Doubly Spectral Stochastic Finite-Element Method for Linear Structural Dynamics

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Abstract: Uncertainties in complex dynamic systems play an important role in the prediction of a dynamic response in the mid- and high-frequency ranges. For distributed parameter systems, parametric uncertainties can be represented by random fields leading to stochastic partial differential equations. Over the past two decades, the spectral stochastic finite-element method has been developed to discretize the random fields and solve such problems. On the other hand, for deterministic distributed parameter linear dynamic systems, the spectral finite-element method has been developed to efficiently solve the problem in the frequency domain. In spite of the fact that both approaches use spectral decomposition (one for the random fields and the other for the dynamic displacement fields), very little overlap between them has been reported in literature. In this paper, these two spectral techniques are unified with the aim that the unified approach would outperform any of the spectral methods considered on their own. An exponential autocorrelation function for the random fields, a frequency-dependent stochastic element stiffness, and mass matrices are derived for the axial and bending vibration of rods. Closed-form exact expressions are derived by using the Karhunen-Loève expansion. Numerical examples are given to illustrate the unified spectral approach. **DOI: 10.1061/(ASCE)AS.1943-5525.0000070.**

CE Database subject headings: Stochastic processes; Finite element method; Frequency response; Structural dynamics.

Author keywords: Random field; Spectral method; Stochastic finite element; Frequency response; Karhunen-Loève expansion.

Introduction

Spectral methods are widely used in various branches of science and engineering. Because of their general nature, the meaning of spectral methods can be very different depending on the applications and the disciplines. In spite of these differences, the unifying factor among the spectral methods in different disciplines is that, generally, they are very powerful tools for the analytical and experimental treatments of wide ranging physical problems. In the context of the stochastic finite-element method (see, for example, Shinozuka and Yamazaki 1988; Ghanem and Spanos 1991; Kleiber and Hien 1992; Matthies et al. 1997; Manohar and Adhikari 1998a, b; Adhikari and Manohar 1999, 2000; Haldar and Mahadevan 2000; Sudret and Der-Kiureghian 2000; Nair and Keane 2002; Elishakoff and Ren 2003; Sachdeva et al. 2006a, b; Stefanou 2009), spectral methods have been used extensively to analytically represent the random fields describing parametric uncertainties of physical systems. In particular, we refer to the recent paper by Nouy (2009). In the context of structural dynamics, spectral methods have been used in random vibration problems (see, for example, Nigam 1983; Lin 1967; Bolotin 1984) and for the discretization of displacement fields in the frequency domain (Doyle 1989; Gopalakrishnan et al. 2007). In spite of the fact that both approaches use spectral decomposition (one for the random fields and the other for the dynamic displacement fields), very little overlap between them has been reported in literature. In this paper, these two spectral techniques are unified with the aim that the unified approach would perform better than any of the spectral methods considered on their own.

In this paper, we focus our attention on structural dynamic systems with parametric uncertainties. Uncertainties should be accounted for by the credible prediction of numerical codes. In the parametric approach, the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient, and geometric parameters are quantified by using statistical methods and propagated, for example, by using the stochastic finite-element method. The effect of uncertainty is significant in the higher frequency ranges. In the higher frequency ranges, as the wavelengths become smaller, a very fine (i.e., static) mesh size is required to capture the dynamic behavior. As a result, the deterministic analysis itself can pose significant computational challenges. One way to address this problem is to use a spectral approach in the frequency domain. The primary idea is that the displacements within an element are expressed by frequency-dependent shape functions. The shape functions adapt themselves with increasing frequency, and consequently, displacements can be obtained accurately without fine remeshing. The spectral approach has the potential to be an efficient method for mid- and high-frequency vibration problems provided the random fields describing parametric uncertainties can be efficiently accounted for. The spectral decomposition of the random files is used in conjunction with the spectral decomposition of the displacements field. It is expected that the simultaneous use of these two types of spectral decomposition will result in an efficient approach for distributed dynamic systems with parametric uncertainties.

The outline of the paper is as follows. The spectral finite-element method in the frequency domain is briefly discussed in the next section. The essential background of spectral representation of stochastic fields is given in the section "Spectral Finite-Element Method for Stochastic Field Problems." The general derivation of the element mass, stiffness, and damping matrices

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Note. This manuscript was submitted on November 9, 2009; approved on July 21, 2010; published online on June 15, 2011. Discussion period open until December 1, 2011; separate discussions must be submitted for individual papers. This paper is part of the ***Journal of Aerospace Engineering***, Vol. 24, No. 3, July 1, 2011. ©ASCE, ISSN 0893-1321/2011/3-264–276/$25.00.

by using the doubly spectral stochastic finite-element method is given in the section "General Derivation of Doubly Spectral-Element Matrices." In the section "DSSFEM for Damped Rods in Axial Vibration," this general theory is applied to axially vibrating rods with uncertain properties. The method is further applied to bending vibration of Euler-Bernoulli beams with random properties in the section "DSSFEM for Damped Beams in Bending Vibration." Finally, some conclusions are drawn in the last section.

Spectral Finite-Element Method in the Frequency Domain

Spectral methods for deterministic dynamic systems have been in use for more than three decades (see, for example, Paz 1980). This approach, or approaches very similar to this, are known by various names such as the dynamic stiffness method (Banerjee and Williams 1985, 1992, 1995; Banerjee 1989, 1997; Banerjee and Fisher 1992; Ferguson and Pilkey 1993a, b), spectral finite-element method (Doyle 1989; Gopalakrishnan et al. 2007), and dynamic finite-element method (Hashemi et al. 1999; Hashemi and Richard 2000). Some of the notable features of the method are

- The mass distribution of the element is treated in an exact manner in deriving the element dynamic stiffness matrix;
- The dynamic stiffness matrix of one dimensional structural elements accounting for the effects of flexure, torsion, axial motion, shear deformation effects, and damping are exactly determinable, which, in turn, enables the exact vibration analysis of skeletal structures by an inversion of the global dynamic stiffness matrix;
- The method does not employ eigenfunction expansions and, consequently, a major step of the traditional finite-element analysis, namely, the determination of natural frequencies and mode shapes, is eliminated, which automatically avoids the errors attributable to series truncation; this makes the method attractive for situations in which a large number of modes participate in vibration;
- Because the modal expansion is not employed, ad hoc assumptions about the damping matrix proportional to mass and/or stiffness are not necessary;
- The method is essentially a frequency domain approach suitable for steady-state harmonic or stationary random excitation problems; the generalization to other types of problems, such as aeroelastic problems and the dynamics of laminate composite materials through the use of Laplace and Fourier transforms is also available (Gopalakrishnan et al. 2007); and
- The static stiffness matrix and the consistent mass matrix appear as the first two terms in the Taylor expansion of the dynamic stiffness matrix in the frequency parameter.

Spectral Finite-Element Method for Stochastic Field Problems

Problems of structural dynamics in which the uncertainty in specifying the stiffness and mass of the structure are modeled within the framework of random fields and can be treated by using the stochastic finite-element method (Ghanem and Spanos 1991; Sudret and Der-Kiureghian 2000; Stefanou 2009; Nouy 2009). The application of the stochastic finite-element method to linear structural dynamics problems typically consists of the following key steps:

1. The selection of appropriate probabilistic models for parameter uncertainties and boundary conditions.
2. The replacement of the element property random fields by an equivalent set of a finite number of random variables. This step, known as the "discretisation of random fields," is a major step in the analysis.
3. The formulation of the system equations of motion of the form $\mathbf{D}(\omega)\mathbf{u} = \mathbf{f}$ where $\mathbf{D}(\omega)$ is the random dynamic stiffness matrix; $\mathbf{u}$ is the vector of random nodal displacement; and $\mathbf{f}$ is the vector of applied forces. In general, $\mathbf{D}(\omega)$ is a random symmetric complex matrix.
4. The solution of the set of the complex random algebraic equation to obtain the statistics of the response vectors. Alternatively, the response statistics can be obtained by solving the underlying random eigenvalue problem (see, for example, Scheidt and Purkert 1983; Adhikari and Friswell 2007; Benaroya 1992; Adhikari 2007; and references therein).

We consider $(\Theta, \mathcal{F}, P)$ a probability space with $\theta \in \Theta$ denoting a sampling point in the sampling space Θ; $\mathcal{F}$ is the complete σ-algebra over the subsets of Θ; and P is the probability measure. Suppose the spatial coordinate vector, $\mathbf{r} \in \mathbb{R}^d$, where $d \in \mathcal{I} \leq 3$ is the spatial dimension of the problem. Consider $H:(\mathbb{R}^d \times \Theta) \to \mathbb{R}$ is a random field with a covariance function, $C_H:(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$, defined in a space, $\mathcal{D} \in \mathbb{R}^d$. Because the covariance function is finite, symmetric, and positive definite, it can be represented by a spectral decomposition. By using this spectral decomposition, the random process, $H(\mathbf{r}, \theta)$, can be expressed in a generalized Fourier-type of series as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j}\xi_j(\theta)\varphi_j(\mathbf{r}) \tag{1}$$

where $\xi_j(\theta)$ = uncorrelated random variables; and λ_j and $\varphi_j(\mathbf{r})$ = eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_H(\mathbf{r}_1, \mathbf{r}_2)\varphi_j(\mathbf{r}_1)d\mathbf{r}_1 = \lambda_j\varphi_j(\mathbf{r}_2) \quad \forall j = 1, 2, \ldots \tag{2}$$

The spectral decomposition in Eq. (1) is known as the Karhunen-Loève expansion. The series in Eq. (1) can be ordered in a decreasing series so that it can be truncated by using a finite number of terms with a desired accuracy. We refer the books by Ghanem and Spanos (1991), Papoulis and Pillai (2002), and references therein for further discussions about the Karhunen-Loève expansion.

