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Distributed parameter model updating using the Karhunen–Loève expansion

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ABSTRACT

Discrepancies between experimentally measured data and computational predictions are unavoidable for complex engineering dynamical systems. To reduce this gap, model updating methods have been developed over the past three decades. Current methods for model updating often use discrete parameters, such as thickness or joint stiffness, for model updating. However, there are many parameters in a numerical model which are spatially distributed in nature. Such parameters include, but are not limited to, thickness, Poisson's ratio, Young's modulus, density and damping. In this paper a novel approach is proposed which takes account of the distributed nature of the parameters to be updated, by expressing the parameters as spatially correlated random fields. Based on this assumption, the random fields corresponding to the parameters to be updated have been expanded in a spectral decomposition known as the Karhunen–Loève (KL) expansion. Using the KL expansion, the mass and stiffness matrices are expanded in series in terms of discrete parameters. These parameters in turn are obtained using a sensitivity based optimization approach. A numerical example involving a beam with distributed updating parameters is used to illustrate this new idea.

1. Introduction

The deterministic finite element model updating problem [1,2] is well established, both in the development of methods and in application to industrial-scale structures. The proposed methods can be broadly divided into two categories, namely, the non-parametric (or direct) and parametric approaches. In the non-parametric approach, developed during early eighties, the system matrices (namely, mass, stiffness and damping matrices) are updated directly so that the differences between predicted data (natural frequencies, damping ratios, and mode shapes) and measured data are minimum according to a suitable norm. The major problem with these methods is the lack of physical insight into the modeling errors that are corrected, and this has led to the popularity of parametric model updating methods. Here physical parameters (for example, joint stiffnesses, thicknesses) are selected and usually updated based on some kind on sensitivity analysis that minimizes the error between predicted results and test data from a single physical structure. The choice of updating parameters is an important aspect of the process and should always be justified physically. Model uncertainties should be located and parameterized sensitively to the predictions. Finally, the model should be validated by assessing the model quality within its range of operation and its robustness to modifications in the loading configuration, design changes, coupled structure analysis and different boundary conditions.

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Collins et al. [3] developed a Bayesian approach to model updating used linearized sensitivities based on knowledge of the statistics of the unknown parameters and the vibration measurements. In these approaches, the randomness arises only from the measurement noise and the updating parameters take unique values, to be found by iterative correction to the estimated means, whilst the variances are minimized [4]. These statistical approaches have been extended to update parameter distributions using measured response distributions from multiple measurements. These include Bayesian methods [5–8], perturbation based methods [9] and the maximum likelihood method [10]. Thus randomness due to manufacturing and material variability in a number of nominally identical test structures was not considered, and this latter variability is often much more significant than measurement noise.

Determining a suitable parametrization is a key issue in model updating [11]. It is important that the chosen parameters should be able to clarify the ambiguity of the model, and in that case it is necessary for the model output to be sensitive to the parameters. Elements in the mass and stiffness matrices perform very poorly as candidate parameters and are rarely used. Element parameters, such as the flexural rigidity of a beam element, may be used provided there is some justification as to why the element properties should be in error. Mottershead et al. [12] used geometric parameters, such as beam offsets, for the updating of mechanical joints and boundary conditions. Gladwell and Ahmadian [13], Ahmadian et al. [14] and Terrell et al. [15] demonstrated the effectiveness of parametrizing the modes at the element level, and used the generic element method to update mechanical joints. Teughels et al. [16,17] parameterized distributed damage in concrete beams and highway bridges using damage functions to determine the spatial bending stiffness distribution. The updating parameters were then the multiplication factors of the damage functions. The objective of this paper is to use model updating to estimate distributed parameters modeled as realizations of random fields. The main motivation of considering this approach is arising from the fact that in many cases the deviations of the parameter values from the assumed constant values are spatially distributed in nature. Examples include thickness, Young's modulus, Poisson's ratio and mass density of a system. Such distributed deviations are obviously a priori unknown and therefore can be considered to be samples from a random field. Such random fields can be discretized into random variables using the Karhunen–Loève (KL) expansion [18–21]. The discretized variables can in turn be used as updating parameters.

Karhunen–Loève (KL) expansion based model updating has not been widely used in the context of structural dynamics. Khalil et al. [22] proposed a system identification method based on proper orthogonal decomposition. Later Khalil et al. [23,24] proposed Kalman filter base techniques for identification on nonlinear uncertain systems. Yeong and Torquato [25,26] and Cule and Torquato [27] have proposed methodologies for reconstructing random media described by two-point correlation function. In the chemical engineering literature some authors have used the Karhunen–Loève expansion for system identification. Rigopoulos et al. [28] proposed a KL expansion based method for the on-line identification of full profile disturbance models for sheet forming processes. Park and Cho [29] proposed a KL based modeling approach for nonlinear heat conduction problems in two dimensions. Krischer et al. [30] used a KL expansion for the identification of a catalytic spatiotemporally varying reaction. Qi and Li [31] used a KL based approach for a nonlinear distributed parameter processes.

We propose a Karhunen–Loève (KL) expansion based model updating for linear structural dynamical systems. The outline of the paper is as follows. In Section 2 the spectral decomposition of random fields using the Karhunen–Loève expansion is discussed. Using this expansion, the mass and stiffness matrices of an Euler–Bernoulli beam are expanded in Section 3 in terms of parameters that will be used for model updating. Based on the expansion of the system matrices, an eigen-sensitivity based updating approach is proposed in Section 4 and illustrated numerically in Section 5. Finally, based on the results and analytical formulations, a set of conclusions are drawn in Section 6.

2. Spectral decomposition of random fields

Problems of structural dynamics in which the uncertainty in specifying mass, damping and stiffness of the structure is modeled within the framework of random fields can be treated using the stochastic finite element method, see for example [32–38]. Here we consider that the difference between the measured data and the model arises from the differences between the assumed values of the distributed parameters and the 'true' values. This difference, which is unknown a priori, is distributed in nature and is modeled by random fields. To obtain a reliable statistical description of the random fields one needs multiple measurements from nominally identical systems. When such multiple measurements are not available, then the difference between the model outputs and the experiment can be considered due to a single realization of the underlying random fields. The model updating approach proposed in this paper is based on this ideological framework.

Suppose $F(\mathbf{r},\theta)$ is a random field with a covariance function $C_F(\mathbf{r}_1,\mathbf{r}_2)$ defined in a space $\mathcal{D}$. Here θ denotes an element of the (random) sample space Ω so that $\theta \in \Omega$. Mathematically and numerically it is very difficult to deal with random fields directly in the equations of motion which are often expressed by partial differential equations. For this reason it is required to discretize a random filed in terms of random variables. Once this is done, then a wide range of mathematical and numerical techniques can be used to solve the resulting discrete stochastic differential equations. Among the many discretization techniques, the spectral decomposition of random fields using the Karhunen–Loève expansion turns out to be very useful in practice. In this paper this approach has been applied to model updating.

