamma_j \in \mathbb{C}$ is some non-zero constant.

Equation [1.115] cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues this matrix has a null space of dimension 1. Following Nelson's approach the eigenvector derivative is written as

$$\frac{\partial \mathbf{z}_j}{\partial p} = \mathbf{v}_j + d_j \mathbf{z}_j \tag{1.120}$$

where $\mathbf{v}_j$ and d_j have to be determined. These quantities are not unique since any multiple of the eigenvector may be added to $\mathbf{v}_j$. A convenient choice is to identify the element of maximum magnitude in $\mathbf{z}_j$ and make the corresponding element in $\mathbf{v}_j$ equal to zero. Although other elements of $\mathbf{v}_j$ could be set to zero, this choice is

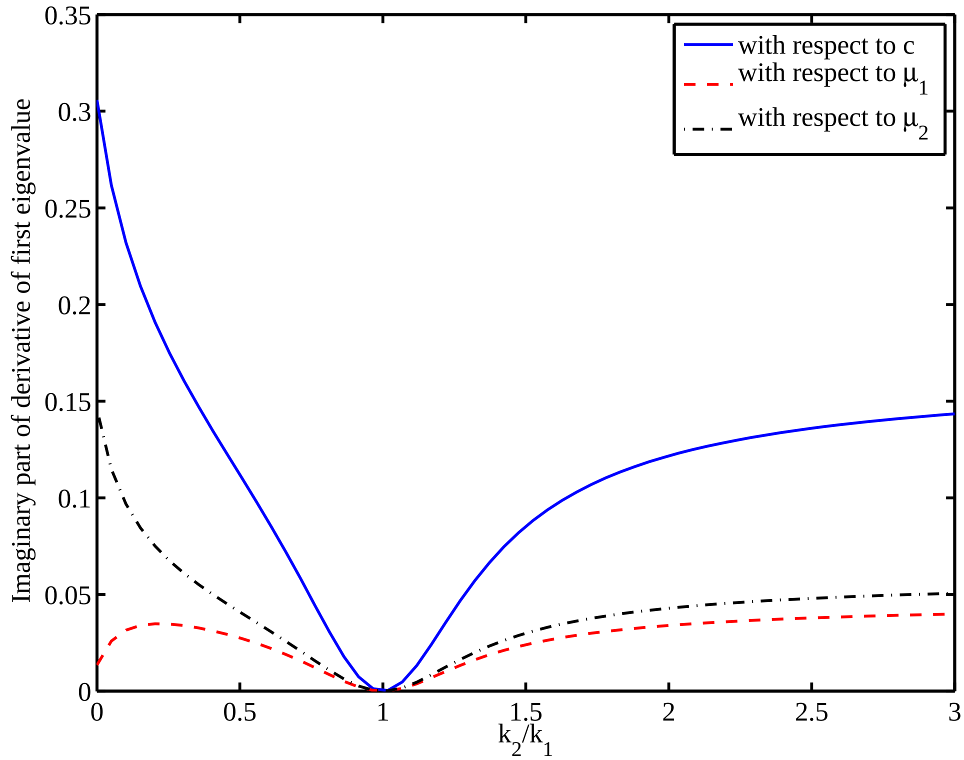

Figure 1.11: Imaginary part of the derivative of the first eigenvalue with respect to the damping parameters c, μ_1 and μ_2

most likely to produce a numerically well-conditioned problem. Because $\mathbf{D}(s_j)\mathbf{z}_j = \mathbf{0}$ due to equation [1.69], substituting equation [1.120] into equation [1.115], gives

$$\mathbf{D}_j\mathbf{v}_j = \mathbf{h}_j \tag{1.121}$$

where

$$\mathbf{D}_j = \mathbf{D}(s_j) = \left[s_j^2\mathbf{M} + s_j\,\mathbf{G}(s_j) + \mathbf{K}\right] \in \mathbb{C}^{N\times N}. \tag{1.122}$$

This may be solved, including the constraint on the zero element of $\mathbf{v}_j$, by solving the equivalent problem,

$$\begin{bmatrix} \mathbf{D}_{j11} & \mathbf{0} & \mathbf{D}_{j31} \\ 0 & 1 & 0 \\ \mathbf{D}_{j31} & \mathbf{0} & \mathbf{D}_{j33} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_{j1} \\ x_{j2}\,(=0) \\ \mathbf{v}_{j3} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_{j1} \\ 0 \\ \mathbf{h}_{j3} \end{Bmatrix} \tag{1.123}$$

where the $\mathbf{D}_j$ is defined in equation [1.122], and has the row and column corresponding to the zeroed element of $\mathbf{v}_j$ replaced with the corresponding row and column of the identity matrix. This approach maintains the banded nature of the structural matrices, and hence is computationally efficient.