In this paper, one-dimensional systems were considered. Moreover, Gaussian random fields with an exponentially decaying autocorrelation function were considered. The autocorrelation function can be expressed as

$$C(x_1, x_2) = e^{-c|x_1 - x_2|} \tag{3}$$

where the quantity $1/c$ = proportional to the correlation length, and it plays an important role in the description of a random field. If the correlation length is very small, then the random process becomes close to a delta-correlated process, often known as white noise. If the correlation length is very large compared to the domain under consideration, the random process effectively becomes a random variable. The underlying random process, $H(x, \theta)$, can be expanded by using the Karhunen-Loève expansion (Ghanem and Spanos 1991; Papoulis and Pillai 2002) in the interval $-l \leq x \leq l$ as

$$H(x, \theta) = \sum_{j=1}^{\infty} \xi_j(\theta)\sqrt{\lambda_j}\varphi_j(x) \tag{4}$$

Because $H(x, \theta)$ is assumed to be a Gaussian random variable, without any loss of generality, we assumed the mean in zero in Eq. (4). The eigenvalues and eigenfunctions for odd j are given by

$$\lambda_j = \frac{2c}{\alpha_j^2 + c^2} \qquad \varphi_j(x) = \frac{\cos(\alpha_j x)}{\sqrt{l + \frac{\sin(2\alpha_j l)}{2\alpha_j}}} \tag{5}$$

where $\tan(\alpha_j l) = c/\alpha_j$ and for even j are given by

$$\lambda_j = \frac{2c}{\alpha_j^2 + c^2} \qquad \varphi_j(x) = \frac{\sin(\alpha_j x)}{\sqrt{l - \frac{\sin(2\alpha_j l)}{2\alpha_j}}} \tag{6}$$

where $\tan(\alpha_j l) = \alpha_j/(-c)$.

These eigenvalues and eigenfunctions will be used to obtain the element mass, stiffness, and damping matrices. For all practical purposes, the infinite series in Eq. (4) needs to be truncated by using a finite number of terms. The number of terms could be selected from the "amount of information" to be retained. This, in turn, can be related to the number of eigenvalues retained because the eigenvalues, λ_j, in Eq. (4) are arranged in decreasing order. For example, if 90% of the information is to be retained, then one can choose the number of terms, M, such that $\lambda_M/\lambda_1 = 0.1$. The value of M primarily depends on the correlation length of the underlying random field. One needs more terms for cases in which the correlation length is small. Intuitively, this indicates that more independent variables are needed for fields with smaller correlation lengths and vice versa.

General Derivation of Doubly Spectral-Element Matrices

A linear damped distributed parameter dynamic system in which the displacement variable, $U(\mathbf{r}, t)$, where $\mathbf{r} \in \mathbb{R}^d$ is the spatial position vector, $d \leq 3$ is the dimension of the model, and t is time, specified in some domain $\mathcal{D}$, as shown in Fig. 1, is governed by a linear partial differential equation (Meirovitch 1997):

$$\rho(\mathbf{r}, \theta)\frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} + L_1(\theta)\frac{\partial U(\mathbf{r}, t)}{\partial t} + L_2(\theta)U(\mathbf{r}, t) = p(\mathbf{r}, t) \qquad \mathbf{r} \in \mathcal{D} \qquad t \in T \tag{7}$$

with linear boundary-initial conditions of the form

$$M_{1j}\frac{\partial U(\mathbf{r}, t)}{\partial t} = 0 \qquad M_{2j}U(\mathbf{r}, t) = 0 \qquad \mathbf{r} \in \partial\mathcal{D} \qquad t = t_0 \qquad j = 1, 2, \ldots \tag{8}$$

specified on some boundary surface, $\partial\mathcal{D}$. In Eq. (8), $T \in \mathbb{R}$ is the domain of the time variable, t; $\rho(\mathbf{r}, \theta)$ is the random mass distribution of the system; $p(\mathbf{r}, t)$ is the distributed time-varying forcing function; L_1 is the random spatial self-adjoint damping operator; L_2 is the random spatial self-adjoint stiffness operator; and M_{1j} and M_{2j} are some linear operators defined on the boundary surface $\partial\mathcal{D}$. When parametric uncertainties are considered, the mass density $\rho(\mathbf{r}, \theta)$:$(\mathbb{R}^d \times \Theta) \rightarrow \mathbb{R}$, and the damping and stiffness operators involve random processes. Frequency-dependent random element stiffness matrices were derived by various writers by using the dynamic weighted integral approach (Manohar and Adhikari 1998a, b; Adhikari and Manohar 2000; Gupta and Manohar 2002), the energy operator approach (Ghanem and Sarkar 2003), substructure approach (Sarkar and Ghanem 2003), and a series expansion approach (Ostoja-Starzewski and Woods 2003). In the context of uncertainty modeling with a fuzzy approach, Nunes et al. (2006) have combined fuzzy sets with the spectral approach. Moens and Vandepitte (2005, 2007), De Gersem et al. (2005), and Giannini and Hanss (2008) have used a fuzzy parametric approach for uncertainty quantification of the dynamic response.

Xiu and Karniadakis (2002, 2003) and Wan and Karniadakis (2005) have proposed a generalized polynomial chaos approach that can be used for the spectral decomposition of random displacement fields. The method proposed is motivated by the energy operator approach proposed by Sarkar and Ghanem (2002) and Ghanem and Sarkar (2003) for the probabilistic case and the spectral approach proposed by Nunes et al. (2006) for fuzzy uncertain variables. Whereas numerical methods were used in these studies, exact closed-form analytical expressions will be derived for the element matrices in this paper. Suppose the underlying homogeneous system corresponding to System 7 without any forcing (see, for example, Meirovitch 1997) is given by

$$\rho_0\frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} + L_{10}\frac{\partial U(\mathbf{r}, t)}{\partial t} + L_{20}U(\mathbf{r}, t) = 0 \qquad \mathbf{r} \in \mathcal{D} \tag{9}$$

together with a suitable homogeneous boundary and initial conditions. Eq. (9) is a deterministic equation. Taking the Fourier transform of Eq. (9) and considering zero initial conditions, one has

$$-\omega^2\rho_0 u(\mathbf{r}, \omega) + \mathrm{i}\omega L_{10}\{u(\mathbf{r}, \omega)\} + L_{20}\{u(\mathbf{r}, \omega)\} = 0 \tag{10}$$

where $\omega \in [0, \Omega]$ = frequency and $\Omega \in \mathbb{R}$ = maximum frequency.

Like the classical finite-element method, suppose that frequency-dependent displacement within an element is interpolated from the nodal displacements as

$$u_e(\mathbf{r}, \omega) = \mathbf{N}^T(\mathbf{r}, \omega)\hat{\mathbf{u}}_e(\omega) \tag{11}$$

where $\hat{\mathbf{u}}_e(\omega) \in \mathbb{C}^n$ = nodal displacement vector; $\mathbf{N}(\mathbf{r}, \omega) \in \mathbb{C}^n$ = the vector of frequency-dependent shape functions; and n = number of the nodal degrees-of-freedom. Suppose the $s_j(\mathbf{r}, \omega) \in \mathbb{C}$, $j = 1, 2, \ldots, m$ are the basis functions that exactly satisfy Eq. (10) where m = order of the ordinary differential Eq. (10). The shape function vector can be expressed as

$$\mathbf{N}(\mathbf{r}, \omega) = \boldsymbol{\Gamma}(\omega)\mathbf{s}(\mathbf{r}, \omega) \tag{12}$$

where the vector $\mathbf{s}(\mathbf{r}, \omega) = \{s_j(\mathbf{r}, \omega)\}^T$; $\forall j = 1, 2, \ldots, m$; and the complex matrix, $\boldsymbol{\Gamma}(\omega) \in \mathbb{C}^{n \times m}$, depend on the boundary conditions. The derivation of $\boldsymbol{\Gamma}(\omega)$ for the axial vibration of rods and bending vibration of beams are given in the next two sections.

Extending the weak-form of the finite-element approach to the complex domain, the frequency-dependent $n \times n$ complex random stiffness, mass, and damping matrices can be obtained as

$$\mathbf{K}_e(\omega, \theta) = \int_{\mathcal{D}_e} k_s(\mathbf{r}, \theta)\mathcal{L}_2\{\mathbf{N}(\mathbf{r}, \omega)\}\mathcal{L}_2\{\mathbf{N}^T(\mathbf{r}, \omega)\}d\mathbf{r} \tag{13}$$

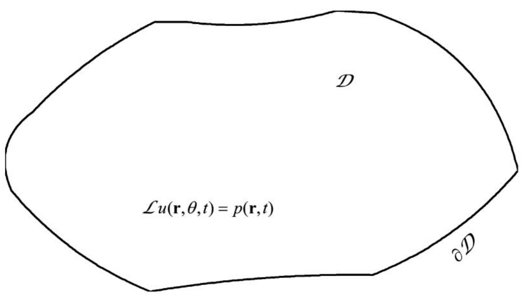

Fig. 1. Domain and boundary surface of differential operator describing stochastic dynamic system

$$\mathbf{M}_e(\omega,\theta) = \int_{\mathcal{D}_e} \rho(\mathbf{r},\theta)\mathbf{N}(\mathbf{r},\omega)\mathbf{N}^T(\mathbf{r},\omega)d\mathbf{r} \tag{14}$$

and

$$\mathbf{C}_e(\omega,\theta) = \int_{\mathcal{D}_e} c(\mathbf{r},\theta)\mathcal{L}_1\{\mathbf{N}(\mathbf{r},\omega)\}\mathcal{L}_1\{\mathbf{N}^T(\mathbf{r},\omega)\}d\mathbf{r} \tag{15}$$

where $(\bullet)^T$ = matrix transpose; $k_s(\mathbf{r},\theta):(\mathbb{R}^d \times \Theta) \to \mathbb{R}$ = random distributed stiffness parameter; $\mathcal{L}_2\{\bullet\}$ = strain energy operator; and $c(\mathbf{r},\theta):(\mathbb{R}^d \times \Theta) \to \mathbb{R}$ = energy dissipation operator. The derivation of the element matrices follows a method similar to the conventional spectral stochastic finite-element method (see, for example, Ghanem and Spanos 1991). The primary difference is that the real shape functions need to be replaced by the equivalent complex shape functions given by Eq. (12). We refer to the papers by Manohar and Adhikari (1998a) and Adhikari and Manohar (1999) for further details, including the derivation of the complex element matrices that use energy principles. In the previous equations, $\mathcal{D}_e \in \mathcal{D}$ is the domain of an element such that $\mathcal{D} = \bigcup \ldots \bigcup \mathcal{D}_e$ and $\mathcal{D}_e \bigcap \mathcal{D}_{e'} = \emptyset$; $\forall e$ and e'. The random fields $k_s(\mathbf{r},\theta)$, $\rho(\mathbf{r},\theta)$, and $c(\mathbf{r},\theta)$ are expanded by using the Karhunen-Loève expansion [Eq. (1)]. By using a finite number of terms, each of the complex element matrices can be expanded in a spectral series as

$$\mathbf{K}_e(\omega,\theta) = \mathbf{K}_{0e}(\omega) + \sum_{j=1}^{M_{\mathrm{K}}} \xi_{\mathrm{K}_j}(\theta)\mathbf{K}_{je}(\omega) \tag{16}$$

$$\mathbf{M}_e(\omega,\theta) = \mathbf{M}_{0e}(\omega) + \sum_{j=1}^{M_{\mathrm{M}}} \xi_{\mathrm{M}_j}(\theta)\mathbf{M}_{je}(\omega) \tag{17}$$

and

$$\mathbf{C}_e(\omega,\theta) = \mathbf{C}_{0e}(\omega) + \sum_{j=1}^{M_{\mathrm{C}}} \xi_{\mathrm{C}_j}(\theta)\mathbf{C}_{je}(\omega) \tag{18}$$

