Structural Dynamic Analysis with Generalised Damping Models: Analysis

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To Sonia Adhikari
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Preface

Among the various ingredients of structural dynamics, damping remains one of the least understood topics. The main reason is that unlike the stiffness and inertia forces, the damping forces cannot be always obtained from 'first principles'. The past two decades have seen significant developments in the modelling and analysis of damping in the context of engineering dynamic systems. Developments in composite materials including nanocomposites and their applications in advanced structures, such as new generation of aircrafts and large wind-turbines, have lead to the need for understanding damping in a superior manner. Additionally, the rise of vibration energy harvesting technology using piezoelectric and electromagnetic principles further enhanced the importance of looking at damping more rigorously. The aim of this book is to systematically present the latest developments in the area modelling and analysis of damping in the context of general linear dynamic systems with multiple degrees of freedom. The focus has been on the mathematical and computational aspects. This book will be relevant to aerospace, mechanical and civil engineering disciplines and various sub-disciplines within them. The intended readers of this book include senior undergraduate students and graduate students doing projects or doctoral research in the filed of damped vibration. Researchers, Professors and and practicing engineers working in the field of advanced vibration will find this book useful. This book will also be useful for researchers working in the fields of aeroelasticity and hydroelasticity, where complex eigenvalue problems routinely arise due to fluid-structure interactions.

There are some excellent books which already exist in the filed of damped vibration. The book by Nashif *et al.* [NAS 85] covers various material damping models and their applications in the design and analysis of dynamic systems. A valuable reference on dynamic analysis of damped structures is the book by Sun and Lu [SUN 95]. The book by Beards [BEA 96] takes a pedagogical approach towards structural vibration of damped systems. The handbook by Jones [JON 01] focuses on viscoelastic damping and analysis of structures with such damping models. These books represent the sate-of-the art at the time of their publications. Since these publications significant research works have gone into the dynamics of damped systems. The aim of this book is to cover some of these latest developments. The attention is mainly limited to theoretical and computational aspects, although some reference to experimental works are given.

One of the key feature of this book is the consideration of general nonviscous damping and how such general models can be seamlessly integrated into the framework of conventional structural dynamic analysis. New results are illustrated by numerical examples and wherever possible connections were made to well-known concepts of viscously damped systems. The book is divided into two volumes. The first volume deals with analysis of linear systems with general damping models. The second volume deals with identification and quantification of damping. There are ten chapters and one appendix in the book - covering analysis and identification of dynamic systems with viscous and nonviscous damping. Chapter 1 gives an introduction to the various damping models. Dynamics of viscously damped systems are discussed in chapter 2. Chapter 3 considers dynamics of nonviscously damped single-degree-of-freedom systems in details. Chapter 4 discusses nonviscously damped multiple-degree-of-freedom systems. Linear systems with general nonviscous damping are studied in Chapter 5. Chapter 6 proposes reduced computational methods for damped systems. Chapter 7 describes parametric sensitivity of damped systems. Chapter 8 takes up the problem of identification of viscous damping. The identification of nonviscous damping is detailed in Chapter 9. Chapter 10 gives some tools for the quantification of damping. A method to deal with general asymmetric systems is described in the appendix.

This book is a result of last 15 years of research and teaching in the area of damped vibration problems. Initial chapters started taking shape when I offered a course on advanced vibration at the University of Bristol. The later chapters originated from research works with numerous colleagues, students, collaborators and mentors. I am deeply indebted to all of them for numerous stimulating scientific discussions, exchanges of ideas and in many occasions direct contributions towards the intellectual content of the book. I am grateful to my teachers Professor C. S. Manohar (Indian Institute of Science, Bangalore), Professor R. S. Langley (University of Cambridge) and in particular Professor J. Woodhouse (University of Cambridge), who was heavily involved with the works reported in chapters 8-10. I am very thankful to my colleague Professor M. I. Friswell with whom I have a long-standing collaboration. Some joint works are directly related to the content of this book (chapter 7 in particular). I would also like to thank Professor D. J. Inman (University of Michigan) for various scientific discussions during his visits to Swansea. I am thankful to Professor A. Sarkar (Carleton University) and his doctoral student M. Khalil for joint research works. I am deeply grateful to Dr A. S. Phani (University of British Columbia) for various discussions related to damping identification and contributions towards chapters 2, 5 and 8. Particular thanks to Dr N. Wagner (Intes GmbH, Stuttgart) for joint works on nonviscously damped systems and contributions in chapter 4. I am also grateful to Professor F. Papai for involving me on research works on damping identification. My former PhD students B Pascual (contributed in chapter 6), J. L. du Bois, F. A. Diaz De la O deserves particular thanks for various contributions throughout their time with me ad putting up with my busy schedules. I am grateful to Dr Y. Lei (University of Defence Technology, Changsha) for carrying out joint research with me on nonviscously damped continuous systems. I am grateful to Professor A. W. Lees (Swansea University), Professor N. Lieven, Professor F. Scarpa (University of Bristol), Professor D. J. Wagg (University of Sheffield), Professor S. Narayanan (IIT Madras), Professor G. Litak (Lublin University), E. Jacquelin (Université Lyon), Dr A. Palmeri (Loughborough University), Professor S. Bhattacharya (University of Surrey), Dr S. F. Ali (IIT Madras), Dr R. Chowdhury (IIT Roorkee), Dr P. Duffour (University College London) and Dr P. Higino, Dr G. Caprio & Dr A. Prado (Embraer Aircraft) for their intellectual contributions and discussions at different times. Beside the names taken here, I am thankful to many colleagues, fellow researchers and students working in this field of research around the world, whose name cannot be listed here for page limitations. The lack of explicit mentions by no means implies that their contributions are any less. The opinions presented in the book are entirely mine, and none of my colleagues, students, collaborators and mentors has any responsibility for any shortcomings.

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Sondipon Adhikari
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Nomenclature

C'_{jj}	diagonal element of the modal damping matrix
$\alpha_k^{(j)}$	terms in the expansion of approximate complex modes
α_1, α_2	proportional damping constants
α_j	coefficients in Caughey series, $j = 0, 1, 2 \cdots$
$\mathbf{0}_j$	a vector of j zeros
$\mathbf{A}$	state-space system matrix
$\mathbf{a}_j$	a coefficient vector for the expansion of j-th complex mode
$\boldsymbol{\alpha}$	a vector containing the constants in Caughey series
$\bar{h}(\mathrm{i}\omega)$	frequency response function of a SDOF system
$\mathbf{B}$	state-space system matrix
$\mathbf{b}_j$	a vector for the expansion of j-th complex mode
$\bar{\mathbf{f}}(s)$	forcing vector in the Laplace domain
$\bar{\mathbf{f}}'(s)$	modal forcing function in the Laplace domain
$\bar{\mathbf{p}}(s)$	effective forcing vector in the Laplace domain
$\bar{\mathbf{q}}(s)$	response vector in the Laplace domain
$\bar{\mathbf{u}}(s)$	Laplace transform of the state-vector of the first-order system
$\bar{\mathbf{y}}(s)$	modal coordinates in the Laplace domain
$\bar{\mathbf{y}}_k$	Laplace transform of the internal variable $\mathbf{y}_k(t)$
$\mathbb{R}^+$	positive real line
$\mathbf{C}$	viscous damping matrix
$\mathbf{C}'$	modal damping matrix
$\mathbf{C}_0$	viscous damping matrix (with a nonviscous model)
$\mathbf{C}_k$	coefficient matrices in the exponential model for $k = 0, ..., n$, where n is the number of kernels
$\boldsymbol{\mathcal{G}}(t)$	nonviscous damping function matrix in the time domain
$\boldsymbol{\Delta}\mathbf{K}$	error in the stiffness matrix
$\boldsymbol{\Delta}\mathbf{M}$	error in the mass matrix
β	nonviscous damping factor
β_c	critical value of β for oscillatory motion, $\beta_c = \frac{1}{3\sqrt{3}}$
$\beta_i(\bullet)$	proportional damping functions (of a matrix)
$\beta_k(s)$	Coefficients in the state-space modal expansion
β_{mU}	the value of β above which the frequency response function always has a maximum
$\mathbf{F}$	linear matrix pencil with time step in state-space, $\mathbf{F} = \mathbf{B} - \frac{h}{2}\mathbf{A}$
$\mathbf{F}_1, \mathbf{F}_2$	linear matrix pencils with time step in the configuration space
$\mathbf{F}_j$	Regular linear matrix pencil for the j-th mode
$\mathbf{f}'(t)$	forcing function in the modal coordinates
$\mathbf{f}(t)$	forcing function

$\mathbf{G}(s)$	nonviscous damping function matrix in the Laplace domain
$\mathbf{G}_0$	the matrix $\mathbf{G}(s)$ at $s \to 0$
$\mathbf{G}_\infty$	the matrix $\mathbf{G}(s)$ at $s \to \infty$
$\mathbf{H}(s)$	frequency response function matrix
$\hat{\mathbf{u}}_j$	Real part of $\hat{\mathbf{z}}_j$
$\hat{\mathbf{v}}_j$	Imaginary part of $\hat{\mathbf{z}}_j$
$\hat{\mathbf{z}}_j$	j-th measured complex mode
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{O}_{ij}$	a null matrix of dimension $i \times j$
$\mathbf{\Omega}$	diagonal matrix containing the natural frequencies
$\mathbf{p}$	parameter vector (in **??**)
$\mathbf{P}_j$	a diagonal matrix for the expansion of j-th complex mode
$\boldsymbol{\phi}_j$	eigenvectors in the state-space
$\boldsymbol{\psi}_j$	left eigenvectors in the state-space
$\mathbf{q}(t)$	displacement response in the time-domain
$\mathbf{q}_0$	vector of initial displacements
$\mathbf{Q}_j$	an off-diagonal matrix for the expansion of j-th complex mode
$\mathbf{r}(t)$	forcing function in the state-space
$\mathbf{R}_k$	rectangular transformation matrices (in chapter 4)
$\mathbf{R}_k$	residue matrix associated with pole s_k
$\mathbf{S}$	a diagonal matrix containing eigenvalues s_j
$\mathbf{T}$	a temporary matrix, $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$ (**??**)
$\mathbf{T}_k$	Moore-Penrose generalised inverse of $\mathbf{R}_k$
$\mathbf{T}_k$	a transformation matrix for the optimal normalisation of the k-th complex mode
$\mathbf{\Theta}$	Normalisation matrix
$\mathbf{u}(t)$	the state-vector of the first-order system
$\mathbf{u}_0$	vector of initial conditions in the state-space
$\mathbf{u}_j$	displacement at the time step j
$\mathbf{v}(t)$	velocity vector $\mathbf{v}(t) = \dot{\mathbf{q}}(t)$
$\mathbf{v}_j$	a vector of the j-modal derivative in Nelson's methods (in **??**)
$\mathbf{v}_j$	velocity at the time step j
$\boldsymbol{\varepsilon}_j$	Error vector associated with j-th complex mode
$\boldsymbol{\varphi}_k(s)$	eigenvectors of the dynamic stiffness matrix
$\mathbf{W}$	coefficient matrix associated with the constants in Caughey series
$\mathbf{X}$	matrix containing the undamped normal modes $\mathbf{x}_j$
$\mathbf{x}_j$	undamped eigenvectors, $j = 1, 2, \cdots, N$
$\mathbf{y}(t)$	modal coordinate vector (in chapter 2)
$\mathbf{y}_k(t)$	vector of internal variables, $k = 1, 2, \cdots n$
$\mathbf{y}_{k,j}$	internal variable $\mathbf{y}_k$ at the time step j
$\mathbf{Z}$	matrix containing the complex eigenvectors $\mathbf{z}_j$
$\mathbf{z}_j$	complex eigenvectors in the configuration space
$\boldsymbol{\zeta}$	diagonal matrix containing the modal damping factors
$\boldsymbol{\zeta}_v$	a vector containing the modal damping factors
χ	merit function of a complex mode for optimal normalisation
χ_R, χ_I	merit functions for real and imaginary parts of a complex mode
Δ	perturbation in the real eigenvalues
δ	perturbation in complex conjugate eigenvalues