It only remains to compute the scalar constant, d_j, to obtain the eigenvector derivative. For this the normalisation equation [1.118] must be used. Differentiating equation [1.118] and using the symmetry property of $\mathbf{D}'(s)$ we have

$$\mathbf{z}_j^T\frac{\partial\mathbf{D}'(s_j)}{\partial p}\mathbf{z}_j + 2\mathbf{z}_j^T\mathbf{D}'(s_j)\frac{\partial\mathbf{z}_j}{\partial p} = 0. \tag{1.124}$$

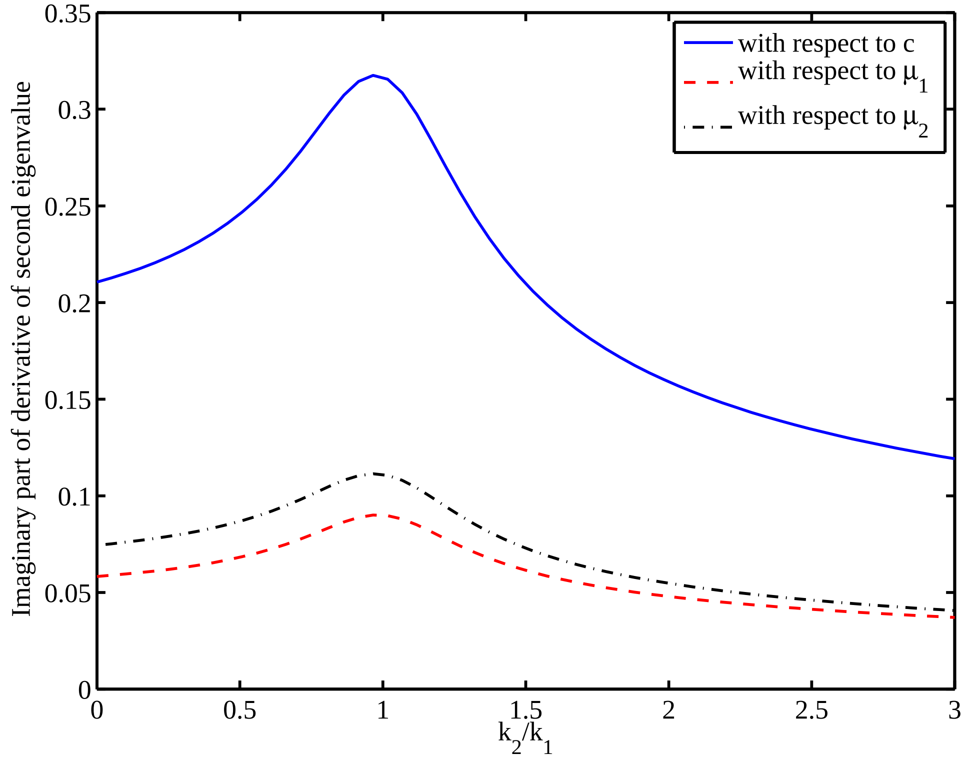

Figure 1.12: Imaginary part of the derivative of the second eigenvalue with respect to the damping parameters c, μ_1 and μ_2

Substituting $\frac{\partial \mathbf{z}_j}{\partial p}$ from equation [1.120] one has

$$\frac{1}{2}\mathbf{z}_j^T \frac{\partial \mathbf{D}'(s_j)}{\partial p}\mathbf{z}_j + \mathbf{v}_j^T \mathbf{D}'(s_j)\mathbf{z}_j + d_j \mathbf{z}_j^T \mathbf{D}'(s_j)\mathbf{z}_j = 0. \qquad [1.125]$$