The complex deterministic symmetric matrices, for example in the case of the stiffness matrix, can be obtained as

$$\mathbf{K}_{0e}(\omega) = \int_{\mathcal{D}_e} k_{s_0}(\mathbf{r})\mathcal{L}_2\{\mathbf{N}(\mathbf{r},\omega)\}\mathcal{L}_2\{\mathbf{N}^T(\mathbf{r},\omega)\}d\mathbf{r} \tag{19}$$

and

$$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{\mathrm{K}_j}} \int_{\mathcal{D}_e} \varphi_{\mathrm{K}_j}(\mathbf{r})\mathcal{L}_2\{\mathbf{N}(\mathbf{r},\omega)\}\mathcal{L}_2\{\mathbf{N}^T(\mathbf{r},\omega)\}d\mathbf{r} \quad \forall j = 1,2,\ldots,M_{\mathrm{K}} \tag{20}$$

The equivalent terms corresponding to the mass and damping matrices can also be obtained in a similar manner. Substituting the shape function from Eq. (12), into Eqs. (19) and (20), one obtains

$$\mathbf{K}_{0e}(\omega) = \mathbf{\Gamma}(\omega)\tilde{\mathbf{K}}_{0e}(\omega)\mathbf{\Gamma}^T(\omega) \tag{21}$$

and

$$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{\mathrm{K}_j}}\mathbf{\Gamma}(\omega)\tilde{\mathbf{K}}_{je}(\omega)\mathbf{\Gamma}^T(\omega) \qquad \forall j = 1,2,\ldots,M_{\mathrm{K}} \tag{22}$$

where

$$\tilde{\mathbf{K}}_{0e}(\omega) = \int_{\mathcal{D}_e} k_{s_0}(\mathbf{r})\mathcal{L}_2\{\mathbf{s}(\mathbf{r},\omega)\}\mathcal{L}_2\{\mathbf{s}^T(\mathbf{r},\omega)\}d\mathbf{r} \in \mathbb{C}^{mm} \tag{23}$$

and

$$\tilde{\mathbf{K}}_{je}(\omega) = \int_{\mathcal{D}_e} \varphi_{\mathrm{K}_j}(\mathbf{r})\mathcal{L}_2\{\mathbf{s}(\mathbf{r},\omega)\}\mathcal{L}_2\{\mathbf{s}^T(\mathbf{r},\omega)\}d\mathbf{r} \in \mathbb{C}^{mm} \quad \forall j = 1,2,\ldots,M_{\mathrm{K}} \tag{24}$$

The expressions of the eigenfunctions given in the previous section are valid within the specific domains defined. One needs to change the coordinateto use them in Eq. (24). Once the element stiffness, mass, and damping matrices are obtained in this manner, calculations for the global matrices can be achieved by summing the element matrices with suitable coordinate transformations as in the standard finite-element method. A closed-form expression of the eigenfunctions appearing in Eq. (24) are available for only a few specific correlation functions and with simple boundaries only. For such cases, shown subsequently, the integral in Eq. (24) may be obtained in a closed-form. However, in general, the integral equation governing the eigenfunctions in Eq. (2) has to be solved numerically. For such general cases, the element matrices should be obtained by using numerical integration techniques.

Because of the use of the spectral element in the frequency domain, only one finite element is required per physical "element" of a built-up system. For this reason, the dimension of the global assembled matrices becomes small, even for cases in which high-frequency vibration is considered. However, for the deterministic system, the element matrices are not exact because the Karhunen-Loève expansion [Eq. (1)] needs to be truncated after a finite number of terms. The global spectral matrix can be expressed as

$$\mathbf{D}(\omega,\theta) = -\omega^2\mathbf{M}(\omega,\theta) + \mathrm{i}\omega\mathbf{C}(\omega,\theta) + \mathbf{K}(\omega,\theta) \in \mathbb{C}^{N\times N} \tag{25}$$

where N = dynamic degrees-of-freedom. Following the proposed DSSFEM approach, in general, the matrix $\mathbf{D}(\omega,\theta)$ can be expressed as

$$\mathbf{D}(\omega,\theta) = \mathbf{D}_0(\omega) + \sum_j \xi_j(\theta)\mathbf{D}_j(\omega) \tag{26}$$

In this equation $\mathbf{D}:(\Omega \times \Theta) \to \mathbb{C}^{N\times N}$ = complex random symmetric matrix and it needs to be inverted for every ω to obtain the dynamic response. In this case, Ω denotes the space of frequency. Unlike the inversion of real symmetric random matrices or complex Hermitian matrices, relatively little literature is available about complex symmetric matrices. Adhikari and Manohar (1999) and, more recently, Ghanem and Das (2009) have considered complex random matrices arising in structural dynamics. In principle, analytical approaches such as the perturbation-based methods (Kleiber and Hien 1992) and projections methods (Ghanem and Spanos 1991) can be applied for the inversion of $\mathbf{D}(\omega,\theta)$. In practice, however, difficulties may arise because of the fact that $\mathbf{D}(\omega,\theta)$ becomes close to singular when ω approaches a system natural frequency. This can be a major problem, particularly for cases in which the damping of the system is low. Reliable and computationally efficient methods for the derivation of dynamic response by using the proposed DSSFEM approach is an outstanding problem and is currently a limitation of this approach. It is beyond the scope of this paper to address this issue in detail. In this paper, a direct Monte Carlo simulation is used to obtain the response statistics in the subsequent numerical examples.

DSSFEM for Damped Rods in Axial Vibration

Equation of Motion

The equation of motion of a damped stochastically nonhomogeneous rod under axial vibration is given by

$$\frac{\partial}{\partial x}\left[AE(x)\frac{\partial U(x,t)}{\partial x} + c_1\frac{\partial^2 U(x,t)}{\partial x \partial t}\right] = m(x)\frac{\partial^2 U(x,t)}{\partial t^2} + c_2\frac{\partial U(x,t)}{\partial t} \tag{27}$$

where $U(x,t)$ = axial displacement; c_1 = strain rate-dependent viscous damping coefficient; and c_2 = velocity-dependent viscous damping coefficient. These quantities are assumed to be deterministic constants. The axial rigidity, $AE(x)$, and the mass per unit length, $m(x)$, are assumed to be random fields of the following form:

$$AE(x,\theta) = AE_0[1 + \epsilon_{AE} H_{AE}(x,\theta)] \tag{28}$$

$$m(x) = m_0[1 + \epsilon_m H_m(x,\theta)] \tag{29}$$

It is assumed that $H_{AE}(x,\theta)$ and $H_m(x,\theta)$ are homogeneous Gaussian random fields with a zero mean and an exponentially decaying autocorrelation function of the form given by Eq. (3). The "strength parameters" ϵ_{AE} and ϵ_m effectively quantify the amount of uncertainty in the axial rigidity and mass per unit length of the rod. The constants AE_0 and m_0 are the mass per unit length and axial rigidity of the underlying baseline model, respectively. The equation of motion of the baseline model is given by

$$AE_0 \frac{\partial^2 U(x,t)}{\partial x^2} + c_1 \frac{\partial^3 U(x,t)}{\partial x^2 \partial t} = m_0 \frac{\partial^2 U(x,t)}{\partial t^2} + c_2 \frac{\partial U(x,t)}{\partial t} \tag{30}$$

With the spectral expansion of the axial displacement, $U(x,t)$, in the frequency-wavenumber space, one has

$$U(x,t) = u(x)e^{\mathrm{i}\omega t} = e^{kx}e^{\mathrm{i}\omega t} \tag{31}$$

where $\mathrm{i} = \sqrt{-1}$ and k = wavenumber for the baseline model in Eq. (30). Substituting $U(x,t)$ from Eq. (31) in Eq. (30) and simplifying, we have

$$k^2 + a^2 = 0 \quad \text{or} \quad k = \pm \mathrm{i}a \tag{32}$$

where

$$a^2 = \frac{m_0\omega^2 - \mathrm{i}\omega c_2}{AE_0 + \mathrm{i}\omega c_1} \tag{33}$$

An element for the damped axially vibrating rod is shown in Fig. 2.