$\dot{q}_0$	initial velocity (SDOF systems)
ϵ	small error
η	ratio between the real and imaginary parts of a complex mode
$\mathcal{F}$	dissipation function
γ	Non-dimensional characteristic time constant
γ_j	complex mode normalisation constant
γ_R, γ_I	weights for the normalisation of the real and imaginary parts of a complex mode
$\hat{\theta}(\omega)$	Frequency dependent estimated characteristic time constant
$\hat{\theta}_j$	Estimated characteristic time constant for j-th mode
$\hat{t}$	an arbitrary independent time variable
κ_j	real part of the complex optimal normalisation constant for the j-th mode
λ	complex eigenvalue corresponding to the oscillating mode (in chapter 3)
λ_j	complex frequencies MDOF systems
$\mathcal{M}_r$	moment of the damping function
$\mathcal{D}$	dissipation energy
$\mathcal{G}(t)$	nonviscous damping kernel function in a SDOF system
$\mathcal{T}$	kinetic energy
$\mathcal{U}$	potential energy
μ	relaxation parameter
μ_k	relaxation parameters associated with coefficient matrix $\mathbf{C}_k$ in the exponential nonviscous damping model
ν	real eigenvalue corresponding to the overdamped mode
$\nu_k(s)$	eigenvalues of the dynamic stiffness matrix
ω	driving frequency
ω_d	damped natural frequency of SDOF systems
ω_j	undamped natural frequencies of MDOF systems, $j = 1, 2, \cdots, N$
ω_n	undamped natural frequency of SDOF systems
$\omega_{\max}$	frequency corresponding to the maximum amplitude of the response function
ω_{d_j}	damped natural frequency of MDOF systems
ρ	mass density
i	unit imaginary number, $\mathrm{i} = \sqrt{-1}$
τ	dummy time variable
θ_j	characteristic time constant for j-th nonviscous model
$\tilde{\mathbf{f}}(t)$	forcing function in the modal domain
$\tilde{\omega}$	normalised frequency ω/ω_n
ς_j	imaginary part of the complex optimal normalisation constant for the j-th mode
ϑ	phase angle of the response of SDOF systems
ϑ_j	phase angle of the modal response
ψ	a trail complex eigenvector (in chapter 2)
$\widehat{\mathbf{A}}$	asymmetric state-space system matrix
$\widehat{\mathbf{C}}$	fitted damping matrix
$\widehat{f}(\omega_j)$	fitted generalised proportional damping function (in **??**)
$\widetilde{\mathbf{A}}$	state-space system matrix for rank deficient systems
$\widetilde{\mathbf{B}}$	state-space system matrix for rank deficient systems
$\widetilde{\mathbf{i}}_r$	integration of the forcing function in the state-space for rank deficient systems
$\widetilde{\mathbf{i}}_r$	integration of the forcing function in the state-space
$\widetilde{\mathbf{\Phi}}$	matrix containing the state-space eigenvectors for rank deficient systems
$\widetilde{\boldsymbol{\phi}}_j$	eigenvectors in the state-space for rank deficient systems
$\widetilde{\mathbf{r}}(t)$	forcing function in the state-space for rank deficient systems

$\widetilde{\mathbf{u}}(t)$	the state vector for rank deficient systems
$\widetilde{\mathbf{y}}_k(t)$	vector of internal variables for rank deficient systems, $k = 1, 2, \cdots n$
$\widetilde{\mathbf{y}}_{k,j}$	internal variable $\mathbf{y}_k$ at the time step j for rank deficient systems
$\widetilde{\boldsymbol{y}}_{k_j}$	jth eigenvector corresponding to the k-th the internal variable for rank deficient systems
ξ	a function of ζ defined in equation [3.132]
ζ	viscous damping factor
ζ_c	critical value of ζ for oscillatory motion, $\zeta_c = \frac{4}{3\sqrt{3}}$
ζ_j	modal damping factors
ζ_L	lower critical damping factor
ζ_n	equivalent viscous damping factor
ζ_U	upper critical damping factor
ζ_{mL}	the value of ζ below which the frequency response function always has a maximum
a_k, b_k	nonviscous damping parameters in the exponential model
B	response amplitude of SDOF systems
B_j	modal response amplitude
c	viscous damping constant of a SDOF system
c_k	coefficients of exponential damping in a SDOF system
c_{cr}	critical damping factor
d_j	a constant of the j-modal derivative in Nelson's methods
E	Young's modulus
$f(t)$	forcing function (SDOF systems)
$f_d(t)$	nonviscous damping force
$G(\mathrm{i}\omega)$	non-dimensional frequency response function
$G(s)$	nonviscous damping kernel function in the Laplace domain (SDOF systems)
$g_{(i)}$	scalar damping functions, $i = 1, 2, \cdots$
h	constant time step
$h(t)$	impulse response function of SDOF systems
$h(t)$	impulse response function
I_k	non-proportionally indices, $k1 = 1, 2, 3, 4$
k	spring stiffness of a SDOF system
L	length of the rod
l_e	length of an element
m	dimension of the state-space for nonviscously damped MDOF systems
m	mass of a SDOF system
N	number of degrees of freedom
n	number of exponential kernels
n_d	number of divisions in the time axis
p	any element in the parameter vector **p** (in **??**)
$q(t)$	displacement in the time domain
q_0	initial displacement (SDOF systems)
Q_{nc_k}	non-conservative forces
$R(\mathbf{x})$	Rayleigh quotient for a trail vector **x**
R_1, R_2, R_3	three new Rayleigh quotients
r_j	normalised eigenvalues of nonviscously damped SDOF systems (in chapter 3)
r_k	rank of $\mathbf{C}_k$ matrices
s	Laplace domain parameter
s_j	eigenvalues of dynamic systems
t	time
T_n	natural time period of an undamped SDOF system

T_{min}	Minimum time period for the system
$varrho_j$	complex optimal normalisation constant for the j-th mode
x	normalised frequency-squared, $x = \omega^2/\omega_n^2$ (in chapter 3)
y_j	modal coordinates (in chapter 2)
$\bar{f}(s)$	forcing function in the Laplace domain
$\bar{q}(s)$	displacement in the Laplace domain
$\hat{\mathbf{U}}$	Matrix containing $\hat{\mathbf{u}}_j$
$\hat{\mathbf{V}}$	Matrix containing $\hat{\mathbf{v}}_j$
$\mathbf{\Phi}$	matrix containing the eigenvectors $\boldsymbol{\phi}_j$
$\dot{\mathbf{q}}_0$	vector of initial velocities
$\mathcal{F}_i(\bullet, \bullet)$	nonviscous proportional damping functions (of a matrix)
$\boldsymbol{Y}_k$	a matrix of internal eigenvectors
$\boldsymbol{y}_{kj}$	jth eigenvector corresponding to the k-th the internal variable
PSD	Power spectral density
$\mathbf{0}$	a vector of zeros
$\mathcal{L}$	Lagrangian (in chapter 3)
$\delta(t)$	Dirac-delta function
δ_{jk}	Kroneker-delta function
$\Gamma(\bullet)$	Gamma function
γ	Lagrange multiplier (in chapter 3)
$(\bullet)^*$	complex conjugate of $(\bullet)$
$(\bullet)^T$	matrix transpose
$(\bullet)^{-1}$	matrix inverse
$(\bullet)^{-T}$	matrix inverse transpose
$(\bullet)^H$	Hermitian transpose of $(\bullet)$
$(\bullet)_e$	elastic modes
$(\bullet)_{nv}$	nonviscous modes
$(\dot{\bullet})$	derivative with respect to time
$\mathbb{C}$	space of complex numbers
$\mathbb{R}$	space of real numbers
$\perp$	orthogonal to
$\mathcal{L}(\bullet)$	Laplace transform operator
$\mathcal{L}^{-1}(\bullet)$	inverse Laplace transform operator
$\det(\bullet)$	determinant of $(\bullet)$
$\mathrm{diag}\,[\bullet]$	a diagonal matrix
$\forall$	for all
$\Im(\bullet)$	imaginary part of $(\bullet)$
$\in$	belongs to
$\notin$	does not belong to
$\otimes$	Kronecker product
$\overline{(\bullet)}$	Laplace transform of $(\bullet)$
$\Re(\bullet)$	real part of $(\bullet)$
vec	vector operation of a matrix
$O(\bullet)$	in the order of
ADF	Anelastic Displacement Field model
$\mathrm{adj}(\bullet)$	adjoint matrix of $(\bullet)$
GHM	Golla, McTavish and Hughes model
MDOF	multiple-degree-of-freedom
SDOF	single-degree-of-freedom

Chapter 1

Introduction to Damping Models and Analysis Methods

It is true that the grasping of truth is not possible without empirical basis. However, the deeper we penetrate and the more extensive and embracing our theories become, the less empirical knowledge is needed to determine those theories.

Albert Einstein, December 1952.

Problems involving vibration occur in many areas of mechanical, civil and aerospace engineering: wave loading of offshore platforms, cabin noise in aircrafts, earthquake and wind loading of cable stayed bridges and high rise buildings, performance of machine tools – to pick only few random examples. Quite often vibration is not desirable and the interest lies in reducing it by dissipation of vibration energy or *damping*. Characterisation of damping forces in a vibrating structure has long been an active area of research in structural dynamics. Since the publication of Lord Rayleigh's classic monograph 'Theory of Sound (1877)', a large body of literature can be found on damping. Although the topic of damping is an age old problem, the demands of modern engineering have led to a steady increase of interest in recent years. Studies of damping have a major role in vibration isolation in automobiles under random loading due to surface irregularities and buildings subjected to earthquake loadings. The developments in the fields of robotics and active structures have provided impetus towards developing procedures for dealing with general dissipative forces in the context of structural dynamics. Beside these, in the last few decades, the sophistication of modern design methods together with the development of improved composite structural materials instilled a trend towards lighter structures. At the same time, there is also a constant demand for larger structures, capable of carrying more loads at higher speeds with minimum noise and vibration level as the safety/workability and environmental criteria become more stringent. Examples include very large wind turbines, which are being used increasingly for superior energy generation. Unfortunately, these two demands are conflicting and the problem cannot be solved without proper understanding of energy dissipation or damping behaviour. Recent advances in vibration energy harvesting [ERT 11, ELV 13] demands further understanding of damping [LES 04, ALI 13] as it is crucial for the quantification of harvested energy. It is the aim of this book is to provide fundamental techniques for the analysis and identification of damped structural systems.

In spite of a large amount of research, understanding of damping mechanisms is quite basic compared to the other aspects of modelling. A major reason for this is that, by contrast with inertia and stiffness forces, it is not in general clear which *state variables* are relevant to determine the damping forces. Moreover, it seems that in a realistic situation it is often the structural joints [SEG 06] which are more responsible for the energy dissipation than the (solid) material. There have been detailed studies on the material damping (see, [BER 73]) and also on energy dissipation mechanisms in the joints [EAR 66, BEA 77]. But here difficulty lies in representing all these tiny mechanisms in different parts of the structure in a unified manner. Even in many cases these mechanisms turn out be locally non-linear, requiring an equivalent linearisation technique for a global analysis [BAN 83]. A well-known method to get rid of all these problems is to use the so called 'viscous damping'. This approach was first introduced by Lord Rayleigh [RAY 77] via his famous 'dissipation function', a quadratic expression for the energy dissipation rate with a symmetric matrix of coefficients, the 'damping matrix'. A further idealisation, also pointed out by Rayleigh, is to assume the damping matrix to be a linear combination of the mass and stiffness matrices. Since its introduction this model has been used extensively and is now usually known as 'Rayleigh damping', 'proportional damping' or 'classical damping'. With such a damping model, the *modal analysis* procedure, originally developed for undamped systems, can be used to analyse damped systems in a very similar manner.

In this chapter we review some existing works on damping and set the scene for this book. Attention of this book is on mathematical analysis and identification of damped linear dynamic systems. We also look mainly at discrete or discretised continuous systems. This can be done by employing conventional finite element approximation to the original boundary value problem. This aspect is not discussed here as there already many excellent books which the readers can refer [BAT 76, DAW 84, ZIE 91, BAT 95, FRI 95b, PET 98, HUG 00, COO 01]. Therefore, for the purpose of this book we consider that the mass and stiffness matrices are available so that we mainly focus on the damping aspects.

Different mathematical models of damping used in literature are discussed in section 1.1. Damping models used for single-degree-of-freedom, multiple-degrees-of-freedom and continuous systems have been included. The concepts of viscous and nonviscous damping are introduced. A brief review of modal analysis method for viscously damped systems is given in section 1.2. The state-space method and approximate methods based on the configuration space are reviewed. Analysis methods for nonviscously damped systems section 1.3 are discussed. State-space based methods, time-domain based methods and approximate methods in the configuration space are reviewed. After the review of different analysis methods, we move to review various damping identification methods.

The methods for identification of viscous damping in discussed in section 1.4. Both single and multiple-degrees-of-freedom systems are considered. In section 1.5, the identification methods for nonviscous damping model in linear dynamic systems have been reviewed. For successful modelling and model updating of a dynamic system it is necessary to know how much the eigenvalue and eigenvectors are sensitive to the parameters [MOT 93, FRI 95b, FRI 01]. Therefore, different methods for computing parametric sensitivity of eigenvalues and eigenvectors are reviewed in section 1.6. Sensitivity of undraped, viscously damped and nonviscously damped systems are discussed. Based on the review of existing works, the motivation behind this book in explained in section 1.7. Finally, in section 1.8 the scope of the book is outlined. Here a summary of the topics that are covered in the following chapters are given.

1.1. Models of damping

Damping is the dissipation of energy from a vibrating structure. In this context, the term dissipate is used to mean the transformation of energy into the other form of energy and, therefore, a removal of energy from the vibrating system. The type of energy into which the mechanical energy is transformed is dependent on the system and the physical mechanism that cause the dissipation. For most vibrating system, a significant part of the energy is converted into heat.