Since the covariance function is finite, symmetric and positive definite it can be represented by a spectral decomposition. Using this spectral decomposition, the random field $F(\mathbf{r},\theta)$ can be expressed in a generalized Fourier type of series as

$$F(\mathbf{r},\theta) = F_0(\mathbf{r}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j}\xi_j(\theta)\varphi_j(\mathbf{r}) \tag{1}$$

where $\xi_j(\theta)$ are uncorrelated random variables and throughout the paper $(\bullet)_0$ implies the deterministic part corresponding to $(\bullet)$. The constants λ_j and functions $\varphi_j(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_F(\mathbf{r}_1,\mathbf{r}_2)\varphi_j(\mathbf{r}_1)\,\mathrm{d}\mathbf{r}_1 = \lambda_j\varphi_j(\mathbf{r}_2), \quad \forall j = 1,2,\ldots \tag{2}$$

The spectral decomposition in Eq. (1), which discretizes a random field into random variables, is known as the Karhunen–Loève expansion. The series in Eq. (1) can be ordered in a decreasing series so that it can be truncated after a finite number of terms with a desired accuracy. Fukunaga [20] and Papoulis and Pillai [21] and the references therein give further discussions on the Karhunen–Loève expansion.

In this paper one dimensional systems are considered. To demonstrate the approach a Gaussian random field with an exponentially decaying autocorrelation function is considered. Such a model is representative of many physical systems and closed form expressions for the Karhunen–Loève expansion may be obtained. The autocorrelation function can be expressed as

$$C(x_1,x_2) = e^{-|x_1-x_2|/b} \tag{3}$$

Here the constant b is known as the correlation length and it plays an important role in the description of a random field. If the correlation length is very small, then the random field becomes close to a delta-correlated field, often known as white noise. If the correlation length is very large compared to domain under consideration, then the random field effectively becomes a random variable. Assuming the mean is zero, then the underlying random field $F(x,\theta)$ can be expanded using the Karhunen–Loève expansion [20,21] in the interval $-a \le x \le a$ as

$$F(x,\theta) = \sum_{j=1}^{\infty} \xi_j(\theta)\sqrt{\lambda_j}\varphi_j(x) \tag{4}$$

Using the notation $c = 1/b$, the corresponding eigenvalues and eigenfunctions for odd j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \quad \varphi_j(x) = \frac{\cos(\omega_j x)}{\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}} \quad \text{where } \tan(\omega_j a) = \frac{c}{\omega_j} \tag{5}$$

and for even j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \quad \varphi_j(x) = \frac{\sin(\omega_j x)}{\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}} \quad \text{where } \tan(\omega_j a) = \frac{\omega_j}{-c} \tag{6}$$

These eigenvalues and eigenfunctions will now be used to obtain the element mass and stiffness matrices.

For all practical purposes, the infinite series in Eq. (4) needs to be truncated using a finite numbers of terms. The number of terms could be selected based on the 'amount of information' to be retained. This in turn can be related to the number of eigenvalues retained, since the eigenvalues, λ_j, in Eq. (4) are arranged in decreasing order. For example, if 90% of the

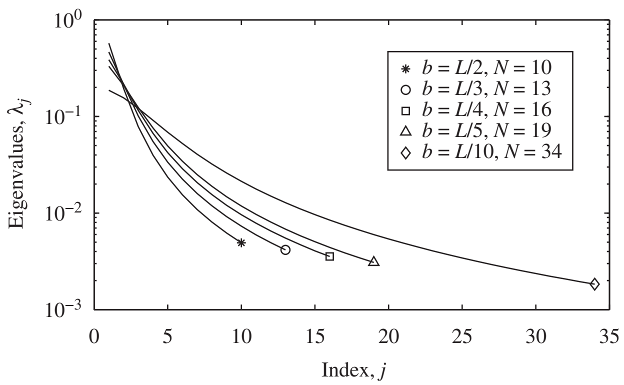

Fig. 1. The eigenvalues of the Karhunen–Loève expansion for different correlation lengths, b, and the number of terms, N, required to capture 90% of the infinite series. An exponential correlation function with unit domain (i.e., $a = \frac{1}{2}$) is assumed for the numerical calculations. The values of N are obtained such that $\lambda_N/\lambda_1 = 0.1$ for all correlation lengths. Only eigenvalues greater than λ_N are plotted.

information is to be retained, then one can choose the number of terms, N, such that $\lambda_N/\lambda_1 = 0.1$. The value of N depends on the correlation length of the underlying random field. In Fig. 1, the number of terms required to capture 90% of the infinite series for different correlation lengths are shown. Observe that one needs more terms when the correlation length is small. Intuitively this means that more independent variables are needed for fields with smaller correlation lengths and vice versa.

3. Parametric expansion of system matrices

The equation of motion of an undamped Euler–Bernoulli beam of length L with random bending stiffness and mass distribution can be expressed as

$$\frac{\partial^2}{\partial x^2}\left[EI(x,\theta)\frac{\partial^2 Y(x,t)}{\partial x^2}\right] + \rho A(x,\theta)\frac{\partial^2 Y(x,t)}{\partial t^2} = p(x,t) \tag{7}$$

Here $Y(x,t)$ is the transverse flexural displacement, $EI(x)$ is the flexural rigidity, $\rho A(x)$ is the mass per unit length, and $p(x,t)$ is the applied forcing. It is assumed that the bending stiffness, EI, and mass per unit length, ρA, are random fields of the form

$$EI(x,\theta) = EI_0(1+\varepsilon_1 F_1(x,\theta)) \tag{8}$$

and

$$\rho A(x,\theta) = \rho A_0(1+\varepsilon_2 F_2(x,\theta)) \tag{9}$$

The subscript 0 indicates the mean values, $0 < \varepsilon_i \ll 1$ $(i = 1,2)$ are deterministic constants and the random fields $F_i(x,\theta)$ are taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$. Since, $EI(x,\theta)$ and $\rho A(x,\theta)$ are strictly positive, $F_i(x,\theta)$ (i=1,2) are required to satisfy the conditions $P[1+\varepsilon_i F_i(x,\theta) \leq 0] = 0$. This requirement, strictly speaking, rules out the use of Gaussian models for these random fields. However, for small ε_i, it is expected that Gaussian models still can be used if the primary interest of the analysis is to estimate the first few response moments and not the response behavior near tails of the probability distributions.

For notational convenience, we express the shape functions for the finite element analysis of Euler–Bernoulli beams as

$$\mathbf{N}(x) = \boldsymbol{\Gamma}\mathbf{s}(x) \tag{10}$$

where

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & \frac{-3}{\ell_e^2} & \frac{2}{\ell_e^3} \\ 0 & 1 & \frac{-2}{\ell_e} & \frac{1}{\ell_e^2} \\ 0 & 0 & \frac{3}{\ell_e^2} & \frac{-2}{\ell_e^3} \\ 0 & 0 & \frac{-1}{\ell_e} & \frac{1}{\ell_e^2} \end{bmatrix} \quad \text{and} \quad \mathbf{s}(x) = [1, x, x^2, x^3]^T \tag{11}$$

Using the expression for $EI(x,\theta)$ given by Eq. (8), the element stiffness matrix can be obtained as

$$\mathbf{K}_e(\theta) = \int_0^{\ell_e} \mathbf{N}''(x) EI(x,\theta) \mathbf{N}''^T(x)\,dx = \int_0^{\ell_e} EI_0(1+\varepsilon_1 F_1(x,\theta))\mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \tag{12}$$