In view of the solutions in Eq. (32), the complex displacement field within the element can be expressed by the linear combination of the basic functions $e^{-\mathrm{i}ax}$ and $e^{\mathrm{i}ax}$ so that, in our notations, $\mathbf{s}(x,\omega) = \{e^{-\mathrm{i}ax}, e^{\mathrm{i}ax}\}^T$. We have expressed the KL expansion by trigonometric functions in Eqs. (5) and (6). Therefore, it is more convenient to express $\mathbf{s}(x,\omega)$ with trigonometric functions. Considering $e^{\pm \mathrm{i}ax} = \cos(ax) \pm \mathrm{i}\sin(ax)$, the vector $\mathbf{s}(x,\omega)$ can be alternatively expressed as

$$\mathbf{s}(x,\omega) = \left\{ \begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix} \right\} \in \mathbb{C}^2 \tag{34}$$

Considering the unit axial displacement boundary condition as $u(x=0) = 1$ and $u(x=L) = 1$, after some elementary algebra, the shape function vector can be expressed in the form of Eq. (12) as

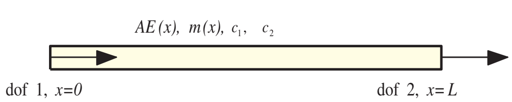

Fig. 2. Element for axially vibrating rod with damping; axial rigidity, $AE(x)$, and mass per unit length, $m(x)$, are assumed to be random fields; strain rate-dependent viscous damping coefficient, c_1, and velocity-dependent viscous damping coefficient, c_2, are assumed to be deterministic; element has two degrees-of-freedom and displacement field within element is complex and frequency-dependent

$$\mathbf{N}(x,\omega) = \boldsymbol{\Gamma}(\omega)\mathbf{s}(x,\omega), \\ \text{where } \boldsymbol{\Gamma}(\omega) = \begin{bmatrix} -\cot(aL) & 1 \\ \operatorname{cosec}(aL) & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}\sigma \tag{35}$$

Now we need to substitute $\mathbf{s}(x,\omega)$ in Eqs. (23) and (24) to obtain the deterministic and random part of the element matrices. In this paper, damping is assumed to be deterministic. Therefore, only the stiffness and mass matrices of the system will be derived.

Derivation of Element Stiffness and Mass Matrices

For the axial vibration, the stiffness operator is given by $\mathcal{L}_2(\bullet) = \partial(\bullet)/\partial x$. Because constant nominal values are assumed, we have $k_{s_0}(\mathbf{r}) = AE_0$. With these, from Eq. (23), one obtains

$$\tilde{\mathbf{K}}_{0e}(\omega) = AE_0 \int_{x=0}^{L} \left\{ \frac{\partial \mathbf{s}(x,\omega)}{\partial x} \right\} \left\{ \frac{\partial \mathbf{s}(x,\omega)}{\partial x} \right\}^T \mathrm{d}x \tag{36}$$

$$= \frac{AE_0 a}{2} \begin{bmatrix} cs + aL & -1 + c^2 \\ -1 + c^2 & aL - cs \end{bmatrix} \tag{37}$$

where

$$c = \cos(aL) \quad \text{and} \quad s = \sin(aL) \tag{38}$$

The deterministic part of the stiffness matrix can be obtained from Eq. (21) by using the $\boldsymbol{\Gamma}(\omega)$ matrix defined in Eq. (35). The term $\tilde{\mathbf{M}}_{0e}(\omega)$ can be obtained in a similar way as

$$\tilde{\mathbf{M}}_{0e}(\omega) = m_0 \int_{x=0}^{L} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega)dx \tag{39}$$

$$= \frac{m_0}{2a} \begin{bmatrix} aL - cs & 1 - c^2 \\ 1 - c^2 & cs + aL \end{bmatrix} \tag{40}$$

The deterministic mass matrix can be obtained from Eq. (40) as $\mathbf{M}_{0e}(\omega) = \boldsymbol{\Gamma}(\omega)\tilde{\mathbf{M}}_{0e}(\omega)\boldsymbol{\Gamma}^T(\omega)$.

To obtain the matrices associated with the random components, for each j, two different matrices correspond to the two eigenfunctions defined in Eqs. (5) and (6). Following Eq. (16), we can express the element stiffness matrix as

$$\mathbf{K}_e(\omega,\theta) = \mathbf{K}_{0e}(\omega) + \Delta\mathbf{K}_e(\omega,\theta) \tag{41}$$

where $\Delta\mathbf{K}_e(\omega,\theta)$ = random part of the matrix. Following Eq. (22), this matrix can be conveniently expressed as

$$\Delta\mathbf{K}_e(\omega,\theta) = \boldsymbol{\Gamma}(\omega)\widetilde{\Delta\mathbf{K}}_e(\omega,\theta)\boldsymbol{\Gamma}^T(\omega) \tag{42}$$

The matrix $\widetilde{\Delta\mathbf{K}}_e(\omega)$ can be expanded by using the Karhunen-Loève expansion as

$$\widetilde{\Delta\mathbf{K}}_e(\omega,\theta) = \sum_{j=1}^{M_{\mathrm{K}}} \xi_{\mathrm{K}_j}(\theta)\sqrt{\lambda_{\mathrm{K}_j}}\tilde{\mathbf{K}}_{je}(\omega) \tag{43}$$

where $\sqrt{\lambda_{\mathrm{K}_j}}$ = eigenvalues corresponding to the random field $H_{AE}(x,\theta)$. The matrices $\tilde{\mathbf{K}}_{je}(\omega)$ can be obtained by using the integrals of Eq. (24). By using the expression of the eigenfunction for the odd values of j, as in Eq. (5), one has

$$\tilde{\mathbf{K}}_{je}(\omega) = \int_0^L \frac{\epsilon_{AE} AE_0 \cos[\alpha_j(-L/2 + x)]}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{ \frac{\partial \mathbf{s}(x,\omega)}{\partial x} \right\} \left\{ \frac{\partial \mathbf{s}(x,\omega)}{\partial x} \right\}^T dx \tag{44}$$

$$= \frac{\epsilon_{AE} A E_0}{\sqrt{(L/2 + c_\alpha s_\alpha/\alpha_j)}} \frac{a^2}{\alpha_j(4a^2 - \alpha_j^2)} \times \begin{bmatrix} 2\alpha_j a c_{\alpha_j} cs + (-\alpha_j^2 + 4a^2 - \alpha_j^2 c^2) s_{\alpha_j} & (-2\alpha_j a + 2\alpha_j a c^2) c_{\alpha_j} + \alpha_j^2 s_{\alpha_j} cs \\ (-2\alpha_j a + 2\alpha_j a c^2) c_{\alpha_j} + \alpha_j^2 s_{\alpha_j} cs & -2\alpha_j a c_{\alpha_j} cs + (4a^2 - \alpha_j^2 + \alpha_j^2 c^2) s_{\alpha_j} \end{bmatrix} \tag{45}$$

In Eq. (45),

$$c_{\alpha_j} = \cos(\alpha_j L/2) \quad \text{and} \quad s_{\alpha_j} = \sin(\alpha_j L/2) \tag{46}$$

and the eigenvalues α_j should be obtained by solving the transcendental Eq. (5) with $l = L/2$. In Eq. (44), the KL eigenfunction is shifted to account for the fact that Eq. (5) is defined for $-L/2 \leq x \leq L/2$ whereas the element shape functions are defined over $0 \leq x \leq L$. In Eq. (44), we have used the identity $\sin(\alpha_j L) = 2\cos(\alpha_j L/2)\sin(\alpha_j L/2) = 2c_\alpha s_\alpha$. In a similar manner, by using the expression of the eigenfunction for the even values of j, as in Eq. (6), one has

$$\tilde{\mathbf{K}}_{je}(\omega) = \int_0^L \frac{\epsilon_{AE} A E_0 \sin[\alpha_j(-L/2 + x)]}{\sqrt{L/2 - \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{\frac{\partial \mathbf{s}(x,\omega)}{\partial x}\right\} \left\{\frac{\partial \mathbf{s}(x,\omega)}{\partial x}\right\}^T dx \tag{47}$$

$$= \frac{\epsilon_{AE} A E_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \frac{a^2}{\alpha_j(4a^2 - \alpha_j^2)} \times \begin{bmatrix} (-\alpha_j^2 + \alpha_j^2 c^2) c_{\alpha_j} + 2\alpha_j a s_{\alpha_j} cs & -\alpha_j^2 c_{\alpha_j} cs + 2\alpha_j a s_{\alpha_j} c^2 \\ -\alpha_j^2 c_{\alpha_j} cs + 2\alpha_j a s_{\alpha_j} c^2 & (\alpha_j^2 - \alpha_j^2 c^2) c_{\alpha_j} - 2\alpha_j a s_{\alpha_j} cs \end{bmatrix} \tag{48}$$

The mass matrix can also be represented as Eqs. (41)–(43). The eigenvalues and eigenfunctions corresponding to the random field $H_m(x,\theta)$ need to be used to obtain the elements of $\tilde{\mathbf{M}}_{je}(\omega)$. By using the expression of the eigenfunction for the odd values of j, as in Eq. (5), one has

$$\tilde{\mathbf{M}}_{je}(\omega) = \int_0^L \frac{\epsilon_m m_0 \cos[\alpha_j(-L/2 + x)]}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega) dx \tag{49}$$

$$= \frac{\epsilon_m m_0}{\sqrt{(L/2 + c_\alpha s_\alpha/\alpha_j)}} \frac{1}{\alpha_j(4a^2 - \alpha_j^2)} \times \begin{bmatrix} -2\alpha_j a c_{\alpha_j} cs + (4a^2 - \alpha_j^2 + \alpha_j^2 c^2) s_{\alpha_j} & (2\alpha_j a - 2\alpha_j a c^2) c_{\alpha_j} - \alpha_j^2 s_{\alpha_j} cs \\ (2\alpha_j a - 2\alpha_j a c^2) c_{\alpha_j} - \alpha_j^2 s_{\alpha_j} cs & 2\alpha_j a c_{\alpha_j} cs + (-\alpha_j^2 + 4a^2 - \alpha_j^2 c^2) s_{\alpha_j} \end{bmatrix} \tag{50}$$

In Eq. (50), the eigenvalues α_j should be obtained by solving the transcendental Eq. (5). In a similar manner, by using the expression of the eigenfunction for the even values of j ,as in Eq. (6), one has

$$\tilde{\mathbf{M}}_{je}(\omega) = \int_0^L \frac{\epsilon_m m_0 \sin[\alpha_j(-L/2 + x)]}{\sqrt{L/2 - \frac{\sin(\alpha_j L)}{2\alpha_j}}} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega) dx \tag{51}$$

$$= \frac{\epsilon_m m_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \frac{1}{\alpha_j(4a^2 - \alpha_j^2)} \times \begin{bmatrix} (\alpha_j^2 - \alpha_j^2 c^2) c_{\alpha_j} - 2\alpha_j a s_{\alpha_j} cs & \alpha_j^2 c_{\alpha_j} cs - 2\alpha_j a s_{\alpha_j} c^2 \\ \alpha_j^2 c_{\alpha_j} cs - 2\alpha_j a s_{\alpha_j} c^2 & (-\alpha_j^2 + \alpha_j^2 c^2) c_{\alpha_j} + 2\alpha_j a s_{\alpha_j} cs \end{bmatrix} \tag{52}$$

Eqs. (44)–(51) completely define the random parts of the element stiffness and mass matrices. The exact closed-form expression of the elements of these four matrices further reduces the computational cost in deriving these matrices.