The specific ways in which energy is dissipated in vibration are dependent upon the physical mechanisms active in the structure. These physical mechanisms are complicated physical process that are not totally understood. The types of damping that are present in the structure will depend on which mechanisms predominate in the given situation. Thus, any mathematical representation of the physical damping mechanisms in the equation of motion of a vibrating system will have to be a generalisation and approximation of the true physical situation. As Scanlan [SCA 70] has observed, any mathematical damping model is really only a crutch which does not give a detailed explanation of the underlying physics. Majority of vibration books, for example, [MEI 67, MEI 80, PAZ 80, NEW 89, BAT 95, MEI 97, PET 98, GÉR 97, INM 03, FRI 10b], consider the classical viscous damping model. However, there are some excellent books and papers which specifically focus on vibration damping in engineering structures. The books by Bland [BLA 60] and Lazan [LAZ 68] give a detailed account or earlier works on viscoelastic damping and damped systems. The book by Nashif *et al.* [NAS 85] covers various material damping models and their applications in the design and analysis of dynamic systems. A valuable reference on dynamics analysis of damped structures is the book by Sun and Lu [SUN 95]. The book by Beards [BEA 96] takes a pedagogical approach towards structural vibration of damped systems. The handbook by Jones [JON 01] focuses on viscoelastic damping and analysis of structures with such damping models. The important role of damping n the context of earthquake engineering was illustrated in the book by Liang *et al.* [LIA 11]. The recent book by Veselic [VES 11] focuses on mathematical aspects of dynamics of multiple-degree-of-freedom damped systems. The paper by Gaul [GAU 99] gives a comprehensive overview of viscoelastically damped systems. The review paper by Mead [MEA 02] give an overview of damping modelling structural dynamics. The two linked review papers by Vasques *et al.* [VAS 10a, VAS 10b] and the article by Vasques and Cardoso [VAS 11] discuss both mathematical aspects and experimental identification of linear dynamic systems with viscoelastic damping.

For our mathematical convenience, we divide the elements that dissipate energy into three classes: (a) damping in single degree-of-freedom (SDOF) systems, (b) damping in continuous systems, and (c) damping in multiple degree-of-freedom (MDOF) systems. Elements such as dampers of a vehicle-suspension fall in the first class. Dissipation within a solid body, on the other hand, falls in the second class, demands a representation which accounts for both its intrinsic properties and its spatial distribution. Damping models for MDOF systems can be obtained by discretisation of the equation of motion. There have been attempt to mathematically describe the damping in SDOF, continuous and MDOF systems.

1.1.1. *Single degree-of-freedom systems*

s Free oscillation of an undamped SDOF system never die out and the simplest approach to introduce dissipation is to incorporate an ideal viscous dashpot in the model. The damping force (F_d) is assumed to be proportional to the instantaneous velocity, that is

$$F_d(t) = c\,\dot{q}(t) \qquad [1.1]$$

and the coefficient of proportionality, c is known as the dashpot-constant or viscous damping constant. The loss factor, which is the energy dissipation per radian to the peak potential energy in the cycle, is widely accepted as a basic measure of the damping. For a SDOF system this loss factor can be given by

$$\eta = \frac{c|\omega|}{k} \qquad [1.2]$$

where k is the stiffness. The expression similar to this equation have been discussed in Ungar and Kerwin [UNG 62] in the context of viscoelastic systems. Equation [1.2] shows a linear dependence of the loss factor on the driving frequency. This dependence has been discussed by Crandall [CRA 70] where it has been pointed out that the frequency dependence, observed in practice, is usually not of this form. In such cases one often resorts to an equivalent ideal dashpot. Theoretical objections to the approximately constant value of damping over a range of frequency, as observed in aeroelasticity problems, have been raised by Naylor [NAY 70]. On the lines of equation [1.2] one is tempted to define the frequency-dependent dashpot as

$$c(\omega) = \frac{k\eta(\omega)}{|\omega|}. \qquad [1.3]$$

This representation, however has some serious physical limitations. In references [CRA 70, CRA 91, NEW 89, SCA 70] it has been pointed out that such a representation violates causality, a principle which asserts that the states of a system at a given point of time can be affected only by the events in the past and not by those of the future.

Now for the SDOF system, the frequency domain description of the equation of motion can be given by

$$\left[-m\omega^2 + \mathrm{i}\omega c(\omega) + k\right]\bar{q}(\mathrm{i}\omega) = \bar{f}(\mathrm{i}\omega) \qquad [1.4]$$

where $\bar{q}(\mathrm{i}\omega)$ and $\bar{f}(\mathrm{i}\omega)$ are the response and excitation respectively, represented in the frequency domain. Note that the dashpot is now allowed to have frequency dependence. Inserting equation [1.3] into [1.4] we obtain

$$\left[-m\omega^2 + k\left\{1 + \mathrm{i}\eta(\omega)\mathrm{sgn}(\omega)\right\}\right]\bar{q}(\mathrm{i}\omega) = \bar{f}(\mathrm{i}\omega) \qquad [1.5]$$

where sgn(•) represents the sign function. The 'time-domain' representations of equations [1.4] and [1.5] are often taken as

$$m\ddot{q} + c(\omega)\dot{q} + kq = f \qquad [1.6]$$

and

$$m\ddot{q} + kq\left\{1 + \mathrm{i}\eta(\omega)\mathrm{sgn}(\omega)\right\} = f \qquad [1.7]$$

respectively. It has been pointed out by Crandall [CRA 70] that these are not the correct Fourier inverses of equations [1.4] and [1.5]. The reason is that the inertia, the stiffness and the forcing function are inverted properly, while the damping terms in equations [1.6] and [1.7] are obtained by mixing the frequency-domain and time-domain operations. Crandall [CRA 70] calls [1.6] and [1.7] the 'non-equations' in time domain. It has been pointed out by Newland [NEW 89] that only certain forms of frequency dependence for $\eta(\omega)$ are allowed in order to to satisfy causality. Crandall [CRA 70] has shown that the impulse response function for the ideal hysteretic dashpot (η independent of frequency), is given by

$$h(t) = \frac{1}{\pi k \eta_0}.\frac{1}{t}, \quad -\infty < t < \infty. \qquad [1.8]$$

This response function is clearly non-causal since it states that the system responds before the excitation (or the cause) takes place. This non-physical behaviour of the hysteretic damping model is a flaw, and further attempts have been made to cure this problem. Bishop and Price [BIS 86] introduced the band limited hysteretic damper and suggested that it might satisfy the causality requirement. However, Crandall [CRA 91] has further shown that the band-limited hysteretic dashpot is also non-causal. In view of this discussion it can be said that the most of the hysteretic damping model fails to satisfy the casualty condition. Based on the analyticity of the transfer function, Makris [MAK 99] has shown that for causal hysteretic damping the real and imaginary parts of the dynamic stiffness matrix must form a Hilbert transform pair. The Hilbert transform relation is also known as Kramers-Kronig result. He has shown that the causal hysteretic damping model is the limiting case of a linear viscoelastic model with nearly frequency-independent dissipation that was proposed by Biot [BIO 58]. It was also shown that there is a continuous transition from the linear viscoelastic model to the ideally hysteretic damping model. More recently Chen and Zhang [CHE 08] showed that ideal linear hysteretic damper possesses a non-causal impulse response precursor.

The physical mechanisms of damping, including various types of external friction, fluid viscosity, and internal material friction, have been studied rather extensively in some detail and are complicated physical phenomena. However, a certain simplified mathematical formulation of damping forces and energy dissipation can be associated with a class of physical phenomenon. Coulomb damping, for example is used to represent dry friction present in sliding surfaces, such as structural joints. For this kind of damping, the force resisting the motion is assumed to be proportional to the normal force between the sliding surfaces and independent of the velocity except for the sign. The damping force is thus

$$F_d = \frac{\dot{q}}{|\dot{q}|}F_r = \mathrm{sgn}(\dot{q})F_r \qquad [1.9]$$

where F_r is the frictional force. In the context of finding an equivalent viscous damping, Bandstra [BAN 83] has reported several mathematical models of physical damping mechanisms in SDOF systems. For example, velocity squared damping, which is present when a mass vibrates in a fluid or when fluid is forced rapidly through an orifice. The damping force in this case is

$$F_d = \mathrm{sgn}(\dot{q})a\dot{q}^2; \quad \text{or, more generally} \quad F_d = c\dot{q}|\dot{q}|^{n-1} \qquad [1.10]$$

where c is the damping proportionality constant. Viscous damping is a special case of this type of damping. If the fluid flow is relatively slow, i.e. laminar, then by letting $n = 1$ the above equation reduces to the case of viscous damping [1.1].

In the context of viscoelastically damped SDOF systems, there are several studies which analyse the dynamics in details. Free and forced vibration of viscoelastic systems were considered in [MUR 98b, MUR 98a]. Muller [MUL 05] and Adhikari [ADH 05] considered the conditions of oscillatory motion for a viscoelastically damped SDOF system. Sieber *et al.* [SIE 08] considered exponential nonviscous damping with weak nonlinearities in a Duffing oscillator. Equation of motion of such a system can be given by

$$m\frac{\mathrm{d}^2 q}{\mathrm{d}\hat{t}^2} + c\int_{\hat{\tau}=0}^{\hat{\tau}=\hat{t}} \mu \mathrm{e}^{-\mu(\hat{t}-\hat{\tau})}\frac{\mathrm{d}q}{\mathrm{d}\hat{\tau}}\mathrm{d}\hat{\tau} + kq + \alpha k q^3 = A\cos(\Omega \hat{t}), \tag{1.11}$$

Both hardening and softening type of nonlinearities were considered and the stability of the system were discussed. In reference [ADH 08], the dynamic response characteristics of a nonviscously damped oscillator was discussed in details. Genta and Amati10 [GEN 10] considered dynamics of nonviscously damped SDOF system and proposed a general state-space approach. In [ADH 09a] some approximate methods (non state-space approach) for the calculation of eigenvalues of nonviscously damped SDOF system were proposed. Palmeri and Giuseppe [PAL 11b] proposed a Laguerre's polynomial approximation (LPA) technique for time-domain analysis of an oscillator with the generalised Maxwell's model. Some of the techniques can be extended to continuous and multiple-degree-of-freedom systems as discussed next.

1.1.2. *Continuous systems*

Construction of damping models becomes more difficult for continuous systems. Inman [INM 89] applied the GHM approach to simple beams and used the separation of variables approach in conjunction with modal analysis. Banks and Inman [BAN 91] have considered four different damping models for a composite beam. These models of damping are:

1) *Viscous air damping:* For this model the damping operator in the Euler-Bernoulli equation for beam vibration becomes

$$L_1 = \gamma\frac{\partial}{\partial t} \tag{1.12}$$

where γ is the viscous damping constant.

2) *Kelvin-Voigt damping:* For this model the damping operator becomes

$$L_1 = c_d I \frac{\partial^5}{\partial x^4 \partial t} \tag{1.13}$$

where I is the moment of inertia and c_d is the strain-rate dependent damping coefficient. A similar damping model was also used in [MAN 98, ADH 99c] in the context of randomly parametered Euler-Bernoulli beams.

3) *Time hysteresis damping:* For this model the damping operator is assumed as

$$L_1 = \int_{-\infty}^{t} g(\tau) q_{xx}(x, t+\tau)\mathrm{d}\tau \quad \text{where } g(\tau) = \frac{\alpha}{\sqrt{-\tau}}\exp(\beta\tau) \tag{1.14}$$

where α and β are constants. Later, this model will be discussed in detail.

4) Spatial hysteresis damping:

$$L_1 = \frac{\partial}{\partial x}\left[\int_0^L h(x,\xi)\{q_{xx}(x,t) - q_{xt}(\xi,t)\}\mathrm{d}\xi\right] \tag{1.15}$$

The kernel function $h(x, \xi)$ is defined as

$$h(x, \xi) = \frac{a}{b\sqrt{\pi}} \exp[-(x-\xi)^2/2b^2] \qquad [1.16]$$

where b is some constant.

It was observed by them that the spatial hysteresis model combined with a viscous air damping model results in the best quantitative agreement with the experimental time histories. Again, in the context of Euler-Bernoulli beams, Bandstra [BAN 83] has considered two damping models where the damping term is assumed to be of the forms $\{\operatorname{sgn} q_t(x,t)\}\, b_1 q^2(x,t)$ and $\{\operatorname{sgn} q_t(x,t)\}\, b_2 |q(x,t)|$.

Lesieutre [LES 92] considered the dynamics of uniaxial rods with frequency dependent material properties. Lei *et al.* [LEI 06] proposed a Galerkin method for dynamics of beam with distributed nonviscous damping. Friswell *et al.* [FRI 07a, FRI 07b] considered dynamics of Euler-Bernoulli beams with nonlocal and nonviscous damping. They considered the following integro-partial-differential equation as the equation of motion for the beam

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 q(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 q(x,t)}{\partial t^2} + Q_N(x,t) = f(x,t) \qquad [1.17]$$

with the damping force $Q_N(x,t)$ is given by

$$Q_N(x,t) = \frac{\partial^2}{\partial x^2}\left(\int_0^L \int_{-\infty}^t C(x,\xi,t-\tau)\, \frac{\partial^2 \dot{q}(\xi,\tau)}{\partial \xi^2} d\tau d\xi\right) \qquad [1.18]$$

Dynamic analysis of beams with general nonlocal and nonviscous damping [1.18] has been considered by several authors [ADH 07b, DIP 13, FAI 13, GON 12, DIP 11, CHE 11, TSA 09, CHI 09, PAN 13, POT 13, ABU 12]. Yuksel and Dalli [YUK 05] considered longitudinally vibrating elastic rods with locally and non-locally reacting viscous dampers. Cortes and Elejabarrieta [COR 06b] considered longitudinal vibration of a rod with exponential nonviscous damping model. They obtained expressions for complex natural frequencies and mode shapes. Damped vibration of spatially curved one dimensional structures was considered by Otrin and Boltezar [OTR 07b, OTR 07a, OTR 09a]. Xue-chuan *et al.* [XUE 08] studied axial vibration of nonlocal viscoelastic Kelvin bars in-tension. Cortes *et al.* [COR 08] proposed a frequency-domain approach for the axial vibration problem of a uniform elastic rod with a viscoelastic end damper. They derived an analytical solution for the frequency response functions. Calim [CAL 09] analysed the dynamic behaviour of Timoshenko beams on Pasternak-type viscoelastic foundation subjected to time-dependent loads. A Galerkin-type state-space approach for transverse vibrations of slender double-beam systems with viscoelastic inner layer was proposed by Palmeri and Adhikari [PAL 11a]. Calim and Akkurt [CAL 11] considered free vibration analysis of straight and circular Timoshenko beams on elastic foundation. Garcia-Barruetabena *et al.* [GAR 12] proposed both analytical solution and finite element approach for axial vibration of rods with exponential nonviscous damping. A rotating Timoshenko beam with Maxwell-Weichert viscoelastic damping model is used in [SKA 12]. Lei *et al.* [LEI 13b, LEI 13a] studied free vibration of nonlocal Euler-Bernoulli and Timoshenko beams with nonviscous damping. A generalised one-dimensional elastoplastic model based on fractional calculus is presented in [MEN 12a]. Recently, Wang and Inman [WAN 13] considered (Golla, McTavish and Hughes) GHM and Anelastic Displacement Field (ADF) models of viscoelastic damping for the dynamics of constrained layer sandwich beams. Finite element approach and an experimental validation have been reported by the authors.