Expanding the random field $F_1(x,\theta)$ in the Karhunen–Loève spectral decomposition given by Eq. (4) we have

$$\mathbf{K}_e(\theta) = \mathbf{K}_{e0} + \Delta\mathbf{K}_e(\theta) \tag{13}$$

where the deterministic part is given by

$$\mathbf{K}_{e0} = EI_0 \int_0^{\ell_e} \mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \tag{14}$$

and the random part is given by

$$\Delta\mathbf{K}_e(\theta) = \varepsilon_1 \sum_{j=1}^{N_K} \xi_{Kj}(\theta)\sqrt{\lambda_{Kj}}\mathbf{K}_{ej} \tag{15}$$

The constant N_K is the number of terms retained in the Karhunen–Loève expansion and $\xi_{Kj}(\theta)$ are uncorrelated Gaussian random variables with zero mean and unit standard deviation. The constant matrices $\mathbf{K}_{ej}$ can be expressed as

$$\mathbf{K}_{ej} = EI_0 \int_0^{\ell_e} \varphi_{Kj}(x_e + x)\mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \tag{16}$$

where the functions $\varphi_{\mathrm{K}j}$ are defined in Eqs. (5) and (6), and x_e is the position of the left node of the element. Closed-form expressions of these matrices are derived in Appendix A.1.

Using the same approach, the mass matrix can be obtained as

$$\mathbf{M}_e(\theta) = \mathbf{M}_{e_0} + \mathbf{\Delta M}_e(\theta) \tag{17}$$

where the deterministic part is given by

$$\mathbf{M}_{e_0} = \rho A_0 \int_0^{\ell_e} \mathbf{N}(x)\mathbf{N}^T(x)\, dx \tag{18}$$

and the random part is given by

$$\mathbf{\Delta M}_e(\theta) = \varepsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_{ej} \tag{19}$$

The constant N_{M} is the number of terms retained in the Karhunen–Loève expansion and the constant matrices $\mathbf{M}_{ej}$ can be expressed as

$$\mathbf{M}_{ej} = \rho A_0 \int_0^{\ell_e} \varphi_{\mathrm{M}j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\, dx \tag{20}$$

The closed-form expressions of these matrices are given in Appendix A.2.

Using the conventional approach, these element matrices can be assembled to form the global random stiffness and mass matrices of the form

$$\mathbf{K}(\theta) = \mathbf{K}_0 + \mathbf{\Delta K}(\theta) \tag{21}$$

and

$$\mathbf{M}(\theta) = \mathbf{M}_0 + \mathbf{\Delta M}(\theta) \tag{22}$$

Here the deterministic parts $\mathbf{K}_0$ and $\mathbf{M}_0$ are the usual global stiffness and mass matrices obtained from the conventional finite element method. The random parts can be expressed as

$$\mathbf{\Delta K}(\theta) = \varepsilon_1 \sum_{j=1}^{N_{\mathrm{K}}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_j \tag{23}$$

and

$$\mathbf{\Delta M}(\theta) = \varepsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_j \tag{24}$$

and the element matrices $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ have been assembled into the global matrices $\mathbf{K}_j$ and $\mathbf{M}_j$. The total number of random variables depend on the number of terms used for the truncation of the infinite series, Eq. (4). This in turn depends on the respective correlation lengths of the underlying random fields; the smaller the correlation length, the higher the number of terms required and vice versa.

4. Eigen-sensitivity based model updating

Sensitivity based methods allow a wide choice of physically meaningful parameters and these advantages has led to their widespread use in model updating. The approach is very general and relies on minimizing a penalty function, which usually consists of the error between the measured quantities and the corresponding predictions from the model. Here we consider natural frequencies as measured data. Parameters are then chosen that are assumed uncertain, and here we choose the coefficients of the KL expansion. These are estimated by approximating the penalty function using a truncated Taylor series and iterating to obtain a converged solution. If there are sufficient measurements and a restricted set of parameters then the identification may be well-conditioned. Often some form of regularization must be applied [39], and this is considered further in Section 4.2. Other optimization methods may be used, such as quadratic programming, simulated annealing or genetic algorithms, but these are not considered further in this paper. Problems will also arise if an incorrect or incomplete set of parameters is chosen, or even worse, if the structure of the model is wrong. Friswell and Mottershead [1] discussed sensitivity based methods in detail.

4.1. Parametric sensitivity of eigenvalues

The random eigenvalue problem corresponding to an undamped stochastic system can be expressed as

$$[\mathbf{K}_0 + \mathbf{\Delta K}(\theta)]\boldsymbol{\phi}_i = \omega_i^2[\mathbf{M}_0 + \mathbf{\Delta M}(\theta)]\boldsymbol{\phi}_i \tag{25}$$

Several authors [40–50] proposed mean-centered perturbation methods. Recently Adhikari [51,52] and Adhikari and Friswell [53] developed an asymptotic approach and an optimal series-expansion methods to obtain second and higher-order joint statistics of eigenvalues. Here, the first-order perturbation approach is proposed for updating the system.

Following Fox and Kapoor [54] the derivative of an eigenvalue with respect to a general parameter α can be obtained as

$$\frac{\partial(\omega_i^2)}{\partial\alpha} = \boldsymbol{\phi}_{0i}^T\left[\frac{\partial\mathbf{K}}{\partial\alpha} - \omega_{0i}^2\frac{\partial\mathbf{M}}{\partial\alpha}\right]\boldsymbol{\phi}_{0i} \tag{26}$$

or

$$\frac{\partial\omega_i}{\partial\alpha} = \frac{1}{2}\boldsymbol{\phi}_{0i}^T\left[\frac{1}{\omega_{0i}}\frac{\partial\mathbf{K}}{\partial\alpha} - \omega_{0i}\frac{\partial\mathbf{M}}{\partial\alpha}\right]\boldsymbol{\phi}_{0i} \tag{27}$$

In the above expressions ω_{0i} and $\boldsymbol{\phi}_{0i}$ correspond to the natural frequency and mass normalized mode shape of the underlying baseline system satisfying

$$\mathbf{K}_0\boldsymbol{\phi}_{0i} = \omega_{0i}^2\mathbf{M}_0\boldsymbol{\phi}_{0i} \quad \text{and} \quad \boldsymbol{\phi}_{0i}^T\mathbf{M}_0\boldsymbol{\phi}_{0i} = 1 \;\; \forall i \tag{28}$$

Using the Karhunen–Loève expansion of the stiffness and mass matrix in Eqs. (21)–(24) and the first-order perturbation method, each eigenvalue can be expressed as

$$\omega_i \approx \omega_{0i} + \sum_{j=1}^{N_\mathrm{K}}\frac{\partial\omega_i}{\partial\xi_{\mathrm{K}j}}\xi_{\mathrm{K}j}(\theta) + \sum_{j=1}^{N_\mathrm{M}}\frac{\partial\omega_i}{\partial\xi_{\mathrm{M}j}}\xi_{\mathrm{M}j}(\theta) \tag{29}$$