Numerical Illustrations

We consider a numerical example to illustrate the application of the expressions derived in the previous subsection. The mean material properties are considered $\rho_0 = 2{,}700\ \text{kg/m}^3$ and $E_0 = 69$ GPa, values corresponding to aluminum. The length and cross-section of the rod are $L = 30$ m and $A_0 = 1\ \text{cm}^2$, respectively. By using these values, we have $AE_0 = 6.9 \times 10^6$ and $m_0 = \rho_0 A_0 = 0.27$. A clamped-free boundary condition is considered. The standard deviations of both the random fields are assumed to be 10% of the mean values of the random fields, that is, $\epsilon_{AE} = 0.1AE_0$ and $\epsilon_m = 0.1m_0$. The damping coefficients are assumed to be $c_1 = 1.5 \times 10^{-5} AE_0$ and $c_2 = 11.15 m_0$. The correlation length of the random fields describing $AE(x)$ and $m(x)$ is assumed to be $L/5$. We consider the response at the free end of the rod attributable to the unit harmonic force at that end. The response is calculated up to 500 Hz covering the first six vibration modes of the system. The response of the deterministic system, the mean, and the standard deviation of the absolute value of the response are shown in Fig. 3.

These results are obtained by using a Monte Carlo simulation with 4,000 samples. In total, 36 terms are used for the KL expansion. With this number of terms, the last eigenvalue of the KL expansion becomes less than 5% of the first eigenvalue. The element matrices associated with 36 random variables are obtained by using the closed-form expression derived in the previous section. The phase of the frequency response function at the free end of the rod is shown in Fig. 4.

The phase does not change sign because we are considering the driving point response. In both Figs. 3 and 4, the mean curve is different from the deterministic curve. This difference is larger at higher frequencies. At lower frequencies, the standard deviation is biased by the mean. But as the frequency increases, the standard deviation curve flattens. These results are obtained by using a single spectral element, although six modes of vibration exist within the frequency range considered.

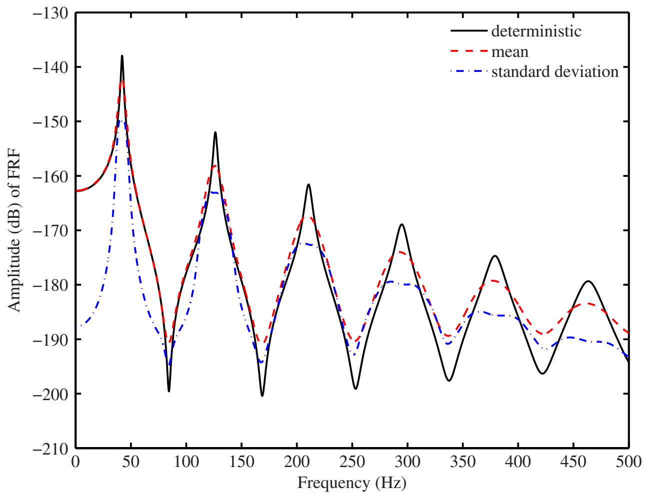

Fig. 3. Amplitude of frequency response function at free end of damped axially vibrating rod with random properties

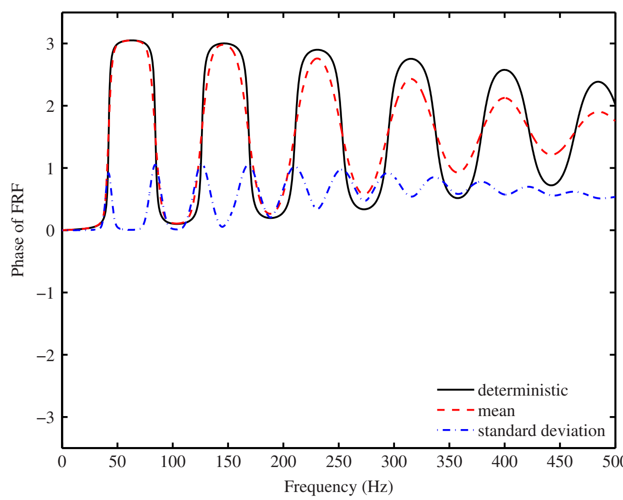

Fig. 4. Phase of frequency response function at free end of damped axially vibrating rod with random properties

DSSFEM for Damped Beams in Bending Vibration

Equation of Motion

The equation of motion of a damped stochastically nonhomogeneous Euler-Bernoulli beam under bending vibration is given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 Y(x,t)}{\partial x^2} + c_3\frac{\partial^3 Y(x,t)}{\partial x^2 \partial t}\right] + m(x)\frac{\partial^2 Y(x,t)}{\partial t^2} + c_4\frac{\partial Y(x,t)}{\partial t} = 0 \tag{53}$$

where $Y(x,t)$ = transverse flexural displacement; c_3 = strain rate-dependent viscous damping coefficient; and c_4 = velocity-dependent viscous damping coefficient. These quantities are assumed to be deterministic constants. The mass per unit length, $m(x)$, is assumed to be a random field of the form given by Eq. (29), and the bending rigidity, $EI(x)$, is assumed to be

$$EI(x,\theta) = EI_0[1 + \epsilon_{EI} H_{EI}(x,\theta)] \tag{54}$$

As in the case of the axially vibrating rod, we consider that $H_{EI}(x,\theta)$ is a homogeneous Gaussian random field with a zero mean and an exponentially decaying autocorrelation function of the form given by Eq. (3). The "strength parameter" ϵ_{EI} quantifies the amount of uncertainty in the bending rigidity of the beam. The constant EI_0 is the bending rigidity of the underlying baseline model. The equation of motion of the baseline model is given by

$$EI_0\frac{\partial^4 Y(x,t)}{\partial x^4} + c_3\frac{\partial^5 Y(x,t)}{\partial x^2 \partial t} + m_0\frac{\partial^2 Y(x,t)}{\partial t^2} + c_4\frac{\partial Y(x,t)}{\partial t} = 0 \tag{55}$$

By using the spectral representation of the transverse displacement, $Y(x,t)$, one has

$$Y(x,t) = y(x)e^{\mathrm{i}\omega t} = e^{kx}e^{\mathrm{i}\omega t} \tag{56}$$

where k = wavenumber for the baseline model in Eq. (55). Substituting $Y(x,t)$ from Eq. (56) in Eq. (55) we have

$$k^4 - b^4 = 0 \quad \text{or} \quad k = \pm \mathrm{i}b \qquad \pm b \tag{57}$$

where

$$b^4 = \frac{m_0\omega^2 - \mathrm{i}\omega c_4}{EI_0 + \mathrm{i}\omega c_3} \tag{58}$$

An element for the damped beam under bending vibration is shown in Fig. 5. The degrees-of-freedom for each nodal point include a vertical and a rotational degrees-of-freedom.

In view of the solutions in Eq. (57), the displacement field with the element can be expressed by a linear combination of the basic functions e^{-bx}, e^{bx}, $e^{-\mathrm{i}bx}$, and $e^{\mathrm{i}bx}$ so that, in our notations, $\mathbf{s}(x,\omega) = \{e^{-bx}, e^{bx}, e^{-\mathrm{i}bx}, e^{\mathrm{i}bx}\}^T$. We have expressed the KL expansions by trigonometric functions in Eqs. (5) and (6). Therefore, as in the previous section, we aim to express $\mathbf{s}(x,\omega)$ with trigonometric functions. Considering $e^{\pm \mathrm{i}bx} = \cos(bx) \pm \mathrm{i}\sin(bx)$ and $e^{\pm bx} = \cosh(bx) \pm \mathrm{i}\sinh(bx)$, the vector $\mathbf{s}(x,\omega)$ can be alternatively expressed as

$$\mathbf{s}(x,\omega) = \begin{Bmatrix} \sin(bx) \\ \cos(bx) \\ \sinh(bx) \\ \cosh(bx) \end{Bmatrix} \in \mathbb{C}^4 \tag{59}$$

The displacement field within the element can be expressed as

$$y(x) = \mathbf{s}(x,\omega)^T \mathbf{y}_e \tag{60}$$

where $\mathbf{y}_e \in \mathbb{C}^4$ = vector of constants to be determined from the boundary conditions.

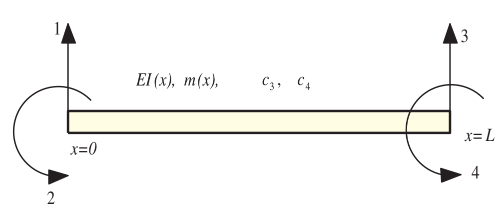

Fig. 5. Element for damped beam under bending vibration; bending rigidity, $EI(x)$, and mass per unit length, $m(x)$, are assumed to be random fields; strain rate-dependent viscous damping coefficient, c_3, and velocity-dependent viscous damping coefficient, c_4, are assumed to be deterministic; element has four degrees-of-freedom and displacement field within element is complex and frequency-dependent

The relationship between the shape functions and the boundary conditions can be represented as in Table 1, in which the boundary conditions in each column give rise to the corresponding shape function.