Noticing that the coefficient associated with d_j is the normalisation constant given by equation [1.118], we have

$$d_j = -\frac{1}{\gamma_j}\left\{\frac{1}{2}\mathbf{z}_j^T \frac{\partial \mathbf{D}'(s_j)}{\partial p}\mathbf{z}_j + \mathbf{z}_j^T \mathbf{D}'(s_j)\mathbf{v}_j\right\}. \qquad [1.126]$$

The first term in the right-hand side can be obtained by substituting $s = s_j$ into equation [1.119] and differentiating

$$\frac{\partial \mathbf{D}'(s_j)}{\partial p} = 2\frac{\partial s_j}{\partial p}\mathbf{M} + 2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} + \frac{\partial s_j}{\partial p}\mathbf{G}'(s_j) + s_j\frac{\partial\left[\mathbf{G}'(s_j)\right]}{\partial p} \qquad [1.127]$$

where $\frac{\partial[\mathbf{G}(s_j)]}{\partial p}$ is given in equation [1.73] and

$$\begin{aligned}\frac{\partial\left[\mathbf{G}'(s_j)\right]}{\partial p} &= \frac{\partial s_j}{\partial p}\frac{\partial \mathbf{G}'(s)}{\partial s}|_{s=s_j} + \frac{\partial \mathbf{G}'(s)}{\partial p}|_{s=s_j} \\ &= \frac{\partial s_j}{\partial p}\frac{\partial^2 \mathbf{G}(s)}{\partial s^2}|_{s=s_j} + \frac{\partial^2 \mathbf{G}(s)}{\partial p\,\partial s}|_{s=s_j}.\end{aligned} \qquad [1.128]$$

Equation [1.120], combined with $\mathbf{v}_j$ obtained by solving equation [1.123] and d_j obtained from equation [1.126] completely defines the derivative of the eigenvectors.

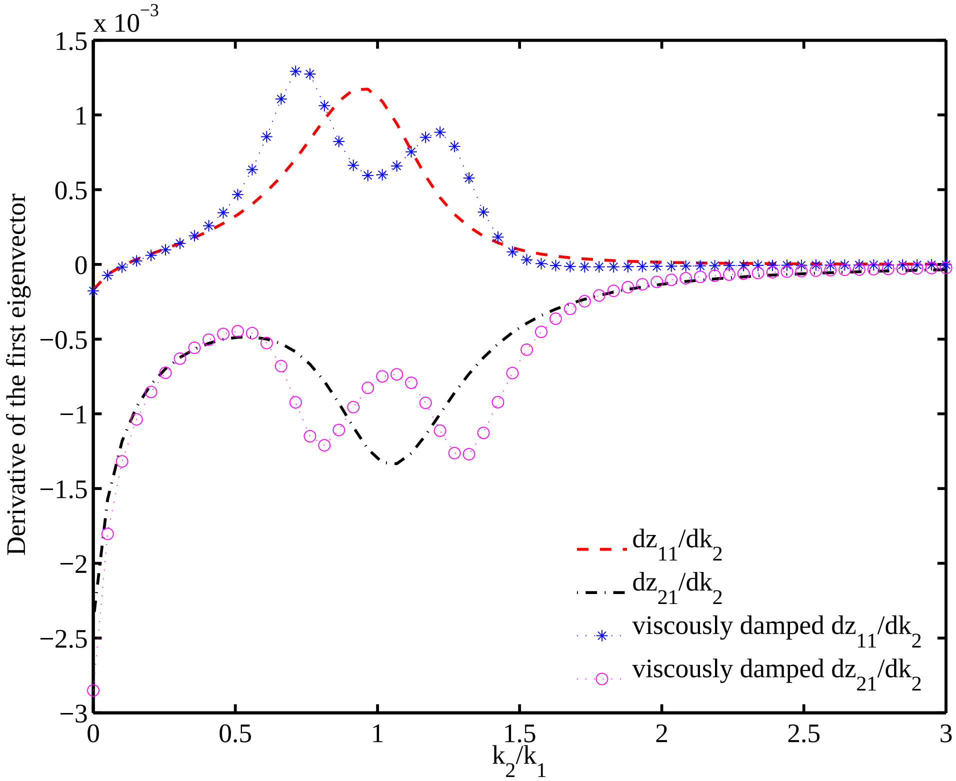

Figure 1.13: Real part of the derivative of the first eigenvector with respect to k_2

