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Impulse response of stochastic oscillators: Ensemble damping and phase decoherence

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ABSTRACT

The impulse response function is the Green's function of the equation of motion and the fundamental building block of linear dynamic analysis. When the natural frequency of an oscillator is uncertain, the impulse response becomes a stochastic process whose statistical properties govern the response to any subsequent forcing. Deriving a closed-form analytical characterisation of the ensemble mean of this process in the time domain has remained an open problem since the foundations of stochastic structural dynamics were established. This paper resolves that problem by deriving an exact closed-form expression for the ensemble mean impulse response of a stochastic single-degree-of-freedom oscillator with Gaussian natural frequency uncertainty. The stochastic expectation is represented as a parametric Gaussian integral through an integral identity for the sinc kernel, and the resulting integral is evaluated exactly using the complex error function. The analysis reveals that stochastic frequency variability introduces a fundamentally new physical mechanism, ensemble damping, by which phase dispersion across the ensemble attenuates the mean response independently of viscous dissipation. This phenomenon, entirely absent from deterministic oscillator theory, is characterised here for the first time through two closed-form analytical descriptors: the ensemble damping ratio and the characteristic phase decoherence time. The latter defines an intrinsic timescale of the stochastic system governing the transition from coherent to incoherent ensemble behaviour. Numerical validation across four probability distributions confirms the accuracy and distributional robustness of all analytical closed-form results.

1. Introduction

The dynamic response of mechanical and structural systems to external excitation is a central concern across engineering disciplines, from the design of aerospace vehicles and civil infrastructure to precision instruments and energy harvesting devices [1–4]. For linear time-invariant systems, the impulse response function occupies a privileged position in this analysis. It is the Green's function of the equation of motion, encoding the complete dynamic character of the system in a single time-domain quantity, and through the convolution integral it determines the response to any externally applied excitation from first principles. The classical theory of random vibration builds upon this foundation by modelling the applied excitation as a stochastic process and deriving the statistical properties of the response through convolution with the loading statistics; it constitutes a mature analytical discipline providing systematic tools for the computation of response power spectral densities, response variance, and level-crossing statistics [5–9]. A defining postulate of this theory is that the dynamical system itself is deterministic: the mass, stiffness, and damping coefficients are prescribed constants, and stochasticity enters exclusively through the applied force. Physical structures and mechanical systems

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rarely conform to this postulate. Manufacturing tolerances, spatial variability of material properties, geometric imperfections, and idealisation errors in the mathematical model introduce inherent scatter into the system parameters, rendering the natural frequency, the damping ratio, and the modal stiffness coefficients effectively random quantities whose variability cannot be suppressed in a rigorous probabilistic treatment of the dynamic response [10–13]. When the system parameters carry uncertainty, the impulse response becomes a stochastic process, and characterising its statistical properties requires an analytical framework that is entirely separate from the classical theory of random vibration.

The statistical analysis of structural and mechanical systems with uncertain parameters has attracted sustained research activity over several decades. Early stochastic finite element formulations established perturbation-based and spectral frameworks for propagating parameter randomness through the equations of motion [10,11,14,15]. The subsequent development of polynomial chaos expansions provided a systematic means of representing stochastic response fields and has been applied extensively in structural and mechanical dynamics [16–19]. Stochastic collocation and sparse grid methods extended the tractability of these spectral representations to higher-dimensional parameter spaces [20]. More recently, the focus of the field has shifted considerably toward surrogate modelling, machine learning, and computational uncertainty quantification. Multi-fidelity emulators, active learning schemes, and general-purpose software frameworks have been developed to reduce the cost of Monte Carlo estimation and to handle complex nonlinear systems [21–25]. These developments have undoubtedly advanced the practical ability to quantify response statistics in complex engineering models. However, the emphasis on computational efficiency and high-dimensional parameter spaces has come at a specific cost: the structural properties of the time-domain mean response of the stochastic system, including the mechanisms governing its decay and the intrinsic timescales of stochastic attenuation, are not revealed by these approaches. Surrogate and machine learning methods yield accurate pointwise predictions of response statistics but do not, in general, produce closed-form expressions that identify how physical parameters govern the statistical structure of the dynamic response.

Analytical approaches that do yield physical insight have been pursued, though predominantly within the frequency domain. For the single-degree-of-freedom stochastic oscillator, it was shown in [26] that uncertainty in the natural frequency produces an apparent increase in the effective damping of the mean-square frequency response function, a phenomenon identified there as the damping effect, with exact analytical expressions derived for the equivalent damping ratio under a uniform frequency distribution. For multi-degree-of-freedom systems with uncertain parameters, approaches based on projection of the stochastic response onto the deterministic eigenmodes of the nominal system reduce the stochastic problem to coupled stochastic modal equations that can be treated through Galerkin projection schemes [27–29]. These formulations illuminate the role of individual modes in the stochastic response but operate entirely in the frequency domain and characterise the response in terms of frequency response functions and their statistical moments. The complementary time-domain problem, characterising the statistics of the stochastic impulse response directly in terms of the ensemble mean and the characteristic timescales of attenuation, has received comparatively little analytical attention. Numerical evidence of the phenomenon now identified here as ensemble damping is in fact visible in the published computational literature. The ensemble mean deflection of a randomly parametered cantilever beam under impulse loading, computed by Monte Carlo simulation in [30], exhibits precisely the accelerated amplitude decay relative to the deterministic response that is the signature of ensemble damping, across all levels of parametric uncertainty considered in that study. That result, obtained for a multi-degree-of-freedom system, was not identified or characterised as a distinct physical mechanism at the time, and the analytical structure governing the rate and timescale of the attenuation was not established. This retrospective observation is significant in two respects: it confirms that ensemble damping is not an artefact of the single-degree-of-freedom idealisation adopted here for analytical tractability, but a generic property of uncertain dynamical systems under impulse loading, and it illustrates why a closed-form analytical treatment is indispensable for identifying and understanding phenomena that remain invisible to computational approaches, however accurate their pointwise predictions. It should be emphasised that the single-degree-of-freedom oscillator is adopted here not as a restrictive idealisation but as the canonical Green's-function building block of linear dynamics. Through modal decomposition, each mode of a multi-degree-of-freedom system with uncertain modal frequencies reduces precisely to a problem of the present form, so that the ensemble damping and phase decoherence identified here propagate directly to the modal response of larger systems, as the multi-degree-of-freedom cantilever results of [30] already demonstrate. The objective of the present work is therefore not to introduce a new computational scheme; many accurate approximate and simulation-based methods already exist and are entirely adequate for routine engineering prediction. The aim is instead to expose, in exact closed form, a physical mechanism that those methods reproduce numerically without ever revealing, and which consequently has remained unnamed in the literature despite being implicit in published simulation data.

Despite the extensive literature surveyed above, a fundamental analytical gap remains. The study of damped structural dynamics has a distinguished tradition of exact analytical results that have yielded genuine physical understanding. Caughey and O'Kelly established the conditions for classical normal modes in dissipative systems [31,32]; Adhikari [33] subsequently extended the idea to generalised proportional damping. Elishakoff derived exact solutions for the bending response of beams with spatially stochastic properties [34–36]. Grigoriu developed rigorous analytical frameworks for statistically equivalent representations of stochastic mechanical problems [37,38]. These contributions succeeded precisely because they sought closed-form results capable of illuminating the mechanisms governing stochastic behaviour, rather than merely computing numerical estimates. The surveys of Ibrahim [39] and Manohar and Ibrahim [40] document this analytical tradition and chart its gradual displacement by simulation-based methods. No closed-form analytical expression for the ensemble mean impulse response of a stochastic oscillator with uncertain natural frequency has previously been reported. The present work returns to the analytical philosophy of exact results, with the aim of deriving such an expression in terms of standard special functions and using it to identify and characterise physical phenomena that are entirely absent from the deterministic theory.

The main contributions of this paper are as follows. First, an exact closed-form analytical expression is derived for the ensemble mean impulse response of a stochastic oscillator with Gaussian frequency uncertainty, using an integral identity for the sinc kernel to convert the stochastic expectation into a Gaussian integral evaluable in terms of the complex error function. Second, the analysis identifies *ensemble damping* as a distinct physical mechanism by which phase dispersion across the ensemble attenuates the mean response independently of viscous dissipation. Third, a closed-form amplitude envelope of the mean response is obtained, and two intrinsic analytical descriptors are introduced: the *ensemble damping ratio*, which quantifies the additional attenuation within the classical viscous damping framework, and the *characteristic phase decoherence time*, which marks the transition from coherent to incoherent ensemble behaviour and constitutes a fundamental new timescale of the stochastic system. Fourth, numerical comparisons across four probability distributions confirm that these results are robust well beyond the Gaussian model used in the analytical derivation. The significance of these contributions lies in the physical concepts they establish, namely ensemble damping and phase decoherence, rather than in the solution of the single-degree-of-freedom equation itself, which is classical. These concepts carry direct engineering relevance. The ensemble damping ratio provides a closed-form design chart (Fig. 3) for estimating the apparent amplification of damping across populations of nominally identical components manufactured to finite tolerances, while the characteristic phase decoherence time supplies a simple criterion for whether a measured or simulated response window lies in the coherent or the incoherent regime. In particular, the ensemble damping effect explains why averaged measurements over a batch of nominally identical structures, as routinely performed in modal testing and quality assurance, can exhibit substantially higher apparent damping than any individual specimen, an effect that would otherwise be misattributed to physical dissipation.

The remainder of the paper is organised as follows. The stochastic oscillator model is established in Section 2, where the stochastic natural frequency formulation is defined, the standardised uncertainty parameter is introduced, and the impulse response is cast as a random process indexed by time. The exact closed-form ensemble mean impulse response is derived in Section 3, followed by a systematic discussion of its qualitative behaviour and distributional robustness across four probability models. The concept of ensemble damping is introduced and developed in Section 4, where the amplitude envelope of the mean response is derived, the ensemble damping ratio and the characteristic phase decoherence time are defined as new analytical descriptors, and all results are validated against Monte Carlo simulation across four uncertainty levels. The main analytical results and the methodology are collected in Section 5. The conclusions are presented in Section 6. The detailed derivation of the mean response integral is provided in Appendix A.