1.1.3. *Multiple degrees-of-freedom systems*

The most popular approach to model damping in the context of multiple degrees-of-freedom (MDOF) systems is to assume viscous damping. This approach was first introduced by Lord Rayleigh [RAY 77]. By analogy with the potential energy and the kinetic energy, Rayleigh assumed the *dissipation function*, given by

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}}. \qquad [1.19]$$

In the above expression $\mathbf{C} \in \mathbb{R}^{N\times N}$ is a non-negative definite symmetric matrix, known as the viscous damping matrix. It should be noted that not all forms of the viscous damping matrix can be handled within the scope of classical modal analysis. Based on the solution method, viscous damping matrices can be further divided into classical and non-classical damping. Further discussions on viscous damping will follow in section 1.2.

It is important to avoid the widespread misconception that viscous damping is the *only* linear model of vibration damping in the context of MDOF systems. Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model. There have been several efforts to incorporate nonviscous damping models in MDOF systems. References [BAG 83, TOR 87, GAU 91, MAI 98] considered damping modelling in terms of fractional derivatives of the displacements. Following Maia *et al.* [MAI 98], the damping force using such models can be expressed by

$$\mathbf{F}_d = \sum_{j=1}^{l} \mathbf{g}_j D^{\nu_j}[\mathbf{q}(t)]. \tag{1.20}$$

Here $\mathbf{g}_j$ are complex constant matrices and the fractional derivative operator

$$D^{\nu_j}[\mathbf{q}(t)] = \frac{\mathrm{d}^{\nu_j}\mathbf{q}(t)}{\mathrm{d}t^{\nu_j}} = \frac{1}{\Gamma(1-\nu_j)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathbf{q}(t)}{(t-\tau)^{\nu_j}}\mathrm{d}\tau \tag{1.21}$$

where ν_j is a fraction and $\Gamma(\bullet)$ is the Gamma function. The familiar viscous damping appears as a special case when $\nu_j = 1$. We refer the readers to the review papers [SLA 93, ROS 97, GAU 99] for further discussions on this topic. The physical justification for such models, however, may not be always very clear for engineering problems.

Possibly the most general way to model damping within the linear range is to consider nonviscous damping models which depend on the past history of motion via convolution integrals over some kernel functions. A *modified dissipation function* for such damping model can be defined as

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\dot{q}_k\int_0^t \mathcal{G}_{jk}(t-\tau)\dot{q}_j(\tau)\mathrm{d}\tau = \frac{1}{2}\dot{\mathbf{q}}^T\int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau. \tag{1.22}$$

Here $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is a symmetric matrix of the damping kernel functions, $\mathcal{G}_{jk}(t)$. The kernel functions, or others closely related to them, are described under many different names in the literature of different subjects: for example, retardation functions, heredity functions, after-effect functions, relaxation functions. In the special case when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, equation [1.22] reduces to the case of viscous damping as in equation [1.19]. The damping model of this kind is a further generalisation of the familiar viscous damping. By choosing suitable kernel functions, it can also be shown that the fractional derivative model discussed before is also a special case of this damping model. Thus, as pointed by Woodhouse [WOO 98], this damping model is the most general damping model within the scope of a linear analysis.

Golla and Hughes [GOL 85] and McTavish and Hughes [MCT 93] have effecgovely used damping model of the form [1.22] in the context of viscoelastic structures. The damping kernel functions are commonly defined in the frequency/Laplace domain. Conditions which $\mathbf{G}(s)$, the Laplace transform of $\boldsymbol{\mathcal{G}}(t)$, must satisfy in order to produce dissipative motion were given by Golla and Hughes [GOL 85]. The approach pioneered by Lesieutre [LES 90, LES 92, LES 95, LES 96b, LES 96a] usesd a first-order state-space method called the Anelastic Displacement Fields (ADF) method. A selection of different damping models proposed in literature is summarised in Table 1.1. Adhikari and Woodhouse [ADH 03b] proposed four indexes to quantify nonviscous damping when the kernel function can have any form as given in Table 1.1.

1.1.4. *Other studies*

Another major source of damping in a vibrating structure is the structural joints, see [TAN 97] for a recent review. Here, a major part of the energy loss takes place through air-pumping. The air-pumping phenomenon is

Model number	Damping functions	Author and reference
1	$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s+b_k}$	Biot [BIO 55], [BIO 58]
2	$G(s) = as \int_0^{\infty} \frac{\gamma(\rho)}{s+\rho} \mathrm{d}\rho$ $\gamma(\rho) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \leq \gamma \leq \beta \\ 0 & \text{otherwise} \end{cases}$	Buhariwala [BUH 82]
3	$G(s) = \frac{E_1 s^{\alpha} - E_0 b s^{\beta}}{1+bs^{\beta}}$ $0 < \alpha < 1, \quad 0 < \beta < 1$	Bagley and Torvik [BAG 83]
4	$sG(s) = G^{\infty} \left[1 + \sum_k \alpha_k \frac{s^2+2\xi_k\omega_k s}{s^2+2\xi_k\omega_k s+\omega_k^2}\right]$	Golla and Hughes [GOL 85] and McTavish and Hughes [MCT 93]
5	$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s+\beta_k}$	Lesieutre and Mingori [LES 90]
6	$G(s) = c\frac{1-e^{-st_0}}{st_0}$	Adhikari [ADH 98]
7	$G(s) = c\frac{1+2(st_0/\pi)^2-e^{-st_0}}{1+2(st_0/\pi)^2}$	Adhikari [ADH 98]
8	$G(s) = c\, e^{s^2/4\mu} \left[1 - \operatorname{erf}\left(\frac{s}{2\sqrt{\mu}}\right)\right]$	Adhikari and Woodhouse [ADH 01c]

Table 1.1: Summary of damping functions in the Laplace domain

associated with damping when air is entrapped in pockets in the vicinity of a vibrating surface. In these situations, the entrapped air is 'squeezed out' and 'sucked-in' through any available hole. Dissipation of energy takes place in the process of air flow and coulomb-friction dominates around the joints. This damping behaviour has been studied by many authors in some practical situations, for example by Cremer and Heckl [CRE 73]. Earls [EAR 66] has obtained the energy dissipation in a lap joint over a cycle under different clamping pressure. Beards and Williams [BEA 77] have noted that significant damping can be obtained by suitably choosing the fastening pressure at the interfacial slip in joints.

Energy dissipation within the material is attributed to a variety of mechanisms such as thermoelasticity, grain-boundary viscosity, point-defect relaxation etc, see [LAZ 59, LAZ 68, BER 73]. Such effects are in general called material damping. In an imperfect elastic material, the stress-strain curve forms a closed hysteresis loop rather that a single line upon a cyclic loading. Much effort has been devoted by numerous investigators to develop models of hysteretic restoring forces and techniques to identify such systems. For a recent review on this literature we refer the readers to [CHA 98]. Most of these studies are motivated by the observed fact that the energy dissipation from materials is only a weak function of frequency and almost directly proportional to q^n. The exponent on displacement for the energy dissipation of material damping ranges from 2 to 3, for example 2.3 for mild steel [BAN 83]. In this context, another large body of literature can be found on composite materials where many researchers have evaluated a material's specific damping capacity (SDC). Baburaj and Matsukai [BAB 94] and the references therein give an account of research that has been conducted in this area.

1.2. Modal analysis of viscously damped systems

Equation of motion of a viscously damped system can be obtained from the Lagrange's equation, see for example [MEI 67, GÉR 97, MEI 97] for further details. Using the Rayleigh's dissipation function given by [1.19]. The damped forces can be obtained as

$$Q_{\mathrm{nc}_k} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k}, \quad k = 1, \cdots, N \tag{1.23}$$

and consequently the equation of motion can expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{1.24}$$

The aim is to solve this equation, together with the initial conditions, by modal analysis (to be described in details in subsection 2.3.1). Using the modal transformation in [2.69], premultiplying equation [1.24] by the transpose of the modal matrix $\mathbf{X}^T$ and using the mode orthogonality relationships in [2.65] and [2.66], equation of motion of a damped system in the modal coordinates may be obtained as

$$\ddot{\mathbf{y}}(t) + \mathbf{X}^T\mathbf{C}\mathbf{X}\dot{\mathbf{y}}(t) + \mathbf{\Omega}^2\mathbf{y}(t) = \tilde{\mathbf{f}}(t). \quad [1.25]$$

Clearly, unless $\mathbf{X}^T\mathbf{C}\mathbf{X}$ is a diagonal matrix, no advantage can be gained by employing modal analysis because the equations of motion will still be coupled. To solve this problem, it it common to assume *proportional damping*, that is $\mathbf{C}$ is simultaneously diagonalisable with $\mathbf{M}$ and $\mathbf{K}$. Such damping model allows to analyse damped systems in very much the same manner as undamped systems. Later, Caughey and O'Kelly [CAU 65] have derived the condition which the system matrices must satisfy so that viscously damped linear systems possess classical normal modes. Adhikari [ADH 06a] introduced the concept of generalised proportional damping by which the damping matrix can be expressed as matrix-functions of mass and stiffness matrices. This can significantly help in identifying the damping matrix from measured damping factors for multiple modes [ADH 09b, PAP 12]. Several authors have used proportional damping modelling approach in wide ranging applications [BIL 06, SUL 13, CAR 11, SUL 10, OTR 09b, LIN 09]. Phani [PHA 03] discussed the necessary and sufficient conditions for the existence of classical normal modes in damped linear dynamic systems. Recently Chang [CHA 13] investigated the performance of proportional damping in the context of nonlinear multiple-degree-of-freedom (MDOF) systems. In chapter 2, the concept of proportional damping or classical damping will be analysed in more details.

Modes of proportionally damped systems preserve the simplicity of the real normal modes as in the undamped case. Unfortunately there is no physical reason why a general system should behave like this. In fact practical experience in modal testing shows that most real-life structures do not do so, as they possess complex modes instead of real normal modes. This implies that in general linear systems are non-classically damped. When the system is non-classically damped, some or all of the N differential equations in [1.25] are coupled through the $\mathbf{X}^T\mathbf{C}\mathbf{X}$ term and can not be reduced to N second-order uncoupled equation. This coupling brings several complication in the system dynamics – the eigenvalues and the eigenvectors no longer remain real and also the eigenvectors do not satisfy the classical orthogonality relationships. The methods for solving this kind of problem follow mainly two routes, the state-space method and the methods in configuration space or configuration space. A brief discussion of these two approaches is taken up in the following subsections.

1.2.1. *The state-space method*

The state-space method is based on transforming the N second-order coupled equations into a set of $2N$ first-order coupled equations by augmenting the displacement response vectors with the velocities of the corresponding coordinates, see [NEW 89]. Equation [1.24] can be recast as

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{r}(t) \quad [1.26]$$

where $\mathbf{A} \in \mathbb{R}^{2N\times 2N}$ is the system matrix, $\mathbf{r}(t) \in \mathbb{R}^{2N}$ the force vector and $\mathbf{u}(t) \in \mathbb{R}^{2N}$ is the response vector in the state-space given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_N & \mathbf{I}_N \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{u}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix}, \quad \text{and} \quad \mathbf{r}(t) = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{f}(t). \end{Bmatrix} \quad [1.27]$$

In the above equation $\mathbf{O}_N$ is the $N \times N$ null matrix and $\mathbf{I}_N$ is the $N \times N$ identity matrix. The eigenvalue problem associated with the above equation is in term of an asymmetric matrix now. Uncoupling of equations in the state-space is again possible and has been considered by many authors, for example, [MEI 80, NEW 89, VEL 86]. This analysis was further generalised by Newland [NEW 87] for the case of systems involving singular matrices. In the formulation of equation [1.26] the matrix $\mathbf{A}$ is no longer symmetric, and so eigenvectors are no longer orthogonal with respect to it. In fact, in this case, instead of an orthogonality relationship, one obtains a biorthogonality relationship, after solving the adjoint eigenvalue problem. The complete procedure for uncoupling the equations now involves solving two eigenvalue problems, each of which is double the size of an eigenvalue problem in the modal space. The details of the relevant algebra can be found in [MEI 80, MEI 97]. It should be noted that these solution procedures are exact in nature. One disadvantage of such an exact method is that it requires significant numerical effort to determine the eigensolutions. The effort required is evidently intensified by the fact that the eigensolutions of a non-classically damped system are complex. From the analyst's view point another disadvantage is the lack of physical insight afforded by this method which is intrinsically numerical in nature.