1.3.2.4. *Numerical example*

We consider a two-degree-of-freedom system shown in Figure 1.8 to illustrate the use of the expressions derived here. Here the dissipative element connected between the two masses is not a simple viscous dashpot but a nonviscous damper. The equation of motion describing the free vibration of the system can be expressed by equation [1.66], with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \qquad [1.129]$$

and

$$\boldsymbol{\mathcal{G}}(t) = g(t)\hat{\mathbf{I}}, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \qquad [1.130]$$

The damping function $g(t)$ is assumed to be a 'double exponential model', with

$$g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right); \quad c, \mu_1, \mu_2 \geq 0, \qquad [1.131]$$

where c is a constant and μ_1 and μ_2 are known as the relaxation parameters. In equation [1.131] if the function associated with c was a delta-function, c would be the familiar viscous damping constant. Taking the Laplace transform of equation [1.130] one obtains

$$\mathbf{G}(s) = c\hat{\mathbf{I}}\left\{(1 + s/\mu_1)^{-1} + (1 + s/\mu_2)^{-1}\right\}. \qquad [1.132]$$

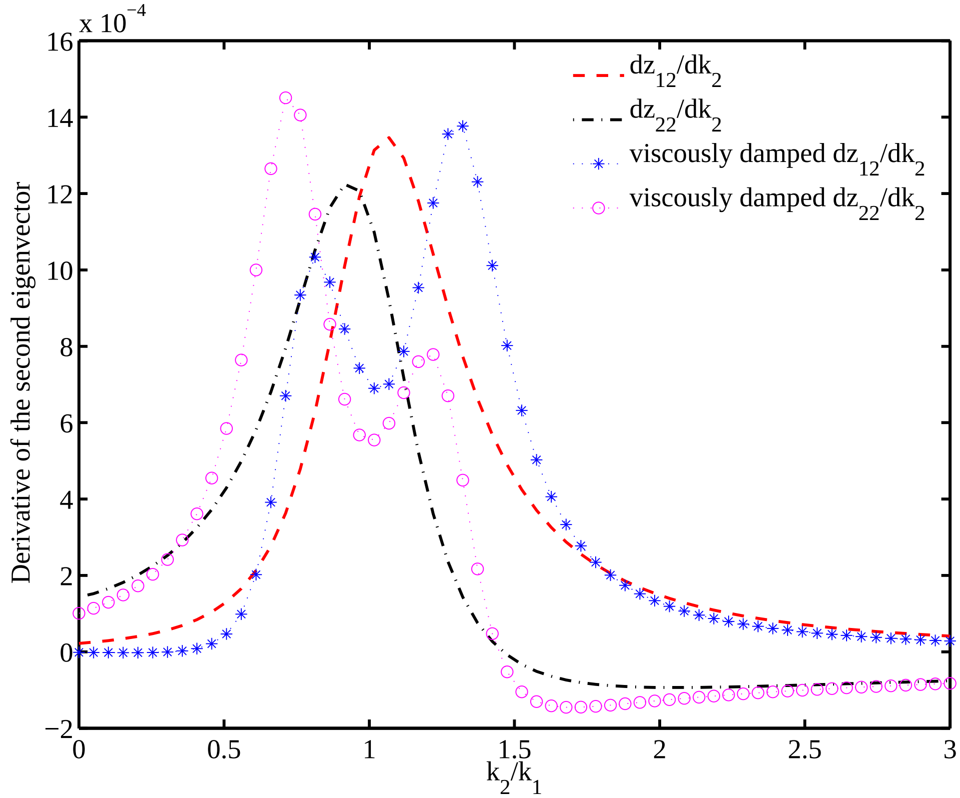

Figure 1.14: Real part of the derivative of the second eigenvalue with respect to k_2

Table 1.1: Eigenvalues and eigenvectors for the example

Quantity	Elastic Mode 1	Elastic Mode 2	Nonviscous Mode 1	Nonviscous Mode 2
s_j	$-0.0387 \pm 38.3232\mathrm{i}$	$-1.5450 \pm 97.5639\mathrm{i}$	-2.8403	-5.9923
$\mathbf{z}_j$	$\left\{\begin{matrix}-0.7500 \pm 0.0043\mathrm{i} \\ -0.6616 \mp 0.0041\mathrm{i}\end{matrix}\right\}$	$\left\{\begin{matrix}0.6622 \mp 0.0035\mathrm{i} \\ -0.7501 \pm 0.0075\mathrm{i}\end{matrix}\right\}$	$\left\{\begin{matrix}-0.0165 \\ 0.0083\end{matrix}\right\}$	$\left\{\begin{matrix}0.0055 \\ -0.0028\end{matrix}\right\}$