Writing Eq. (60) for the four sets of boundary conditions, one obtains

$$[\mathbf{R}][\mathbf{y}_e^1, \mathbf{y}_e^2, \mathbf{y}_e^3, \mathbf{y}_e^4] = \mathbf{I}\mu \tag{61}$$

where

$$\mathbf{R} = \begin{bmatrix} s_1(0) & s_2(0) & s_3(0) & s_4(0) \\ \frac{ds_1}{dx}(0) & \frac{ds_2}{dx}(0) & \frac{ds_3}{dx}(0) & \frac{ds_4}{dx}(0) \\ s_1(L) & s_2(L) & s_3(L) & s_4(L) \\ \frac{ds_1}{dx}(L) & \frac{ds_2}{dx}(L) & \frac{ds_3}{dx}(L) & \frac{ds_4}{dx}(L) \end{bmatrix} \tag{62}$$

and $\mathbf{y}_e^k$ = vector of constants, giving rise to the kth shape function. In view of the boundary conditions represented in Table 1 and Eq. (61), the shape functions for the bending vibration can be given by Eq. (12) where

$$\mathbf{\Gamma}(\omega) = [\mathbf{y}_e^1, \mathbf{y}_e^2, \mathbf{y}_e^3, \mathbf{y}_e^4]^T = [\mathbf{R}^{-1}]^T$$
$$= \begin{bmatrix} \frac{1}{2}\frac{cS+Cs}{cC-1} & -\frac{1}{2}\frac{1+sS-cC}{cC-1} & -\frac{1}{2}\frac{cS+Cs}{cC-1} & \frac{1}{2}\frac{cC+sS-1}{cC-1} \\ \frac{1}{2}\frac{cC+sS-1}{b(cC-1)} & \frac{1}{2}\frac{-Cs+cS}{b(cC-1)} & -\frac{1}{2}\frac{1+sS-cC}{b(cC-1)} & -\frac{1}{2}\frac{-Cs+cS}{b(cC-1)} \\ -\frac{1}{2}\frac{S+s}{cC-1} & \frac{1}{2}\frac{C-c}{cC-1} & \frac{1}{2}\frac{S+s}{cC-1} & -\frac{1}{2}\frac{C-c}{cC-1} \\ \frac{1}{2}\frac{C-c}{b(cC-1)} & -\frac{1}{2}\frac{S-s}{b(cC-1)} & -\frac{1}{2}\frac{C-c}{b(cC-1)} & -\frac{1}{2}\frac{S-s}{b(cC-1)} \end{bmatrix} \tag{63}$$

where

Table 1. Relationship between Boundary Conditions and Shape Functions for Bending Vibration of Beams

	$N_1(x,\omega)$	$N_2(x,\omega)$	$N_3(x,\omega)$	$N_4(x,\omega)$
$y(0)$	1	0	0	0
$(dy/dx)(0)$	0	1	0	0
$y(L)$	0	0	1	0
$(dy/dx)(L)$	0	0	0	1

$$\begin{aligned} C &= \cosh(bL) \qquad c = \cos(bL) \\ S &= \sinh(bL) \qquad s = \sin(bL) \end{aligned} \tag{64}$$

are frequency-dependent quantities because b is a function of ω. We need to substitute $\mathbf{s}(x,\omega)$ in Eqs. (23) and (24) to obtain the deterministic and random part of the element matrices. Because damping is assumed to be deterministic, we will only derive the stiffness and mass matrices of the system.

Derivation of Element Stiffness and Mass Matrices

For the bending vibration, the stiffness operator can be given as $\mathcal{L}_2(\bullet) = \partial^2(\bullet)/\partial x^2$. Because constant nominal values are assumed, we have $k_{s_0}(\mathbf{r}) = EI_0$. By using these, from Eq. (23), one obtains

$$\tilde{\mathbf{K}}_{0e}(\omega) = EI_0 \int_{x=0}^{L} \left\{\frac{\partial^2 \mathbf{s}(x,\omega)}{\partial x^2}\right\}\left\{\frac{\partial^2 \mathbf{s}(x,\omega)}{\partial x^2}\right\}^T dx \tag{65}$$

$$= \frac{EI_0 b^3}{2}\begin{bmatrix} bL - cs & 1-c^2 & cS - sC & -1 + cC - sS \\ 1 - c^2 & cs + bL & 1 - cC - sS & -cS - sC \\ cS - sC & 1 - cC - sS & CS - bL & -1 + C^2 \\ -1 + cC - sS & -cS - sC & -1 + C^2 & CS + bL \end{bmatrix} \tag{66}$$

The deterministic part of the stiffness matrix can be obtained from Eq. (21) by using the $\mathbf{\Gamma}(\omega)$ matrix defined in Eq. (63). The term $\tilde{\mathbf{M}}_{0e}(\omega)$ can be obtained in a similar way as

$$\tilde{\mathbf{M}}_{0e}(\omega) = m_0 \int_{x=0}^{L} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega)dx \tag{67}$$

$$= \frac{m_0}{2b}\begin{bmatrix} bL - cs & 1 - c^2 & -cS + sC & 1 - cC + sS \\ 1 - c^2 & cs + bL & -1 + cC + sS & cS + sC \\ -cS + sC & -1 + cC + sS & CS - bL & -1 + C^2 \\ 1 - cC + sS & cS + sC & -1 + C^2 & CS + bL \end{bmatrix} \tag{68}$$

The deterministic mass matrix can be obtained from Eq. (68) as $\mathbf{M}_{0e}(\omega) = \mathbf{\Gamma}(\omega)\tilde{\mathbf{M}}_{0e}(\omega)\mathbf{\Gamma}^T(\omega)$.

To obtain the matrices associated with the random components, for each j, there will be two different matrices corresponding to the two eigenfunctions defined in Eqs. (5) and (6). As in the case of the axial vibration of rods, the element stiffness matrix can be expressed as Eq. (42) whereas the matrix $\Delta\tilde{\mathbf{K}}_e(\omega)$ can be expanded with the Karhunen-Loève expansion as Eq. (43).

The matrices $\tilde{\mathbf{K}}_{je}(\omega)$ can be obtained by using the integrals of the form Eq. (24). By using the expression of the eigenfunction for the odd values of j, as in Eq. (5), one has

$$\begin{aligned} \tilde{\mathbf{K}}_{je}(\omega) &= \int_0^L \frac{\epsilon_{EI} EI_0 \cos[\alpha_j(-L/2 + x)]}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{\frac{\partial^2 \mathbf{s}(x,\omega)}{\partial x^2}\right\}\left\{\frac{\partial^2 \mathbf{s}(x,\omega)}{\partial x^2}\right\}^T dx \\ &= \frac{\epsilon_{EI} EI_0}{\sqrt{(L/2 + c_\alpha s_\alpha/\alpha_j)}} \widehat{\mathbf{K}}_j \end{aligned} \tag{69}$$

where c_α and s_α = defined in Eq. (46); and $\widehat{\mathbf{K}}_j \in \mathbb{C}^{4\times 4}$ = symmetric matrix obtained in the appendix. In a similar manner, by using the expression of the eigenfunction for the even values of j, as in Eq. (6), one has

$$\begin{aligned} \tilde{\mathbf{K}}_{je}(\omega) &= \int_0^L \frac{\epsilon_{EI} EI_0 \sin[\alpha_j(-L/2 + x)]}{\sqrt{L/2 - \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{\frac{\partial^2 \mathbf{s}(x,\omega)}{\partial x^2}\right\}\left\{\frac{\partial^2 \mathbf{s}(x,\omega)}{\partial x^2}\right\}^T dx \\ &= \frac{\epsilon_{EI} EI_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \widehat{\mathbf{K}}_j \end{aligned} \tag{70}$$

The mass matrix can also be represented as Eq. (70). The eigenvalues and eigenfunctions corresponding to the random field $H_m(x,\theta)$ need to be used to obtain the elements of $\tilde{\mathbf{M}}_{je}(\omega)$. By using the expression of the eigenfunction for the odd values of j, as in

Eq. (5), one has

$$\tilde{\mathbf{M}}_{je}(\omega) = \int_0^L \frac{\epsilon_m m_0 \cos[\alpha_j(-L/2+x)]}{\sqrt{L/2+\frac{\sin(\alpha_j L)}{2\alpha_j}}} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega)dx$$
$$= \frac{\epsilon_m m_0}{\sqrt{(L/2+c_\alpha s_\alpha/\alpha_j)}}\widehat{\mathbf{M}}_j \tag{71}$$

In the Eq. (71), the eigenvalues α_j should be obtained by solving the transcendental Eq. (5). In a similar manner, by using the expression of the eigenfunction for the even values of j, as in Eq. (6), one has

$$\tilde{\mathbf{M}}_{je}(\omega) = \int_0^L \frac{\epsilon_m m_0 \sin[\alpha_j(-L/2+x)]}{\sqrt{L/2-\frac{\sin(2\alpha_j a)}{2\alpha_j}}} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega)dx$$
$$= \frac{\epsilon_m m_0}{\sqrt{(L/2-c_\alpha s_\alpha/\alpha_j)}}\widehat{\mathbf{M}}_j \tag{72}$$

Eqs. (69)–(72) completely define the random parts of the element stiffness and mass matrices. The definite integrals appearing in these expressions can be evaluated in closed form. This further reduces the computational cost in deriving the element matrices. The exact closed-form expression of the elements of these four matrices are given in the appendix.

Numerical Illustrations

A simple numerical example is considered to illustrate the application of the matrices derived for the Euler-Bernoulli beam. The mean material properties are considered as $\rho_0 = 7{,}800\ \mathrm{kg/m^3}$ and $E_0 = 210$ GPa, values corresponding to steel. The length of the beam is $L = 1.5$ m and the rectangular cross section has a width of 40.06 mm and a thickness of 2.05 mm. The area moment of inertia of the cross section $I = 2.876 \times 10^{-11}\ \mathrm{m}^4$. A clamped-free boundary condition is considered for this example. By using these values, we have $EI_0 = 5.752\ \mathrm{Nm}^2$ and $m_0 = \rho_0 A_0 = 0.6406\ \mathrm{kg/m}$. The standard deviations of both the random fields are assumed to be 10% of their mean values, that is, $\epsilon_{EI} = 0.1EI_0$ and $\epsilon_m = 0.1m_0$. The damping coefficients are assumed to be $c_1 = 6.15 \times 10^{-5}EI_0$ and $c_2 = 0.09m_0$. The correlation length of the random fields describing $EI(x)$ and $m(x)$ is assumed to be $L/2$. We consider the displacement response at the free end of the beam attributable to a unit harmonic vertical force at that end. The response is calculated up to 200 Hz covering the first 10 vibration modes of the system. The response of the deterministic system, the mean, and the standard deviation of the absolute value of the response are shown in Fig. 6.

These results are obtained by using a Monte Carlo simulation with 4,000 samples. In total, 18 terms are used for the KL expansion. With this number of terms, the last eigenvalue of the KL expansion becomes less than 5% of the first eigenvalue. The element matrices associated with 18 random variables are obtained by using the closed-form expression derived the previous section. The phase of the frequency response function at the free end of the beam is shown in Fig. 7.