Noting that

$$\frac{\partial\mathbf{K}}{\partial\xi_{\mathrm{K}j}} = \varepsilon_1\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_j \quad \text{and} \quad \frac{\partial\mathbf{M}}{\partial\xi_{\mathrm{M}j}} = \varepsilon_2\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_j \tag{30}$$

the derivative of the eigenvalues can be obtained using Eq. (27) as

$$\frac{\partial\omega_i}{\partial\xi_{\mathrm{K}j}} = s_{ij} = \varepsilon_1\sqrt{\lambda_{\mathrm{K}j}}\frac{\boldsymbol{\phi}_{0i}^T\mathbf{K}_j\boldsymbol{\phi}_{0i}}{2\omega_{0i}} \tag{31}$$

and

$$\frac{\partial\omega_i}{\partial\xi_{\mathrm{M}j}} = s_{i(N_\mathrm{K}+j)} = -\varepsilon_2\frac{1}{2}\omega_{0i}\sqrt{\lambda_{\mathrm{M}j}}\boldsymbol{\phi}_{0i}^T\mathbf{M}_j\boldsymbol{\phi}_{0i} \tag{32}$$

Suppose m number of natural frequencies have been measured. Combining the preceding four equations for all m we can express

$$\boldsymbol{\omega} \approx \boldsymbol{\omega}_0 + \mathbf{S}\boldsymbol{\xi} \tag{33}$$

Here the elements of the $m \times (N_\mathrm{K} + N_\mathrm{M})$ sensitivity matrix $\mathbf{S}$ are given by Eqs. (31) and (32) and the $(N_\mathrm{K} + N_\mathrm{M})$ dimensional vector of updating parameters $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi} = [\xi_{\mathrm{K}1} \;\; \xi_{\mathrm{K}2} \;\; \ldots \;\; \xi_{\mathrm{K}N_\mathrm{K}} \;\; \xi_{\mathrm{M}1} \;\; \xi_{\mathrm{M}2} \;\; \ldots \;\; \xi_{\mathrm{M}N_\mathrm{M}}]^T \tag{34}$$

The vector $\boldsymbol{\xi}$ in Eq. (33) is a deterministic vector when a single structure is considered. However, it should be recalled from the KL expansion that the elements of $\boldsymbol{\xi}$ are sampled from independent and identically distributed standard Gaussian random variables (i.e., with zero-mean and unit standard deviation). Following the cumulative distribution function of the Gaussian random variable, this in turn implies that there is about 99% probability that the elements of the vector $\boldsymbol{\xi}$ will lie within three standard deviations of the mean. We now use Eq. (33) to update the system.

4.2. Optimization methods

Minimization of the error between the measured and predicted natural frequencies may be expressed as the minimization of the cost function J_e, defined as

$$J_\mathrm{e}(\boldsymbol{\xi}) = \boldsymbol{\varepsilon}^T\mathbf{W}_\mathrm{e}\boldsymbol{\varepsilon} \tag{35}$$

where

$$\boldsymbol{\varepsilon} = \boldsymbol{\omega}_\mathrm{m} - \boldsymbol{\omega}(\boldsymbol{\xi}) \tag{36}$$

and $\mathbf{W}_\mathrm{e}$ is a weighting matrix. A suitable choice of weighting matrix is the inverse of the variance of the natural frequencies, or a matrix with the inverse of natural frequency squared along the diagonal. $\boldsymbol{\omega}_\mathrm{m}$ is the vector of measured natural frequencies corresponding to the predicted natural frequencies $\boldsymbol{\omega}(\boldsymbol{\xi})$, $\boldsymbol{\xi}$ is the vector of unknown parameters, and $\boldsymbol{\varepsilon}$ is the modal residual vector.

The modal residual in Eq. (36) is a nonlinear function of the parameters and the minimization is solved using a truncated linear Taylor series and iteration. Suppose at the j th iteration the parameter estimate is ξ_j and the corresponding vector of natural frequencies is $\boldsymbol{\omega}_j$. The iteration is initialized with $\xi_0 = 0$. The truncated Taylor series centered on the current parameter estimate is based on Eq. (33), and given by

$$\boldsymbol{\omega} \approx \boldsymbol{\omega}_j + \mathbf{S}_j(\xi - \xi_j) \tag{37}$$

where $\mathbf{S}_j$ is the sensitivity matrix evaluated at the current parameter estimate ξ_j. Assuming there are more measurements than parameters then the updated parameter estimate ξ_{j+1} is obtained using the pseudo-inverse as

$$\xi_{j+1} = \xi_j + [\mathbf{S}_j^T \mathbf{W}_e \mathbf{S}_j]^{-1} \mathbf{S}_j^T \mathbf{W}_e (\boldsymbol{\omega}_m - \boldsymbol{\omega}_j) \tag{38}$$

Often calculating the solution in Eq. (38) will be ill-conditioned, even if there are more measurements than parameters. In this case the solution may be regularized. Since we know that the KL coefficients should be Gaussian with zero mean, a suitable side constraint is

$$J_p(\xi) = \xi^T \mathbf{W}_p \xi \tag{39}$$

where the weighting matrix $\mathbf{W}_p$ will be chosen as the identity matrix here so that each parameter is equally weighted. A combined penalty function is then defined based on the natural frequency residuals and the side constraint as

$$J(\xi) = (1-\mu) J_e(\xi) + \mu J_p(\xi) \tag{40}$$

where $\mu \in [0\ 1]$ is a regularization parameter that determines the relative weight between the residual and the side constraint. If $\mu = 0$ then no weight is given to the parameter changes, and if $\mu = 1$ the parameters are not updated so that $\xi = 0$. Minimizing Eq. (40) at the j th iteration, based on the Taylor series in Eq. (37), gives the updated parameter vector as

$$\xi_{j+1} = \xi_j + [(1-\mu)\mathbf{S}_j^T \mathbf{W}_e \mathbf{S}_j + \mu \mathbf{W}_p]^{-1} \{(1-\mu)\mathbf{S}_j^T \mathbf{W}_e (\boldsymbol{\omega}_m - \boldsymbol{\omega}_j) - \mu \mathbf{W}_p \xi_j\} \tag{41}$$

We refer to Titurus and Friswell [39] for the derivation of the above result. The iterations continue until convergence, which is determined when the change in parameters, $\|\xi_{j+1} - \xi_j\|$, falls below a given constant.

The remaining question is how to choose the regularization parameter μ. The procedure adopted here is to vary the regularization parameter, while recording the cost functions J_e and J_p representing the error in the natural frequencies and the change in the parameters, respectively. These cost functions are then plotted to give the so-called L-curve [11]. This will be a monotonic function and will often have a sharp corner that represents the optimum trade-off between the minimization of the residual and the parameter changes. In the next section we apply this approach numerically as an illustration.

5. Numerical example

We consider an Euler–Bernoulli beam with variability in the bending rigidity $EI(x)$ only. The beam is assumed to be clamped at one end. The unperturbed physical and geometrical properties of the beam are shown in Table 1. It is assumed that the variations from the unperturbed value of EI can be modeled by a homogeneous Gaussian random field. For numerical calculations we considered 20% variation with a correlation length of $b = L/3$. Some random realizations of the type of variability considered are shown in Fig. 2. Note that the deviations from the baseline value are distributed in nature. The data were generated using 13 terms in the KL expansion. In this example we treat the first 16 natural frequencies of the perturbed system as 'measured' values ($m = 16$). This simulates a realistic situation where the true model parameters (EI in this example) can deviate from the baseline assumed values in an a priori unknown manner. The objective is to reconstruct the distributed EI function from the measured natural frequencies of a sample beam.

