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Matrix Variate Distributions for Probabilistic Structural Dynamics

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Matrix variate distributions are proposed to quantify uncertainty in the mass, stiffness, and damping matrices arising in linear structural dynamics. The proposed approach is based on the so-called Wishart random matrices. It is assumed that the mean of the system matrices are known. A new optimal Wishart distribution is proposed to model the random system matrices. The optimal Wishart distribution is such that the mean of the matrix and its inverse produce minimum deviations from their respective deterministic values. The method proposed here gives a simple nonparametric approach for uncertainty quantification and propagation for complex aerospace structural systems. The new method is illustrated using a numerical example. It is shown that Wishart random matrices can be used to model uncertainty across a wide range of excitation frequencies.

Nomenclature

$\mathbb{C}$	=	space of complex numbers
$\mathbf{D}(\omega)$	=	dynamic stiffness matrix
$E[(\Omega)]$	=	an $n \times n$ real symmetric matrix
$E[(\bullet)]$	=	mathematical expectation operator
$E[\bullet]$	=	expectation operator
etr$\{\bullet\}$	=	exp{Trace$(\bullet)$}
$\mathbf{F}$	=	symbol for the inverse of a system matrix, $\mathbf{F} \equiv \{\mathbf{M^{-1}}, \mathbf{C^{-1}}, \mathbf{L^{-1}}\}$
$\boldsymbol{f}(t)$	=	forcing vector
$\mathbf{G}$	=	symbol for a system matrix, $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$
$\mathbf{H}(\omega)$	=	frequency response function matrix
$\mathbf{I}_{\mathrm{n}}$	=	identity matrix of dimension n
i	=	unit imaginary number, $i = \sqrt{-1}$
$\mathbb{L}\{(\bullet)\}$	=	Laplace transform of $(\bullet)$
$\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$	=	mass, damping, and stiffness matrices, respectively
m, $\boldsymbol{\Psi}$	=	scalar and matrix parameters of the inverted Wishart distribution
n	=	number of degrees of freedom
$\mathbf{O}_{\mathrm{n,m}}$	=	null matrix of dimension $n \times m$
p, $\boldsymbol{\Sigma}$	=	scalar and matrix parameters of the Wishart distribution
$p_{(\bullet)}(\mathbf{X})$	=	probability density function of $(\bullet)$ in (matrix) variable $\mathbf{X}$
$\boldsymbol{q}(t)$	=	response vector
$\mathbb{R}$	=	space of real numbers
$\mathbb{R}_n^+$	=	space $n \times n$ real positive-definite matrices
$\mathbb{R}_{n,m}$	=	space $n \times m$ real matrices
Trace$(\bullet)$	=	sum of the diagonal elements of a matrix
$\mathbf{Z}$	=	an $n \times n$ symmetric complex matrix
α	=	consonant for the optimal Wishart distribution
$\Gamma_n(a)$	=	multivariate gamma function
θ	=	a consonant, $\theta = 2\nu$
ν	=	order of the inverse-moment constraint
$\phi_{(\bullet)}$	=	characteristic function of $(\bullet)$
ω	=	excitation frequency
$(\bullet)^T$	=	matrix transposition
$\lvert \bullet \rvert$	=	determinant of a matrix
$\lVert \bullet \rVert_F$	=	Frobenius norm of a matrix, $\lVert \bullet \rVert_F = \{\mathrm{Trace}[(\bullet)(\bullet)^T]\}^{1/2}$
$\otimes$	=	Kronecker product
$\sim$	=	distributed as

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I. Introduction

UNCERTAINTIES are unavoidable in the description of real-life engineering systems. The quantification of uncertainties plays a crucial role in establishing the credibility of a numerical model. Uncertainties can be broadly divided into two categories. The first type is due to the inherent variability in the system parameters, for example, different cars manufactured from a single production line are not exactly the same. This type of uncertainty is often referred to as *aleatoric uncertainty*. If enough samples are present, it is possible to characterize the variability using well-established statistical methods and consequently the probability density functions (pdf) of the parameters can be obtained. The second type of uncertainty is mainly due to the lack of knowledge regarding a system, often referred to as *epistemic uncertainty*. This kind of uncertainty generally arises in the modeling of complex systems, for example, in the modeling of cabin noise in helicopters. Because of its very nature, it is comparatively difficult to quantify and consequently model this type of uncertainty.

Broadly speaking, there are two complimentary approaches to quantify uncertainties in a model. The first is the parametric approach and the second is the nonparametric approach. In the parametric approach, the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient, and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element method [1–10]. This type of approach is suitable to quantify aleatoric uncertainties. Epistemic uncertainty, on the other hand, does not explicitly depend on the systems parameters. For example, there can be unquantified errors associated with the equation of motion (linear or nonlinear), in the damping model (viscous or nonviscous), in the model of structural joints, and also in the numerical methods (e.g., discretization of displacement fields, truncation and roundoff errors, tolerances in the optimization and iterative algorithms, step sizes in the time-integration methods). It is evident that the parametric approach is not suitable to quantify this type of uncertainty and a nonparametric approach is needed for this purpose.

In this paper, a general nonparametric uncertainty quantification tool for structural dynamic systems is proposed. The method is based

on the random matrix theory and builds upon the existing nonparametric approach proposed by Soize [11]. Uncertainties associated with a variable can be characterized using the probabilistic approach or possibilistic approaches based on interval algebra, convex sets, or fuzzy sets. In this paper, the probabilistic approach has been adopted. The equation of motion of a damped n-degree-of-freedom linear structural dynamic system can be expressed as

$$\mathbf{M}\,\ddot{\boldsymbol{q}}(t) + \mathbf{C}\dot{\boldsymbol{q}}(t) + \mathbf{K}\boldsymbol{q}(t) = \boldsymbol{f}(t) \tag{1}$$

The importance of considering parametric and/or nonparametric uncertainty also depends on the frequency of excitation. For example, in the high-frequency vibration the wavelengths of the vibration modes become very small. As a result, the vibration response can be very sensitive to the small details of the system. In such situations, a nonparametric uncertainty model may be adequate. Overall, three different approaches are currently available to model stochastic structural dynamic systems across the frequency range:

1) Low-frequency vibration problems use the stochastic finite element method (SFEM) [1–10] which considers parametric uncertainties in details.

2) High-frequency vibration problems use statistical energy analysis [12] (SEA) which does not consider parametric uncertainties in details.

3) Midfrequency vibration problems [13–16] in which both parametric and nonparametric uncertainties need to be considered.

The aim of this paper is to propose a method which will work across the frequency range. Here we will investigate the possibility of using the random matrix theory as the unified uncertainty modeling tool to be valid for low-, medium-, and high-frequency vibration problems. The probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ will be derived to completely quantify the uncertainties associated with system, Eq. (1). In the next section, we briefly outline some aspects of the random matrix theory required for further developments.

II. Background of the Random Matrix Theory

Random matrices were introduced by Wishart [17] in the late 1920s in the context of multivariate statistics. However, random matrix theory (RMT) was not used in other branches until the 1950s when Wigner [18] published his works (leading to the Nobel Prize in physics in 1963) on the eigenvalues of random matrices arising in high-energy physics. Using an asymptotic theory for large dimensional matrices, Wigner was able to bypass the Schrödinger equation and explain the statistics of measured atomic energy levels in terms of the limiting eigenvalues of these random matrices. Since then, research on random matrices has continued to attract interest in multivariate statistics, physics, number theory, and more recently in mechanical and electrical engineering. We refer the readers to the books by Mezzadri and Snaith [19], Tulino and Verdú [20], Eaton [21], Muirhead [22], and Mehta [23] for history and applications of random matrix theory.

The probability density function of a random matrix can be defined in a manner similar to that of a random variable or random vector. If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A})$: $\mathbb{R}_{n,m} \to \mathbb{R}$. Here, we define four types of random matrices which are relevant to this study.

Definition 1. Gaussian random matrix: The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-\frac{np}{2}} |\mathbf{\Sigma}|^{-\frac{p}{2}} |\mathbf{\Psi}|^{-\frac{n}{2}} \text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \tag{2}$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Definition 2. Wishart matrix: An $n \times n$ symmetric positive-definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} \tag{3}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Definition 3. Matrix variate gamma distribution: An $n \times n$ symmetric positive-definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \{\Gamma_n(a)|\mathbf{\Psi}|^{-a}\}^{-1}|\mathbf{W}|^{a-\frac{1}{2}(n+1)}\text{etr}\{-\mathbf{\Psi}\mathbf{W}\} \qquad \Re(a) > \frac{1}{2}(n-1) \tag{4}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. The matrix variate gamma distribution was used by Soize [11] for the random system matrices of linear dynamical systems.

Definition 4. Inverted Wishart matrix: An $n \times n$ symmetric positive-definite matrix random $\mathbf{V}$ is said to have an inverted Wishart distribution with parameters m and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{V}}(\mathbf{V}) = \frac{2^{-\frac{1}{2}(m-n-1)n}|\mathbf{\Psi}|^{\frac{1}{2}(m-n-1)}}{\Gamma_n\left(\frac{1}{2}(m-n-1)\right)|\mathbf{V}|^{\frac{m}{2}}}\text{etr}\{-\mathbf{V}^{-1}\mathbf{\Psi}\} \qquad m > 2n, \qquad \mathbf{\Psi} > 0 \tag{5}$$

This distribution is usually denoted as $\mathbf{V} \sim IW_n(m, \mathbf{\Psi})$.