Substituting Eqs. [1.129] and [1.132] into equation [1.70] shows that the system has six eigenvalues – four of which occur in complex conjugate pairs and correspond to the two elastic modes. The other two eigenvalues are real and negative and they correspond to the two nonviscous modes. The eigenvalues and the eigenvectors of the system are shown in Table 1.1. The normalisation constants γ_j are selected such that $\gamma_j = 2s_j$ for the elastic modes and $\gamma_j = 1$ for the nonviscous modes.

We consider the derivative of eigenvalues with respect to the stiffness parameter k_1 and the relaxation parameter μ_1. The derivative of the relevant system matrices with respect to k_1 may be obtained as

$$\frac{\partial \mathbf{M}}{\partial k_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial k_1}\Big|_{s=s_j} = \mathbf{O}, \quad \frac{\partial \mathbf{K}}{\partial k_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad [1.133]$$

Table 1.2: Derivative of eigenvalues and eigenvectors with respect to the stiffness parameter k_1

Quantity	Elastic Mode 1	Elastic Mode 2	Nonviscous Mode 1	Nonviscous Mode 2
$\dfrac{\partial s_j}{\partial k_1}$	$0.0001 \pm 0.0073\mathrm{i}$	$0.0001 \pm 0.0022\mathrm{i}$	-2.7106×10^{-4}	-2.9837×10^{-5}
$\dfrac{\partial \mathbf{z}_j}{\partial k_1} \times 10^3$	$\begin{Bmatrix} 0.1130 \mp 0.0066\mathrm{i} \\ 0.0169 \pm 0.0041\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} 0.0385 \mp 0.0015\mathrm{i} \\ 0.0494 \mp 0.0026\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} 0.0072 \\ 0.0046 \end{Bmatrix}$	$\begin{Bmatrix} -0.0018 \\ -0.0018 \end{Bmatrix}$

Table 1.3: Derivative of eigenvalues and eigenvectors with respect to the relaxation parameter μ_1

Quantity	Elastic Mode 1	Elastic Mode 2	Nonviscous Mode 1	Nonviscous Mode 2
$\dfrac{\partial s_j}{\partial \mu_1}$	$-0.0034 \pm 0.0196\mathrm{i}$	$-0.2279 \pm 2.0255\mathrm{i}$	-0.0570	-0.4804
$\dfrac{\partial \mathbf{z}_j}{\partial \mu_1} \times 10^3$	$\begin{Bmatrix} 0.0022 \pm 0.0004\mathrm{i} \\ -0.0021 \mp 0.0003\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} -0.0045 \mp 0.0012\mathrm{i} \\ 0.0098 \pm 0.0015\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} -0.0002 \\ 0.0001 \end{Bmatrix}$	$\begin{Bmatrix} 0.0022 \\ -0.0011 \end{Bmatrix}$

$$\frac{\partial \left[\mathbf{D}(s_j)\right]}{\partial k_1} = \left(2 s_j \mathbf{M} + \mathbf{G}(s_j) - c s_j \hat{\mathbf{I}} \left\{\mu_1^{-1} \left(1 + s_j/\mu_1\right)^{-2} + \mu_2^{-1} \left(1 + s_j/\mu_2\right)^{-2}\right\}\right) \frac{\partial s_j}{\partial k_1} + \frac{\partial \mathbf{K}}{\partial k_1} \quad [1.134]$$

and

$$\frac{\partial \left[\mathbf{D}'(s_j)\right]}{\partial k_1} = \left(2\mathbf{M} - 2c\hat{\mathbf{I}} \left\{\mu_1^{-1} \left(1 + s_j/\mu_1\right)^{-3} + \mu_2^{-1} \left(1 + s_j/\mu_2\right)^{-3}\right\}\right) \frac{\partial s_j}{\partial k_1}. \quad [1.135]$$

Using these expressions, the derivative of the eigenvalues and eigenvectors are obtained from Eqs. [1.80] and [1.120] shown in Table 1.2.