Before estimating the coefficients of the KL expansion the sensitivity matrix should be studied to understand any likely ill-conditioning problems. This matrix is shown in Fig. 3 for 20 natural frequencies and 12 parameters. A number of features are immediately apparent. There are several orders of magnitude difference in some sensitivities, and in particular the natural frequencies are much more sensitive to the odd terms in the KL expansion than the even terms. For structures with other boundary conditions this will not necessarily be the case. Second, the sensitivities vary smoothly with the mode

Table 1
Material and geometric properties of the beam considered in the example.

Beam properties	Numerical values
Length (L)	1 m
Correlation length (b)	$L/3$
Width	40 mm
Thickness	2 mm
Mass density (ρ)	7800 kg/m^3
Young's modulus (E)	2.0×10^5 MPa

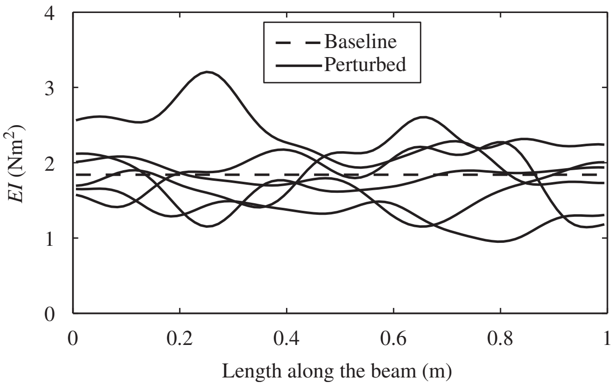

Fig. 2. Some random realizations of the bending rigidity *EI* of the beam for correlation length $b = L/3$ and strength parameter $\varepsilon_1 = 0.2$. Thirteen terms have been used in the KL expansion.

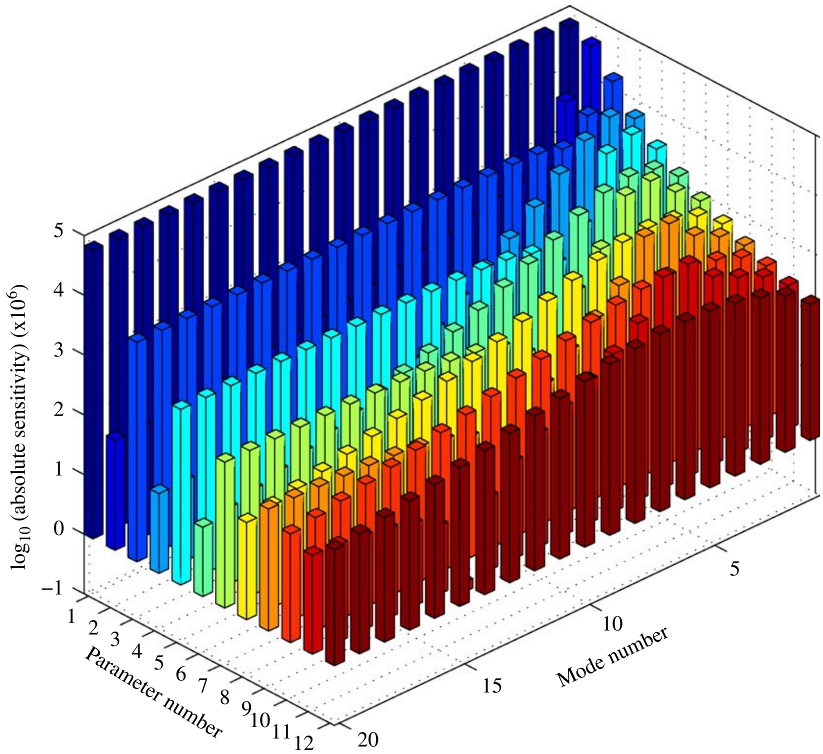

Fig. 3. The sensitivity of the natural frequencies to the coefficients in the KL expansion, normalized by the baseline eigenvalues and multiplied by 10^6.

number, and the sensitivity is maximum when the correlation between the physical mode and the KL mode is greatest. These two features mean that it is highly unlikely that the distributed *EI* can be accurately reconstructed from the lower measured natural frequencies.

For a particular sample of the distributed *EI*, simulated using 13 terms in the KL expansion, updating is performed for a variable number of parameters and various values of regularization parameter. The weighting matrix for the residuals, $\mathbf{W}_e$, is chosen as a matrix with the inverse of the measured natural frequencies squared along the diagonal, so that each natural frequency is essentially weighted equally. The number of parameters is fixed and L-curves for 6, 8, 10 and 12 parameters are shown in Fig. 4. For six and eight parameters there is a distinct corner to the L-curve that gives the optimum choice of regularization parameter. The cross is placed at the corner of the L-curve for six parameters (where $\mu = 5.7 \times 10^{-5}$) and the reconstructed *EI* based on the updated parameters is shown in Fig. 5. For higher numbers of parameters the corner of the L-curve in Fig. 4 is not distinct, and giving less weight to the regularization term continues to slowly improve the fit to the 'measured' data, at the expense of larger parameter values. A typical reconstructed *EI* based on updating 12 parameters is shown in Fig. 5 (given as the circle on the L-curve in Fig. 4, where $\mu = 1.4 \times 10^{-5}$). Note that both reconstructed *EI*

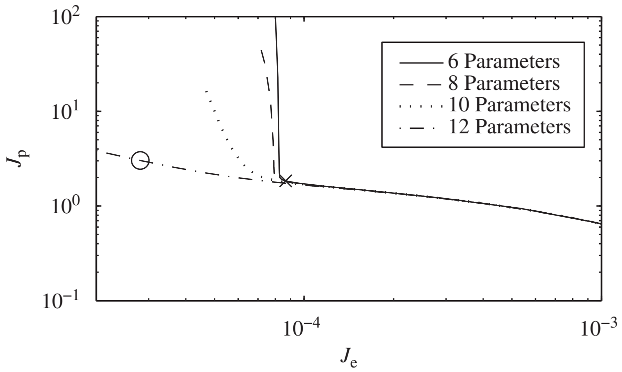

Fig. 4. The L-curve for 6, 8, 10 and 12 parameters. The cross and circle represent the cases for 6 and 12 parameters for which the reconstructed *EI* is given in Fig. 5.

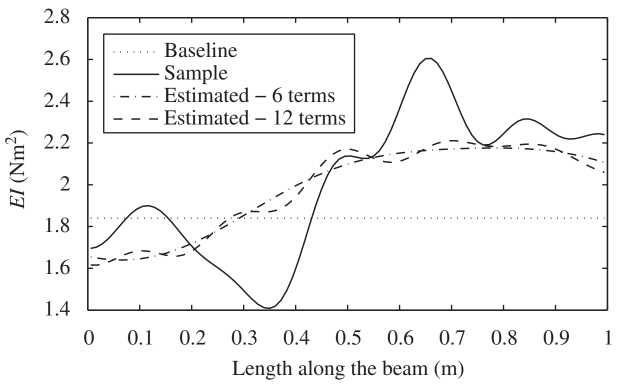

Fig. 5. Baseline, actual and reconstructed values of the bending rigidity along the length of the beam.

Table 2
Initial, measured and updated natural frequencies.