In Eqs. (3–5), the function $\Gamma_n(a)$ can be expressed in terms of products of the univariate gamma functions as

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)}\prod_{k=1}^{n}\Gamma\left[a - \frac{1}{2}(k-1)\right]; \quad \text{for } \Re(a) > \frac{1}{2}(n-1) \tag{6}$$

The multivariate gamma function plays a key role in the random matrix method proposed in this paper. See Appendix A for a proof of Eq. (6) and related mathematical methods. For more details on the matrix variate distributions we refer to the books by Tulino and Verdú [20], Gupta and Nagar [24], Eaton [21], Muirhead [22], and references therein. Among the four types of random matrices introduced here, the distributions given by Eqs. (3–5) always result symmetric and positive-definite matrices. Therefore, they can be possible candidates to model the random system matrices arising in probabilistic structural mechanics.

III. Matrix Variate Distribution for System Matrices

In this section, an information theoretic approach is taken to obtain the matrix variate distributions of the random system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. First, we look at the information available to us and then consider the constraints the matrix variate distributions must satisfy to be physically realistic. Once these steps are completed, the matrix variate distributions will be obtained using the maximum-entropy method. In a series of papers, Soize [11] used this approach to obtain the probability density function of the system matrices.

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$, and $\overline{\mathbf{K}}$, respectively. This information is likely to be available, for example, using the deterministic finite element method. However, there are uncertainties associated with our modeling so that $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are actually random matrices. The distribution of these random matrices should be such that they are 1) symmetric, 2) positive definite, and 3) the moments of the inverse of the dynamic stiffness matrix

$$\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \tag{7}$$

should exist $\forall\ \omega$. That is, if $\mathbf{H}(\omega)$ is the frequency response function (FRF) matrix

$$\mathbf{H}(\omega) = \mathbf{D}^{-1}(\omega) = [-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]^{-1} \tag{8}$$

then the following condition must be satisfied:

$$E[\|\mathbf{H}(\omega)\|_F^{\nu}] < \infty, \quad \forall\ \omega \tag{9}$$

For example, if $\mathbf{H}(\omega)$ is considered to be a second-order (matrix variate) random process, then $\nu = 2$ should be used. This constraint clearly arising from the fact that the moments and the pdf of the response vector must exist for all frequency of excitation. Because the matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ have similar probabilistic characteristics, for notational convenience we will use the notation $\mathbf{G}$ which stands for any one of the system matrices. Suppose the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G})$: $\mathbb{R}_n^+ \to \mathbb{R}$. We have the following information and constraints to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, \mathrm{d}\mathbf{G} = 1 \quad \text{(normalization)} \tag{10}$$

and

$$E[\mathbf{G}] = \int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, \mathrm{d}\mathbf{G} = \bar{\mathbf{G}} \quad \text{(the mean matrix)} \tag{11}$$

The mean matrix $\bar{\mathbf{G}}$ is symmetric and positive definite, and the integrals appearing in these equations are $n(n+1)/2$ dimensional. The maximum-entropy method [25] can be used to obtain the probability density function of the random system matrices. Udwadia [26,27] used an entropy-based method to obtain the probability density functions of the mass, stiffness, and damping constants of a single degree-of-freedom oscillator. For multiple degree-of-freedom systems, using the maximum-entropy method, Soize [11] obtained the matrix variate gamma distribution for the system matrices given by Eq. (4). According to this, for a general system matrix $\mathbf{G}$, the parameters for the matrix variate gamma distribution in Eq. (4) are given by

$$a = \lambda + \frac{1}{2}(n-1) \tag{12}$$

$$\mathbf{\Psi} = \left\{\lambda + \frac{1}{2}(n-1)\right\}\bar{\mathbf{G}}^{-1} \tag{13}$$

where $\lambda \in \mathbb{R}$ is an undetermined coefficient. The main difference between the matrix variate gamma distribution and the Wishart distribution is that historically only integer values were considered for the shape parameter p in the Wishart matrices. However, from an analytical point of view, the gamma and the Wishart distributions are identical and the modern random matrix theory normally does not make any distinctions between them. Because the Wishart random matrix is the oldest [17] and perhaps the most widely used random matrix model [20–22], in this paper we present our results in terms of the Wishart matrices. In the next subsections, the matrix variate distributions are derived considering two separate cases.

A. Distribution Without the Inverse-Moment Constraint

The inverse-moment constraint of the dynamic stiffness matrix mentioned before [condition 3] is difficult to implement analytically. Therefore, in this section we derive the pdf of the system matrices *without* this constraint. This distribution is "maximally uncertain" because a minimum number of constraints are used to derive it. Once the distribution is obtained, the consequence of ignoring the inverse-moment constraint will be discussed. To extend the maximum-entropy method to random matrices, first note that the entropy associated with the matrix variate probability density function $p_{\mathbf{G}}(\mathbf{G})$ can be expressed as

$$\mathcal{S}(p_{\mathbf{G}}) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, \ell n\{p_{\mathbf{G}}(\mathbf{G})\}\, \mathrm{d}\mathbf{G} \tag{14}$$

Using this, together with the constraints in Eqs. (10) and (11), we construct the Lagrangian [25]

$$\begin{aligned}\mathcal{L}(p_{\mathbf{G}}) = &-\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, \ell n\{p_{\mathbf{G}}(\mathbf{G})\}\, \mathrm{d}\mathbf{G} \\ &- (\lambda 0 - 1)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, \mathrm{d}\mathbf{G} - 1\right) \\ &- \mathrm{Trace}\left[\mathbf{\Lambda}_1\left(\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, \mathrm{d}\mathbf{G} - \bar{\mathbf{G}}\right)\right]\end{aligned} \tag{15}$$

The scalar $\lambda_0 \in \mathbb{R}$ and the symmetric matrix $\mathbf{\Lambda}_1 \in \mathbb{R}_{n,n}$ are the unknown Lagrange multiplies which need to be determined. Using the variational calculus, it can be shown that the optimal condition is given by

$$\frac{\partial \mathcal{L}(p_{\mathbf{G}})}{\partial p_{\mathbf{G}}} = 0 \tag{16}$$

or

$$-(1 + \ell n\{p_{\mathbf{G}}(\mathbf{G})\}) - (\lambda_0 - 1) - \mathrm{Trace}(\mathbf{\Lambda}_1\mathbf{G}) = 0 \tag{17}$$

or

$$-\ell n\{p_{\mathbf{G}}(\mathbf{G})\} = \lambda_0 + \mathrm{Trace}(\mathbf{\Lambda}_1\mathbf{G}) \tag{18}$$

or

$$p_{\mathbf{G}}(\mathbf{G}) = \exp\{-\lambda_0\}\mathrm{etr}\{-\mathbf{\Lambda}_1\mathbf{G}\} \tag{19}$$

Using the matrix calculus [28–31], the Lagrange multipliers λ_0 and $\mathbf{\Lambda}_1$ can be obtained exactly by substituting $p_{\mathbf{G}}(\mathbf{G})$ from Eq. (19) into the constraint Eqs. (10) and (11). After some algebra (see Appendix B for the details) it can be shown that

$$p_{\mathbf{G}}(\mathbf{G}) = \frac{r^{nr}|\bar{\mathbf{G}}|^{-r}}{\Gamma_n(r)}\mathrm{etr}\{-r\bar{\mathbf{G}}^{-1}\mathbf{G}\}, \quad \text{where } r = \frac{1}{2}(n+1) \tag{20}$$

This distribution can be viewed as the matrix variate generalization of the single degree-of-freedom case [26]. If we consider the special case when $\mathbf{G}$ is a one-dimensional ($n = 1$) matrix (that is a scalar, say G), then from Eq. (20) we obtain $p_G(G) = \exp(-G/\bar{G})/\bar{G}$. This implies that G becomes an exponentially distributed random variable, which is well known [25] that if we know *only* the mean of a random variable, then the maximum-entropy distribution of that random variable becomes exponential. Therefore, the distribution in Eq. (20) can be viewed as the matrix generalization of the familiar exponential distribution.

Comparing Eq. (20) with the Wishart distribution in Eq. (3) it can be shown [see Eqs. (B12) and (B13) for the details] that $\mathbf{G}$ has the Wishart distribution with parameters $p = n + 1$ and $\mathbf{\Sigma} = \bar{\mathbf{G}}/(n+1)$. Therefore, we have the following fundamental result regarding the uncertainty modeling of linear structural dynamic systems.

Theorem 1. If only the mean of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ is available, say $\bar{\mathbf{G}}$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $(n+1)$ and $\bar{\mathbf{G}}/(n+1)$, that is $\mathbf{G} \sim W_n[n+1, \bar{\mathbf{G}}/(n+1)]$.

B. Distribution with the Inverse-Moment Constraint

In the previous section we have ignored the constraint that the inverse moments of the dynamic stiffness matrix should exist for all frequencies. The exact application of this constraint requires the derivation of the joint probability density function of the random matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$, which is analytically intractable within the scope of the present state of developments in the random matrix theory. Therefore, we consider a simpler problem where it is required that inverse moments of each of the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ must exist. Provided the system is damped, this condition will always guarantee the existence of the moments of the frequency response function matrix. This is only a sufficient condition and not a necessary condition. As a result, the distributions arising from this approach will be more constrained than is necessary.