2. Stochastic oscillator model

2.1. Deterministic single-degree-of-freedom oscillator

Consider a damped single-degree-of-freedom (SDOF) oscillator subjected to an external forcing. The equation of motion can be written in terms of the physical time τ as

$$m\ddot{u}(\tau) + c\dot{u}(\tau) + ku(\tau) = f(\tau) \tag{1}$$

where $u(\tau)$ denotes the displacement, m is the mass, c is the damping coefficient, k is the stiffness, and $f(\tau)$ is the applied force. Dividing Eq. (1) by m gives

$$\ddot{u}(\tau) + 2\zeta\omega_n\dot{u}(\tau) + \omega_n^2 u(\tau) = \frac{f(\tau)}{m} \tag{2}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}, \qquad \zeta = \frac{c}{2\sqrt{km}} \tag{3}$$

are respectively the natural frequency and damping ratio of the oscillator. The impulse response function plays a fundamental role in the analysis of forced dynamic systems. For linear time-invariant systems, the response to an arbitrary forcing can be expressed using the convolution integral

$$u(\tau) = \int_0^\tau h(\tau - s) f(s)\, ds \tag{4}$$

where $h(\tau)$ is the impulse response function.

The impulse response is obtained by considering a unit impulse excitation

$$f(\tau) = \delta(\tau) \tag{5}$$

where $\delta(\tau)$ is the Dirac delta function. Substituting this forcing into Eq. (2) gives

$$\ddot{u}(\tau) + 2\zeta\omega_n\dot{u}(\tau) + \omega_n^2 u(\tau) = \frac{\delta(\tau)}{m}. \tag{6}$$

For $\tau > 0$ the response satisfies the homogeneous equation with initial conditions obtained by integrating across the impulse at $\tau = 0$. This yields

$$u(0^+) = 0, \qquad \dot{u}(0^+) = \frac{1}{m}. \tag{7}$$

Solving the homogeneous equation with these initial conditions gives the impulse response function

$$h(\tau) = \frac{1}{m\omega_d} e^{-\zeta\omega_n\tau} \sin(\omega_d \tau) \tag{8}$$

where

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \tag{9}$$

is the damped natural frequency.

Eq. (8) fully characterises the forced response of the deterministic oscillator. In the following sections we introduce uncertainty in the natural frequency and examine the resulting statistical properties of the impulse response.

2.2. Stochastic natural frequency formulation

Suppose the natural frequency is expressed as

$$\omega_n = \omega_{n_0} z \tag{10}$$

where ω_{n_0} is the deterministic frequency and z is a random variable with a given probability distribution function. We assume that the mean of z is 1 and the standard deviation is σ. Stochastic modelling of this kind can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency. In the special case when the standard deviation of the random variable is close to zero, the stochastic oscillator approaches the classical deterministic oscillator. Several probability distributions are commonly used to represent uncertainty in engineering parameters. In this work, we consider the uniform, normal, lognormal, and chi-square distributions. These distributions are selected to span a representative range of probability models commonly encountered in engineering uncertainty quantification: the uniform distribution provides bounded support essential for analytical tractability, the Gaussian distribution represents the most widely used model in practice, the lognormal distribution is standard for positive-valued quantities, and the chi-square distribution naturally represents squared random variables.

Without any loss of generality, we introduce a standardised random variable x as

$$z = (1 + \epsilon x) \tag{11}$$

where x has zero mean and unit standard deviation with a pdf $p_X(x)$. This can be obtained using the standard procedure of transformation of random variables [41]. For most random variables z, the quantity ϵ can be identified as the standard deviation.

Among the probability models considered above, the Gaussian distribution plays a particularly important role because of its analytical tractability. When x follows a standard normal distribution $x \sim N(0,1)$, its probability density function is given by

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{12}$$

This choice greatly simplifies the analytical evaluation of statistical quantities involving the stochastic impulse response, since expectations can be expressed as Gaussian integrals which admit closed-form solutions in terms of special functions.

Although the Gaussian distribution has unbounded support and therefore allows negative values of z through Eq. (11), this limitation is not significant for the present analysis. In practical applications, the coefficient of variation ϵ associated with uncertainty in natural frequency is typically small. Consequently, the probability of obtaining negative values of z becomes negligibly small. The Gaussian distribution often arises naturally as an approximation to the aggregate effect of multiple independent sources of uncertainty through the central limit theorem.

For these reasons, the Gaussian model provides a convenient and widely used framework for analytical studies of stochastic dynamical systems. In the present work, it serves as the primary model used for the derivation of analytical expressions for the statistical properties of the impulse response. The other probability distributions introduced earlier will be used to illustrate the generality of the formulation through numerical simulations.

It is worth clarifying the scope of the single-parameter uncertainty model adopted here in relation to systems in which the mass, stiffness, and damping are simultaneously random and possibly correlated. The natural frequency $\omega_n = \sqrt{k/m}$ is a deterministic function of the mass and stiffness, so randomness in m and k propagates into randomness in ω_n through a standard transformation of random variables [41,42]. Whatever the joint distribution of k and m, including any correlation between them, the induced distribution of ω_n can be obtained and then mapped onto the standardised variable x of Eq. (11). For small coefficients of variation, a first-order propagation gives the effective frequency-uncertainty level as

$$\epsilon^2 \approx \frac{1}{4}\left(\delta_k^2 + \delta_m^2 - 2\rho_{km}\,\delta_k\,\delta_m\right), \tag{13}$$

where δ_k and δ_m are the coefficients of variation of the stiffness and mass and ρ_{km} their correlation coefficient. Uncertainty in m and k therefore introduces no new mechanism: it merely fixes the effective value of ϵ, and this step may be regarded as a pre-processing of the input statistics that leaves the subsequent analysis unchanged. Uncertainty in the damping ratio is qualitatively different. Because the damping enters the impulse response through the amplitude envelope $e^{-\zeta\omega_n\tau}$ rather than through the oscillatory phase, random variation of ζ produces dispersion in the decay amplitude of individual realisations but does not cause the progressive phase drift that is responsible for ensemble damping and phase decoherence. The mechanism characterised in this paper is thus controlled specifically by frequency uncertainty, which is precisely why the present analysis isolates it; an uncertainty-driven attenuation of the same frequency-scatter origin was reported for carbon nanotube terahertz oscillators in [42].

2.3. Impulse response as a stochastic process

The impulse response derived in Eq. (8) corresponds to a deterministic oscillator with a fixed natural frequency. When the natural frequency becomes uncertain according to Eq. (10), the impulse response also becomes stochastic because the system parameters depend on the random variable z. In this case, the random impulse response function can be written as

$$h(\tau, z) = \frac{e^{-\zeta\omega_n\tau}\sin(\omega_d\tau)}{\omega_d} \tag{14}$$

Using the transformation introduced in Eq. (11), the stochastic natural frequency becomes

$$\omega_n = \omega_{n_0}(1 + \epsilon x) \tag{15}$$

To obtain a convenient dimensionless representation, we introduce the non-dimensional time using the reference deterministic frequency ω_{n_0}

$$t = \omega_{n_0}\tau \tag{16}$$

Substituting these relations into the impulse response expression yields

$$h(t, x) = \frac{\exp[-\zeta(1 + \epsilon x)t]\ \sin((1 + \epsilon x)\beta t)}{(1 + \epsilon x)\beta} \tag{17}$$

where

$$\beta = \sqrt{1 - \zeta^2} \tag{18}$$

Eq. (17) shows that the impulse response depends explicitly on the random variable x. Because the same random variable governs the response at all times, the quantity $h(t, x)$ is not simply a random variable but a *random process*. Each realisation of x corresponds to the impulse response of a deterministic oscillator with a particular natural frequency, while the ensemble of realisations represents the statistical variability of the dynamic system.

The stochastic impulse response plays a central role in the analysis of uncertain dynamical systems. For linear systems, the response to an arbitrary forcing can be expressed through the convolution integral in Eq. (4). Consequently, once the statistical properties of the impulse response are known, the response to deterministic or stochastic forcing can be determined. In the following sections, we therefore focus on the analytical characterisation of the statistical structure of the stochastic impulse response defined in Eq. (17), including its mean behaviour and the time-domain mechanisms associated with stochastic frequency uncertainty.

3. Analytical characterisation of the mean response

3.1. Ensemble mean formulation

The stochastic impulse response derived in Eq. (17) depends on the random variable x. Since the same random variable governs the response at all times, the quantity $h(t, x)$ represents a random process indexed by time. The statistical properties of this process can be characterised through ensemble averages taken over the probability distribution of x.

The ensemble mean impulse response is defined as

$$\bar{h}(t) = \mathbb{E}[h(t, x)] \tag{19}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator. Using the probability density function $p_X(x)$ of the random variable x, the ensemble mean can be written in integral form as

$$\bar{h}(t) = \int_{-\infty}^{\infty} h(t, x)\, p_X(x)\, dx. \tag{20}$$

Substituting the stochastic impulse response given by Eq. (17) into Eq. (20) gives

$$\bar{h}(t) = \int_{-\infty}^{\infty} \frac{\exp[-\zeta(1 + \epsilon x)t]\ \sin((1 + \epsilon x)\beta t)}{(1 + \epsilon x)\beta}\, p_X(x)\, dx \tag{21}$$

Eq. (21) represents the ensemble mean response of the stochastic oscillator. The evaluation of this integral depends on the probability distribution of the random variable x. In this work, particular attention is given to the Gaussian distribution because it leads to analytically tractable expressions. When x follows a standard normal distribution, the probability density function is given by Eq. 12.

Substituting Eq. (12) into Eq. (21) yields the Gaussian integral

$$\bar{h}(t) = \frac{1}{\beta\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp[-\zeta(1 + \epsilon x)t]\sin((1 + \epsilon x)\beta t)}{(1 + \epsilon x)} \exp\left(-\frac{x^2}{2}\right) dx \tag{22}$$

The analytical evaluation of the integral in Eq. (22) forms the basis of the ensemble mean analysis developed in the following subsection.