Natural frequencies (Hz)			
Initial	Measured	Updated	
		Six parameters	12 parameters
10.262 (−2.8473)	9.9779	9.9923 (−0.1443)	9.9813 (−0.0341)
64.311 (3.0205)	66.314	66.288 (0.0392)	66.305 (0.0136)
180.07 (3.2090)	186.04	186.24 (−0.1075)	186.09 (−0.0269)
352.87 (3.6480)	366.23	365.44 (0.2157)	366.00 (0.0628)
583.32 (3.0160)	601.46	604.18 (−0.4522)	602.08 (−0.1031)
871.38 (4.1376)	908.99	902.48 (0.7162)	905.31 (0.4048)
1217.1 (3.1973)	1257.3	1260.4 (−0.2466)	1261.0 (−0.2943)
1620.3 (3.3638)	1676.7	1677.9 (−0.0716)	1678.2 (−0.0895)
2081.2 (3.3932)	2154.3	2155.0 (−0.0325)	2155.2 (−0.0418)
2599.8 (3.4034)	2691.4	2691.7 (−0.0111)	2691.8 (−0.0149)
3175.9 (3.4152)	3288.2	3288.1 (0.0030)	3288.2 (0.0000)
3809.7 (3.4223)	3944.7	3944.1 (0.0152)	3944.2 (0.0127)
4501.0 (3.4307)	4660.9	4659.7 (0.0257)	4659.8 (0.0236)
5250.1 (3.4340)	5436.8	5435.1 (0.0313)	5435.1 (0.0313)
6056.7 (3.4389)	6272.4	6270.0 (0.0383)	6270.1 (0.0367)
6921.0 (3.4418)	7167.7	7164.7 (0.0419)	7164.8 (0.0405)

The percentage error with respect to the measured values are shown in brackets.

functions are smoother than the simulated function that generated the data. This is because the higher terms in the KL expansion cannot be estimated from the data. The initial, measured and updated natural frequencies (for the two cases in Fig. 5) are shown in Table 2.

6. Conclusions

In this paper we proposed a new technique for updating of distributed parameters in structural dynamics. The proposed approach is based on KL expansion and eigen-sensitivity analysis. The main motivation behind considering this approach is arising from the fact that in many cases the deviations of the parameter values from the assumed constant values are spatially distributed in nature. Such distributed deviations are a priori unknown and therefore may be considered to be samples from a random field. Such random fields are discretized into random variables using the Karhunen–Loève (KL) expansion. A subset of these random variables are in turn considered as parameters for model updating. Some notable features of the proposed method are

- The parameter selection process in the model updating becomes automatic and largely problem independent.
- The KL expansion automatically generates the basis matrices for the expansion of the mass and stiffness matrices.
- The variables to be updated are a set of abstract variables as opposed to physical variables (although the underlying distributed parameters are physical parameters). These variables are naturally arranged in a decreasing order.
- Since these variables are essentially samples from standard normal variables, they are generally expected to be within the range ± 5. This may be used in the optimization problem (e.g. a constrained optimization can be used).

The proposed method is illustrated using an example of a beam with spatially varying bending rigidity. It was demonstrated that the variability can be reconstructed using the proposed method. Further research is needed to apply, extend, validate and test this method to complex dynamical systems.

Acknowledgments

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Appendix A. Stochastic parts of the element mass and stiffness matrices

This appendix derives closed-form expressions for the random part of the element stiffness and mass matrices.

A.1. Stochastic element stiffness matrix

Using the expression for φ_{K_j} from Eq. (5), the matrix $\mathbf{K}_{e_j}$ can be expressed from Eq. (16) as

$$\mathbf{K}_{e_j} = EI_0 \int_0^{\ell_e} \varphi_{K_j}(x_e + x)\mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \tag{A.1}$$

$$= \frac{EI_0}{\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\boldsymbol{\Gamma}\left[\int_0^{\ell_e} \cos(\omega_j(x_e + x))\mathbf{s}''(x)\mathbf{s}''^T(x)\,dx\right]\boldsymbol{\Gamma}^T \tag{A.2}$$

$$= \frac{4EI_0}{\omega_j^3 \ell_e^6 \sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\tilde{\mathbf{K}}_{e_j} \tag{A.3}$$

Here $a = L/2$ where L is the length of the whole beam and

$$\tilde{\mathbf{K}}_{e_j} = \frac{\omega_j^3 \ell_e^6}{4}\boldsymbol{\Gamma}\left[\int_0^{\ell_e} \cos(\omega_j(x_e + x))\mathbf{s}''(x)\mathbf{s}''^T(x)\,dx\right]\boldsymbol{\Gamma}^T = \frac{\omega_j^3 \ell_e^6}{4}\boldsymbol{\Gamma}\int_0^{\ell_e} \cos(\omega_j(x_e + x))\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 12x \\ 0 & 0 & 12x & 36x^2 \end{bmatrix} dx\,\boldsymbol{\Gamma}^T \tag{A.4}$$

The elements of the symmetric matrix $\tilde{\mathbf{K}}_{e_j}$ can be expressed as

$$\tilde{\mathbf{K}}_{e_j} = \begin{bmatrix} \hat{K}_{ej_{11}} & \hat{K}_{ej_{12}} & -\hat{K}_{ej_{11}} & \hat{K}_{ej_{12}} \\ & \hat{K}_{ej_{22}} & -\hat{K}_{ej_{12}} & \hat{K}_{ej_{23}} \\ & & \hat{K}_{ej_{11}} & -\hat{K}_{ej_{12}} \\ & & & \hat{K}_{ej_{22}} \end{bmatrix} \tag{A.5}$$

Performing the integrals appearing in Eq. (A.4), the expressions of the four independent terms appearing in Eq. (A.5) are

$$\hat{K}_{ej_{11}} = -9\omega_j^2\ell_e^2\sin(\omega_j x_e) + 9\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 36\omega_j\ell_e\cos(\omega_j x_e) + 36\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e + 72\sin(\omega_j x_e) - 72\sin(\omega_j\ell_e + \omega_j x_e) \tag{A.6}$$

$$\hat{K}_{ej_{12}} = -3\ell_e(2\omega_j^2\ell_e^2\sin(\omega_j x_e) - \sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 7\omega_j\ell_e\cos(\omega_j x_e) - 5\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 12\sin(\omega_j x_e) + 12\sin(\omega_j\ell_e + \omega_j x_e)) \tag{A.7}$$

$$\hat{K}_{ej_{22}} = -\ell_e^2(4\omega_j^2\ell_e^2\sin(\omega_j x_e) - \sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 12\omega_j\ell_e\cos(\omega_j x_e) - 6\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 18\sin(\omega_j x_e) + 18\sin(\omega_j\ell_e + \omega_j x_e)) \tag{A.8}$$

$$\hat{K}_{ej_{23}} = 3\ell_e(2\omega_j^2\ell_e^2\sin(\omega_j x_e) - \sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 7\omega_j\ell_e\cos(\omega_j x_e) - 5\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 12\sin(\omega_j x_e) + 12\sin(\omega_j\ell_e + \omega_j x_e)) \tag{A.9}$$