Another variation of the state-space method available in the literature is through the use of 'Duncan form'. This approach was introduced in [FOS 58] and later several authors, for example, [BÉL 77, NEL 79, VIG 86, SUA 87, SUA 89, SES 94, REN 97, ELB 09] have used this approach to solve a wide range of interesting problems. The advantage of this approach is that the system matrices in the state-space retain symmetry as in the configuration space.

1.2.2. *Methods in the configuration space*

It has been pointed out that the state-space approach towards the solution of equation of motion in the context of linear structural dynamics is not only computationally expensive but also fails to provide the physical insight which modal analysis in configuration space or configuration space offers. The eigenvalue problem associated with equation [1.24] can be represented by the $\lambda-$matrix problem [LAN 66]

$$s_j^2\mathbf{M}\mathbf{z}_j + s_j\mathbf{C}\mathbf{z}_j + \mathbf{K}\mathbf{z}_j = \mathbf{0} \qquad [1.28]$$

where $s_j \in \mathbb{C}$ is the j-th latent root (eigenvalue) and $\mathbf{z}_j \in \mathbb{C}^N$ is the j-th latent vector (eigenvector). The eigenvalues, s_j, are the roots of the characteristic polynomial

$$\det\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] = 0. \qquad [1.29]$$

The order of the polynomial is $2N$ and the roots appear in complex conjugate pairs. Several authors have studied non-classically damped linear systems by approximate methods. In this section we briefly review the existing methods for this kind of analysis.

1.2.2.1. *Approximate decoupling method*

Consider the equation of motion of a general viscously damped system in the modal coordinates given by [1.25]. Earlier it has been mentioned that due to non-classical nature of the damping this set of N differential equations are coupled through the $\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X}$ term. An usual approach in this case is simply to ignore the off-diagonal terms of the modal damping matrix $\mathbf{C}'$ which couple the equation of motion. This approach is termed the decoupling approximation. For large-scale systems, the computational effort in adopting the decoupling approximation is an order of magnitude smaller than the methods of complex modes. The solution of the decoupled equation would be close to the exact solution of the coupled equations if the non-classical damping terms are sufficiently small. Analysis of this question goes back to Rayleigh [RAY 77]. A preliminary discussion on this topic can be found in [MEI 67, MEI 97]. Thomson *et al.* [THO 74] have studied the effect of neglecting off-diagonal entries of the modal damping matrix through numerical experiments and have proposed a method for improved accuracy. Warburton and Soni [WAR 77] have suggested a criterion for such a diagonalization so that the computed response is acceptable. Using the frequency domain approach, Hasselsman [HAS 76] proposed a criterion for determining whether the equations of motion might be considered practically decoupled if non-classical damping exists. The criterion suggested by him was to have adequate frequency separation between the natural modes.

Using matrix norms, Shahruz and Ma [SHA 88] have tried to find an optimal diagonal matrix $\mathbf{C}_d$ in place of $\mathbf{C}'$. An important conclusion emerging from their study is that if $\mathbf{C}'$ is diagonally dominant, then among all approximating diagonal matrices $\mathbf{C}_d$, the one that minimises the error bound is simply the diagonal matrix obtained by omitting the off-diagonal elements of $\mathbf{C}'$. Using a time-domain analysis Shahruz [SHA 90] has rigorously proved that if $\mathbf{C}_d$ is obtained form $\mathbf{C}'$ by neglecting the off-diagonal elements of $\mathbf{C}'$, then the error in the solution of the approximately decoupled system will be small as long as the off-diagonal elements of $\mathbf{C}'$ are not too large. Udwadia [UDW 09] proved that for systems with non-repeated eigenvalues, the best approximation of a diagonal modal damping matrix is simply to consider the diagonal of the $\mathbf{C}'$ matrix. Mentrasti [MEN 12b] considered complex modal analysis for a proportionally damped structure equipped with linear non-proportionally damped viscous elements.

Ibrahimbegovic and Wilson [IBR 89] have developed a procedure for analysing non-proportionally damped systems using a subspace with a vector basis generated from the mass and stiffness matrices. Their approach avoids the use of complex eigensolutions. An iterative approach for solving the coupled equations is developed in [UDW 90] based on updating the forcing term appropriately. Felszeghy [FEL 93] presented a method which searches for another coordinate system in the neighborhood of the normal coordinate system so that in the new coordinate system removal of coupling terms in the equations of motion produces a minimum bound on the relative error introduced in the approximate solution. Hwang and Ma [HWA 93] have shown that the error due to the decoupling approximation can be decomposed into an infinite series and can be summed exactly in the Laplace domain. They also concluded that by solving a small number of additional coupled equations in an iterative fashion, the accuracy of the approximate solution can be greatly enhanced. Felszeghy [FEL 94] developed a formulation based on biorthonormal eigenvector for modal analysis of non-classically damped discrete systems. The analytical procedure take advantage of simplification that arises when the modal analysis of the motion separated into a classical and non-classical modal vector expansion.

From the above mentioned studies it has been believed that either frequency separation between the normal modes [HAS 76], often known as 'Hasselsman's criteria', or some form of diagonal dominance [SHA 88], in the modal damping matrix $\mathbf{C}'$ is sufficient for neglecting modal coupling. In contrast to these widely accepted beliefs [PAR 92a, PAR 92b, PAR 94] have shown using Laplace transform methods that within the practical range of engineering applications neither the diagonal dominance of the modal damping matrix nor the frequency separation between the normal modes would be sufficient for neglecting modal coupling. They have also given examples when the effect of modal coupling may even increase following the previous criterion. Phani and Adhikari [PHA 08] proposed three Rayleigh quotients for nonproportionally damped systems based on approximate complex modes. It was shown that the stationarity can only be obtained when the modal damping matrix is diagonally dominant.

In the context of approximate decoupling, Shahruz and Srimatsya [SHA 97] considered error vectors in modal and physical coordinates, say denoted by $\mathbf{e_N}(\bullet)$ and $\mathbf{e_P}(\bullet)$ respectively. They have shown that based on the norm (denoted here as $\parallel (\bullet) \parallel$) of these error vectors three cases may arise:

1) $\parallel \mathbf{e_N}(\bullet) \parallel$ is small (respectively, large) and $\parallel \mathbf{e_P}(\bullet) \parallel$ is small (respectively, large)

2) $\parallel \mathbf{e_N}(\bullet) \parallel$ is large but $\parallel \mathbf{e_P}(\bullet) \parallel$ is small

3) $\parallel \mathbf{e_N}(\bullet) \parallel$ is small but $\parallel \mathbf{e_P}(\bullet) \parallel$ is large

From this study, especially in view of case 3, it is clear that the error norms based on the modal coordinates are not reliable to use in the actual physical coordinates. However, they have given conditions when $\parallel \mathbf{e_N}(\bullet) \parallel$ will lead to a reliable estimate of $\parallel \mathbf{e_P}(\bullet) \parallel$. For a flexible structure with light damping, it was shown [GAW 97] that neglecting off-diagonal terms of the modal damping matrix in most practical cases imposes negligible errors in the system dynamics. They also concluded that the requirement of diagonal dominance of the damping matrix is not necessary in the case of small damping, which relaxes the criterion earlier given by [SHA 88].

In order to quantify the extent of non-proportionality, several authors have proposed 'non-proportionality indices'. References [PAR 86, NAI 86] have developed several indices based on modal phase difference, modal polygon areas, relative magnitude of coupling terms in the modal damping matrix, system response, Nyquist plot. Based on the idea related to the modal polygon area, Bhaskar [BHA 99] has proposed two more indices of non-proportionality. Another index based on driving frequency and elements of the modal damping matrix is given in [BEL 90]. Bhaskar [BHA 95] has proposed a non-proportionality index based on the error introduced by ignoring the coupling terms in the modal damping matrix. An analytical index for the quantification of non-proportionality for discrete vibrating systems was developed in [TON 92, TON 94]. It has been shown that the fundamental nature of non-proportionality lies in finer decompositions of the damping matrix. Shahruz [SHA 95] have shown that the analytical index given by [TON 94] solely based on the damping matrix may lead to erroneous results when the driving frequency lies close to a system natural frequency. They have suggested that a suitable index for non-proportionality should include the damping matrix and natural frequencies as well as the excitation vector. Prells and Friswell [PRE 00] have shown that the (complex) modal matrix of a non-proportionally damped system depends on an orthonormal matrix, which represents the phase between different degrees of freedom of the system. For proportionally damped systems this matrix becomes an identity matrix and consequently they have used this orthonormal matrix as an indicator of non-proportionality. Three indices to measure the damping non-proportionality was proposed in [LIU 00]. The first index measures the correlation between the real and imaginary parts of the complex modes, the second index measures the magnitude of the imaginary parts of the complex modes and the third index quantifies the degree of modal coupling. These indices are based on the fact that the complex modal matrix can be decomposed to a product of a real and and complex matrix. Adhikari [ADH 04a] proposed the optimal normalisation of complex modes and suggested a mode-by-mode non-proportionality index. Koruk and Sanliturk [KOR 13] quantified mode shape complexity based on conservation of energy principle when a structure is vibrating at a specific mode during a period of vibration.

In an another line of work, some researchers aimed at diagonalising a linear dynamic system *exactly* using real transformations even when it is non-proportionally damped. Works by Garvey *et al.* [GAR 02b, GAR 02a, GAR 04, ABU 09, PRE 09, TIS 11] proposed the structure preserving transformation and for viscously damped systems and extended the idea to more general linear dynamical systems. In a series of papers, Ma *et al.* [KAW 11, MOR 11a, MOR 11b, MA 10, MA 09, MOR 09, MOR 08b, MOR 08a, MA 04] considered the possibility of decoupling the equation of motion using real modes. They showed that it is possible to diagonalise the **M**, **C**, **K** system using a real transformation even when these matrices are general in nature (i.e., not proportionally damped). These works have the potential to rethink the concept of proportional damping in linear dynamic systems in a new light.

1.2.2.2. *Complex modal analysis*

Other than the approximate decoupling methods, another approach towards the analysis of non-proportionally damped linear systems is to use complex modes. Since the original contribution of Caughey and O'Kelly [CAU 65], many papers have been written on complex modes. Several authors, for example [MIT 90, IMR 95, LAL 95], have given reviews on this subject. Placidi *et al.* [PLA 91] have used a series expansion of complex eigenvectors into the subspace of real modes, in order to identify normal modes from complex eigensolutions. In the context of modal analysis Liang *et al.* [LIA 92] have proposed and analysed the question of whether the existence of complex modes is an indicator of non-proportional damping and how a mode is influenced by damping. Analysing the errors in the use of modal coordinates, [SES 94, IBR 95] have concluded that the complex mode shapes are not necessarily the result of high damping. The complexity of the mode shapes is the result of particular damping distributions in the system and depends upon the proximity of the mode shapes. Liu and Sneckenberger [LIU 94] have developed a complex mode theory for a linear vibrating deficient system based on the assumption that it has a complete set of eigenvectors. Complex mode superposition methods have been used by [OLI 96] in the context of soil structure interaction problems. Balmès [BAL 97] has proposed a method to find normal modes and the associated non-proportional damping matrix from the complex modes. He has also shown that a set of complex modes is complete if it verifies a defined properness condition which is used to find complete approximations of identified complex modes. Garvey *et al.* [GAR 95] have given a relationship between real and imaginary parts of complex modes for general systems whose mass, stiffness and damping can be expressed by real symmetric matrices. They have also observed that the relationship becomes most simple when all roots are complex and the real part of all the roots have same sign. Bhaskar [BHA 99] has analysed complex modes in detail and addressed the problem of visualising the deformed modes shapes when the motion is not synchronous.

While the above mentioned works concentrate on the properties of the complex modes, several authors have considered the problem of determination of complex modes in the configuration space. Cronin [CRO 76] has obtained an approximate solution for a non-classically damped system under harmonic excitation by perturbation techniques. Clough and Mojtahedi [CLO 76] considered several methods of treating generally damped systems, and concluded that the proportional damping approximation may give unreliable results for many cases. Similarly, it was shown [DUN 79] that significant errors can be incurred when dynamic analysis of a non-proportionally damped system is based on a truncated set of modes, as is commonly done in modelling continuous systems. Meirovitch and Ryland [MEI 85] have used a perturbation approach to obtain left and right eigenvectors of damped gyroscopic systems. Chung and Lee [CHU 86] applied perturbation techniques to obtain the eigensolutions of damped systems with weakly non-classical damping. Cronin [CRO 90] has developed an efficient perturbation-based series method to solve the eigenproblem for dynamic systems having non-proportional damping matrix. To illustrate the general applicability of this method, Peres-Da-Silva *et al.* [PER 95] have applied it to determine the eigenvalues and eigenvectors of a damped gyroscopic system. In the context of non-proportionally damped gyroscopic systems Malone *et al.* [MAL 97] have developed a perturbation method which uses an undamped gyroscopic system as the unperturbed system. Based on a small damping assumption, Woodhouse [WOO 98] has given the expression for complex natural frequencies and mode shapes of non-proportionally damped linear discrete systems with viscous and nonviscous damping.

Adhikari [ADH 99a] derived approximate expressions of complex modes using a Neumann expansion for each mode. A general expression of the frequency response function when the system matrices are asymmetric were derived. This is particularly useful when it is not possible to transform a asymmetric system [INM 83, AHM 87, SHA 89, AHM 84a, ADH 00c, LIU 05] to a symmetric one. Gallina [GAL 03] discussed the effect of damping on asymmetric systems. Liu and Zheng [LIU 10] proposed a synthesis method for transient response of nonproportionally damped structures. An iterative approach to obtain complex modes for nonproportionally damped systems was proposed in [ADH 11a].