3.2. Analytical evaluation of the mean response

The ensemble mean response defined in Eq. (22) requires the evaluation of the Gaussian integral

$$I = \int_{-\infty}^{\infty} \exp[-\zeta(1+\epsilon x)t] \frac{\sin\left((1+\epsilon x)t\sqrt{1-\zeta^2}\right)}{1+\epsilon x} \phi(x)\, dx \tag{23}$$

where $\phi(x)\,dx$ is the Gaussian pdf given by Eq. (12). To simplify the notation, define

$$a = \zeta t, \qquad b = t\sqrt{1-\zeta^2} \tag{24}$$

The key step is to use the identity

$$\frac{\sin(bu)}{u} = \int_0^b \cos(su)\, ds \tag{25}$$

with $u = 1 + \epsilon x$. Substituting this relation into Eq. (23) transforms the problem into a Gaussian integral that can be evaluated analytically. After straightforward manipulation, the mean response can be written in the exact form

$$I = e^{-a+\frac{\epsilon^2 a^2}{2}} \int_0^b e^{-\frac{\epsilon^2 s^2}{2}} \cos\left((1-\epsilon^2 a)s\right) ds \tag{26}$$

Eq. (26) provides an exact representation of the mean impulse response of the stochastic oscillator. This expression reveals how the statistical variability of the natural frequency modifies both the amplitude and phase of the deterministic impulse response.

The integral in Eq. (26) can be evaluated in closed form using the error function. The resulting expression of the ensemble mean impulse response function is given by

$$\bar{h}(t) = \frac{I}{\beta} = e^{-a+\frac{\epsilon^2 a^2}{2}} \frac{\sqrt{\pi}}{\sqrt{2}\epsilon\beta} e^{-\frac{(1-\epsilon^2 a)^2}{2\epsilon^2}} \Re\left[\operatorname{erf}\left(\frac{\epsilon b}{\sqrt{2}} - i\frac{1-\epsilon^2 a}{\sqrt{2}\epsilon}\right) - \operatorname{erf}\left(-i\frac{1-\epsilon^2 a}{\sqrt{2}\epsilon}\right)\right] \tag{27}$$

The analytical result obtained above is significant for several reasons. The impulse response of linear oscillators has been studied for many decades, particularly in the context of random vibration [5,6,9]. However, most existing studies consider deterministic systems. To the best of the author's knowledge, an exact analytical expression for the ensemble mean impulse response of a stochastic oscillator with Gaussian frequency uncertainty has not previously been reported in the literature.

In Fig. 1, the mean response obtained using direct Monte Carlo Simulation (MCS) is compared with the analytical expression in Eq. (27) for four levels of frequency uncertainty.

The parameter values $\zeta = 0.03$ and $\epsilon \in \{0.05, 0.1, 0.15, 0.2\}$ span the range of randomness levels commonly encountered in structural and mechanical systems. The analytical expression is exact and matches the MCS mean to within graphical precision across all four subfigures. The progression from subfigure (a) to subfigure (d) reveals the systematic effect of increasing frequency uncertainty on the ensemble mean response. For $\epsilon = 0.05$, the individual sample realisations remain tightly clustered and the mean closely tracks the deterministic impulse response for approximately eight oscillation cycles before any appreciable attenuation is visible. As ϵ increases to 0.1 and 0.15, the ensemble mean decays progressively faster and the spread of individual realisations about the mean widens, with phase decoherence becoming visible after fewer cycles. For $\epsilon = 0.2$, the mean response is effectively extinguished within approximately two oscillation cycles, while individual realisations span nearly the full amplitude range of the deterministic response throughout the duration shown.

The derivation presented here shows that the mean response can be expressed in a compact closed form involving the error function. This result is noteworthy because it provides an explicit analytical characterisation of the stochastic impulse response without resorting to numerical averaging. In view of the long history of stochastic vibration theory, it is somewhat remarkable that such a closed-form expression has not been derived earlier. The availability of this analytical representation enables deeper insight into the statistical structure of the impulse response and forms the basis for the subsequent analysis of ensemble damping and phase decoherence.

3.3. Qualitative results, main observations and theoretical explanations

3.3.1. Qualitative results and main observations

In Fig. 2, the mean impulse response is compared across four probability distributions of the natural frequency for four uncertainty levels. For the normal distribution, the analytical expression Eq. (27) is used directly; for the remaining three distributions, the mean is obtained by direct Monte Carlo Simulation (MCS).

Three principal observations emerge from Fig. 2.

- *Ensemble damping.* The mean response decays substantially faster than the deterministic impulse response in all four subfigures, even though the physical damping ratio $\zeta = 0.03$ is identical throughout. The accelerated attenuation arises from phase decoherence across the ensemble and constitutes an effective increase in system damping induced purely by frequency uncertainty.
- *Uncertainty–damping proportionality.* The rate of amplitude decay of the mean response increases systematically with the randomness parameter ϵ. For $\epsilon = 0.05$ in subfigure (a), the mean response tracks the deterministic impulse response closely for approximately eight oscillation cycles before any appreciable attenuation is visible. As ϵ increases through 0.1 and 0.15 in subfigures (b)

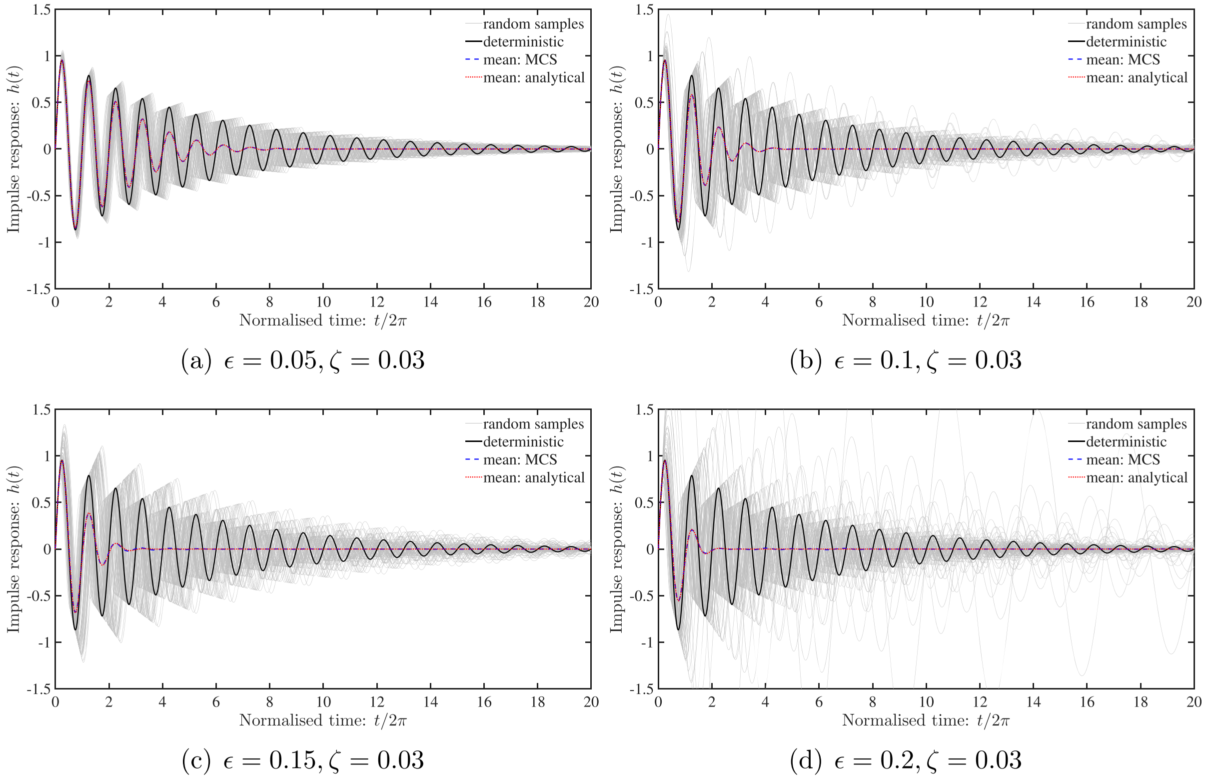


Fig. 1. Validation of the analytical mean impulse response from Eq. (27) against Monte Carlo simulation (10,000 samples) for baseline damping $\zeta = 0.03$ and four randomness levels $\epsilon \in \{0.05, 0.1, 0.15, 0.2\}$.

and (c), decoherence occurs progressively earlier and the mean is suppressed after fewer cycles. For $\epsilon = 0.2$ in subfigure (d), the mean response is effectively extinguished within approximately two oscillation cycles. This monotonic relationship between ϵ and the rate of ensemble attenuation confirms that the ensemble damping is directly controlled by the level of frequency uncertainty.

- *Distribution invariance.* Across all four subfigures, the mean responses for the normal, log-normal, and chi-square distributions are nearly indistinguishable throughout the full time range shown, demonstrating that the qualitative features of the ensemble damping are largely insensitive to the particular choice of probability model. The uniform distribution departs slightly from the other three in the intermediate time range, which is attributable to its sharply bounded support and the absence of tail contributions. This departure is most visible in subfigures (c) and (d) at higher uncertainty levels, where tail behaviour has a greater influence on the ensemble average. The near-invariance across the remaining three distributions validates the use of the Gaussian assumption as the basis for the analytical developments in this paper, even when the true distribution of the natural frequency is not normal.