Suppose the inverse moments (say up to order ν) of a system matrix exist. This implies that $E[\|\mathbf{G}^{-1}\|_F^{\nu}]$ should be finite. Because $\mathbf{G}$ is a symmetric positive-definite matrix, $\mathbf{G}$ can be expressed in terms of its eigenvalues and eigenvectors as $\mathbf{G} = \Phi\lambda\Phi^T$. Here, $\Phi \in \mathbb{R}_{n,n}$ is an orthonormal matrix containing the eigenvectors of $\mathbf{G}$, that is, $\Phi^T\Phi = I_n$ and λ is a real diagonal matrix containing the eigenvalues of $\mathbf{G}$. Note that for any $m \in \mathbb{R}$, $\mathbf{G}^m = \Phi\lambda^m\Phi^T$. Recalling that the Frobenius norm of the matrix $\mathbf{G}$ is given by $\|\mathbf{G}\|_F = [\text{Trace}(\mathbf{G}\mathbf{G}^{\text{T}})]^{1/2}$ and for any three compatible matrices $\text{Trace}(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3) = \text{Trace}(\mathbf{A}_3\mathbf{A}_1\mathbf{A}_2) = \text{Trace}(\mathbf{A}_2\mathbf{A}_3\mathbf{A}_1)$, we have

$$\begin{aligned}\|\mathbf{G}^{-1}\|_F^{\nu} &= \text{Trace}(\sqrt{\Phi\lambda^{-\nu}\Phi^T\Phi\lambda^{-\nu}\Phi^T}) = \text{Trace}(\Phi\lambda^{-\nu}\Phi^T)\\ &= \text{Trace}(\Phi^T\Phi\lambda^{-\nu}) = \text{Trace}(\lambda^{-\nu}) = \lambda_1^{-\nu} + \lambda_2^{-\nu} + \cdots \lambda_n^{-\nu} \quad (21)\end{aligned}$$

Because all the eigenvalues are positive, the condition $E[\|\mathbf{G}^{-1}\|_F^{\nu}] < \infty$ will also be satisfied when

$$E\Big[\ell_n\,\lambda_1^{-\nu} + \ell_n\,\lambda_2^{-\nu} + \cdots \ell_n\,\lambda_n^{-\nu}\Big] < \infty \quad (22)$$

or

$$E[-\nu\,\ell_n(\lambda_1\lambda_2\cdots\lambda_n)] < \infty \quad (23)$$

or

$$E[\ell_n\,|\mathbf{G}|^{-\nu}] < \infty \quad (24)$$

Equation (24) is the new constraint which should be satisfied. The last step is done mainly for analytical convenience. The new Lagrangian [25] becomes

$$\begin{aligned}\mathcal{L}(p_{\mathbf{G}}) = &-\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\,\ell_n\{p_{\mathbf{G}}(\mathbf{G})\}\,\mathrm{d}\mathbf{G}\\ &-(\lambda_0-1)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\,\mathrm{d}\mathbf{G} - 1\right) - \int_{\mathbf{G}>0} \ell_n\,|\mathbf{G}|^{-\nu} p_{\mathbf{G}}\,\mathrm{d}\mathbf{G}\\ &-\text{Trace}\left(\mathbf{\Lambda}_1\left[\int_{\mathbf{G}>0}\mathbf{G}\,p_{\mathbf{G}}(\mathbf{G})\,\mathrm{d}\mathbf{G} - \bar{\mathbf{G}}\right]\right) \quad (25)\end{aligned}$$

Although ν behaves like a Lagrange multiplier, it cannot be obtained uniquely because the constraint in Eq. (24) does not involve finite numbers. As a result, we treat ν as a parameter, rather than an unknown constant, in the optimization procedure. Again using the calculus of variation, we have

$$\frac{\partial\mathcal{L}(p_{\mathbf{G}})}{\partial p_{\mathbf{G}}} = 0 \quad (26)$$

or

$$-(1+\ell_n\{p_{\mathbf{G}}(\mathbf{G})\}) - (\lambda_0-1) + \nu\,\ell_n\,|\mathbf{G}| - \text{Trace}(\mathbf{\Lambda}_1\mathbf{G}) = 0 \quad (27)$$

or

$$-\ell_n\{p_{\mathbf{G}}(\mathbf{G})\} = \lambda_0 + \text{Trace}(\mathbf{\Lambda}_1\mathbf{G}) - \ell_n\,|\mathbf{G}|^{\nu} \quad (28)$$

or

$$p_{\mathbf{G}}(\mathbf{G}) = \exp\{-\lambda_0\}|\mathbf{G}|^{\nu}\text{etr}\{-\mathbf{\Lambda}_1\mathbf{G}\} \quad (29)$$

Substituting $p_{\mathbf{G}}(\mathbf{G})$ from Eq. (29) into the constraint Eqs. (10) and (11), the Lagrange multipliers λ_0 and $\mathbf{\Lambda}_1$ can be obtained exactly using the matrix calculus [28–31]. After some algebra, it can be shown (see Appendix B for the details) that

$$p_{\mathbf{G}}(\mathbf{G}) = \frac{r^{nr}|\bar{\mathbf{G}}|^{-r}}{\Gamma_n(r)}|\mathbf{G}|^{\nu}\text{etr}\{-r\bar{\mathbf{G}}^{-1}\mathbf{G}\}, \quad \text{where } r = \nu + \frac{1}{2}(n+1) \quad (30)$$

Comparing Eq. (30) with the Wishart distribution in Eq. (3), it can be observed [see Eqs. (B12) and (B13) for the details] that $\mathbf{G}$ has the Wishart distribution with parameters $p = 2\nu + n + 1$ and $\mathbf{\Sigma} = \bar{\mathbf{G}}/(2\nu+n+1)$. Therefore, we have the following basic result regarding the uncertainty modeling of linear structural dynamic systems.

Theorem 2. If νth order inverse moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\bar{\mathbf{G}}$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu+n+1)$ and $\mathbf{\Sigma} = \bar{\mathbf{G}}/(2\nu+n+1)$, that is $\mathbf{G} \sim W_n[2\nu+n+1, \bar{\mathbf{G}}/(2\nu+n+1)]$.

Observe that substituting $\nu = 0$ in Eq. (30) gives us the "maximal uncertain distribution" derived in the preceding subsection.

C. Statistical Properties of the System Matrices

From the discussion in the preceding section, it is clear that each system matrix follows a Wishart distribution. The discovery of the Wishart distribution in this context turns out to be very useful because it has been studied extensively in the multivariate statistics literature [21,22,24,32]. Here, we outline some interesting properties of the distribution. For convenience, we present the results in terms of the general Wishart matrix $\mathbf{G} \sim W_n(p, \mathbf{\Sigma})$, where $\mathbf{\Sigma} = \bar{\mathbf{G}}/p$ and $p = 2\nu+n+1$ or $p = n+1$ depending on whether the inverse-moment constraint is considered or not.

Perhaps the most useful property of a Wishart matrix is the fact that it can be decomposed in terms of Gaussian random matrices.

Theorem 3: If $\mathbf{X}$ is a Gaussian random matrix such that $\mathbf{X} \sim N_{n,p}(\mathbf{O}_{n,p}, \mathbf{\Sigma} \otimes \mathbf{I}_p)$, then $\mathbf{G} = \mathbf{X}\mathbf{X}^T$ has the Wishart distribution $\mathbf{G} \sim W_n(p, \mathbf{\Sigma})$.

The proof is given in Appendix C. This result is particularly important because it gives an easy simulation algorithm for the system matrices. From this representation, it is clear that $\mathbf{G}$ is symmetric with probability one. Using Theorem 3.2.1 of Gupta and Nagar [24], we can also say that $\mathbf{G}$ is positive definite with probability one. Therefore, two out of the three requirements outlined in the previous subsections are always satisfied by a Wishart matrix. If we consider the special case when $\mathbf{G}$ is a one-dimensional matrix (that is a scalar), then from Theorem 3, we obtain that $\mathbf{G}$ becomes an χ^2 random variable with p degrees of freedom [33]. Therefore, the Wishart matrix can also be viewed as the matrix generalization of the χ^2 random variable.