1.2.2.3. *Response bounds and frequency response*

Previously we have mainly discussed the calculation of the eigensolutions of non-classically damped systems. Here we briefly consider the problem of obtaining dynamic response of such systems. Nicholson [NIC 87b] and Nicholson and Baojiu [NIC 96] have reviewed the literature on stable response of non-classically damped mechanical systems. Nicholson [NIC 87a] gave upper bounds for the response of non-classically damped systems under impulsive loads and step loads. Yae and Inman [YAE 87] have obtained bound on the displacement response of non-proportionally damped discrete systems in terms of physical parameters of the system and input. They also have observed that the larger the deviation from proportional damping the less accurate their results become.

Bellos and Inman [BEL 90] have given a procedure for computing the transfer functions of a non-proportionally damped discrete system. Their method was based on Laplace transformation of the equation of motion in modal coordinates. A fairly detailed survey of the previous research is made in [BEL 90]. Yang [YAN 93] has developed a iterative procedure for calculation of the transfer functions of non-proportionally damped systems. Bhaskar [BHA 95] has analysed the behaviour of errors in calculating frequency response function when the off-diagonal terms of modal damping matrix are neglected. It has been shown that the exact response can be expressed by an infinite Taylor series and the approximation of ignoring the off-diagonal terms of modal damping matrix is equivalent to retaining one term of the series.

Finally, it should be noted that frequency responses of viscously damped systems with non-proportional damping can be obtained *exactly* in terms of the complex frequencies and complex modes in the configuration space, see for example [LAN 66] (Section 7.5) and [GÉR 97] (pp. 126-128). Similar expressions are also derived in [FAW 76, VIG 86, WOO 98]. This in turn requires determination of complex modes in the configuration space. This problem will be discussed in details later in this book.

1.3. Analysis of nonviscously damped systems

In subsection 1.1.3 it was pointed out that the most general way to model (nonviscous) damping within the scope of linear theory is through the use of the modified dissipation function given by equation [1.22]. Equation of motion of such nonviscously damped systems can be obtained from the Lagrange's equation (see for example [MEI 67, GÉR 97, MEI 97]). The damping forces can be obtained as

$$Q_{\mathrm{nc}_k} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k} = -\sum_{j=1}^{N} \int_0^t \mathcal{G}_{jk}(t-\tau)\dot{q}_j(\tau)\mathrm{d}\tau, \quad k = 1, \cdots, N \tag{1.30}$$

and consequently the equation of motion can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{1.31}$$

This is a set of coupled second-order integro-differential equation. The presence of the 'integral' term in the equations of motion complicates the analysis. Unlike the viscously damped systems, the concept of 'proportional damping' cannot easily be formulated for such systems. The question of the existence of classical normal modes in such systems, i.e., if proportional damping can occur in such systems, will be discussed in chapter 5.

Equations similar to [1.31] occur in many different subjects. Bishop and Price [BIS 79] have considered equation of motion similar to [1.31] in the context of ship dynamics. The convolution term appeared in order to represent the fluid forces and moments. They have discussed the eigenvalue problem associated with equation [1.31] and presented an orthogonality relationship for the right and left eigenvectors. They have also given an expression for the system response due to sinusoidal excitation. Their results were not very efficient because the orthogonality relationship of the eigenvectors were not utilised due to the difficulty associated with the form of the orthogonality equation, which itself became frequency dependent. Here we briefly discuss different numerical methods proposed for linear dynamic systems with nonviscous damping.

1.3.1. *State-space based methods*

Equation of motion like [1.31] arise in the dynamics of *viscoelastic structures*. A method to obtain such equations using a time-domain finite-element formulation was proposed in [GOL 85, MCT 93]. Their approach (the GHM method), which introduces additional dissipation coordinates corresponding to the internal dampers, increases the size of the problem. Dynamic responses of the system were obtained by using the eigensolutions of the augmented problem in the state-space. A method to obtain the time and frequency-domain description of the response by introducing additional coordinates like the GHM method was proposed in [MUR 97a, MUR 98a]. To reduce the order of the problem, [FRI 97, PAR 99, FRI 99] have proposed a state-space approach which employs a modal truncation and uses an iterative approach to obtain the eigensolutions. State-space model reduction approach was considered by YC Yiu [YIU 93, YIU 94] using sub-structuring techniques for linear systems with exponential viscoelastic damping model. Trindade *et al.* [TRI 00] considered frequency-dependent viscoelastic material models for active-passive vibration damping and compared two widely used models. Adhikari [ADH 01a] derived the conditions for existence of proportional damping for nonviscously damped systems. Under such conditions undamped normal modes can diagonalise the dynamic system. Palmeri [PAL 03] considered a state-space approach linear dynamic systems with memory. Time domain approach for viscoelastically damped systems were considered in [MUS 05, PAL 04]. Trindade [TRI 06] proposed a reduced order approach for viscoelastically damped beams through projection of the dissipative modes onto the structural modes. Wagner and Adhikari [WAG 03] proposed a symmetric extended state-space approach for exponentially damped systems using internal dissipation coordinates. Adhikari and Wagner [ADH 03a] considered general asymmetric nonviscously damped systems and explained the structures of the left and right eigenvectors. Zhang and Zheng [ZHA 07a] proposed a state-space approach for general linear dynamic system with Biot viscoelastic model. Vasques *et al.* [VAS 06] considered finite element modelling and experimental validation of a beam with frequency dependent viscoelastic damping. Vasques *et al.* [VAS 10a] discussed computational methods for viscoelastically damped systems using reduced approaches in the extended state-space. Genta and Amati10 [GEN 10] considered dynamics of nonviscously damped MDOF systems in the context of rotor dynamics by a state-space approach. Friswell *et al.* [FRI 10a] used internal variables for the time domain analysis of rotors with frequency-dependent damping. The papers by de Lima *et* *al.* [LIM 10, LIM 09] proposed a novel component mode synthesis approach for general viscoelastic linear dynamic systems. Wang and Inman [WAN 13] used a symmetric state-space formulation linear systems with GHM damping.

1.3.2. *Time-domain based methods*

While the above methods often aimed at determining the eigensolutions of the system, few authors have considered the calculation of the dynamic response in the time-domain. Adhikari and Wagner [ADH 04b] proposed a direct time-domain approach for exponentially damped systems which avoids the use of dissipation coordinates. Shen and Duan [SHE 09] proposed a Guass integration approach in conjunction with state-space representation of the equation of motion for linear MDOF systems with exponential damping. An efficient time-domain approach for linear dynamic systems with fractional damping was proposed by Trinks and Ruge [TRI 02]. Cortes and Elejabarrieta [COR 07a] proposed a time-domain integration approach for linear systems with fractional derivative damping model. Later Cortes *et al.* [COR 09] proposed a direct integration formulation for linear dynamics systems with exponential nonviscous damping model. Pan and Wang [PAN 13] proposed a frequency as well as a time-domain approach using a Discrete Fourier Transform (DFT) method in combination with the Fast Fourier Transform (FFT) for exponentially damped systems.

1.3.3. *Approximate methods in the configuration space*

Computational cost for nonviscously damped systems can be prohibitive for large dimensional problems. To address this, several authors have proposed reduced approximate methods in the configuration space. Using a first-order perturbation approach, Woodhouse [WOO 98] has obtained expressions for the eigensolutions and transfer functions of system [1.31]. His method, although it avoids the state-space representations and additional dissipation coordinates, is valid for small damping terms only. Adhikari [ADH 02b] proposed an approximate method based on Neumann expansion for the eigenvectors of linear systems with general nonviscous damping. Several mathematical properties of the eigensolutions of such systems were derived in [ADH 01b]. Cortes and Elejabarrieta [COR 06a, COR 06c] proposed a new approximate method for the complex eigensolutions of a nonviscously damped system. The key idea proposed used the solution of the undamped system and approximated the complex eigensolutions by finite increments using the eigenvector derivatives and the Rayleigh quotient. Garcia-Barruetabena [GAR 11] demonstrated that nonviscous modes only contribute to the transient response in of a linear system with exponential nonviscous damping. Some approximate methods to obtain the eigensolutions of nonviscously damped systems using the eigensolutions of the underlying undamped systems were proposed in [ADH 09a, ADH 10, ADH 11b]. Lázaro *et al.* [L´ 12, L´ 13a, L´ 13b] proposed an approach for the computation of eigensolutions and dynamic response of MDOF system with exponential damping. The motivation was to approximates the response of the original viscoelastic system using the eigensolutions of the underlying undamped or proportionally damped system. Li *et al.* [LI 13a] approximated the frequency response function (FRF) matrix without using the dissipation modes of the linear MDOF systems with viscoelastic hereditary terms. Pawlak and Lewandowski [PAW 13] proposed a reduced computational approach for nonlinear eigenvalue problems arising in nonviscously damped systems.

1.4. Identification of viscous damping

In section 1.2 we have discussed several methods for *analysis* of viscously damped linear dynamic systems. In this section we focus our attention on the methodologies available for identification of viscous damping parameters from experimental measurements.

1.4.1. *Single degree-of-freedom systems systems*

Several methods are available for identifying the viscous damping parameters for single-degree-of-freedom systems for linear and non-linear damping models, see [NAS 85]. For linear damping models these methods can be broadly described as:

1) *Methods based on transient response of the system:* This is also known as logarithmic decrement method: if q_i and q_{i+i} are heights of two subsequent peaks then the damping ratio ζ can be obtained as

$$\delta = \log_e \left(\frac{q_i}{q_{i+i}} \right) \approx 2\pi\zeta \qquad [1.32]$$

For applicability of this method the decay must be exponential.

2) *Methods based on harmonic response of the system:* These methods are based on calculating the half power points and bandwidth from the frequency response curve. It can be shown that the damping factor ζ can be related to a peak of the normalised frequency response curve by

$$|H|_{\max} \approx \frac{1}{2\zeta} \qquad [1.33]$$

3) *Methods based on energy dissipation:* Consider the force-deflection behaviour of a spring-mass-damper (equivalent to a block of material) under sinusoidal loading at some particular frequency. In steady-state, considering conservation of energy, energy loss per cycle (Δq_{cyc}) can be calculated by equating it with the input power. Here it can be shown that the damping factor ζ can be related as

$$2\zeta = \frac{\Delta q_{\mathrm{cyc}}}{2\pi U_{\max}} \qquad [1.34]$$

where $U_{\max}$ is maximum energy of the system.

The above mentioned methods, although developed for single-degree-of-freedom systems, can be used for separate modes of multiple-degree-of-freedom systems, for example a cantilever beam vibrating in the first mode. Chassiakos *et al.* [CHA 98] proposed an on-line parameter identification technique for a single-degree-of-freedom hysteretic system. Some authors have [KHA 09, KHA 10, SRA 11] proposed methods to identify damping parameters in nonlinear systems.

1.4.2. *Multiple degrees-of-freedom systems*

For multiple degree-of-freedom systems, most of the common methods for experimental determination of the damping parameters use the proportional damping assumption. A typical procedure can be described as follows, see [EWI 84] for details:

1) Measure a set of transfer functions $H_{ij}(\omega)$ at a set of grid points on the structure.

2) Obtain the natural frequencies ω_k by a pole-fitting method.

3) Evaluate the modal half-power bandwidth $\Delta\omega_k$ from the frequency response functions, then the Q-factor $Q_k = \frac{\omega_k}{\Delta\omega_k}$ and the modal damping factor $\zeta_k = \frac{1}{2Q_k}$.

4) Determine the modal amplitude factors a_k to obtain the mode shapes, $\mathbf{x}_k$.

5) Finally reconstruct some transfer functions to verify the accuracy of the evaluated parameters.

Such a procedure does not provide reliable information about the nature or spatial distribution of the damping, though the reconstructed transfer functions may match the measured ones well.

The next stage, followed by many researchers, is to attempt to obtain the full viscous damping matrix from the experimental measurements. Pilkey and Inman [PIL 98] have given a recent survey on methods of viscous damping identification. These methods can be divided into two basic categories [FAB 88]: (a) damping identification from modal testing and analysis, and (b) direct damping identification from the forced response measurements.

The modal testing and analysis method seeks to determine the modal parameters, such as natural frequencies, damping ratio and mode shapes, from the measured transfer functions, and then fit a damping matrix to these data.

In one of the earliest works, Lancaster [LAN 61] has given an expression from which the damping matrix can be constructed from complex modes and frequencies. Unfortunately this expression relies on having all the modes, which is almost impossible in practice. For this reason, several authors have proposed identification methods by considering the modal data to be incomplete or noisy. Hasselsman [HAS 72] has proposed a perturbation method to identify a non-proportional viscous damping matrix from complex modes and frequencies. Béliveau [BÉL 76] has proposed a method which uses eigensolutions, phase angles and damping ratios to identify the parameters of viscous damping matrix. His method utilises a Bayesian framework based on eigensolution perturbation and a Newton-Raphson scheme. Ibrahim [IBR 83b] uses the higher order analytical modes together with the experimental set of complex modes to compute improved mass, stiffness and damping matrices. Minas and Inman [MIN 91] have proposed a method for viscous damping identification in which it is assumed that the mass and stiffness are *a priori* known and modal data, obtained from experiment, allowed to be incomplete. Starek and Inman [STA 97] have proposed an inverse vibration problem approach in which it is assumed that the damping matrix has an *a priori* known structure. Their method yields a positive-definite damping matrix but requires the full set of complex modes. Pilkey and Inman [PIL 97] have developed an iterative method for damping matrix identification by using Lancaster's [LAN 61] algorithm. This method requires experimentally identified complex eigensolutions and the mass matrix. Alvin *et al.* [ALV 97] have proposed a method in which a correction was applied to the proportionally damped matrix by means of an error minimisation approach. Halevi and Kenigsbuch [HAL 99] have proposed a method for updating the damping matrix by using the reference basis approach in which error and incompleteness of the measured modal data were taken into account. As an intermediate step, their method corrects the imaginary parts of the measured complex modes which are more inaccurate than their corresponding real parts.