The term associated with the even elements of the spectral expansion of the element stiffness matrix can be obtained from Eq. (16) as

$$\mathbf{K}_{e_j} = EI_0\int_0^{\ell_e}\varphi_{K_j}(x_e + x)\mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \tag{A.10}$$

$$= \frac{EI_0}{\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\boldsymbol{\Gamma}\left[\int_0^{\ell_e}\sin(\omega_j(x_e + x))\mathbf{s}''(x)\mathbf{s}''^T(x)\,dx\right]\boldsymbol{\Gamma}^T \tag{A.11}$$

$$= \frac{4EI_0}{\omega_j^3\ell_e^6\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\tilde{\mathbf{K}}_{e_j} \tag{A.12}$$

where

$$\tilde{\mathbf{K}}_{e_j} = \frac{\omega_j^3\ell_e^6}{4}\boldsymbol{\Gamma}\left[\int_0^{\ell_e}\sin(\omega_j(x_e + x))\mathbf{s}''(x)\mathbf{s}''^T(x)\,dx\right]\boldsymbol{\Gamma}^T \tag{A.13}$$

The elements of the symmetric matrix $\tilde{\mathbf{K}}_{e_j}$ can be expressed as

$$\tilde{\mathbf{K}}_{e_j} = \begin{bmatrix} \hat{K}_{ej_{11}} & \hat{K}_{ej_{12}} & -\hat{K}_{ej_{11}} & \hat{K}_{ej_{12}} \\ & \hat{K}_{ej_{22}} & -\hat{K}_{ej_{12}} & \hat{K}_{ej_{23}} \\ & & \hat{K}_{ej_{11}} & -\hat{K}_{ej_{12}} \\ & & & \hat{K}_{ej_{22}} \end{bmatrix} \tag{A.14}$$

Performing the integrals appearing in Eq. (A.13), the expressions of the four independent terms appearing in the preceding equation can be obtained as

$$\hat{K}_{e_{11}} = 9\omega_j^2\ell_e^2\cos(\omega_j x_e) - 9\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 36\omega_j\ell_e\sin(\omega_j x_e) + 36\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 72\cos(\omega_j x_e) + 72\cos(\omega_j\ell_e + \omega_j x_e) \tag{A.15}$$

$$\hat{K}_{e_{12}} = 3\ell_e(2\omega_j^2\ell_e^2\cos(\omega_j x_e) - \cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 7\omega_j\ell_e\sin(\omega_j x_e) + 5\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 12\cos(\omega_j x_e) + 12\cos(\omega_j\ell_e + \omega_j x_e)) \tag{A.16}$$

$$\hat{K}_{e_{22}} = \ell_e^2(4\omega_j^2\ell_e^2\cos(\omega_j x_e) - \cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 12\omega_j\ell_e\sin(\omega_j x_e) + 6\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 18\cos(\omega_j x_e) + 18\cos(\omega_j\ell_e + \omega_j x_e)) \tag{A.17}$$

$$\hat{K}_{e_{23}} = -3\ell_e(2\omega_j^2\ell_e^2\cos(\omega_j x_e) - \cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 7\omega_j\ell_e\sin(\omega_j x_e) + 5\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 12\cos(\omega_j x_e) + 12\cos(\omega_j\ell_e + \omega_j x_e)) \tag{A.18}$$

A.2. Stochastic element mass matrix

Using the expression of φ_{M_j} from Eq. (5), the matrix $\mathbf{M}_{e_j}$ can be expressed from Eq. (16) as

$$\mathbf{M}_{e_j} = m_0 \int_0^{\ell_e} \varphi_{M_j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\,dx \tag{A.19}$$

$$= \frac{m_0}{\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\boldsymbol{\Gamma}\left[\int_0^{\ell_e} \cos(\omega_j(x_e + x))\mathbf{s}(x)\mathbf{s}^T(x)\,dx\right]\boldsymbol{\Gamma}^T \tag{A.20}$$

$$= \frac{m_0}{\omega_j^7 \ell_e^6 \sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\tilde{\mathbf{M}}_{e_j} \tag{A.21}$$

where

$$\tilde{\mathbf{M}}_{e_j} = (\omega_j^7 \ell_e^6)\boldsymbol{\Gamma}\left[\int_0^{\ell_e} \cos(\omega_j(x_e + x))\mathbf{s}(x)\mathbf{s}^T(x)\,dx\right]\boldsymbol{\Gamma}^T = (\omega_j^7 \ell_e^6)\boldsymbol{\Gamma}\int_0^{\ell_e} \cos(\omega_j(x_e + x))\begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & x^4 \\ x^2 & x^3 & x^4 & x^5 \\ x^3 & x^4 & x^5 & x^6 \end{bmatrix} dx\,\boldsymbol{\Gamma}^T \tag{A.22}$$

The elements of the symmetric matrix $\tilde{\mathbf{M}}_{e_j}$ can be expressed as

$$\tilde{\mathbf{M}}_{e_j} = \begin{bmatrix} \hat{M}_{ej_{11}} & \hat{M}_{ej_{12}} & -\hat{M}_{ej_{13}} & \hat{M}_{ej_{14}} \\ & \hat{M}_{ej_{22}} & -\hat{M}_{ej_{14}} & \hat{M}_{ej_{24}} \\ & & \hat{M}_{ej_{11}} & -\hat{M}_{ej_{12}} \\ & & & \hat{M}_{ej_{22}} \end{bmatrix} \tag{A.23}$$

Performing the integrals appearing in Eq. (A.22), the expressions of the six independent terms appearing in the preceding equation can be obtained as

$$\begin{aligned}\hat{M}_{ej_{11}} = {} & 24\omega_j^3\ell_e^3\cos(\omega_j x_e) + 1440\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 216\omega_j^2\ell_e^2\sin(\omega_j x_e) + 1440\omega_j\ell_e\cos(\omega_j x_e) \\ & + 216\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 12\omega_j^4\ell_e^4\sin(\omega_j x_e) - \ell_e^6\omega_j^6\sin(\omega_j x_e) - 2880\sin(\omega_j\ell_e + \omega_j x_e) \\ & + 2880\sin(\omega_j x_e)\end{aligned} \tag{A.24}$$

$$\begin{aligned}\hat{M}_{ej_{12}} = {} & -\ell_e(12\omega_j^3\ell_e^3\cos(\omega_j x_e) - 600\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e + 192\omega_j^2\ell_e^2\sin(\omega_j x_e) - 840\omega_j\ell_e\cos(\omega_j x_e) \\ & - 72\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 4\omega_j^4\ell_e^4\sin(\omega_j x_e) + 1440\sin(\omega_j\ell_e + \omega_j x_e) - 1440\sin(\omega_j x_e) + \ell_e^5\omega_j^5\cos(\omega_j x_e))\end{aligned} \tag{A.25}$$

$$\begin{aligned}\hat{M}_{ej_{13}} = {} & -12\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^3\ell_e^3 - 6\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^4\ell_e^4 - 12\omega_j^3\ell_e^3\cos(\omega_j x_e) - 1440\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e \\ & + 216\omega_j^2\ell_e^2\sin(\omega_j x_e) - 1440\omega_j\ell_e\cos(\omega_j x_e) - 216\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 6\omega_j^4\ell_e^4\sin(\omega_j x_e) \\ & + 2880\sin(\omega_j\ell_e + \omega_j x_e) - 2880\sin(\omega_j x_e)\end{aligned} \tag{A.26}$$