The probability density function of $\mathbf{G}$ has been derived exactly in closed form in Eq. (30). Using this characteristic function of $\mathbf{G}$, that is the joint characteristic function of G_{11}, G_{12}, G_{nn} can be obtained (see Appendix C for the derivation) as

$$\Phi_{\mathbf{G}}(\mathbf{\Omega}) = E[\text{etr}\{i\mathbf{\Omega}\mathbf{G}\}] = |\mathbf{I}_n - 2i\mathbf{\Omega}\,\mathbf{\Sigma}|^{-\frac{p}{2}} \quad (31)$$

where $\mathbf{\Omega} \in \mathbb{R}_{n,n}$ is a symmetric matrix. The first moment (mean), the second moment, the elements of the covariance tensor, and the variance of $\mathbf{G}$ can be obtained [24] as

$$E[\mathbf{G}] = p\mathbf{\Sigma} = \bar{\mathbf{G}} \quad (32)$$

$$\begin{aligned}E[\mathbf{G}^2] &= p\mathbf{\Sigma}^2 + p\text{Trace}(\mathbf{\Sigma})\mathbf{\Sigma} + p^2\mathbf{\Sigma}^2\\ &= \frac{1}{2\nu+n+1}[(2\nu+n+2)\bar{\mathbf{G}}^2 + \bar{\mathbf{G}}\text{Trace}(\bar{\mathbf{G}})] \quad (33)\end{aligned}$$

$$\begin{aligned}\text{cov}(G_{ij}, G_{kl}) &= p(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}) = \frac{1}{2\nu+n+1}\\ &\times\left(\bar{G}_{ik}\bar{G}_{jl} + \bar{G}_{il}\bar{G}_{jk}\right) \quad (34)\end{aligned}$$

$$E[(\mathbf{G} - E[\mathbf{G}]^2)] = E[\mathbf{G}^2] - \bar{\mathbf{G}}^2 = \frac{1}{2\nu+n+1}[\bar{\mathbf{G}}^2 + \bar{\mathbf{G}}\text{Trace}(\bar{\mathbf{G}})] \quad (35)$$

It is useful to define the normalized standard deviation of $\mathbf{G}$ (introduced by Soize [11] as the dispersion parameter) as

$$\delta_G^2 = \frac{E[\|\mathbf{G} - E[\mathbf{G}]\|_F^2]}{\|E[\mathbf{G}]\|_F^2} \tag{36}$$

Because both $E[\bullet]$ and $\text{Trace}(\bullet)$ are linear operators, their order can be interchanged. Using Eqs. (32) and (33) we have

$$\begin{aligned} E[\|\mathbf{G} - E(\mathbf{G})\|_F^2] &= E(\text{Trace}\{[\mathbf{G} - E(\mathbf{G})][\mathbf{G} - E(\mathbf{G})]^T\}) \\ &= \text{Trace}(E\{[\mathbf{G}^2 - \mathbf{G}E(\mathbf{G}) - E(\mathbf{G})\mathbf{G} - E(\mathbf{G})^2]\}) \\ &= \text{Trace}[E(\mathbf{G}^2) - E(\mathbf{G})^2] = \text{Trace}[p\boldsymbol{\Sigma}^2 + p\text{Trace}(\boldsymbol{\Sigma})\boldsymbol{\Sigma} \\ &\quad + p^2\boldsymbol{\Sigma}^2 - (p\boldsymbol{\Sigma})^2] = p\text{Trace}(\boldsymbol{\Sigma}^2) + p[\text{Trace}(\boldsymbol{\Sigma})]^2 \end{aligned} \tag{37}$$

Therefore

$$\begin{aligned} \delta_G^2 &= \frac{p\text{Trace}(\boldsymbol{\Sigma}^2) + p\{\text{Trace}(\boldsymbol{\Sigma})\}^2}{p^2\text{Trace}(\boldsymbol{\Sigma}^2)} = \frac{1}{p}\left\{1 + \frac{\{\text{Trace}(\boldsymbol{\Sigma})\}^2}{\text{Trace}(\boldsymbol{\Sigma}^2)}\right\} \\ &= \frac{1}{2\nu + n + 1}\left\{1 + \frac{\{\text{Trace}(\bar{\mathbf{G}})\}^2}{\text{Trace}(\bar{\mathbf{G}}^2)}\right\} \end{aligned} \tag{38}$$

Equation (38) shows that the normalized standard deviation of $\mathbf{G}$ will be smaller for higher values of ν. This equation clearly shows that for a system with fixed dimension, the uncertainty in the system matrices reduces when ν increases. Recall that ν is the order of the inverse moment that we have enforced to exist. Intuitively Eq. (38) implies that if we enforce more constraint (in terms of the order of the inverse moment), the resulting distribution becomes less uncertain. This fact, in turn, allows one to control the amount of uncertainty in the system by choosing different values of ν. It is interesting to observe that the parameter ν, which was originally used as the order of the inverse-moment constraint, now solely controls the amount of variability in the matrices as both n and $\bar{\mathbf{G}}$ are fixed. If δ_G^2 is known (e.g., from experiments, stochastic finite element calculations, or experience) then Eq. (38) can be used to calculate ν. Next, we consider the properties of the inverse of the system matrices.

D. Statistical Properties of the Inverse of the System Matrices

Suppose $\mathbf{F} = \mathbf{G}^{-1}$ denotes the inverse of a system matrix. The Jacobian of this transformation [28] is given by $J(\mathbf{G} \to \mathbf{F}) = |\mathbf{F}|^{-(n+1)}$. Using this, the pdf of the inverse of the system matrices can be obtained from Eq. (30) as

$$\begin{aligned} p_{\mathbf{F}}(\mathbf{F}) &= J(\mathbf{G} \to \mathbf{F}) p_{\mathbf{G}}(\mathbf{G} = \mathbf{F}^{-1}) \\ &= \frac{r^{nr}|\bar{\mathbf{G}}|^{-r}}{\Gamma_n(r)}|\mathbf{F}|^{-(\nu+n+1)}\text{etr}\{-r\bar{\mathbf{G}}^{-1}\mathbf{F}^{-1}\} \\ &\text{where } r = \nu + \frac{1}{2}(n+1) \end{aligned} \tag{39}$$

Comparing this with the inverted Wishart distribution in Eq. (5), we can say that the inverse of a system matrix has an inverted Wishart distribution with parameters $m = 2(\nu + n + 1)$ and $\boldsymbol{\Psi} = (2\nu + n + 1)\bar{\mathbf{G}}^{-1}$. From this discussion we have the following basic result.

Theorem 4. If νth order inverse moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the pdf of $\mathbf{G}^{-1}$ follows the inverted Wishart distribution with parameters $m = 2(\nu + n + 1)$ and $\boldsymbol{\Psi} = (2\nu + n + 1)\bar{\mathbf{G}}^{-1}$, that is $\mathbf{G}^{-1} \sim IW_n[2(\nu + n + 1), (2\nu + n + 1)\bar{\mathbf{G}}^{-1}]$.

The exact pdf of the inverse of the system matrices might be useful, for example, to obtain the pdf of the response. At present, mostly perturbation-based methods are used for this purpose. The first moment (mean) and the second moment can be obtained [24] exactly in closed form as

$$E[\mathbf{G}^{-1}] = \frac{\boldsymbol{\Psi}}{m - 2n - 2} = \frac{2\nu + n + 1}{2\nu}\bar{\mathbf{G}}^{-1} \tag{40}$$

$$\begin{aligned} E[\mathbf{G}^{-2}] &= \frac{\text{Trace}(\boldsymbol{\Psi})\boldsymbol{\Psi} + (m - 2n - 1)\boldsymbol{\Psi}^2}{(m - 2n - 1)(m - 2n - 2)(m - 2n - 4)} \\ &= \frac{(2\nu + n + 1)^2\text{Trace}(\bar{\mathbf{G}}^{-1})\bar{\mathbf{G}}^{-1} + 2\nu\bar{\mathbf{G}}^{-2}}{2\nu(2\nu + 1)(2\nu - 2)} \end{aligned} \tag{41}$$

From Eq. (40), observe that ν must be more than zero for the existence of the mean of the inverse matrices. Similarly, from Eq. (41), for the existence of the second inverse moment, ν must be more than one. This in turn implies that for the distribution given in Theorem 1, neither the mean nor the variance of the inverse of the random system matrix exist. This is, however, not a limitation because we are interested in the existence of the inverse moments of the dynamic stiffness matrix and not the individual matrices. It is perfectly possible (provided there is some damping in the system) that the inverse moments of the dynamic stiffness matrix exists, whereas that for the individual system matrices do not.

Equation (40) also shows one intriguing fact. Suppose the degrees of freedom of a system $n = 100$ and $\nu = 2$. Therefore, from Eq. (32) we have $E[\mathbf{G}] = \bar{\mathbf{G}}$ and from Eq. (40) we have

$$E[\mathbf{G}^{-1}] = \frac{2 \times 2 + 100 + 1}{4}\bar{\mathbf{G}}^{-1} = 26.25\bar{\mathbf{G}}^{-1} \tag{42}$$

This is clearly unacceptable for engineering structural matrices because the randomness of real system are not very large. Of course there is no reason as to why always $E[\mathbf{G}^{-1}] = \bar{\mathbf{G}}^{-1}$. However, we do not expect them to be so far apart. One possible way to reduce this "gap" is to increase the values of ν. However, this implies the reduction of the variance, that is, the assumption of more constraints than necessary. This discrepancy between the "mean of the inverse" and the "inverse of the mean" of the random matrices appears to be a fundamental limitation of the matrix variate distribution derived so far. A new approach based on optimal Wishart matrices is proposed in the next section to address this issue.

IV. Optimal Wishart Distributions

From the discussions so far, it follows that one could have formulated the maximum-entropy approach in terms of the inverse of the system matrices also. In that case, one would obtain a very large difference between $E[\mathbf{G}]$ and $\bar{\mathbf{G}}$. It is not quite obvious whether the maximum-entropy approach should be formulated with respect $\mathbf{G}$ or $\mathbf{G}^{-1}$, or indeed for any other powers of $\mathbf{G}$. Depending on the what information we select, the resulting distribution can differ dramatically from one to another. To avoid this problem of "information dependence," in this section an optimal Wishart distribution is proposed.