The phase does not change sign because we are considering the driving point response. In both Figs. 6 and 7, the mean curve is different from the deterministic curve. This difference is larger at higher frequencies. At lower frequencies, the standard deviation is biased by the mean. But as we approach the higher frequencies, the standard deviation curve flattens. These results are obtained by using a single spectral element, although 10 modes of vibration exist within the frequency range considered.

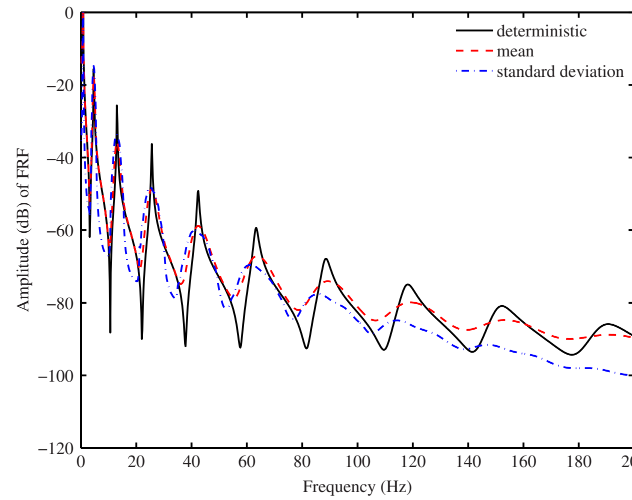

Fig. 6. Amplitude of displacement frequency response function at free end of damped beam with random properties

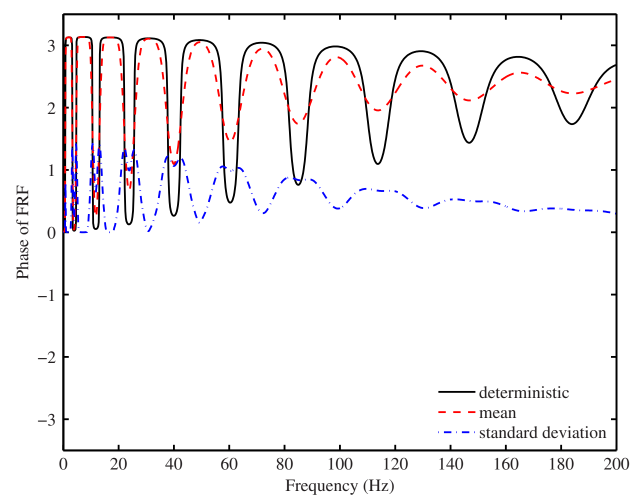

Fig. 7. Phase of displacement frequency response function at free end of damped beam with random properties

Conclusions

The basic formulation for a doubly spectral stochastic finite-element method (DSSFEM) for damped linear dynamic systems with distributed parametric uncertainty was derived. This new approach simultaneously uses the spectral representations in the frequency and random domains. The spatial displacement fields were discretized by using frequency-adaptive complex shape functions; the spatial random fields were discretized by using the Karhunen-Loève expansion. The frequency-adaptive shape functions were obtained from the spectral analysis of the underlying deterministic system; the Karhunen-Loève expansion was obtained from the spectral decomposition of the autocorrelation function of the spatial random field. In spite of the fact that these two spectral approaches existed for well over three decades, not been much overlap between them was in literature. In this paper, these two spectral techniques were unified with the aim that the unified approach would outperform any of the spectral methods considered on their own. The resulting frequency-dependent random element matrices, in general, turned out to be complex symmetric matrices. The primary computational advantage of the proposed approach is that the fine spatial discretization is not necessary for high and mid-frequency vibration analyses. This, in turn, results in smaller matrices to be inverted. The detailed derivations for rods in axial vibration and beams in bending vibration were given. Closed-form expressions of the element stiffness and mass matrices were derived for the stochastic parametric fields with an exponential autocorrelation function. Numerical examples were given to illustrate the applicability of the proposed method.

The calculation of the dynamic response by using the DSSFEM requires the inversion of a complex random symmetric matrix for every frequency. A limitation of the proposed method is that a Monte Carlo simulation is necessary for this step. Although the matrix sizes are smaller by using the DSSFEM than by using the conventional SFEM, this step still requires considerable computational effort. Further research is necessary to develop analytical methods in this direction. Further research is also necessary to extract eigenvalues from the complex random matrices obtained by using the proposed method.

Appendix. Expression of Spectral-Element Matrices Associated with KL Expansion for Bending Vibration of Beam

This appendix gives the explicit expressions for the spectral stiffness and mass matrices associated with the KL expansion for the bending vibration of beam. The elements of the stiffness matrix associated with the odd values of j in Eq. (69) can be obtained as

$$\widehat{K}_{11} = \frac{4b^6 s_{\alpha_j} - 2b^5\alpha_j c_{\alpha_j} cs + (-\alpha_j^2 + \alpha_j^2 c^2)s_{\alpha_j} b^4}{-\alpha_j^3 + 4\alpha_j b^2} \qquad \widehat{K}_{12} = \frac{(2-2c^2)c_{\alpha_j} b^5 - b^4\alpha_j s_{\alpha_j} cs}{-\alpha_j^2 + 4b^2}$$

$$\widehat{K}_{13} = \frac{(2cS - 2sC)c_{\alpha_j} b^7 + (2\alpha_j + 2\alpha_j Cc)s_{\alpha_j} b^6 + (-\alpha_j^2 Cs - \alpha_j^2 cS)c_{\alpha_j} b^5 - b^4\alpha_j^3 s_{\alpha_j} Ss}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{14} = \frac{(-2Ss - 2 + 2Cc)c_{\alpha_j} b^7 + 2b^6\alpha_j s_{\alpha_j} cS + (-\alpha_j^2 Ss + \alpha_j^2 - \alpha_j^2 Cc)c_{\alpha_j} b^5 - b^4\alpha_j^3 Cs_{\alpha_j} s}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{22} = \frac{4b^6 s_{\alpha_j} + 2b^5\alpha_j c_{\alpha_j} cs + (-\alpha_j^2 - \alpha_j^2 c^2)s_{\alpha_j} b^4}{-\alpha_j^3 + 4\alpha_j b^2}$$

$$\widehat{K}_{23} = \frac{(2 - 2Ss - 2Cc)c_{\alpha_j} b^7 - 2b^6\alpha_j Cs_{\alpha_j} s + (-\alpha_j^2 Cc + \alpha_j^2 Ss + \alpha_j^2)c_{\alpha_j} b^5 - b^4\alpha_j^3 s_{\alpha_j} cS}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{24} = \frac{(-2cS - 2sC)c_{\alpha_j} b^7 - 2b^6\alpha_j s_{\alpha_j} Ss + (\alpha_j^2 Cs - \alpha_j^2 cS)c_{\alpha_j} b^5 + (-\alpha_j^3 - \alpha_j^3 Cc)s_{\alpha_j} b^4}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{33} = \frac{-4b^6 s_{\alpha_j} + 2b^5\alpha_j SCc_{\alpha_j} + (\alpha_j^2 C^2 - \alpha_j^2)s_{\alpha_j} b^4}{4\alpha_j b^2 + \alpha_j^3} \qquad \widehat{K}_{34} = \frac{(-2 + 2C^2)c_{\alpha_j} b^5 + b^4\alpha_j SCs_{\alpha_j}}{4b^2 + \alpha_j^2}$$

$$\widehat{K}_{44} = \frac{4b^6 s_{\alpha_j} + 2b^5\alpha_j SCc_{\alpha_j} + (\alpha_j^2 C^2 + \alpha_j^2)s_{\alpha_j} b^4}{4\alpha_j b^2 + \alpha_j^3}$$

The subscript j is omitted in $\widehat{K}$ for notational convenience. Because the matrix is symmetric, only the upper triangular part is shown. All the terms appearing in these expressions have been defined in the body of the paper. The elements of the stiffness matrix associated with the even values of j in Eq. (70) can be obtained as

$$\widehat{K}_{11} = \frac{-2b^5 s_{\alpha_j} cs + (\alpha_j - \alpha_j c^2)c_{\alpha_j} b^4}{-\alpha_j^2 + 4b^2} \qquad \widehat{K}_{12} = \frac{b^4 c\alpha_j c_{\alpha_j} s - 2b^5 c^2 s_{\alpha_j}}{-\alpha_j^2 + 4b^2}$$

$$\widehat{K}_{13} = \frac{(2cS - 2sC)s_{\alpha_j} b^7 + (-2\alpha_j Cc + 2\alpha_j)c_{\alpha_j} b^6 + (-\alpha_j^2 Cs - \alpha_j^2 cS)s_{\alpha_j} b^5 + b^4\alpha_j^3 c_{\alpha_j} Ss}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{14} = \frac{(2Cc - 2Ss + 2)s_{\alpha_j} b^7 - 2b^6\alpha_j c_{\alpha_j} cS + (-\alpha_j^2 Cc - \alpha_j^2 - \alpha_j^2 Ss)s_{\alpha_j} b^5 + b^4\alpha_j^3 Cc_{\alpha_j} s}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{22} = \frac{(-2s_{\alpha_j}\alpha_j \cos(\alpha_j L)cs + 2c_{\alpha_j}\alpha_j \sin(\alpha_j L)cs)b^5 + ((-\alpha_j^2 + \alpha_j^2 \cos(\alpha_j L)c^2)c_{\alpha_j} + s_{\alpha_j}\alpha_j^2 \sin(\alpha_j L)c^2)b^4}{-\alpha_j^3 + 4\alpha_j b^2}$$

$$\widehat{K}_{23} = \frac{(-2 - 2Cc - 2Ss)s_{\alpha_j} b^7 + 2b^6\alpha_j Cc_{\alpha_j} s + (-\alpha_j^2 Cc - \alpha_j^2 + \alpha_j^2 Ss)s_{\alpha_j} b^5 + b^4\alpha_j^3 c_{\alpha_j} cS}{4b^4 + \alpha_j^4}$$

$$\widehat{K}_{24} = \frac{(-2cS - 2sC)s_{\alpha_j} b^7 + 2b^6\alpha_j c_{\alpha_j} Ss + (\alpha_j^2 Cs - \alpha_j^2 cS)s_{\alpha_j} b^5 + (-\alpha_j^3 + \alpha_j^3 Cc)c_{\alpha_j} b^4}{4b^4 + \alpha_j^4} \qquad \widehat{K}_{33} = \frac{2b^5 SCs_{\alpha_j} + (\alpha_j - \alpha_j C^2)c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}$$

$$\widehat{K}_{34} = \frac{2C^2 b^5 s_{\alpha_j} - Cb^4\alpha_j Sc_{\alpha_j}}{4b^2 + \alpha_j^2} \qquad \widehat{K}_{44} = \frac{2b^5 SCs_{\alpha_j} + (\alpha_j - \alpha_j C^2)c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}$$