3.3.2. *Theoretical explanations*

The behaviour observed in Fig. 2 can be interpreted more precisely and explained by examining the small uncertainty structure of the exact mean response derived in Section 3.2. We start from the integral representation Eq. (26). A Taylor expansion with respect to the randomness parameter ϵ can be carried out for small ϵ. First, we observe that

$$e^{-a+\frac{\epsilon^2 a^2}{2}} = e^{-a}\left(1 + \frac{\epsilon^2 a^2}{2}\right) + O(\epsilon^4) \tag{28}$$

Next

$$e^{-\frac{\epsilon^2 s^2}{2}} = 1 - \frac{\epsilon^2 s^2}{2} + O(\epsilon^4) \tag{29}$$

and

$$\cos\big((1-\epsilon^2 a)s\big) = \cos s + \epsilon^2 a s \sin s + O(\epsilon^4) \tag{30}$$

Hence one obtain

$$e^{-\frac{\epsilon^2 s^2}{2}} \cos\big((1-\epsilon^2 a)s\big) = \cos s + \epsilon^2\left(a s \sin s - \frac{s^2}{2}\cos s\right) + O(\epsilon^4) \tag{31}$$

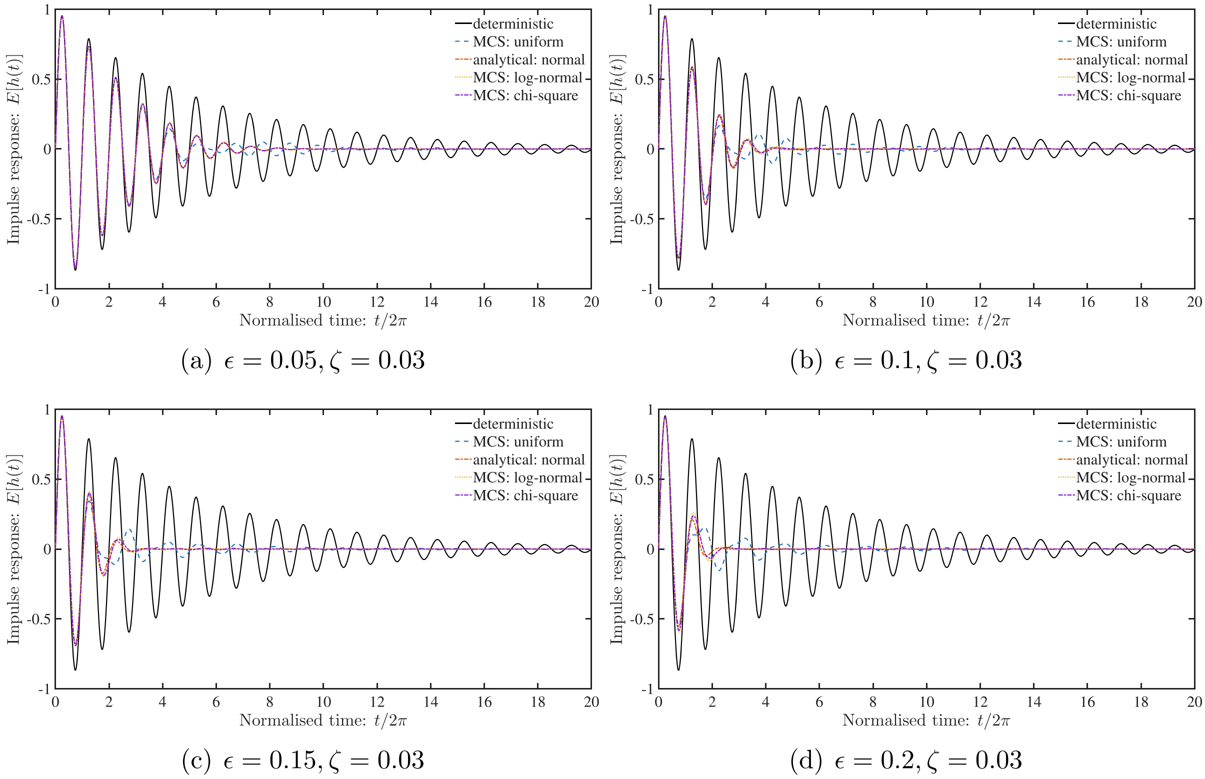


Fig. 2. Mean impulse response computed using four probability distributions of the natural frequency (normal: analytical from Eq. (27); uniform, log-normal, chi-square: MCS with 10,000 samples). Results are shown for baseline damping $\zeta = 0.03$ and four randomness levels $\epsilon \in \{0.05, 0.1, 0.15, 0.2\}$.

Substituting Eqs. (28)–(31) into Eq. (26) gives

$$I \approx e^{-a}\left(1+\frac{\epsilon^2 a^2}{2}\right)\int_0^b \left[\cos s + \epsilon^2\left(as\sin s - \frac{s^2}{2}\cos s\right)\right] ds \tag{32}$$

Now we use these basic integrals for the second part of the above equation

$$\begin{gathered}\int_0^b \cos s\, ds = \sin b, \quad \int_0^b s\sin s\, ds = \sin b - b\cos b \\ \int_0^b s^2 \cos s\, ds = b^2 \sin b + 2b\cos b - 2\sin b\end{gathered} \tag{33}$$

Therefore

$$I \approx e^{-a}\left[\sin b + \epsilon^2\left(a(\sin b - b\cos b) - \frac{1}{2}(b^2\sin b + 2b\cos b - 2\sin b) + \frac{a^2}{2}\sin b\right)\right] \tag{34}$$

Collecting terms gives

$$I \approx e^{-a}\left[\sin b + \epsilon^2\left(\left(a+1+\frac{a^2}{2}-\frac{b^2}{2}\right)\sin b - (a+1)b\cos b\right)\right] \tag{35}$$

Substituting back $a = \zeta t$ and $b = \beta t$

$$I \approx e^{-\zeta t}\left[\sin(\beta t) + \epsilon^2\left(\left(\zeta t + 1 + \frac{\zeta^2 t^2}{2} - \frac{\beta^2 t^2}{2}\right)\sin(\beta t) - (\zeta t+1)\beta t\cos(\beta t)\right)\right] \tag{36}$$

Since the ensemble mean impulse response satisfies

$$\bar{h}(t) = \frac{I}{\beta} \tag{37}$$

it follows that

$$\bar{h}(t) \approx \frac{e^{-\zeta t}}{\beta}\left[\sin(\beta t) + \epsilon^2\left(\left(\zeta t + 1 + \frac{\zeta^2 t^2}{2} - \frac{\beta^2 t^2}{2}\right)\sin(\beta t) - (\zeta t+1)\beta t\cos(\beta t)\right)\right] \tag{38}$$

Indeed, when $\epsilon = 0$ Eq. (38) reduces to the deterministic impulse response

$$\bar{h}(t) = \frac{e^{-\zeta t}\sin(\beta t)}{\beta} \tag{39}$$

Eq. (38) provides useful insight into the behaviour observed in Fig. 2. First, it shows that the deterministic impulse response is recovered when $\epsilon = 0$, confirming that the stochastic formulation is consistent with the classical oscillator in the absence of uncertainty. More importantly, the leading stochastic correction appears at order ϵ^2. This is expected because the random variable x has zero mean, so the first-order contribution proportional to ϵ vanishes upon ensemble averaging. Consequently, the influence of stochastic frequency fluctuations on the mean response is fundamentally a second-order effect.

The ϵ^2 terms introduce two important modifications to the deterministic impulse response. The first group of terms multiplies $\sin(\beta t)$ and contains polynomial factors in t and t^2. These terms modify the amplitude envelope of the oscillatory response and grow with time, which explains the progressive attenuation of the mean response relative to the deterministic case observed in Fig. 2. The second contribution is proportional to $\cos(\beta t)$ and represents a phase correction induced by stochastic frequency variability. This phase perturbation causes individual realisations of the oscillator to gradually lose synchronisation, leading to partial cancellation when averaged across the ensemble.

Eq. (38) also clarifies why the decay of the mean response increases with the uncertainty level ϵ. Since the stochastic corrections scale with ϵ^2, doubling ϵ increases the magnitude of the correction terms by approximately a factor of four. This quadratic dependence explains the markedly faster attenuation observed in panel (b) of Fig. 2 compared with panel (a). Finally, because the derivation of Eq. (38) depends only on the first two statistical moments of the random variable x, the resulting mean response is largely insensitive to the precise form of the probability distribution. This explains the close agreement observed between the normal, log-normal and chi-square distributions in Fig. 2.

This argument can be made precise and constitutes the basis for the generality of the present results beyond the Gaussian model. The small-uncertainty expansion in Eq. (38) is obtained by expanding the stochastic impulse response in powers of ϵx and taking the expectation term by term, so the coefficients of the resulting series are the raw moments $\mathbb{E}[x^k]$ of the standardised random variable. Since x is defined to have zero mean and unit variance for every distribution considered, the leading deterministic term and the entire $O(\epsilon^2)$ correction, and hence the amplitude envelope, the ensemble damping ratio, and the phase decoherence time derived from it, depend only on these first two moments and are identical for all distributions sharing them. Distribution-specific features enter only through the third and higher moments, which first appear at $O(\epsilon^3)$ and are weighted by correspondingly higher powers of the small parameter ϵ. This explains analytically why the normal, log-normal, and chi-square results in Fig. 2 are nearly indistinguishable, and why the bounded uniform distribution departs only modestly and only at the higher uncertainty levels, where the neglected higher-order moments become appreciable. Thus, although the closed-form expression in Eq. (27) is exact specifically for the Gaussian model, the physically significant descriptors it yields are, to leading order in ϵ, universal across distributions. In the next section, we explore this idea in depth by introducing a new concept of Ensemble damping.

4. Ensemble damping

4.1. Envelope of the mean response

The results in Fig. 2 reveal a systematic and reproducible phenomenon: the ensemble mean impulse response decays substantially faster than the deterministic impulse response even though the physical damping ratio ζ remains unchanged. This accelerated attenuation is not caused by additional physical dissipation but rather by the statistical dispersion of the natural frequency across the ensemble of oscillators. We refer to this effect as *ensemble damping*.

Insight into this mechanism can be obtained from the asymptotic expansion of the mean response derived in the previous section. Eq. (38) shows that the stochastic corrections to the deterministic impulse response appear at order ϵ^2. In particular, the correction terms contain polynomial factors in time that grow as t and t^2. Consequently, although the instantaneous response of each realisation remains oscillatory, the ensemble average experiences progressive attenuation as time increases.