Direct damping identification methods attempt to fit the equations of motion to the measured forced response data at several time/frequency points. Caravani and Thomson [CAR 74] have proposed a least-square error minimisation approach to obtain the viscous damping matrix. Their method uses measured frequency response at a set of chosen frequency points and utilises an iterative method to successively improve the identified parameters. Fritzen [FRI 86] has used the instrumental variable method for identification of the mass, damping and stiffness matrices. It was observed that the identified values are less sensitive to noise compared to what obtained from least-square approach. Fabunmi *et al.* [FAB 88] has presented a damping matrix identification scheme that uses forced response data in the frequency domain and assumes that the mass and stiffness matrices are known. Mottershead [MOT 90] has used the inverse of the frequency response functions to modify the system matrices so that the modified model varies minimally from an initial finite-element model. Using a different approach, Roemer and Mook [ROE 92] have developed methods in the time domain for simultaneous identification of the mass, damping and stiffness matrices. It was observed that the identified damping matrix has larger relative error than that of the mass and stiffness matrices. Chen *et al.* [CHE 96a] have proposed a frequency domain technique for identification of the system matrices in which the damping matrix was determined independently. It was shown that separate identification of the damping matrix improves the result as relative magnitude of the damping matrix is less than those of the mass and stiffness matrices. Later, Baruch [BAR 97] has proposed a similar approach in which the damping matrix was identified separately from the mass and stiffness matrices.

Adhikari and Woodhouse [ADH 01c, ADH 02e] proposed a complex mode based approach for the identification of viscous damping matrix. Later, a method to identify symmetric damping matrices [ADH 02d] were proposed. Li [LI 05] used modulations of the responses to identify damping. Damping identification in pneumatic tyres was discussed by Geng *et al.* [GEN 07] uisng complex modes. A pattern recognition approach was used to identify damping in sucker-rod pumping system [LIU 07]. Erlicher and Argoul [ERL 07] proposed a wavelet transform based method for damping identification. Lin and Zhu [LIN 06] and Phani and Woodhouse [PHA 07, PHA 09] discussed methods for damping matrix identification from measured frequency response functions. Khalil *et al.* [KHA 07] proposed a proper orthogonal decomposition approach for the identification of the damping matrix along with the mass and stiffness matrices. Arora *et al.* [ARO 10, ARO 09a, ARO 09d, ARO 09b, ARO 09c] considered updating of the damping matrix using analytical and experimental approaches. Prandina *et al.* [PRA 09] discussed the philosophy and performance of different damping identification approaches and investigated the role of the first-order perturbation methods and modal truncation on damping identification. A system identification algorithm based on the free vibration response of structures was proposed to identify damping in [ROY 09, CHA 10]. Cavacece *et al.* [CAV 09] used a Moore-Penrose pseudo-inverse for the identification of damping. Pradhan and

Modak [PRA 12a, PRA 12b] considered the determination of damping matrices from the frequency response function data. They developed an updating formulation that seeks to separate updating of the damping matrix from the updating of the stiffness and the mass matrix. Holland *et al.* [HOL 12a, HOL 12b] considered identification of damping in bladed disks. In a series of work Liu *et al.* [LIU 08c, LIU 08d, LIU 08b, LIU 08a, LIU 09] pioneered the Lie-group estimation method for the inverse problem and damping identification in linear structural dynamics. Li and Law [LI 09] proposed a time-domain approach for damping identification using the sensitivity of the acceleration response of the analytical model along with a model updating technique. Algorithms for the mass normalisation of the mode shapes in the context of experimental modal analysis was proposed by Yang *et al.* [YAN 12]. A common-base proper orthogonal decomposition approach was used by Andrianne and Dimitriadis [AND 12]. A pattern recognition approach was proposed to identify damping in sucker-rod pumping system [LIU 07]. Cheonhong *et al.* [MIN 12] discussed a direct method for the identification of non-proportional damping matrix using modal parameters. A two-step model updating algorithm for parameter identification of linear elastic damped structures was proposed by García-Palencia and Santini-Bell [GAR 13]. More recently, Holland and Epureanu [HOL 13] suggested a technique to identify the overall damping matrix utilising identified (simple) damping matrices from different components. They demonstrated the approach for a mistuned blisk with varying levels of measurement noise.

1.5. Identification of nonviscous damping

Unlike viscous damping, there is little available in the literature which discusses generic methodologies for identification of nonviscous damping. Most of the methods proposed in the literature are system-specific. Banks and Inman [BAN 91] have considered the problem of estimating damping parameters in a non-proportionally damped beam. They have taken four different models of damping: viscous air damping, Kelvin-Voigt damping, time hysteresis damping and spatial hysteresis damping, and used a spline inverse procedure to form a least-square fit to the experimental data. A procedure for obtaining hysteretic damping parameters in free-hanging pipe systems is given by Fang and Lyons [FAN 94]. Assuming material damping is the only source of damping they have given a theoretical expression for the loss factor of the n-th mode. Their theory predicts higher modal damping ratios in higher modes. Maia *et al.* [MAI 97b] have emphasised the need for development of identification methodologies of general damping models and indicated several difficulties that might arise. Dalenbring [DAL 99] has proposed a method for identification of (exponentially decaying) damping functions from the measured frequency response functions and finite element displacement modes. A limitation of this method is that it neglects the effect of modal coupling, that is, the identified nonviscous damping model is effectively proportional.

Adhikari and Woodhouse [ADH 01d] proposed a complex mode based approach for the identification of exponential nonviscous damping model. Zhang and Zheng [ZHA 07a] considered the Biot model in the context of MDOF systems and experimentally identified the model parameters. Vasques *et al.* [VAS 10b] discussed experimental identification and model validation of viscoelastically damped systems. They have compared several viscoelastic models in their study. Adhikari [ADH 02c] used a modified Lancaster's method to identify viscous and nonviscous damping matrix from the frequency response function matrix. Cortes and Elejabarrieta [COR 07b] characterised the viscoelastic damping properties of cantilever beams using the seismic response. Parameter identification of dynamical systems with fractional derivative damping models using methods based on inverse sensitivity analysis of damped eigensolutions and frequency response functions was proposed by Sivaprasad *et al.* [SIV 09]. Ding and Law [DIN 11] proposed an iterative regularisation method for the identification of structural damping. Wang and Inman [WAN 13] proposed methods to identify parameters of GHM and ADF models from vibration experiments.

1.6. Parametric sensitivity of eigenvalues and eigenvectors

As seen so far, the characterisation of eigenvalues and eigenvectors constitutes a central role in the design, analysis and identification of damped dynamic systems. As a result, the study of the variation of the eigenvalues and eigenvectors due to variations in the system parameters, or more precisely the sensitivity of eigensolutions, has emerged as an important area of research. For physically representative damping modelling and model updating of a dynamic system, it is necessary to know how much the eigenvalue and eigenvectors might change due to the changes in the parameters [MOT 93, FRI 95b, FRI 01]. For generally damped systems, this tantamounts to computing sensitivity of complex eigenvalues and eigenvectors in general. Sensitivity of eigenvalues and eigenvectors with respect to some system parameters may be represented by their derivatives with respect to those parameters. We briefly review some of the existing works on sensitivity of eigensolutions of undamped and damped systems.

1.6.1. *Undamped systems*

In one of the earliest work, Fox and Kapoor [FOX 68] gave exact expressions for the first derivative of eigenvalues and eigenvectors with respect to any design variable. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure, and have been used extensively in a wide range of application areas of structural dynamics. The expressions derived in [FOX 68] are valid for symmetric undamped systems. In many problems in dynamics the inertia, stiffness and damping properties of the system cannot be represented by symmetric matrices or self-adjoint differential operators. These kind of problems typically arise in the dynamics of actively controlled structures and in many general damped dynamic systems, for example – moving vehicles on roads, missile following trajectories, ship motion in sea water or the study of aircraft flutter. The asymmetry of damping and stiffness terms are often addressed in the context of gyroscopic and follower forces. Many authors [ROG 70, PLA 73, GAR 73, RUD 74] have extended Fox and Kapoor's [FOX 68] approach to determine eigensolution derivatives for more general asymmetric *conservative* systems. For these kind of systems, Nelson [NEL 76] proposed an efficient method to calculate the first-order derivative of eigenvectors which requires only the eigenvalue and eigenvector under consideration. Murthy and Haftka [MUR 88] have written an excellent review on calculating the derivatives of eigenvalues and eigenvectors associated with general (non-Hermitian) matrices. Eigensensitivity analysis of a defective matrix with zero first-order eigenvalue derivatives was considered in [ZHA 04]. A method for modal reanalysis due to topological modifications (which changes the degree of freedom of the system) of structures was discussed by Zhi *et al.* [ZHI 06]. A new eigensolution reanalysis method was developed by Chen *et al.* [CHE 06] based on the Neumann series expansion and epsilon-algorithm. A general approach for incorporating the eigenvector normalisation condition in the computation of eigenvector design sensitivities was proposed in [SMI 06]. Cha and Sabater [CHA 11] considered eigenvalue sensitivities of a linear structure carrying lumped attachments.

First-order derivatives are useful for practical problems as long as the perturbations of the system parameters remain 'small'. To consider a wide range of changes in the design parameters the linear approximation intrinsic with the first-order derivatives may not be sufficient. Apart from large perturbations of system parameters, Brandon [BRA 84] has shown that the second-order eigensolution derivatives are not negligible compared to the first-order derivatives when the system has closely spaced natural frequencies. Second-order eigensolution derivatives are also required in design optimisation to calculate the so called 'Hessian Matrix'. For these reasons there has been considerable interest in obtaining the second-order derivatives of the eigensolutions. Plaut and Huseyin [PLA 73] gave an expression for the second derivative of the eigenvalues for asymmetric systems. Rudisill [RUD 74] suggested a similar expression for the second derivative of the eigenvalues and went on to derive the second derivative of the eigenvectors. Brandon [BRA 91] derived the second derivative of the eigenvalues and eigenvectors for the case when the system matrices are linear functions of the design variables. Chen *et al.* [CHE 94a, CHE 94b] derived the second-order derivative of eigenvectors in terms of a series in the eigenvectors. Friswell [FRI 95a] proposed a method, similar to [NEL 76], to obtain the second-order derivative of the eigenvectors which employs only the eigensolutions of interest. Most of the methods discussed so far do not explicitly consider damped systems. In order to apply these results to obtain the second derivatives of the eigensolutions of general (non-proportionally) damped systems, the state-space formalism is required.

1.6.2. *Damped systems*

The work discussed so far does not explicitly consider the damping present in the system. In order to apply these results to systems with general non-proportional damping it is required to convert the equations of motion into state-space form (see, [ZEN 95] for example). Although exact in nature, the state-space methods require significant numerical effort as the size of the problem doubles. Moreover, these methods also lack some of the intuitive simplicity of the analysis based on configuration space. For these reasons the determination of the derivatives of eigenvalues and eigenvectors in the configuration space for damped systems is very desirable. Unlike undamped systems, in damped systems the eigenvalues and eigenvectors, and consequently their derivatives, become complex in general. Some authors have considered the problem of the calculation of first-order derivatives of eigensolutions of viscously damped symmetric systems. Lee *et al.* [LEE 99a, LEE 99b] have proposed first-order formalism to determine natural frequency and mode shape sensitivities of damped systems. Adhikari [ADH 99b] derived an exact expression for the first-order derivative of complex eigenvalues and eigenvectors. The results were expressed in terms of the complex eigenvalues and eigenvectors of the second-order system and the first-order representation of the equation of motion was avoided. Later Adhikari [ADH 00a] suggested an approximate method to calculate the first derivative of complex modes using a modal series involving only classical normal modes. An expression for the derivatives of eigenvalues and eigenvectors of non-conservative systems is presented by Choi *et al.* [CHO 04] in the configuration space. Moon *et al.* [MOO 04] proposed modified modal methods for calculating eigenpair sensitivity of asymmetric damped system. They have used few lowest sets of modes to reduce the computational time. Guedria *et al.* [GUE 06] presented a new approach for calculating simultaneously the derivatives of the eigenvalues and their associated derivatives of the left and right eigenvectors for asymmetric damped systems. Friswell and Adhikari [FRI 00] extended Nelson's method to symmetric and asymmetric viscously damped systems. Guedria *et al.* [GUE 07] considered the computation of the second-order derivatives of the eigenvalues and eigenvectors of symmetric and asymmetric damped systems using Nelson's method. A modal approach for efficient calculation of complex eigenvector derivatives were proposed by Zhang-Ping and Jin-Wu [ZHA 07b]. Derivatives of repeated complex eigenvalues and corresponding eigenvectors of nonproportionally damped systems were considered in [HUI 07]. Chouchane *et al.* [CHO 07] proposed an algebraic approach for the calculation of eigensensitivity of asymmetric damped systems. Calculation of derivatives of multiple eigenvalues and eigenvectors of general unsymmetrical quadratic eigenvalue problems was considered in [XIE 08]. Abuazoum and Garvey [ABU 09] used structure-preserving equivalences to obtain eigenvalue and eigenvector derivatives of general second-order systems. Burchett [BUR 09] proposed a QZ-Based algorithm for calculating derivatives of the system pole, transmission zero and residues. For the case when the system matrices are defective, efficient approaches to calculate the sensitivity of the eigensolutions was proposed in [XU 10, ZHA 11]. In the context of bridge deck flutter problems, Omenzetter [OME 12] considered sensitivity analysis of the eigenvalues for general dynamic systems. Some iterative methods for the derivatives of eigenvectors of quadratic eigenvalue problems arising in damped systems were suggested by Xie [XIE 12, XIE 13]. Li *et al.* [LI 12b, LI 13e, LI 13b] proposed efficient computational methods for the problem of eigensensitivity analysis of damped systems with both distinct and repeated eigenvalues.