Here, the main idea is that the distribution of $\mathbf{G}$ must be such that $E[\mathbf{G}]$ and $E[\mathbf{G}^{-1}]$ be closest to $\bar{\mathbf{G}}$ and $\bar{\mathbf{G}}^{-1}$, respectively. As a result, the resulting distribution will not be biased on the choice of $\mathbf{G}$ or $\mathbf{G}^{-1}$. Suppose $\mathbf{G}$ has Wishart distribution with parameters $p = n + 1 + \theta$ and $\boldsymbol{\Sigma} = \bar{\mathbf{G}}/\alpha$, that is, $\mathbf{G} \sim W_n(n + 1 + \theta, \bar{\mathbf{G}}/\alpha)$. There are two undetermined parameters α and θ in the problem. Recall that the variance of the system is dependent *only* on $\theta = 2\nu$. Therefore, this parameter must not be changed. This leaves us to determine only the parameter $\alpha \in \mathbb{R}^+$ such that $E[\mathbf{G}]$ and $E[\mathbf{G}^{-1}]$ become closest to $\bar{\mathbf{G}}$ and $\bar{\mathbf{G}}^{-1}$. To obtain the optimal value of α, we define the normalized errors as

$$\varepsilon_1 = \|\bar{\mathbf{G}} - E[\mathbf{G}]\|_F / \|\bar{\mathbf{G}}\|_F \tag{43}$$

and

$$\varepsilon_2 = \|\bar{\mathbf{G}}^{-1} - E[\mathbf{G}^{-1}]\|_F / \|\bar{\mathbf{G}}^{-1}\|_F \tag{44}$$

Because $\mathbf{G} \sim W_n(n + 1 + \theta, \bar{\mathbf{G}}/\alpha)$, we have

$$E[\mathbf{G}] = \frac{n + 1 + \theta}{\alpha}\bar{\mathbf{G}} \tag{45}$$

and

$$E[\mathbf{G}^{-1}] = \frac{\alpha}{\theta}\bar{\mathbf{G}}^{-1} \tag{46}$$

Substituting the expressions of $E[\mathbf{G}]$ in Eq. (43), one obtains

$$\varepsilon_1 = \sqrt{\left(\bar{\mathbf{G}} - \frac{n+1+\theta}{\alpha}\bar{\mathbf{G}}\right)\left(\bar{\mathbf{G}}^T - \frac{n+1+\theta}{\alpha}\bar{\mathbf{G}}^T\right)} \Big/ \sqrt{\overline{\mathbf{G}\mathbf{G}^T}} \tag{47}$$

$$= \sqrt{\left(1 - \frac{n+1+\theta}{\alpha}\right)^2 \overline{\mathbf{G}\mathbf{G}^T}} \Big/ \sqrt{\overline{\mathbf{G}\mathbf{G}^T}} \tag{48}$$

$$= \left(1 - \frac{n+1+\theta}{\alpha}\right) \tag{49}$$

Similarly, substituting the expressions of $E[\mathbf{G}^{-1}]$ in Eq. (44), one can obtain

$$\varepsilon_2 = \left(1 - \frac{\alpha}{\theta}\right) \tag{50}$$

We define the objective function to be minimized as

$$\chi^2 = \varepsilon_1^2 + \varepsilon_2^2 \tag{51}$$

Using Eqs. (49) and (51), the objective function can be expressed as

$$\chi^2 = \left(1 - \frac{n+1+\theta}{\alpha}\right)^2 + \left(1 - \frac{\alpha}{\theta}\right)^2 \tag{52}$$

The optimal value of α is obtained by setting

$$\frac{\partial \chi^2}{\partial \alpha} = 0 \tag{53}$$

or

$$\alpha^4 - \theta\alpha^3 + \theta^2(n+1+\theta)\alpha - \theta^2(n+1+\theta)^2 = 0 \tag{54}$$

This fourth-order algebraic equation in α has the following four exact solutions:

$$\alpha = \pm\sqrt{\theta(n+1+\theta)} \quad \text{and} \quad \alpha = \theta/2 \pm i\sqrt{\theta(n+1+3\theta/4)} \tag{55}$$

Because α must be real and positive, the only feasible value of $\alpha \in \mathbb{R}^+$ is

$$\alpha = \sqrt{\theta(n+1+\theta)} = \sqrt{2\nu(n+1+2\nu)} \tag{56}$$

The choice of α in Eq. (56) not only minimizes the overall difference, but also the difference between $E[\mathbf{G}]$ and $\bar{\mathbf{G}}$ and $E[\mathbf{G}^{-1}]$ and $\bar{\mathbf{G}}^{-1}$ become the same as

$$\varepsilon_1^2 = \varepsilon_2^2 = \left(1 - \sqrt{\frac{n+1+2\nu}{2\nu}}\right)^2 \tag{57}$$

This result implies that if $\mathbf{G} \sim W_n[n+1+2\nu, \bar{\mathbf{G}}/\sqrt{2\nu(n+1+2\nu)}]$, then the resulting distribution becomes independent of whether $\mathbf{G}$ or $\mathbf{G}^{-1}$ is used for the derivation of the distribution. As a result, we have the following:

Theorem 5. If νth order inverse moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\bar{\mathbf{G}}$, then the unbiased distribution of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\boldsymbol{\Sigma} = \bar{\mathbf{G}}/\sqrt{2\nu(2\nu+n+1)}$, that is $\mathbf{G} \sim W_n[2\nu + n + 1, \bar{\mathbf{G}}/\sqrt{2\nu(2\nu+n+1)}]$.

To give a numerical illustration, again consider $n = 100$ and $\nu = 2$, so that $\theta = 2\nu = 4$. For the distribution in the previous section $\alpha = 2\nu + n + 1 = 105$. For the optimal distribution, $\alpha = \sqrt{\theta(\theta+n+1)} = 2\sqrt{105} = 20.49$. Using these values, we have

$$E[\mathbf{G}] = \frac{105}{2\sqrt{105}}\bar{\mathbf{G}} = 5.12\bar{\mathbf{G}}$$

and

$$E[\mathbf{G}^{-1}] = \frac{2\sqrt{105}}{4}\bar{\mathbf{G}}^{-1} = 5.12\bar{\mathbf{G}}^{-1}$$

The overall normalized difference for the previous case is $\chi^2 = 0 + (1 - 105/4)^2 = 637.56$. The same for the optimal distribution is $\chi^2 = 2(1 - \sqrt{105}/2)^2 = 34.01$, which is considerably smaller compared with the nonoptimal distribution.

V. Summary of the Proposed Method

The discussion so far leads to a simple simulation algorithm for probabilistic structural dynamics. The method can be implemented by following these steps:

1) Form the deterministic matrices $\bar{\mathbf{G}} \equiv \{\bar{\mathbf{M}}, \bar{\mathbf{C}}, \bar{\mathbf{K}}\}$ using the standard finite element method. Obtain n, the dimension of the system matrices.

2) Obtain the normalized standard deviations or the "dispersion parameters" $\delta_G \equiv \{\delta_M, \delta_C, \delta_K\}$ corresponding to the system matrices. This can be obtained from experiment, experience, or using the stochastic finite element method. This is the *only* information regarding the system uncertainty used in this approach.

3) Calculate

$$\theta_G = \frac{1}{\delta_G^2}\left\{1 + \frac{\{\text{Trace}(\bar{\mathbf{G}})\}^2}{\text{Trace}(\bar{\mathbf{G}}^2)}\right\} - (n+1) \tag{58}$$

If the value of δ_G is high, then θ_G will be small. The minimum value of θ_G is four for $\mathbf{G}^{-1}$ to be a second-order random matrix.

4) Calculate

$$\alpha_G = \sqrt{\theta_G(n+1+\theta_G)} \tag{59}$$

5) Obtain the samples of Wishart random matrices $W_n(n+1+\theta_G, \bar{\mathbf{G}}/\alpha_G)$. The simulation can be based on the simulation of the Gaussian random matrices as described in Theorem 3. To use this approach, θ_G must be an integer. Because θ_G obtained from Eq. (58) is in general fractional, we need to approximate it to its nearest integer value. For large n approximating θ_G to its nearest integer produces negligible error. Alternatively, MATLAB command wishrnd can be used to generate the samples of Wishart matrices. MATLAB can handle fractional values of $(n+1+\theta_G)$ so that the approximation to its nearest integer may be avoided.

The preceding procedure can be implemented very easily. A sample code in MATLAB is given in Appendix C to illustrate a typical implementation. Once the samples of the system matrices are generated, the rest of the analysis is identical to any Monte Carlo simulation-based approach. If one implements this approach in conjunction with a commercial finite element software, unlike the stochastic finite element method, the commercial software needs to be accessed only once to obtain the mean matrices. This simulation procedure is therefore "nonintrusive." In the next section, the proposed approach is illustrated thorough two examples.

VI. Numerical Example: Dynamic Response of a Clamped Plate with Random Properties

A cantilever steel plate with a slot is considered in this section. The diagram of the plate, together with the deterministic numerical values assumed for the system parameters, are shown in Fig. 1. The plate is excited by an unit harmonic force and the response is calculated at the points shown in the diagram.