A more transparent interpretation can be obtained by examining the dominant amplitude modulation of the oscillatory response. The small-uncertainty expansion of the mean impulse response derived in Eq. (38) allows the amplitude envelope to be extracted analytically. To isolate the envelope, it is convenient to write Eq. (38) in the form

$$\bar{h}(t) \approx \frac{e^{-\zeta t}}{\beta}\left[A(t)\sin(\beta t) - B(t)\cos(\beta t)\right] \tag{40}$$

where the coefficients of the sine and cosine terms are respectively

$$A(t) = 1 + \epsilon^2\left(\zeta t + 1 + \frac{\zeta^2 t^2}{2} - \frac{\beta^2 t^2}{2}\right) \tag{41}$$

$$B(t) = \epsilon^2(\zeta t + 1)\beta t. \tag{42}$$

Using $\beta^2 - \zeta^2 = 1 - 2\zeta^2$, the coefficient in Eq. (41) simplifies to

$$A(t) = 1 + \epsilon^2\left(\zeta t + 1 - \frac{(1 - 2\zeta^2)t^2}{2}\right). \tag{43}$$

The envelope of the mean response is the amplitude of the combined sinusoidal expression in Eq. (40), given by

$$\mathcal{E}(t) = \frac{e^{-\zeta t}}{\beta}\sqrt{A(t)^2 + B(t)^2}. \tag{44}$$

Retaining terms up to $O(\epsilon^2)$ and using $\sqrt{1+\delta} \approx 1 + \delta/2$ for small δ, the envelope simplifies to

$$\mathcal{E}(t) \approx \frac{e^{-\zeta t}}{\beta}\left[1 + \epsilon^2\left(\zeta t + 1 - \frac{(1-2\zeta^2)t^2}{2}\right)\right]. \tag{45}$$

When $\epsilon = 0$, Eq. (45) recovers the deterministic envelope $e^{-\zeta t}/\beta$ consistent with Eq. (39). For $\epsilon > 0$, the correction term introduces both a linear growth at rate $\epsilon^2(\zeta t + 1)$ at short times and a quadratic decay at rate $\epsilon^2(1-2\zeta^2)t^2/2$ at longer times. The quadratic decay term dominates for $t \gg 1$, producing the accelerated amplitude attenuation observed in Fig. 2. This behaviour is entirely absent from the deterministic impulse response and arises purely from the statistical spread of natural frequencies across the ensemble.

For times beyond the decoherence scale $t \gg 1$, the quadratic term dominates and the linear corrections satisfy $\epsilon^2(\zeta t + 1) \ll \frac{(1-2\zeta^2)}{2}\epsilon^2 t^2$, so that Eq. (45) reduces to

$$\mathcal{E}(t) \approx \frac{e^{-\zeta t}}{\beta}\left[1 - \frac{(1-2\zeta^2)}{2}\epsilon^2 t^2\right]. \tag{46}$$

The factor in brackets is precisely the leading-order truncation of the exponential

$$e^{-\frac{(1-2\zeta^2)}{2}\epsilon^2 t^2} = 1 - \frac{(1-2\zeta^2)}{2}\epsilon^2 t^2 + O(\epsilon^4 t^4), \tag{47}$$

valid to $O(\epsilon^2)$. Resuming this polynomial into the exponential yields the amplitude envelope

$$\mathcal{E}(t) = \frac{1}{\beta}\exp\left[-\zeta t - \frac{(1-2\zeta^2)}{2}\epsilon^2 t^2\right]. \tag{48}$$

When $\epsilon = 0$, Eq. (48) reduces to the deterministic envelope $e^{-\zeta t}/\beta$. For $\epsilon > 0$, the envelope acquires an additional Gaussian decay beyond the classical exponential, which is responsible for the accelerated amplitude attenuation of the mean response observed in Fig. 2.

4.2. Physical interpretation: phase decoherence

The additional amplitude decay in Eq. (48) has a transparent physical interpretation. Each realisation of the stochastic oscillator oscillates at its own damped frequency $(1+\epsilon x)\beta$ determined by the random variable x. At non-dimensional time t, the accumulated phase of a realisation with frequency deviation x relative to the nominal oscillator is $\epsilon x \beta t$. Since x follows a zero-mean Gaussian distribution with unit variance, the standard deviation of the relative phase at time t is

$$\sigma_\phi(t) = \epsilon \beta t. \tag{49}$$

For $\sigma_\phi(t) \ll 1$, all realisations remain nearly in phase, and the ensemble mean approximates the deterministic response. As time advances, σ_ϕ grows linearly, and the phases spread over increasing fractions of the unit circle, producing destructive interference in the ensemble average. When $\sigma_\phi \sim 1$, the mean response amplitude falls substantially below the deterministic envelope. The *characteristic phase decoherence time* at which this transition occurs is therefore

$$t_{\mathrm{d}} = \frac{1}{\epsilon\beta} \tag{50}$$

In physical time, $\tau_{\mathrm{d}} = 1/(\epsilon\omega_d)$ where $\omega_d = \omega_{n0}\beta$ is the nominal damped natural frequency, that is

$$\tau_{\mathrm{d}} = \frac{1}{\epsilon\omega_{n0}\sqrt{1-\zeta^2}} \tag{51}$$

The decoherence time is inversely proportional to both the uncertainty level and the damped frequency. This can be rewritten in terms of the nominal system parameters as

$$\tau_{\mathrm{d}} = \frac{2m}{\epsilon\sqrt{4km - c^2}} \tag{52}$$

For $t \ll t_{\mathrm{d}}$ the mean response closely tracks the deterministic impulse response; for $t \gg t_{\mathrm{d}}$ the mean response is effectively extinguished by phase decoherence. This mechanism is entirely distinct from viscous dissipation: it arises purely from the randomness of the natural frequency and persists even when $\zeta = 0$.

In the undamped or very lightly damped limit, $\zeta^2 \to 0$, we have $\beta = \sqrt{1-\zeta^2} \to 1$, so the phase decoherence time becomes

$$t_{\mathrm{d}} \to \frac{1}{\epsilon} \tag{53}$$

in dimensionless form. In physical time, this gives

$$\tau_{\mathrm{d}} \to \frac{1}{\epsilon\omega_{n0}} = \frac{1}{\epsilon}\sqrt{\frac{m}{k}} \tag{54}$$

since $\omega_{n0} = \sqrt{k/m}$. Thus, in the undamped limit, the decoherence time depends only on the mass, stiffness, and uncertainty level and is entirely independent of viscous dissipation. Physically, this shows that phase decoherence remains a genuine stochastic attenuation mechanism even when no energy is dissipated: the ensemble mean decays purely because the phases of the individual undamped oscillator realisations progressively drift apart.

This observation admits a precise energetic interpretation, which clarifies how ensemble damping differs fundamentally from classical viscous damping in terms of energy balance and reversibility. Classical damping is a dissipative mechanism: in each individual oscillator the viscous force $c\dot{u}$ removes mechanical energy irreversibly and converts it into heat, so that the energy of every realisation decays monotonically and the process cannot be undone. Ensemble damping involves no such dissipation. Each realisation of the stochastic oscillator conserves or dissipates energy exactly as its own deterministic equation of motion prescribes; in the undamped limit $\zeta = 0$ every realisation oscillates with constant energy indefinitely. The decay of the *ensemble mean* arises solely from the progressive phase dispersion across realisations, which produces destructive interference when the responses are averaged. No energy is removed from any oscillator, and none is exchanged between them, since the realisations are dynamically independent. The attenuation is therefore a property of the statistical average rather than of any individual system, and it is in principle reversible: a rephasing of the ensemble, for instance by conditioning on the realised frequency, recovers the full coherent amplitude. In this respect ensemble damping is an interference phenomenon, closely analogous to the free-induction decay of an inhomogeneously broadened ensemble in resonance physics, and is conceptually distinct from the irreversible energy dissipation that defines classical damping.

The quantity t_d defines a new timescale intrinsic to the stochastic system that is entirely absent from the deterministic theory: it represents the time beyond which the spread of natural frequencies across the ensemble produces sufficient phase dispersion to suppress the mean response, and it is independent of the viscous damping ratio ζ to leading order. This decoherence timescale has not previously been identified explicitly in the context of stochastic structural dynamics, where the focus has traditionally been on response dispersion rather than on the decay structure of the ensemble mean.

4.3. *Equivalent damping: the ensemble damping ratio*

The envelope representation in Eq. (48) shows that stochastic frequency variability introduces an additional decay mechanism in the ensemble mean response. Although this attenuation is not caused by viscous dissipation, it is useful to characterise its overall effect in terms of an equivalent damping ratio that quantifies the average decay rate over one oscillation cycle. We define the quantity ζ_en as the *ensemble damping ratio*, representing the effective attenuation of the ensemble mean response induced by stochastic variability of the natural frequency.

Let the deterministic period of oscillation be

$$T_0 = \frac{2\pi}{\beta} \tag{55}$$

The instantaneous logarithmic decay rate implied by the envelope in Eq. (48) can be written as

$$\gamma_\mathrm{eff}(t) = -\frac{d}{dt}\ln \mathcal{E}(t) = \zeta + (1 - 2\zeta^2)\epsilon^2 t \tag{56}$$

which represents the effective damping acting on the mean response at time t.

To obtain a single representative quantity, we define the *ensemble damping ratio* as a cycle-averaged equivalent damping ratio

$$\zeta_\mathrm{en} = \frac{1}{T_0}\int_0^{T_0} \gamma_\mathrm{eff}(t)\,dt \tag{57}$$

Substituting the expression for $\gamma_\mathrm{eff}(t)$ gives

$$\zeta_\mathrm{en} = \zeta + (1 - 2\zeta^2)\epsilon^2 \frac{1}{T_0}\int_0^{T_0} t\,dt \tag{58}$$

Evaluating the integral

$$\int_0^{T_0} t\,dt = \frac{T_0^2}{2} \tag{59}$$

yields

$$\zeta_\mathrm{en} = \zeta + \frac{1 - 2\zeta^2}{2}\epsilon^2 T_0 \tag{60}$$

Using $T_0 = 2\pi/\beta$ finally gives the closed-form expression of the ensemble damping ratio

$$\zeta_\mathrm{en} = \zeta + \frac{\pi(1 - 2\zeta^2)}{\sqrt{1 - \zeta^2}}\epsilon^2 \tag{61}$$

The equation above provides a compact measure of the additional attenuation induced by stochastic frequency variability. The ensemble damping ratio increases quadratically with the uncertainty level ϵ, consistent with the ϵ^2 scaling observed in the asymptotic expansion of the mean response. The expression also shows that the stochastic contribution depends on the nominal damping ratio ζ through the factor $(1 - 2\zeta^2)$ and through the oscillation period T_0.