Most of the above studies consider viscously damped system. Adhikari [ADH 02a] proposed a modal approach for the eigensensitivity of linear systems with general nonviscous and non-proportional damping. It was shown that the eigenvector derivative can be expressed as a liner combination of other eigenvectors even when they do not satisfy any simple orthogonality relationships. Later Adhikari and Friswell [ADH 06b] used Nelson's method to calculate the eigenvector derivatives of general nonviscously damped systems. Li *et al.* [LI 12a] proposed an algebraic method to compute the eigensolution derivatives for nonviscously damped systems. Later they extended the formulation to asymmetric nonviscous systems [LI 13d]. More recently Li *et al.* [LI 13c] discussed sensitivity analysis for general nonlinear eigenproblems arising in non-proportional and nonviscously damped systems.

1.7. Motivation behind this book

From the discussions so far in this chapter, it emerges that significant developments in the analysis of damped systems has taken place in the past two decades. This is fuelled by the emergence of new materials such as composite and nanocomposite materials and the need to predict the system response ever more accurately in an efficient way. Based on the existing literature it is clear that there are some pressing questions of general interest. These questions include, but not limited to:

1) What damping model has to be used for a given structure, i.e., viscous or nonviscous, and if nonviscous then what kind of model should it be?

2) How can conventional modal analysis be extended to systems with nonviscous damping?

3) Can one physically understand the role of nonviscous damping in structural dynamics, as we do for viscous damping?

4) How is it possible to determine the damping parameters by conventional modal testing if a system is non-viscous?

5) How can we efficiently calculate the dynamic response of a large complex system in an efficient manner if the damping is nonproportional and nonviscous?

6) How sensitive is the dynamics of a system to the damping parameters? Does it mater if we get errors in some damping parameters?

7) How can we quantify damping in a system? What measures and tools can we use when the damping is in general nonproportional and nonviscous?

This book is motivated by these type of questions. We do not necessarily provide precise answers to these questions. The aim is to develop mathematical tools so that we can at least appreciate and investigate these type of questions for practically relevant engineering problems.

The first question is a major issue, and in the context of general vibration analysis, has been 'settled' by assuming viscous damping, although has been pointed out in the literature that in general it will not be the correct model. The next three questions are related to each other in the sense that for the identification of nonviscous damping parameters, a reliable method of modal analysis is also required. The fifth question on computational efficiency is becoming an issue as structural dynamic finite element models are getting larger. The consideration of parametric sensitivity of dynamic system is important to due to the recent drive towards model validation and uncertainty quantification of computational models. Finally, the last question regarding the quantification of damping is related to conceptual and intuitive feeling about how much damping is there in a system provided by certain parametric model.

Most of the techniques for detecting damping in a structure either consider the structure to be viscously damped or *a priori* assume some particular nonviscous model of damping and try to fit its parameters with regard to some specific structure. This *a priori* selection of damping no doubt hides the physics of the system and there has not been any indication in the literature on how to find a damping model by doing conventional vibration testing. However another relevant question in this context is whether this *a priori* selection of damping model matters from an engineering point of view: it may be possible that a pre-assumed damping model with a 'correct' set of parameters may represent the system response quite well, although the actual physical mechanism behind the daemping may be different. These issues will be discussed in this book. Next, the scope of the book is discussed together with brief overview of the chapters.

1.8. Scope of the book

Motivated by the pressing questions identified in the last section, a systematic study on the *analysis* and *identification* of damped discrete linear dynamic systems has been carried out in this book. The book is divided into two volumes. The first volume deals with analysis of linear systems with general damping models. The second volume deals with identification and quantification of damping. The focus of the book is towards theoretical and computational aspects. However, some limited experimental results are given to support the theoretical developments. In section 1.1 it has been brought out that the convolution integral model is the most general damping model for multiple-degrees-of-freedom linear systems. Attention is specifically focused on this kind of general damping model. However, for comparing and establishing the relationship with current practice, viscously damped systems are also discussed. The book is divided into ten chapters and one Appendix.

In chapter 2, we begin by reviewing the theory of dynamics of single-degree-of-freedom undamped systems. The concept of resonance frequency is explained and methods to calculate dynamic response with initial conditions are discussed. Next viscously damped single-degree-of-freedom systems are considered. Fundamental ideas such as damped natural frequency, damping ratio, frequency response function and impulse response function are discussed. A general expression of the forced dynamic response with nonzero initial condition is derived. Undamped vibration of multiple-degrees-of-freedom system is discussed next. Classical concepts of eigenfrequencies, eigenmodes and mode orthogonality are introduced. Expressions of the dynamic response in the frequency domain and time domain are derived using the eigensolutions. The discussions are then extended to viscously damped multiple-degrees-of-freedom systems. The idea of classical damping or proportional damping is critically reviewed and generalised proportional damping is introduced. Expressions of the dynamic response of proportionally damped systems are derived in terms of the classical normal modes and modal damping factors. General nonproportionally damped multiple-degrees-of-freedom systems is discussed within the scope of the state-space method. Expressions of the dynamic response in the frequency and time domain due to general forcing and initial conditions are derived. The idea of Rayleigh quotient for damped systems is discussed. Stationarity properties for systems with proportional damping and non-proportional damping are derived. Numerical examples are provided to illustrate the theoretical developments.

Dynamics of single-degree-of-freedom nonviscously damped oscillators is considered in chapter 3. It is assumed that the nonviscous damping force depends on the history of velocity via a convolution integral over an exponentially decaying kernel function. Classical qualitative dynamic properties known for viscously damped oscillators have been generalised to such nonviscously damped oscillators. The following questions of fundamental interest have been addressed: (i) under what conditions can a nonviscously damped oscillator sustain oscillatory motions? (ii) how does the natural frequency of a nonviscously damped oscillator compare with that of an equivalent undamped oscillator? and (iii) how does the decay rate compare with that of an equivalent viscously damped oscillator? Next the characteristics of the frequency response function is discussed. The classical dynamic response properties known for viscously damped oscillators have been generalised to such nonviscously damped oscillators. Following questions of wide interest have been investigated: (a) under what conditions can the amplitude of the frequency response function reach a maximum value? (b) at what frequency will it occur?, and (c) what will be the value of the maximum amplitude of the frequency response function? Introducing two non-dimensional factors, namely, the viscous damping factor and the nonviscous damping factor, answers to these questions are provided. Wherever possible, attempts have been made to relate the new results with equivalent classical results for a viscously damped oscillator. It is shown that the classical concepts based on viscously damped systems can be extended to a nonviscously damped system only under certain conditions. Such conditions have been explicitly determined and illustrated numerically.

Chapter 4 extends the study in chapter 3 to multiple-degrees-of-freedom systems. Possible choices of nonviscous kernel functions are discussed. A general nonproportionally damped MDOF system with exponential nonviscous damping is considered. The traditional state-space approach, well known for viscously damped systems, is extended to such nonviscously damped systems using a set of internal variables. Two physically realistic cases, namely, (a) when all the damping coefficient matrices are of full rank, and (b) when the damping coefficient matrices have rank deficiency, are presented. For both cases the equation of motion has been represented in terms of two symmetric matrices. The eigenvalues and the corresponding eigenvectors of the system are obtained by solving the state-space eigenvalue problem. It is shown that unlike viscously damped systems, the number of eigensolutions is more than $2N$ and depends on the rank of the damping coefficient matrices. The idea of elastic modes and nonviscous modes are introduced. The nature of these eigensolutions in the extended state-space has been explored. Some useful results relating the modal matrix in the extended state-space and the modal matrix in the original space are derived. Closed-form expressions of the dynamic response in the time domain and frequency domain due to arbitrary forcing and initial conditions are derived. It is shown than even for general nonviscously damped systems, the response can be obtained using an approach similar to classical modal superposition method. A direct time-domain analysis of linear systems with exponentially decaying damping memory kernels is also considered. The method is based on the extended state-space representation of the equations of motion. Numerical examples are provided to illustrate the theoretical expressions.

Chapter 5 is aimed at extending classical modal analysis to treat lumped-parameter *general* nonviscously damped linear dynamic systems. This chapter extends the results of the last chapter where the *special case* of exponential damping was considered. The analytical approach adopted here is very different as the state-space approach has not been used. The nature of the eigenvalues and eigenvectors are discussed under certain simplified but physically realistic assumptions concerning the system matrices and the damping kernel functions. A numerical method based on Neumann series expansion for the calculation of the eigenvectors is suggested. The transfer function matrix of the system is derived in terms of the eigenvectors of the second-order system. Exact closed-form expressions for the dynamic response due to general forces and initial conditions are derived. The mode-orthogonality relationships, known for undamped or viscously damped systems, have been generalised to such nonviscously damped systems. Some expressions are suggested for the normalisation of the complex eigenvectors. A number of useful results which relate the system matrices with the eigensolutions are established. The approach taken in this chapter neither uses the state-space approach nor employs additional dissipation coordinates. The concept of the Rayleigh quotient for nonviscously damped systems is discussed. Three new Rayleigh quotients are proposed and their stationary properties are investigated. Suitable examples are given throughout the chapter to illustrate the derived analytical results.

Chapter 6 is devoted to reduced computational methods for damped dynamic systems. First, nonproportionally damped system with viscous model is considered. An iterative method to calculate complex modes from classical normal modes is proposed. A simple numerical algorithm is given to implement the iterative method. The calculation of eigenvalues of single-degree-of-freedom linear nonviscously damped systems with exponential model is considered next. An approximate non-state-space based approach is proposed for this type of problem. The proposed approximations are based on certain physical assumptions which simplify the underlying characteristic equation to be solved. Closed-form approximate expressions of the complex and real eigenvalues of the system are derived. These approximate expressions are obtained as functions of the undamped eigenvalues only. The methods are then extended to exponentially damped multiple-degrees-of-freedom systems. This technique enables one to approximately calculate the eigenvalues of nonviscously damped systems by simple post-processing of the undamped eigenvalues. Beside these reduced modal methods, another model reduction approach based on an equivalent second-order form is discussed. This method is applicable to any general nonviscous model and *not* only the exponential model. The proposed approximation utilises the idea of generalised proportional damping and expressions of approximate eigenvalues of the system. A closed-form expression of the equivalent second-order system has been derived. The new expression is obtained by elementary operations involving the mass, stiffness and the kernel function matrix only. This enables one to approximately calculate the dynamic response of general nonviscously damped systems using the standard tools for conventional second-order viscously damped systems. Representative numerical examples are given throughout to verify the accuracy of the derived expressions.

Theory of dynamics of multiple-degrees-of-freedom *symmetric* systems has been studied in this book. However, dynamical behaviour of some systems encountered in practice can be asymmetric in nature. In A methods are proposed by which an asymmetric dynamic systems can be transformed into symmetric systems. In this way, the methods proposed in the book can in turn be applied to asymmetric systems also. Under what conditions multiple-degrees-of-freedom linear dynamical systems can be transformed into equivalent symmetric systems by non-singular linear transformations are discussed. An approach is proposed to transform asymmetric systems into symmetric systems by an equivalence transformation. The existing approach of symmetrisation by similarity transformation is the ‘first kind’ and proposed approach by equivalence transformation is the ‘second kind’. Because equivalence transformations are the most general non-singular linear transformations, conditions of symmetrisability obtained here are more ‘liberal’ compared to the first kind and numerical calculations also become more straightforward. Several examples are provided to illustrate this approach.

The intended readers of this book are primarily senior undergraduate students, graduate students and practicing engineers working in the field of advanced vibration. Limited examples are provided to support of the theoretical developments. The book is written with the aim of being a self-contained book. However, a recommended prerequisite is an undergraduate level vibration course. There are many excellent books which cover the fundamentals of the theory of vibration, for example [MEI 67, MEI 80, PAZ 80, NEW 89, CLO 93, BAT 95, MEI 97, PET 98, GÉR 97, INM 03, RAO 11]. Readers will highly benefit by familiarising themselves with the basics of the theory of vibration.

In spite of the attempt of being exhaustive at the time of writing, clearly many relevant and possibly important bibliographic references are missed. This is inevitable as huge amount of literature were published recently due to the significant rise in the interest in this topic. However, the author expects that the book covers the necessary background so that at least the readers will appreciate existing publications and future research works and developments in the field of damping. It is hoped that the readers will not only gain an understanding of the material presented in the book, but also will be able do their personal research and take this field forward.

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