The standard four-noded thin plate bending element (resulting in 12 degrees of freedom per element) is used. This simple element may not yield very accurate results in the high-frequency vibration

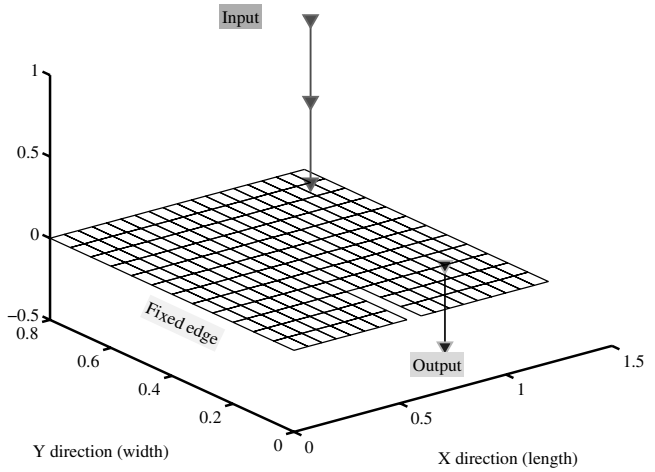


Fig. 1 Steel cantilever plate with slot; $\bar{E} = 200 \times 10^9$ N/m^2, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860$ kg/m^3, $\bar{t} = 7.5$ mm, $L_x = 1.2$ m, $L_y = 0.8$ m.

because the shear deformation and other three-dimensional effect become important in the high frequency. However, the primary focus of this paper is in the quantification and propagation of uncertainty. It is expected that the probabilistic aspect of our results will not change significantly with the use of superior plate elements available in literature. The plate is divided into 18 elements in the x axis and 12 elements in the y axis for the numerical calculations. The resulting system has 702 degrees of freedom so that $n = 702$. A constant modal damping factor of 2% has been assumed for all the modes.

Now we consider uncertainties in the plate structure. It is assumed that the Young's modulus, Poissons ratio, mass density, and thickness are random fields of the form

$$E(\boldsymbol{x}) = \bar{E}[1 + \epsilon_E f_1(\boldsymbol{x})] \tag{60}$$

$$\mu(\boldsymbol{x}) = \bar{\mu}[1 + \epsilon_\mu f_2(\boldsymbol{x})] \tag{61}$$

$$\rho(\boldsymbol{x}) = \bar{\rho}[1 + \epsilon_\rho f_3(\boldsymbol{x})] \tag{62}$$

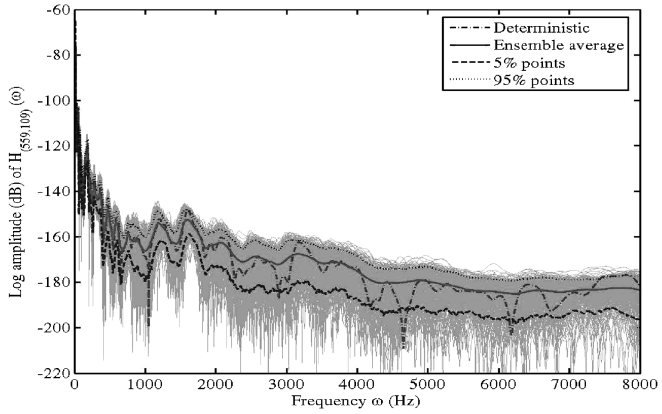
and

$$t(\boldsymbol{x}) = \bar{t}[1 + \epsilon_t f_4(\boldsymbol{x})] \tag{63}$$

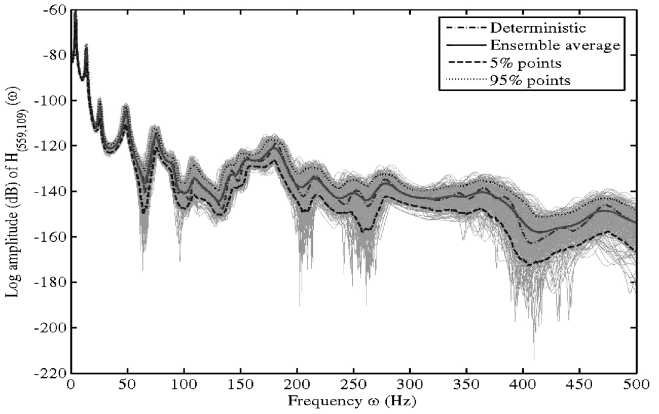
Here, the two-dimensional vector $\boldsymbol{x}$ denotes the spatial coordinates. The strength parameters are assumed to be $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$, and $\epsilon_t = 0.15$. The random fields $f_i(\boldsymbol{x}), i = 1, \ldots, 4$ are assumed to be delta-correlated homogenous Gaussian random fields. A 500-sample Monte Carlo simulation is performed to obtain the FRFs.

Figure 2 shows the direct finite element (FE) Monte Carlo simulation result for the cross-FRF. The realizations of the amplitude of the FRF for each sample are shown together with the ensemble mean, 5 and 95% probability points.

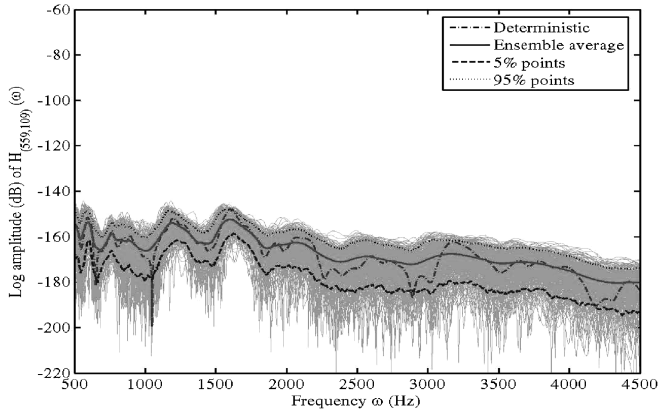
In Figs. 2b–2d we have separately shown the low-, medium-, and high-frequency response, obtained by zooming around the appropriate frequency ranges in Fig. 2a. There are of course no fixed and definite boundaries between the low-, medium-, and high-frequency ranges. Here, we have selected 0–0.5 kHz as the low-frequency vibration, 0.5–4.5 kHz as the medium-frequency vibration, and 4.5–8.0 kHz as the high-frequency vibration. These frequency boundaries are selected on the basis of the qualitative nature of the response (which are fairly obvious, as will be seen later) and devised purely for the purpose of the presentation of our results. The methods discussed here are independent on these selections. The ensemble mean follows the deterministic result closely in the low- and medium-frequency range. Figure 3 shows the direct finite element Monte Carlo simulation result for the driving-point FRF. The realizations of the amplitude of the FRF for each sample are shown together with the ensemble mean, 5 and 95% probability