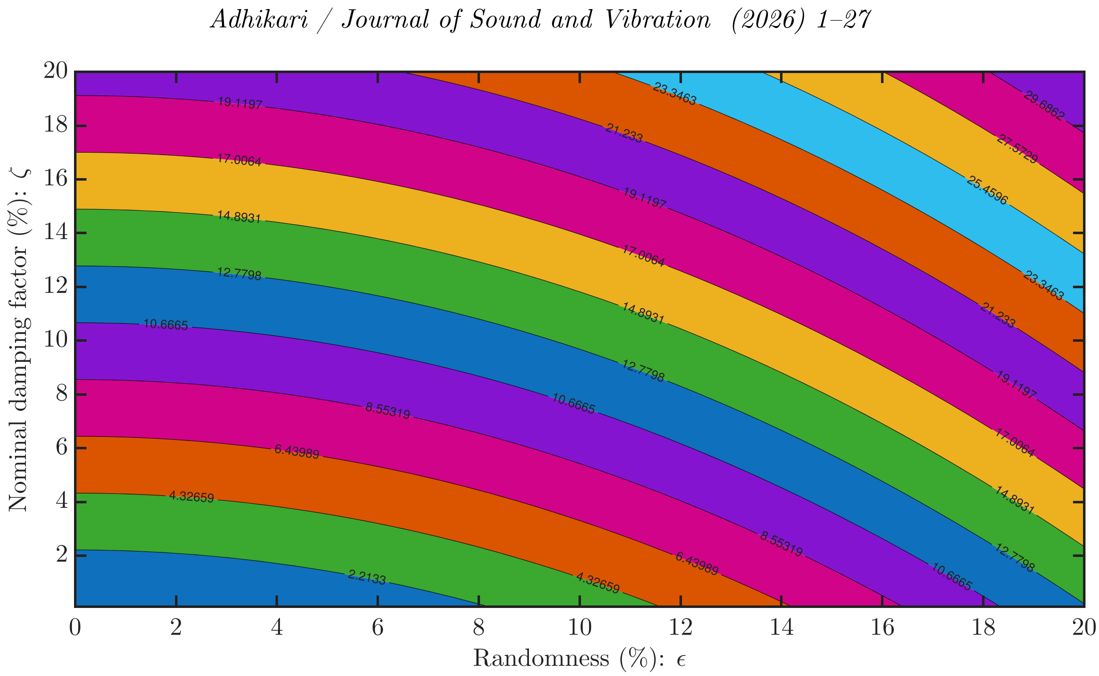


Fig. 3. Contour plot of the ensemble damping ratio ζ_{en} from Eq. (61) as a function of the randomness parameter ϵ and the nominal damping factor ζ, both expressed as percentages. Each contour line corresponds to a constant value of ζ_{en} (in percentage). The contours curve systematically toward higher ζ as ϵ increases, reflecting the quadratic dependence of the stochastic attenuation contribution on ϵ.

It is important to emphasise that this equivalent damping does not correspond to any additional physical dissipation mechanism. Rather, it arises from the progressive loss of phase coherence across the ensemble of oscillators. As discussed in Section 4.2, the spread of natural frequencies causes the phases of individual realisations to diverge with time, producing destructive interference when averaged across the ensemble. The ensemble damping ratio, therefore, provides a convenient macroscopic description of a fundamentally statistical phenomenon.

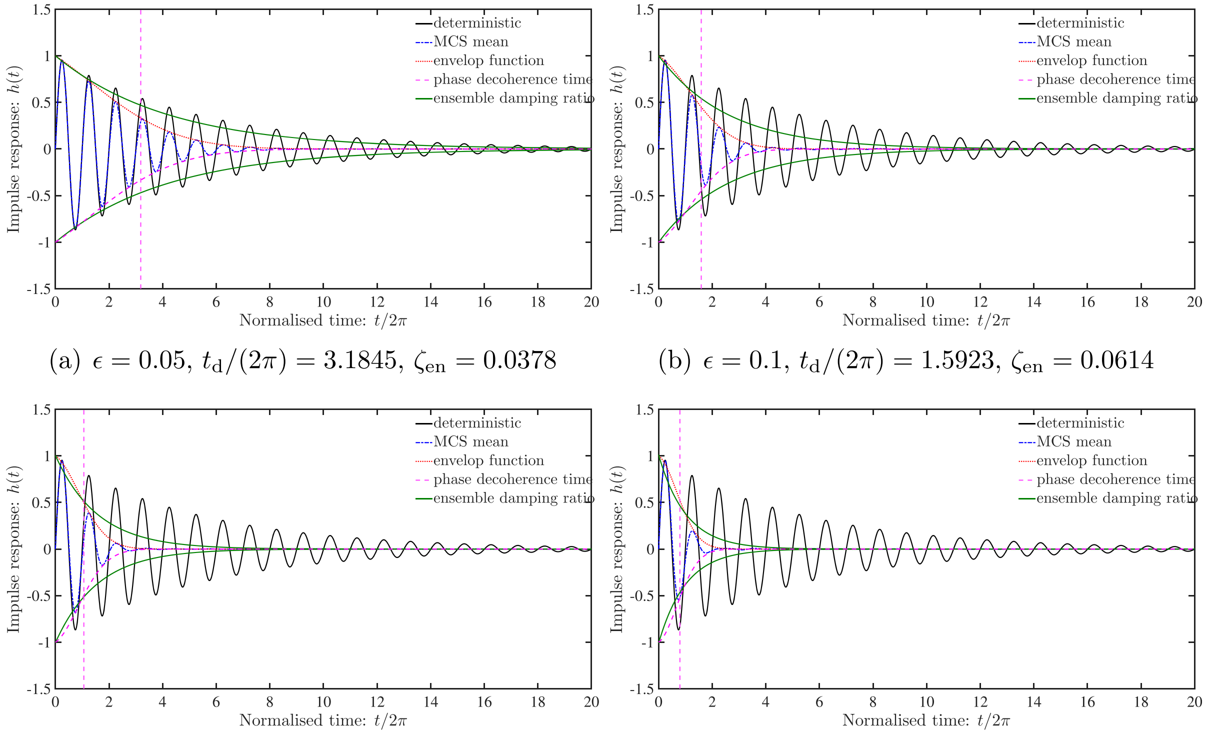
To understand the overall behaviour of the ensemble damping ratio across the full parameter space, in Fig. 3 the expression given by Eq. (61) is plotted as a contour map over the ranges $\epsilon \in [0, 20\%]$ and $\zeta \in [0, 20\%]$.

The contour plot in Fig. 3 reveals several important characteristics of the ensemble damping mechanism. Most significantly, ζ_{en} is always greater than the nominal damping ζ for any non-zero randomness, confirming the accelerated amplitude attenuation of the ensemble mean observed in Fig. 2. The contour lines are nearly horizontal for small ϵ but curve progressively downward as ϵ increases, reflecting the quadratic growth of the stochastic contribution with ϵ. This means that even a modest level of frequency uncertainty can substantially amplify the effective damping: for example, a system with nominal damping $\zeta = 2\%$ subjected to $\epsilon = 10\%$ frequency randomness acquires an ensemble damping ratio of approximately 6.4%, representing more than a threefold increase over the physical value. Conversely, for systems with high nominal damping, the relative contribution of the stochastic term is smaller, since the factor $(1 - 2\zeta^2)$ in Eq. (61) decreases as ζ increases, causing the contour lines to become more closely spaced at lower values of ζ. The figure therefore provides a practical design chart for estimating the effective attenuation induced by natural frequency uncertainty across a wide range of structural and mechanical systems.

Although Eq. (61) provides a convenient single-parameter measure of the stochastic attenuation, it should be interpreted as an approximate characterisation rather than an exact description of the mean envelope. The true envelope derived in Eq. (48) contains a Gaussian-type decay term proportional to $\exp(-\epsilon^2 t^2)$, whereas the ensemble damping ratio represents this time-dependent attenuation by a constant decay rate averaged over one oscillation cycle. Consequently, while ζ_{en} captures the overall scaling of the stochastic attenuation and its quadratic dependence on ϵ, it does not reproduce the full temporal structure of the mean decay. Despite this limitation, this newly proposed ensemble damping ratio remains useful as a compact engineering descriptor of the ensemble damping effect. It provides a simple quantitative estimate of the additional attenuation induced by frequency uncertainty and allows the stochastic oscillator to be interpreted within the familiar framework of classical damping concepts.

4.4. Comparison with numerical simulations

In the previous subsections, three key analytical quantities characterising the mean response have been introduced: the amplitude envelope $\mathcal{E}(t)$, the characteristic phase decoherence time t_d, and the ensemble damping ratio ζ_{en}. In Fig. 4, these three quantities are plotted alongside the deterministic impulse response and the MCS mean for all four uncertainty levels considered in this section.



(a) $\epsilon = 0.05$, $t_{\mathrm{d}}/(2\pi) = 3.1845$, $\zeta_{\mathrm{en}} = 0.0378$
(b) $\epsilon = 0.1$, $t_{\mathrm{d}}/(2\pi) = 1.5923$, $\zeta_{\mathrm{en}} = 0.0614$
(c) $\epsilon = 0.15$, $t_{\mathrm{d}}/(2\pi) = 1.0615$, $\zeta_{\mathrm{en}} = 0.1006$
(d) $\epsilon = 0.2$, $t_{\mathrm{d}}/(2\pi) = 0.7961$, $\zeta_{\mathrm{en}} = 0.1555$

Fig. 4. The mean envelope function $\mathcal{E}(t)$, the exponential decay envelope corresponding to the ensemble damping ratio ζ_{en}, and the characteristic phase decoherence time t_{d} are shown alongside the MCS mean impulse response and the deterministic impulse response. Baseline damping $\zeta = 0.03$ and $\epsilon \in \{0.05, 0.1, 0.15, 0.2\}$.

The envelope function $\mathcal{E}(t)$ from Eq. (48) is shown as a red dotted curve bounding the MCS mean from above and below. Across all four subfigures the envelope tracks the amplitude of the MCS mean response with excellent accuracy, confirming that the Gaussian decay structure $\exp[-(1-2\zeta^2)\epsilon^2 t^2/2]$ correctly captures the progressive amplitude attenuation of the mean. The agreement is particularly close beyond the first few oscillation cycles, where the quadratic term in the exponent dominates and the envelope falls off rapidly relative to the deterministic case. This validates the analytical derivation in Section 4.1 and demonstrates that the envelope formula provides a reliable pointwise bound on the mean response amplitude across the full range of uncertainty levels considered.

The vertical magenta dashed line marks the characteristic phase decoherence time t_{d} from Eq. (50). A clear and systematic trend is visible across the four subfigures: as ϵ increases from 0.05 to 0.2, the decoherence time $t_{\mathrm{d}}/(2\pi)$ decreases from 3.1845 to 0.7961, shifting the vertical line progressively toward the origin. In each subfigure this line coincides closely with the time at which the MCS mean response ceases to track the deterministic impulse response and begins its rapid decay toward zero. For $\epsilon = 0.2$ in subfigure (d), the decoherence time falls within the first oscillation cycle, consistent with the near-immediate suppression of the mean response. This confirms that t_{d} provides a physically meaningful and quantitatively accurate threshold separating the coherent and incoherent regimes of the ensemble mean across all uncertainty levels.