The elements of the mass matrix associated with the odd values of j in Eq. (71) can be obtained as

$$\widehat{M}_{11} = \frac{-2b^5 s_{\alpha_j} cs + (\alpha_j - \alpha_j c^2) c_{\alpha_j} b^4}{-\alpha_j^2 + 4b^2} \qquad \widehat{M}_{12} = \frac{b^4 c\alpha_j c_{\alpha_j} s - 2b^5 c^2 s_{\alpha_j}}{-\alpha_j^2 + 4b^2}$$

$$\widehat{M}_{13} = \frac{(2cS - 2sC)s_{\alpha_j} b^7 + (-2\alpha_j Cc + 2\alpha_j)c_{\alpha_j} b^6 + (-\alpha_j^2 Cs - \alpha_j^2 cS)s_{\alpha_j} b^5 + b^4 \alpha_j^3 c_{\alpha_j} Ss}{4b^4 + \alpha_j^4}$$

$$\widehat{M}_{14} = \frac{(2Cc - 2Ss + 2)s_{\alpha_j} b^7 - 2b^6 \alpha_j c_{\alpha_j} cS + (-\alpha_j^2 Cc - \alpha_j^2 - \alpha_j^2 Ss)s_{\alpha_j} b^5 + b^4 \alpha_j^3 Cc_{\alpha_j} s}{4b^4 + \alpha_j^4}$$

$$\widehat{M}_{22} = \frac{(-2s_{\alpha_j}\alpha_j \cos(\alpha_j L)cs + 2c_{\alpha_j}\alpha_j \sin(\alpha_j L)cs)b^5 + ((-\alpha_j^2 + \alpha_j^2 \cos(\alpha_j L)c^2)c_{\alpha_j} + s_{\alpha_j}\alpha_j^2 \sin(\alpha_j L)c^2)b^4}{-\alpha_j^3 + 4\alpha_j b^2}$$

$$\widehat{M}_{23} = \frac{(-2 - 2Cc - 2Ss)s_{\alpha_j} b^7 + 2b^6 \alpha_j Cc_{\alpha_j} s + (-\alpha_j^2 Cc - \alpha_j^2 + \alpha_j^2 Ss)s_{\alpha_j} b^5 + b^4 \alpha_j^3 c_{\alpha_j} cS}{4b^4 + \alpha_j^4}$$

$$\widehat{M}_{24} = \frac{(-2cS - 2sC)s_{\alpha_j} b^7 + 2b^6 \alpha_j c_{\alpha_j} Ss + (\alpha_j^2 Cs - \alpha_j^2 cS)s_{\alpha_j} b^5 + (-\alpha_j^3 + \alpha_j^3 Cc)c_{\alpha_j} b^4}{4b^4 + \alpha_j^4} \qquad \widehat{M}_{33} = \frac{2b^5 SCs_{\alpha_j} + (\alpha_j - \alpha_j C^2)c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}$$

$$\widehat{M}_{34} = \frac{2C^2 b^5 s_{\alpha_j} - Cb^4 \alpha_j Sc_{\alpha_j}}{4b^2 + \alpha_j^2} \qquad \widehat{M}_{44} = \frac{2b^5 SCs_{\alpha_j} + (\alpha_j - \alpha_j C^2)c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}$$

The elements of the mass matrix associated with the even values of j in Eq. (72) can be obtained as

$$\widehat{M}_{11} = \frac{-2b^5 s_{\alpha_j} cs + (\alpha_j - \alpha_j c^2) c_{\alpha_j} b^4}{-\alpha_j^2 + 4b^2} \qquad \widehat{M}_{12} = \frac{b^4 c\alpha_j c_{\alpha_j} s - 2b^5 c^2 s_{\alpha_j}}{-\alpha_j^2 + 4b^2}$$

$$\widehat{M}_{13} = \frac{(2cS - 2sC)s_{\alpha_j} b^7 + (-2\alpha_j Cc + 2\alpha_j)c_{\alpha_j} b^6 + (-\alpha_j^2 Cs - \alpha_j^2 cS)s_{\alpha_j} b^5 + b^4 \alpha_j^3 c_{\alpha_j} Ss}{4b^4 + \alpha_j^4}$$

$$\widehat{M}_{14} = \frac{(2Cc - 2Ss + 2)s_{\alpha_j} b^7 - 2b^6 \alpha_j c_{\alpha_j} cS + (-\alpha_j^2 Cc - \alpha_j^2 - \alpha_j^2 Ss)s_{\alpha_j} b^5 + b^4 \alpha_j^3 Cc_{\alpha_j} s}{4b^4 + \alpha_j^4}$$

$$\widehat{M}_{22} = \frac{(-2s_{\alpha_j}\alpha_j \cos(\alpha_j L)cs + 2c_{\alpha_j}\alpha_j \sin(\alpha_j L)cs)b^5 + ((-\alpha_j^2 + \alpha_j^2 \cos(\alpha_j L)c^2)c_{\alpha_j} + s_{\alpha_j}\alpha_j^2 \sin(\alpha_j L)c^2)b^4}{-\alpha_j^3 + 4\alpha_j b^2}$$

$$\widehat{M}_{23} = \frac{(-2 - 2Cc - 2Ss)s_{\alpha_j} b^7 + 2b^6 \alpha_j Cc_{\alpha_j} s + (-\alpha_j^2 Cc - \alpha_j^2 + \alpha_j^2 Ss)s_{\alpha_j} b^5 + b^4 \alpha_j^3 c_{\alpha_j} cS}{4b^4 + \alpha_j^4}$$

$$\widehat{M}_{24} = \frac{(-2cS - 2sC)s_{\alpha_j} b^7 + 2b^6 \alpha_j c_{\alpha_j} Ss + (\alpha_j^2 Cs - \alpha_j^2 cS)s_{\alpha_j} b^5 + (-\alpha_j^3 + \alpha_j^3 Cc)c_{\alpha_j} b^4}{4b^4 + \alpha_j^4} \qquad \widehat{M}_{33} = \frac{2b^5 SCs_{\alpha_j} + (\alpha_j - \alpha_j C^2)c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}$$

$$\widehat{M}_{34} = \frac{2C^2 b^5 s_{\alpha_j} - Cb^4 \alpha_j Sc_{\alpha_j}}{4b^2 + \alpha_j^2} \qquad \widehat{M}_{44} = \frac{2b^5 SCs_{\alpha_j} + (\alpha_j - \alpha_j C^2)c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}$$

Acknowledgments

The writer gratefully acknowledges the support of the UK Engineering and Physical Sciences Research Council (EPSRC) through the award of an Advanced Research Fellowship and The Leverhulme Trust for the award of the Philip Leverhulme Prize.

Notation

The following symbols are used in this paper:

$AE(x)$ = axial rigidity;
a = constant for axial vibration;
b = constant for bending vibration;
$\mathbb{C}$ = space of complex numbers;
$C_H(\mathbf{r}_1, \mathbf{r}_2)$ = covariance function of random field H;
$\mathbb{C}^{n\times m}$ = space $n \times m$ complex matrices;
c = inverse of correlation length;
$c(\mathbf{r}, \theta)$ = random distributed damping parameter;
c_1, c_3 = strain rate-dependent viscous damping coefficients;
c_2, c_4 = velocity-dependent viscous damping coefficients;
$\mathcal{D}$ = space of random field H;
$\mathbf{D}(\omega, \theta)$ = global dynamic stiffness matrix;
d = spatial dimension, $d = 1$, 2, or 3;
$EI(x)$ = bending rigidity;
$\mathbf{f}$ = forcing function of discretized system;
$\mathcal{I}$ = space of integers;
k = wave number;
$k_s(\mathbf{r}, \theta)$ = random distributed stiffness parameter;
L = length of element;

$L_1(\bullet)$ = damping operator;
$L_2(\bullet)$ = stiffness operator;
l = domain for KL expansion;
$M_{(\bullet)}$ = number of terms in KL expansion of $(\bullet)$;
m = order of governing differential equation;
N = dimension of global matrices;
$\mathbf{N}(\mathbf{r}, \omega)$ = shape function vector;
n = dimension of element matrices;
$\mathbf{r}$ = spatial coordinate vector;
$\mathbf{s}(\mathbf{r}, \omega)$ = a vector of elementary functions for shape functions;
T = domain of time variable t;
t = time;
$U(\mathbf{r}, t)$ = general response variable;
$U(x, t)$ = axial displacement;
$\mathbf{u}$ = response vector of discretized system;
$u_e(\mathbf{r}, \omega)$ = displacement variable within an element;
$\hat{\mathbf{u}}_e(\omega)$ = nodal displacement vector for an element;
x = length variable;
$Y(x, t)$ = transverse flexural displacement;
$\mathbf{\Gamma}(\omega)$ = constant matrix for shape functions;
$\partial\mathcal{D}$ = boundary surface of domain $\mathcal{D}$;
$\epsilon(\bullet)$ = standard deviation of random field $(\bullet)$;
Θ = sample space;
θ = elements of sample space Θ;
λ_j = jth eigenvalue corresponding to autocovariance kernel;
$\xi_j(\theta)$ = uncorrelated Gaussian random variables;
$\rho(\mathbf{r}, \theta)$ = random mass density;
$\varphi_j(\mathbf{r})$ = jth eigenfunction corresponding to autocovariance kernel;
Ω = maximum frequency;
ω = frequency;
$(\bullet)^T$ = matrix transposition;
$(\bullet)_0$ = deterministic part corresponding to $(\bullet)$; and
$\rightarrow$ = maps into.

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