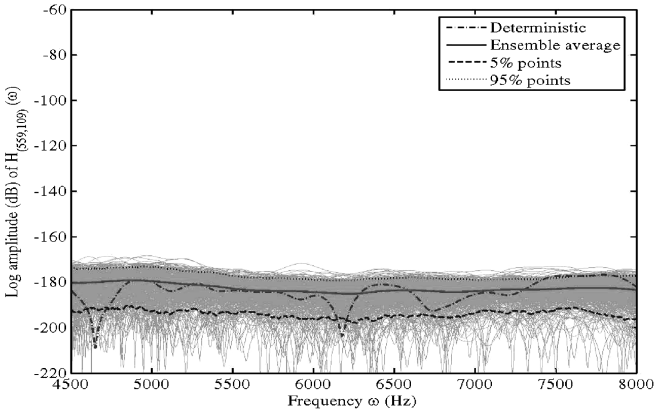
a) Response across the frequency range



b) Low-frequency response

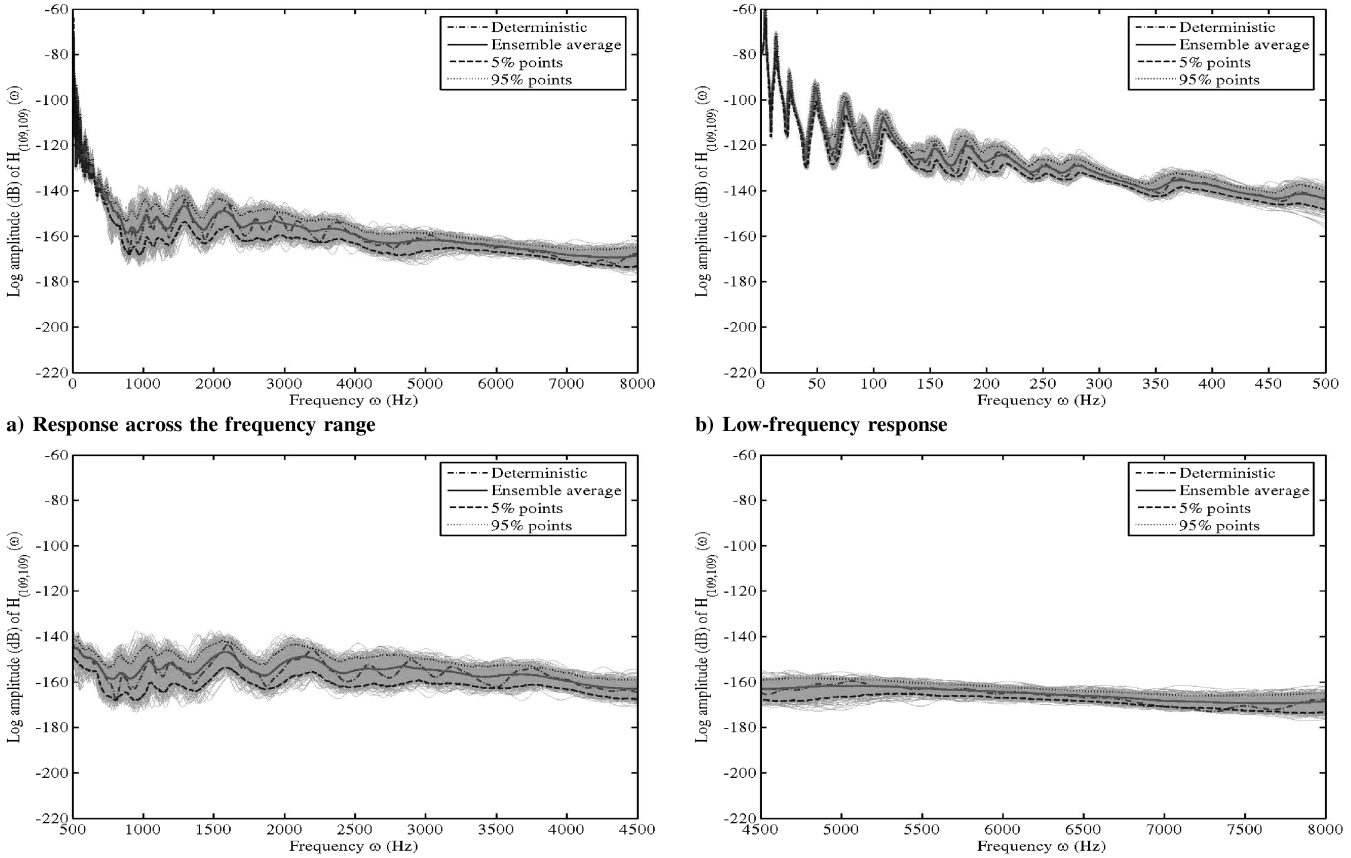


c) Medium-frequency response



d) High-frequency

Fig. 2 Direct stochastic finite element Monte Carlo simulation of amplitude of cross-FRF of plate with randomly distributed material properties.



a) Response across the frequency range

b) Low-frequency response

c) Medium-frequency response

d) High-frequency

Fig. 3 Direct stochastic finite element Monte Carlo simulation of amplitude of driving-point FRF of plate with randomly distributed material properties.

points. In Figs. 3b–3d, we have separately shown the low-, medium-, and high-frequency response, obtained by zooming around the appropriate frequency ranges in Fig. 3a. The amount of variation in this case is smaller compared with the cross-FRF. This is due to the fact that the effect of distributed random material properties is more when we measure the response at a point further from the source. In this case, the ensemble mean follows the deterministic result closely across the frequency range.

Now, we want to see if the results obtained from the direct stochastic finite element Monte Carlo simulation can be reproduced using the random matrix theory. The details of the parametric variations of the random fields will *not* be used in the random matrix approach. From the simulated random mass and stiffness matrices we obtain $n = 702$, $\delta_M = 0.1166$, and $\delta_K = 0.2622$. Here, a 2% constant modal damping factor is assumed for all the modes so that $\delta_C = 0$. The only uncertainty-related information used in the random matrix approach are the values of δ_M and δ_K. That is, the information regarding which element property functions are random fields, nature of these random fields (correlation structure, Gaussian, or non-Gaussian), and the amount of randomness are not used in the random matrix approach. This example is aimed to depict a realistic situation when the detailed information regarding the uncertainty of a complex engineering system may not be available to the analyst.

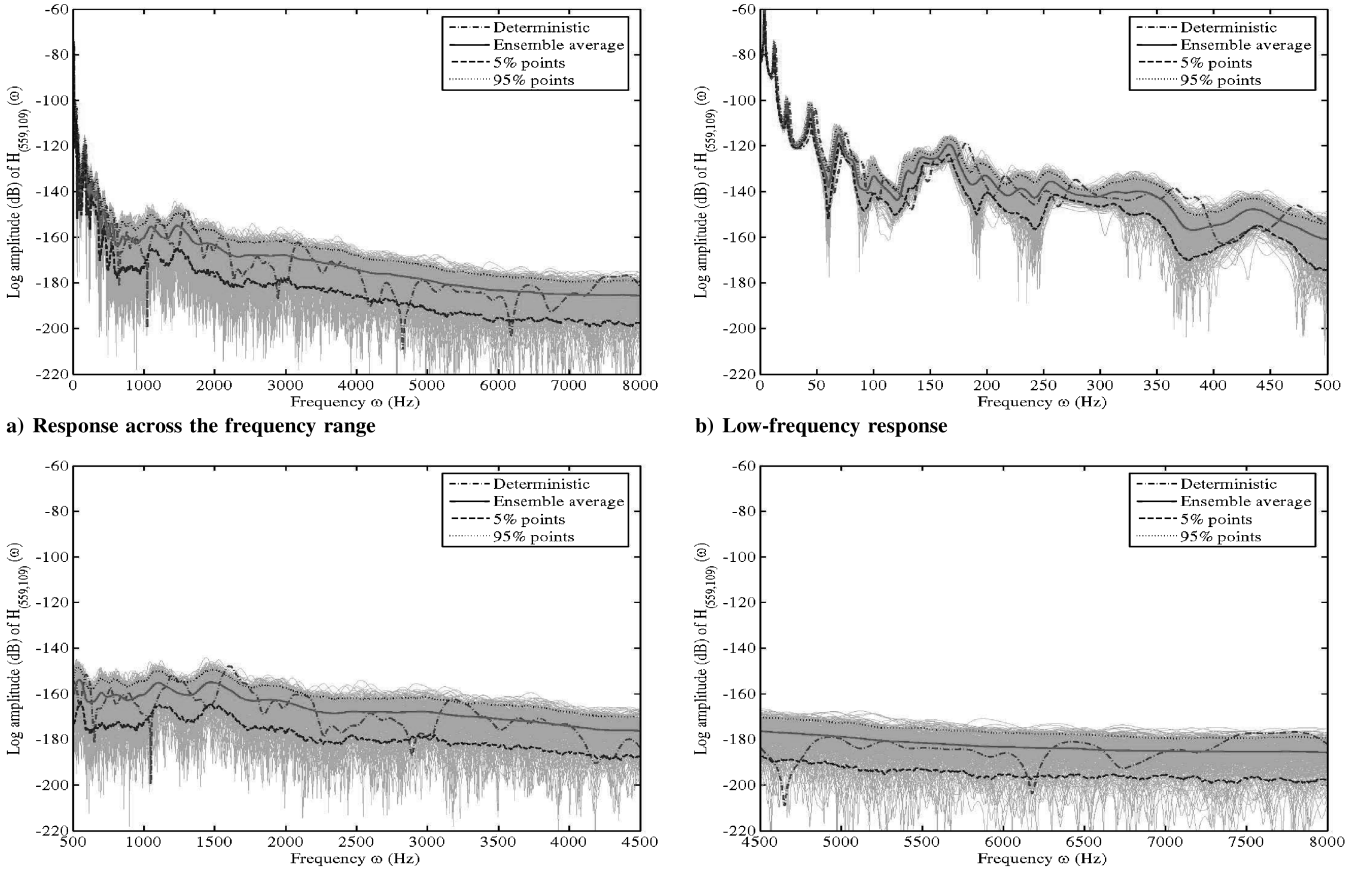
Using $n = 702$, $\delta_M = 0.1166$, and $\delta_K = 0.2622$, together with the deterministic values of $\mathbf{M}$ and $\mathbf{K}$, the samples of the mass matrices are simulated using the optimal Wishart matrices derived in Sec. IV. The simulation procedure can be based on the simulation of the Gaussian random matrices as described in Theorem 3. Here MATLAB command wishrnd is used to generate the samples of Wishart matrices (see Algorithm 1 for the relevant MATLAB code). Figure 4 shows the Monte Carlo simulation result for the cross-FRF using the optimal Wishart mass matrix. The realizations of the amplitude of the FRF of each sample are shown in the figure along with the ensemble mean, 5 and 95% probability points.

In Figs. 4b–4d the low-, medium-, and high-frequency response, obtained by zooming around the appropriate frequency ranges in Fig. 4a, are shown separately. Figure 5 shows the Monte Carlo simulation result for the driving-point FRF using the optimal Wishart mass matrix. The realizations of the amplitude of the FRF of all samples are shown in the figure along with the ensemble mean, 5 and 95% probability points. In Figs. 5b–5d the low-, medium-, and high-frequency response, obtained by zooming around the appropriate frequency ranges in Fig. 5a, are again shown separately. The amount of variation in this case is smaller compared with the cross-FRF, similar to what was predicted by the direct stochastic finite element simulation results in Fig. 3.

The predicted mean and standard deviation values using the direct stochastic finite element simulation and the random matrix theory are compared in Figs. 6 and 7 for the cross-FRF and the driving-point FRF, respectively.

In Figs. 6b–6d and 7b–7d the low-, medium-, and high-frequency response, obtained by zooming around the appropriate frequency ranges in Figs. 6a and 7a, are shown, respectively. Note the difference between these plots and the usual comparison plots in which results from an approximate analytical method is often compared with the results from Monte Carlo simulation. Instead, here both the results are obtained using Monte Carlo simulation. The difference between the two methods are in the amount of information used regarding the uncertainty, not how the uncertainty is propagated using the same information. After the samples of the random system matrices are generated, both methods are identical.