The green solid curves show the exponential envelope $\pm\mathcal{E}_{\mathrm{en}}(t) = \pm\beta^{-1}\exp(-\zeta_{\mathrm{en}}t)$ corresponding to the ensemble damping ratio ζ_{en} from Eq. (61). A second systematic trend is visible: ζ_{en} increases monotonically with ϵ, taking values 0.0378, 0.0614, 0.1006, and 0.1555 for $\epsilon \in \{0.05, 0.1, 0.15, 0.2\}$ respectively, compared with the physical damping ratio $\zeta = 0.03$. The ensemble damping ratio for $\epsilon = 0.2$ is thus more than five times the physical value. As expected, the exponential envelope decays more slowly than the true Gaussian envelope for large t, since a single exponential cannot reproduce the super-exponential decay of the mean response. Nevertheless, the ensemble damping ratio envelope correctly captures the initial rate of amplitude attenuation in each subfigure and provides a useful single-parameter summary of the overall stochastic attenuation effect, interpretable within the familiar framework of classical viscous damping.

5. Summary of main results and the approach taken

This paper has presented an analytical characterisation of the ensemble mean impulse response of a stochastic single-degree-of-freedom oscillator with uncertain natural frequency. The analysis reveals new time-domain properties of stochastic impulse responses that are absent from the classical deterministic theory and are not accessible through purely computational approaches. The main findings are as follows.

1. *Exact analytical mean impulse response.* An exact closed-form expression for the ensemble mean impulse response of a stochastic oscillator with Gaussian frequency uncertainty has been derived. To the best of the author's knowledge, this result has not previously been reported. With $a = \zeta t$, $b = \beta t$, and $\beta = \sqrt{1-\zeta^2}$, the mean impulse response is
$$\bar{h}(t) = e^{-a+\frac{\epsilon^2 a^2}{2}} \frac{\sqrt{\pi}}{\sqrt{2}\,\epsilon\beta} e^{-\frac{(1-\epsilon^2 a)^2}{2\epsilon^2}} \Re\left[\operatorname{erf}\left(\frac{\epsilon b}{\sqrt{2}} - i\frac{1-\epsilon^2 a}{\sqrt{2}\,\epsilon}\right) - \operatorname{erf}\left(-i\frac{1-\epsilon^2 a}{\sqrt{2}\,\epsilon}\right)\right]$$
2. *Envelope of the mean response.* A closed-form amplitude envelope has been derived for the ensemble mean response,
$$\mathcal{E}(t) = \frac{1}{\beta}\exp\left[-\zeta t - \frac{(1-2\zeta^2)}{2}\epsilon^2 t^2\right]$$
The additional Gaussian decay, absent from the deterministic theory, grows quadratically with the uncertainty level ϵ and explains why the mean response collapses far more rapidly than the classical exponential envelope as frequency spread increases.
3. *Ensemble damping and ensemble damping ratio.* The ensemble mean impulse response decays substantially faster than the deterministic impulse response even though the physical damping ratio of each realisation remains unchanged. This phenomenon, identified here as *ensemble damping*, arises from stochastic phase dispersion across the ensemble and represents a statistical rather than a physical dissipation mechanism. A cycle-averaged *ensemble damping ratio* has been introduced as a compact engineering descriptor of this additional attenuation,
$$\zeta_{\rm en} = \zeta + \frac{\pi(1-2\zeta^2)}{\sqrt{1-\zeta^2}}\,\epsilon^2$$
This quantity increases quadratically with ϵ and allows the ensemble damping effect to be interpreted within the familiar framework of classical viscous damping, while recognising that it arises from a purely statistical rather than a dissipative mechanism.
4. *Characteristic phase decoherence time.* A new intrinsic timescale, the *characteristic phase decoherence time*,
$$t_{\rm d} = \frac{1}{\epsilon\beta} = \frac{1}{\epsilon\sqrt{1-\zeta^2}}$$
has been identified. In dimensional time, this becomes $\tau_{\rm d} = 2m/\epsilon\sqrt{4km-c^2}$. This quantity, entirely absent from the deterministic theory, marks the transition from coherent to incoherent ensemble behaviour and is independent of the viscous damping ratio to leading order.
5. *Distributional robustness.* Numerical comparisons across uniform, normal, log-normal, and chi-square models for the natural frequency uncertainty confirm that the principal qualitative features of the ensemble mean response are largely insensitive to the specific probability distribution adopted. The normal, log-normal, and chi-square models are nearly indistinguishable over the full time range considered, while the bounded support of the uniform model produces only modest deviations at higher uncertainty levels. This provides strong support for the use of the Gaussian model as the analytical basis of the present work.

The derivation proceeds by representing the stochastic expectation of the impulse response as a parametric Gaussian integral through an integral identity for the sinc kernel, and then evaluating the resulting integral exactly in terms of the complex error function.

6. Conclusions

This paper derives an exact closed-form analytical expression for the ensemble mean impulse response of a stochastic single-degree-of-freedom oscillator with uncertain natural frequency, establishing a result that has remained absent from the analytical foundations of stochastic structural dynamics since the field was formalised. The stochastic expectation is converted into a parametric Gaussian integral through the integral representation of the sinc kernel $\sin(bu)/u = \int_0^b \cos(su)\,ds$, and the resulting integral is evaluated in exact closed form using the complex error function. This analytical strategy is adopted deliberately in order to extract physical understanding from the mathematical structure of the problem. The analysis advances the theory of uncertain dynamical systems on four fronts. First, the exact closed-form mean impulse response is derived for the first time, providing a complete analytical characterisation of the mean time-domain response of the stochastic oscillator. Second, ensemble damping is identified as a distinct physical mechanism by which phase dispersion across the ensemble attenuates the mean response independently of viscous dissipation, a phenomenon entirely absent from the deterministic theory. Third, two closed-form analytical descriptors that quantify this mechanism are introduced: the ensemble damping ratio $\zeta_{\rm en} = \zeta + \pi(1-2\zeta^2)\epsilon^2/\sqrt{1-\zeta^2}$, which measures the effective decay rate of the ensemble mean within the familiar framework of classical viscous damping, and the characteristic phase decoherence time $t_{\rm d} = 1/(\epsilon\sqrt{1-\zeta^2})$, which defines the intrinsic timescale over which stochastic phase dispersion destroys phase coherence across the ensemble. Fourth, numerical comparisons across uniform, normal, log-normal, and chi-square frequency models confirm that ensemble damping and the decoherence timescale are robust across probability distributions, confirming that these results capture a fundamental dynamical mechanism rather than a distribution-specific effect.

The characteristic phase decoherence time $t_{\rm d}$ constitutes a genuinely new physical quantity: it is intrinsic to the stochastic system, entirely absent from the deterministic theory, and independent of the viscous damping ratio to leading order. In the undamped or very lightly damped limit $\zeta^2 \to 0$, it reduces to the transparent relation $t_{\rm d} \to 1/\epsilon$ in dimensionless form, and in physical time becomes
$$\tau_{\rm d} = \frac{1}{\epsilon}\sqrt{\frac{m}{k}}$$

showing that the loss of phase coherence is governed solely by the natural timescale of the oscillator and the level of frequency uncertainty, and persists even in the complete absence of viscous dissipation. The engineering consequences of these results are direct and quantifiable. The ensemble damping ratio reveals that an ensemble of lightly damped structures with scattered natural frequencies will appear, when averaged, to be substantially more heavily damped than any individual member of the ensemble, an effect that will be misattributed to physical dissipation unless the underlying stochastic mechanism is recognised. The characteristic phase decoherence time provides a closed-form criterion, computable directly from the nominal system parameters, for determining whether a given observation window lies within the coherent or incoherent regime of the ensemble mean.

Since the impulse response is the Green's function of the equation of motion, the analytical results derived here carry significance that reaches beyond the stochastic oscillator problem itself. The sinc-kernel integral identity and the Gaussian integration strategy introduced in this paper provide a transferable analytical route for studying time-domain response quantities of uncertain dynamical systems from first principles, and the two descriptors introduced here, the ensemble damping ratio and the characteristic phase decoherence time, provide the first-order analytical foundation upon which the second-order theory can be developed.

The present study has been confined to uncertainty in the natural frequency with a deterministic damping ratio, in order to isolate the fundamental mechanism and enable exact analytical development. Extension of the closed-form methodology to the covariance structure and forced response statistics of stochastic oscillators, and to multi-degree-of-freedom and continuously distributed systems, represents the natural continuation of this programme of work. For multi-degree-of-freedom systems, the modal coordinates of a structure with uncertain modal frequencies satisfy equations of precisely the present form, so that the ensemble damping ratio and the characteristic phase decoherence time carry over to each mode and govern the decay of the modal ensemble mean. For nonlinear systems, where closed-form impulse responses are generally unavailable, the exact results obtained here provide an analytically rigorous benchmark against which approximate and numerical treatments of stochastic nonlinear oscillators can be calibrated and validated.