$$\begin{aligned}\hat{M}_{ej_{14}} = {} & -2\ell_e(\omega_j^4\ell_e^4\sin(\omega_j x_e) - 3\omega_j^3\ell_e^3\cos(\omega_j x_e) + 9\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^3\ell_e^3 + 36\omega_j^2\ell_e^2\sin(\omega_j x_e) \\ & - 96\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 300\omega_j\ell_e\cos(\omega_j x_e) - 420\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 720\sin(\omega_j x_e) \\ & + 720\sin(\omega_j\ell_e + \omega_j x_e))\end{aligned} \tag{A.27}$$

$$\begin{aligned}\hat{M}_{ej_{22}} = {} & 2\ell_e^2(12\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 240\omega_j\ell_e\cos(\omega_j x_e) - 72\omega_j^2\ell_e^2\sin(\omega_j x_e) + 120\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e \\ & + 360\sin(\omega_j x_e) - 360\sin(\omega_j\ell_e + \omega_j x_e) + \omega_j^4\ell_e^4\sin(\omega_j x_e) - 12\omega_j^3\ell_e^3\cos(\omega_j x_e))\end{aligned} \tag{A.28}$$

$$\begin{aligned}\hat{M}_{ej_{24}} = {} & -6\ell_e^2(\omega_j^3\ell_e^3\cos(\omega_j x_e) + \cos(\omega_j\ell_e + \omega_j x_e)\omega_j^3\ell_e^3 + 12\omega_j^2\ell_e^2\sin(\omega_j x_e) - 12\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 \\ & - 60\omega_j\ell_e\cos(\omega_j x_e) - 60\cos(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 120\sin(\omega_j x_e) + 120\sin(\omega_j\ell_e + \omega_j x_e))\end{aligned} \tag{A.29}$$

The term associated with the even elements of the spectral expansion of the element stiffness matrix can be obtained from Eq. (16) as

$$\mathbf{M}_{e_j} = m_0 \int_0^{\ell_e} \varphi_{M_j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\,dx \tag{A.30}$$

$$= \frac{m_0}{\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\boldsymbol{\Gamma}\left[\int_0^{\ell_e} \sin(\omega_j(x_e + x))\mathbf{s}(x)\mathbf{s}^T(x)\,dx\right]\boldsymbol{\Gamma}^T \tag{A.31}$$

$$= \frac{m_0}{\omega_j^7 \ell_e^6 \sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\tilde{\mathbf{M}}_{e_j} \tag{A.32}$$

where

$$\tilde{\mathbf{M}}_{e_j} = (\omega_j^7 \ell_e^6)\boldsymbol{\Gamma}\left[\int_0^{\ell_e} \sin(\omega_j(x_e + x))\mathbf{s}(x)\mathbf{s}^T(x)\,dx\right]\boldsymbol{\Gamma}^T \tag{A.33}$$

The elements of the symmetric matrix $\tilde{\mathbf{M}}_{e_j}$ can be expressed as

$$\tilde{\mathbf{M}}_{e_j} = \begin{bmatrix} \hat{M}_{ej_{11}} & \hat{M}_{ej_{12}} & -\hat{M}_{ej_{13}} & \hat{M}_{ej_{14}} \\ & \hat{M}_{ej_{22}} & -\hat{M}_{ej_{14}} & \hat{M}_{ej_{24}} \\ & & \hat{M}_{ej_{11}} & -\hat{M}_{ej_{12}} \\ & & & \hat{M}_{ej_{22}} \end{bmatrix} \tag{A.34}$$

Performing the integrals appearing in Eq. (A.33), the expressions of the six independent terms appearing in the preceding equation can be obtained as

$$\begin{aligned}\hat{M}_{e_{11}} &= -216\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 1440\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e + \omega_j^6\ell_e^6\cos(\omega_j x_e) + 12\ell_e^4\omega_j^4\cos(\omega_j x_e) \\ &\quad + 216\omega_j^2\ell_e^2\cos(\omega_j x_e) + 1440\omega_j\ell_e\sin(\omega_j x_e) + 24\ell_e^3\omega_j^3\sin(\omega_j x_e) - 2880\cos(\omega_j x_e) + 2880\cos(\omega_j\ell_e + \omega_j x_e)\end{aligned} \tag{A.35}$$

$$\begin{aligned}\hat{M}_{e_{12}} &= -\ell_e(72\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 600\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 4\ell_e^4\omega_j^4\cos(\omega_j x_e) - 192\omega_j^2\ell_e^2\cos(\omega_j x_e) \\ &\quad - 840\omega_j\ell_e\sin(\omega_j x_e) + 12\ell_e^3\omega_j^3\sin(\omega_j x_e) + 1440\cos(\omega_j x_e) - 1440\cos(\omega_j\ell_e + \omega_j x_e) + \omega_j^5\ell_e^5\sin(\omega_j x_e))\end{aligned} \tag{A.36}$$

$$\begin{aligned}\hat{M}_{e_{13}} &= 216\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 1440\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 12\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^3\ell_e^3 - 6\ell_e^4\omega_j^4\cos(\omega_j x_e) \\ &\quad - 216\omega_j^2\ell_e^2\cos(\omega_j x_e) - 1440\omega_j\ell_e\sin(\omega_j x_e) + 6\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^4\ell_e^4 - 12\ell_e^3\omega_j^3\sin(\omega_j x_e) + 2880\cos(\omega_j x_e) \\ &\quad - 2880\cos(\omega_j\ell_e + \omega_j x_e)\end{aligned} \tag{A.37}$$

$$\begin{aligned}\hat{M}_{e_{14}} &= 2\ell_e(\ell_e^4\omega_j^4\cos(\omega_j x_e) + 3\ell_e^3\omega_j^3\sin(\omega_j x_e) - 9\sin(\omega_j\ell_e + \omega_j x_e)\omega_j^3\ell_e^3 + 36\omega_j^2\ell_e^2\cos(\omega_j x_e) \\ &\quad - 96\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 + 300\omega_j\ell_e\sin(\omega_j x_e) + 420\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 720\cos(\omega_j x_e) \\ &\quad + 720\cos(\omega_j\ell_e + \omega_j x_e))\end{aligned} \tag{A.38}$$

$$\begin{aligned}\hat{M}_{e_{22}} &= -2\ell_e^2(12\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 - 120\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e - 72\omega_j^2\ell_e^2\cos(\omega_j x_e) - 240\omega_j\ell_e\sin(\omega_j x_e) \\ &\quad + 12\ell_e^3\omega_j^3\sin(\omega_j x_e) + 360\cos(\omega_j x_e) - 360\cos(\omega_j\ell_e + \omega_j x_e) + \ell_e^4\omega_j^4\cos(\omega_j x_e))\end{aligned} \tag{A.39}$$

$$\begin{aligned}\hat{M}_{e_{24}} &= -6\ell_e^2(\ell_e^3\omega_j^3\sin(\omega_j x_e) + \sin(\omega_j\ell_e + \omega_j x_e)\omega_j^3\ell_e^3 - 12\omega_j^2\ell_e^2\cos(\omega_j x_e) + 12\cos(\omega_j\ell_e + \omega_j x_e)\omega_j^2\ell_e^2 \\ &\quad - 60\omega_j\ell_e\sin(\omega_j x_e) - 60\sin(\omega_j\ell_e + \omega_j x_e)\omega_j\ell_e + 120\cos(\omega_j x_e) - 120\cos(\omega_j\ell_e + \omega_j x_e))\end{aligned} \tag{A.40}$$

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