The derivatives of the eigensolutions with respect to the relaxation parameter μ_1 may be obtained using similar manner. The derivative of the relevant system matrices with respect to μ_1 may be obtained as

$$\frac{\partial \mathbf{M}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{K}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial \mu_1}\Big|_{s=s_j} = c\hat{\mathbf{I}}\, s_j \mu_1^{-2} \left(1 + s_j/\mu_1\right)^{-2}, \quad [1.136]$$

$$\frac{\partial \left[\mathbf{D}(s_j)\right]}{\partial \mu_1} = \left(2 s_j \mathbf{M} + \mathbf{G}(s_j) - c s_j \hat{\mathbf{I}} \left\{\mu_1^{-1} \left(1 + s_j/\mu_1\right)^{-2} + \mu_2^{-1} \left(1 + s_j/\mu_2\right)^{-2}\right\}\right) \frac{\partial s_j}{\partial \mu_1} + c\hat{\mathbf{I}} s_j^2 \mu_1^{-2} \left(1 + s_j/\mu_1\right)^{-2} \quad [1.137]$$

and

$$\frac{\partial \left[\mathbf{D}'(s_j)\right]}{\partial \mu_1} = \left(2\mathbf{M} - 2c\hat{\mathbf{I}} \left\{\mu_1^{-1} \left(1 + s_j/\mu_1\right)^{-3} + \mu_2^{-1} \left(1 + s_j/\mu_2\right)^{-3}\right\}\right) \frac{\partial s_j}{\partial \mu_1} + 2c\hat{\mathbf{I}}\, s_j \mu_1^{-2} \left(1 + s_j/\mu_1\right)^{-3}. \quad [1.138]$$

Using these expressions, the derivative of the eigenvalues and eigenvectors are obtained from Eqs. [1.80] and [1.120] shown in Table 1.3.

1.4. Summary

Sensitivity of the eigenvalues and eigenvectors of linear damped discrete systems with respect to the system parameters have been derived. In the presence of general non-proportional viscous damping, the eigenvalues and eigenvectors of the system become complex. The results are presented in terms of changes in the mass, damping, stiffness matrices and complex eigensolutions of the second-order system so that the state-space representation of equation of motion can be avoided. The expressions derived hereby generalise earlier results on derivatives of eigenvalues and eigenvectors of undamped systems to the damped systems. It was shown through an example problem that the use of the expression for the derivative of undamped modes can give rise to erroneous results even when the modal damping is quite low. For non-classically damped systems the expressions for the sensitivity of eigenvalues and eigenvectors developed in this chapter should be used. These complex eigensolution derivatives can be useful in various application areas, for example, finite element model updating, damage detection, design optimisation and system stochasticity analysis relaxing the present restriction to use the real undamped modes only.

In general structural systems are expected to be nonviscously damped. The derivative of eigenvalues and eigenvectors of nonviscously damped discrete linear systems have been derived. The assumed nonviscous damping forces depend on the past history of velocities via convolution integrals over suitable kernel functions. The familiar viscous damping model is a special case corresponding to a 'memory-less' kernel. It has been assumed that, in general, the mass and the stiffness matrices as well as the matrix of the kernel functions cannot be simultaneously diagonalised by any linear transformation. The analysis is, however, restricted to systems with non-repetitive eigenvalues and non-singular mass matrices. Eigenvectors of linear dynamic systems with general nonviscous damping do not satisfy any kind of orthogonality relationship (not even in the usual state-space). For this reason none of the established methodologies for determination of the derivative of eigenvectors are applicable to nonviscously damped systems. An approach is shown which utilises the eigenvalue problem of the associated complex dynamic stiffness matrix. The original eigenvalue problem is a limiting case of this eigenvalue problem. The expressions derived for the derivative of eigenvalues and eigenvectors (equations [1.76] and [1.102]) are very general and also valid for undamped and viscously damped systems. This analysis opens up the possibility of extending the conventional modal updating and parameter estimation techniques to nonviscously damped systems.

So far in this book we have discussed dynamics of damped systems with known parameters. In the next two chapters we show how the damping parameters can be identified from structural dynamic experiments.

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Author index

Index

E

F