CRediT authorship contribution statement

S. Adhikari: Writing – review & editing, Writing – original draft, Project administration, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Evaluation of the mean response integral

This appendix provides the analytical evaluation of the Gaussian integral appearing in Eq. (23). Define

$$a = \zeta t, \qquad b = t\sqrt{1-\zeta^2} \tag{A.62}$$

and introduce

$$u = 1 + \epsilon x \tag{A.63}$$

The mean response integral can then be written as

$$I = \int_{-\infty}^{\infty} \phi(x) e^{-au} \frac{\sin(bu)}{u}\, dx \tag{A.64}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{A.65}$$

Although the factor $1/u$ appears singular at $u = 0$, the singularity is removable since

$$\lim_{u\to 0} \frac{\sin(bu)}{u} = b \tag{A.66}$$

The identity

$$\frac{\sin(bu)}{u} = \int_0^b \cos(su)\, ds \tag{A.67}$$

is used to transform the integral into

$$I = \int_{-\infty}^{\infty} \phi(x) e^{-au} \left(\int_0^b \cos(su)\, ds \right) dx \tag{A.68}$$

Interchanging the order of integration gives

$$I = \int_0^b \left[\int_{-\infty}^{\infty} \phi(x) e^{-au} \cos(su)\, dx \right] ds \tag{A.69}$$

Define

$$J(s) = \int_{-\infty}^{\infty} \phi(x) e^{-a(1+\epsilon x)} \cos(s(1+\epsilon x))\, dx \tag{A.70}$$

Writing the cosine as the real part of a complex exponential,

$$\cos(s(1+\epsilon x)) = \Re\left(e^{is(1+\epsilon x)}\right) \tag{A.71}$$

gives

$$J(s) = \Re\left[e^{-(a-is)} \int_{-\infty}^{\infty} \phi(x) e^{-(a-is)\epsilon x}\, dx \right] \tag{A.72}$$

Now, using the Gaussian identity

$$\int_{-\infty}^{\infty} \phi(x) e^{cx}\, dx = e^{c^2/2} \tag{A.73}$$

with $c = -(a - is)\epsilon$, yields

$$J(s) = \Re\left[e^{-(a-is)} e^{\frac{\epsilon^2}{2}(a-is)^2} \right] \tag{A.74}$$

After simplification this becomes

$$J(s) = e^{-a+\frac{\epsilon^2}{2}(a^2-s^2)} \cos\left((1-\epsilon^2 a)s\right) \tag{A.75}$$

Substituting into the outer integral gives the exact representation

$$I = e^{-a+\frac{\epsilon^2 a^2}{2}} \int_0^b e^{-\frac{\epsilon^2 s^2}{2}} \cos\left((1-\epsilon^2 a)s\right) ds \tag{A.76}$$

To obtain a closed form, introduce

$$\alpha = \frac{\epsilon^2}{2}, \qquad \beta = 1 - \epsilon^2 a \tag{A.77}$$

The remaining integral becomes

$$I = e^{-a+\frac{\epsilon^2 a^2}{2}} \int_0^b e^{-\alpha s^2} \cos(\beta s)\, ds \tag{A.78}$$

Using the representation

$$\cos(\beta s) = \Re\left(e^{i\beta s}\right) \tag{A.79}$$

and completing the square in the exponent leads to

$$\int_0^b e^{-\alpha s^2 + i\beta s}\, ds = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} e^{-\beta^2/(4\alpha)} \left[\mathrm{erf}\left(\sqrt{\alpha} b - \frac{i\beta}{2\sqrt{\alpha}} \right) - \mathrm{erf}\left(-\frac{i\beta}{2\sqrt{\alpha}} \right) \right] \tag{A.80}$$

Taking the real part gives the final closed form

$$I = e^{-a+\frac{\epsilon^2 a^2}{2}} \frac{\sqrt{\pi}}{\sqrt{2}\epsilon} e^{-\frac{(1-\epsilon^2 a)^2}{2\epsilon^2}} \Re\left[\mathrm{erf}\left(\frac{\epsilon b}{\sqrt{2}} - i\frac{1-\epsilon^2 a}{\sqrt{2}\epsilon} \right) - \mathrm{erf}\left(-i\frac{1-\epsilon^2 a}{\sqrt{2}\epsilon} \right) \right] \tag{A.81}$$

This expression provides the exact analytical evaluation of the ensemble mean impulse response.

References

[1] J.P.D. Hartog, Mechanical Vibrations, Dover Publications, New York, 1985.
[2] D.J. Inman, Engineering Vibration, Prentice Hall PTR, NJ, USA, 2003.
[3] S.S. Rao, Mechanical Vibrations, 6th ed., Pearson, Upper Saddle River, NJ, 2017.
[4] D.E. Newland, Mechanical Vibration Analysis and Computation, Longman, Harlow and John Wiley, New York, 1989.
[5] Y.K. Lin, Probabilistic Theory of Structural Dynamics, McGraw-Hill Inc, NY, USA, 1967.
[6] N.C. Nigam, Introduction to Random Vibration, The MIT Press, Cambridge, Massachusetts, 1983.
[7] J.B. Roberts, P.D. Spanos, Random Vibration and Statistical Linearization, John Wiley and Sons Ltd, Chichester, England, 1990.
[8] L.D. Lutes, S. Sarkani, Random Vibrations: Analysis of Structural and Mechanical Systems, Butterworth-Heinemann, Burlington, MA, 2004.
[9] J.B. Roberts, P.D. Spanos, Stochastic averaging - an approximate method of solving random vibration problems, J. Appl. Phys. 21 (2) (1986) 111–134.
[10] M. Kleiber, T.D. Hien, The Stochastic Finite Element Method, John Wiley, Chichester, 1992.
[11] R. Ghanem, P. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer-Verlag, New York, USA, 1991.
[12] G.I. Schüller, Computational stochastic mechanics - recent advances, Comput. Struct. 85 (11-14) (2007) 678–693.
[13] G. Stefanou, The stochastic finite element method: past, present and future, Comput. Methods Appl. Mech. Eng. 198 (9-12) (2009) 1031–1051.

[14] E. Vanmarcke, M. Grigoriu, Stochastic finite element analysis of simple beams, J. Eng. Mech. ASCE 109 (5) (1983) 1203–1214.
[15] M. Shinozuka, G. Deodatis, Response variability of stochastic finite element systems, J. Eng. Mech. ASCE 114 (3) (1988) 499–519.
[16] D. Xiu, G.E. Karniadakis, The wiener-askey polynomial chaos for stochastic differential equations, SIAM J. Sci. Comput. 24 (2) (2002) 619–644.
[17] G. Blatman, B. Sudret, Adaptive sparse polynomial chaos expansion based on least angle regression, J. Comput. Phys. 230 (6) (2011) 2345–2367.
[18] E. Jacquelin, S. Adhikari, J.-J. Sinou, M.I. Friswell, Polynomial chaos expansion in structural dynamics: accelerating the convergence of the first two statistical moment sequences, J. Sound Vib. 356 (11) (2015) 144–154.
[19] B. Pascual, S. Adhikari, A reduced polynomial chaos approach for stochastic finite element analysis, Sādhanā Proc. Indian Acad. Eng. Sci. 37 (3) (2012) 319–340.
[20] F. Nobile, R. Tempone, C.G. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data, SIAM J. Numer. Anal. 46 (5) (2008) 2309–2345.
[21] L. Pichler, H. Pradlwarter, G. Schueller, A mode-based meta-model for the frequency response functions of uncertain structural systems, Comput. Struct. 87 (5-6) (2009) 332–341.
[22] H. Pradlwarter, G. Schüller, Uncertain linear structural systems in dynamics: efficient stochastic reliability assessment, Comput. Struct. 88 (1-2) (2010) 74–86.
[23] M. Torzoni, A. Manzoni, S. Mariani, A multi-fidelity surrogate model based on extreme support vector regression: fusing different fidelity data for engineering design, Eng. Comput. 40 (2024) 473–493.
[24] M.-L. Shi, L. Lv, L. Xu, A multi-fidelity surrogate model based on extreme support vector regression: fusing different fidelity data for engineering design, Eng. Comput. 40 (2) (2023) 473–493.
[25] A. Olivier, D.G. Giovanis, B. Aakash, M. Chauhan, L. Vandanapu, M.D. Shields, Uqpy: a general purpose python package and development environment for uncertainty quantification, J. Comput. Sci. 47 (2020) 101204.
[26] S. Adhikari, B. Pascual, The 'damping effect' in the dynamic response of stochastic oscillators, Probabilistic Eng. Mech. 44 (4) (2016) 2–17.
[27] S.E. Pryse, S. Adhikari, A. Kundu, Sample-based and sample-aggregated galerkin projection schemes for structural dynamics, Probabilistic Eng. Mech. 54 (10) (2018) 118–130.
[28] S.E. Pryse, A. Kundu, S. Adhikari, Projection methods for stochastic dynamic systems: a frequency domain approach, Comput. Methods Appl. Mech. Eng. 338 (8) (2018) 412–439.
[29] S. Kasinos, A. Palmeri, M. Lombardo, S. Adhikari, A reduced modal subspace approach for damped stochastic dynamic systems, Comput. Struct. 257 (12) (2021) 106651.
[30] A. Kundu, S. Adhikari, Transient response of structural dynamic systems with parametric uncertainty, ASCE J. Eng. Mech. 140 (2) (2014) 315–331.
[31] T.K. Caughey, Classical normal modes in damped linear dynamic systems, Trans. ASME J. Appl. Mech. 27 (1960) 269–271.
[32] T.K. Caughey, M.E.J. O'Kelly, Classical normal modes in damped linear dynamic systems, Trans. ASME J. Appl. Mech. 32 (1965) 583–588.
[33] S. Adhikari, Damping modelling using generalized proportional damping, J. Sound Vib. 293 (1-2) (2006) 156–170.
[34] I. Elishakoff, Random vibration of structures: a personal perspective, Appl. Mech. Rev. 48 (12) (1995) 809–825.
[35] I. Elishakoff, Y.J. Ren, M. Shinozuka, Some exact solutions for the bending of beams with spatially stochastic properties, Int. J. Solids Struct. 32 (1995) 2315–2327.
[36] I. Elishakoff, Y.J. Ren, N. Impollonia, New exact solution for randomly loaded beams with stochastic flexibility, Int. J. Solids Struct. 36 (1999) 2325–2340.
[37] M. Grigoriu, Statistically equivalent solutions of stochastic mechanics problems, J. Eng. Mech. ASCE 117 (8) (1991) 1906–1918.
[38] M. Grigoriu, Stochastic Calculus: Applications in Science and Engineering, Birkhäuser, Boston, 2002.
[39] R.A. Ibrahim, Structural dynamics with parameter uncertainties, Appl. Mech. Rev. 40 (3) (1987) 309–328.
[40] C.S. Manohar, R.A. Ibrahim, Progress in structural dynamics with stochastic parameter variations: 1987 to 1998, Appl. Mech. Rev. 52 (5) (1999) 177–197.
[41] A. Papoulis, S.U. Pillai, Probability, Random Variables and Stochastic Processes, 4th ed., McGraw-Hill, New York, 2002.
[42] F. Scarpa, S. Adhikari, Uncertainty modelling of carbon nanotube terahertz oscillators, J. Non-Crystalline Solids 354 (35-39) (2008) 4151–4156.