The mean values obtained from the proposed random matrix theory is very close to the results obtained using the stochastic finite element simulation across the frequency range considered. The standard deviations obtained from the proposed random matrix theory match very well to the standard deviations obtained using the stochastic finite element simulation for the medium- and high-frequency response, as can be observed in Figs. 6c, 6d, 7c, and 7d. It



a) Response across the frequency range

b) Low-frequency response

c) Medium-frequency response

d) High-frequency

Fig. 4 Monte Carlo simulation of amplitude of cross-FRF of plate using optimal Wishart mass and stiffness matrices, $n = 702$, $\delta_M = 0.1166$, and $\delta_K = 0.2622$.

is interesting to note that the proposed random matrix approach produces accurate results in the medium-frequency range. This is an important feature of the proposed method. For the standard deviations in the low-frequency regions in Figs. 6a and 7a, the proposed method produces a similar trend but the results do not agree as well as they do in the higher frequency regions. This is expected because it is known that the response variations in the low-frequency regions depend on the details of the nature of the uncertainty. Because this information is not used in the proposed random matrix approach, we do not expect it to work very well in the low-frequency regions.

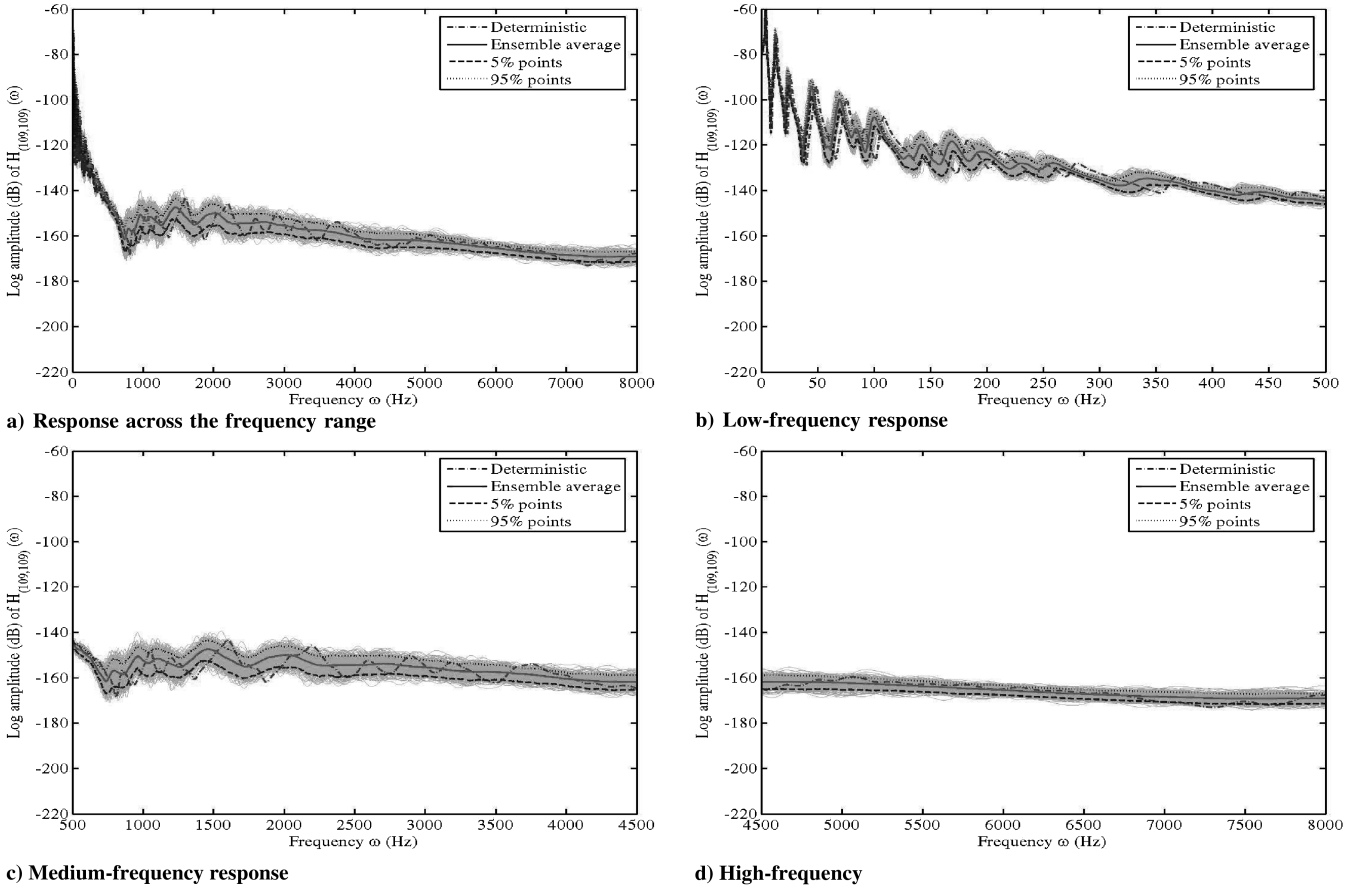
The predicted 5 and 95% probability points using the direct stochastic finite element simulation and proposed random matrix method are compared in Figs. 8 and 9. The essential feature of these plots are similar to the standard deviation plots shown before, that is, the results from both approaches match well in the medium- and high-frequency regions and do not match so well in the low-frequency regions. Considering the fact that the only uncertainty-related information used in the random matrix method are the values of δ_M and δ_K, these results are encouraging.

Although we have obtained excellent agreements between the proposed random matrix method and the direct stochastic finite element simulation, care must be taken to interpret these results. In the medium- and high-frequency ranges, model uncertainties (such as unmodeled dynamics, incorrect damping models, nonlinearities, and joints) play a significant role in the response variability. In our stochastic finite element simulation, we have not simulated such model uncertainties. It is also not quite obvious what type of uncertainties are being modeled by the maximum-entropy approach used in this study. Therefore, we cannot claim with certainty that the Wishart matrix is *the* random matrix model in the medium- and high-frequency ranges. However, one fact that emerges from these results is that the response variability in the medium- and high-frequency ranges is not very sensitive to the details of the nature of the uncertainty in the system. It appears that a measure (in this case, the normalized standard deviations of the system matrices) of the overall uncertainty, whether arising from model uncertainty or data uncertainty, or both, is enough to predict the response variability. This is the main motivation behind the fact that an approach similar to what is proposed here might be useful in the medium- and high-frequency vibration problems.

VII. Summary and Conclusions

A method based on the random matrix theory is proposed for uncertainty quantification in linear dynamic systems across the frequency range of excitation. The central outcome of this paper is that if only the mean value of a system matrix is known, then the matrix follows a Wishart distribution with proper parameters. The optimal parameters of the Wishart matrices are obtained such that the mean of the matrix and its inverse produce minimum deviations from their deterministic values. The derived probability density function of the random system matrices is characterized by the dimension of the matrices, their mean values, and a scalar parameter defining their overall randomness. The random matrix based uncertainty quantification tool proposed here does not require explicit information regarding the detailed descriptions of the uncertainties in the system.

The discovery of the Wishart distribution in the context of structural dynamics is important. This is not only because the Wishart matrices are fundamental to the random matrix theory, but also due to the fact that the analysis becomes simpler with this distribution. The moments and elements of the covariance tensor of the matrix elements are given exactly in closed form. The exact expression of the probability density function of the inverse of a random system matrix is obtained in closed form. The application of



a) Response across the frequency range

b) Low-frequency response

c) Medium-frequency response

d) High-frequency

Fig. 5 Monte Carlo simulation of amplitude of driving-point FRF of plate using optimal Wishart mass and stiffness matrices, $n = 702$, $\delta_M = 0.1166$ and $\delta_K = 0.2622$.

the derived matrix variate distribution is illustrated by a random plate problem with 702 degrees of freedom. It was shown that it is possible to predict the variation of the dynamic response using the optimal Wishart matrices across a wide range of driving frequency. These results suggest that the Wishart matrices may be used as a consistent and unified uncertainty quantification tool valid for medium- and high-frequency vibration problems.

Appendix A: Multivariate Gamma Function and Matrix Variate Laplace Transforms

The multivariate gamma function $\Gamma_n(a)$ is defined as

$$\Gamma_n(a) = \int_{\mathbf{X}>0} \text{etr}\{-\mathbf{X}\}|\mathbf{X}|^{a-\frac{1}{2}(n+1)}\,\text{d}\mathbf{X} \tag{A1}$$

Here $\Re(a) > 1/2(n-1)$ and the integral in Eq. (A1) is over the space of $n \times n$ symmetric positive-definite matrices. Therefore, Eq. (A1) represents an $n(n+1)/2$ dimensional integral. Fortunately, this integral can be evaluated exactly in closed form [24], which forms the basis of the analytical results given in the paper. Because $\mathbf{X}$ is a symmetric positive-definite matrix, we can factor it as

$$\mathbf{X} = \mathbf{T}\mathbf{T}^{\text{T}} \tag{A2}$$

where $\mathbf{T}$ is a lower triangular matrix with $t_{ii} > 0, \forall\ i$. The Jacobian of the matrix transformation in Eq. (A2) can be obtained from Theorem 1.28 in Mathai [28] as

$$\text{d}\,\mathbf{X} = 2^n \prod_{i=1}^{n} t_{ii}^{n-i+1}\,\text{d}\mathbf{T} \tag{A3}$$

Because of the factorization in Eq. (A2), we also have

$$\text{Trace}\,(\mathbf{X}) = \text{Trace}(\mathbf{T}\mathbf{T}^T) = \sum_{j\le i}^{n} t_{ij}^2 \tag{A4}$$

$$|\mathbf{X}| = |\mathbf{T}\mathbf{T}^T| = |\mathbf{T}^2| = \prod_{i=1}^{n} t_{ii}^2 \tag{A5}$$

Substituting Eqs. (A3–A5) in the integral (A1) one has

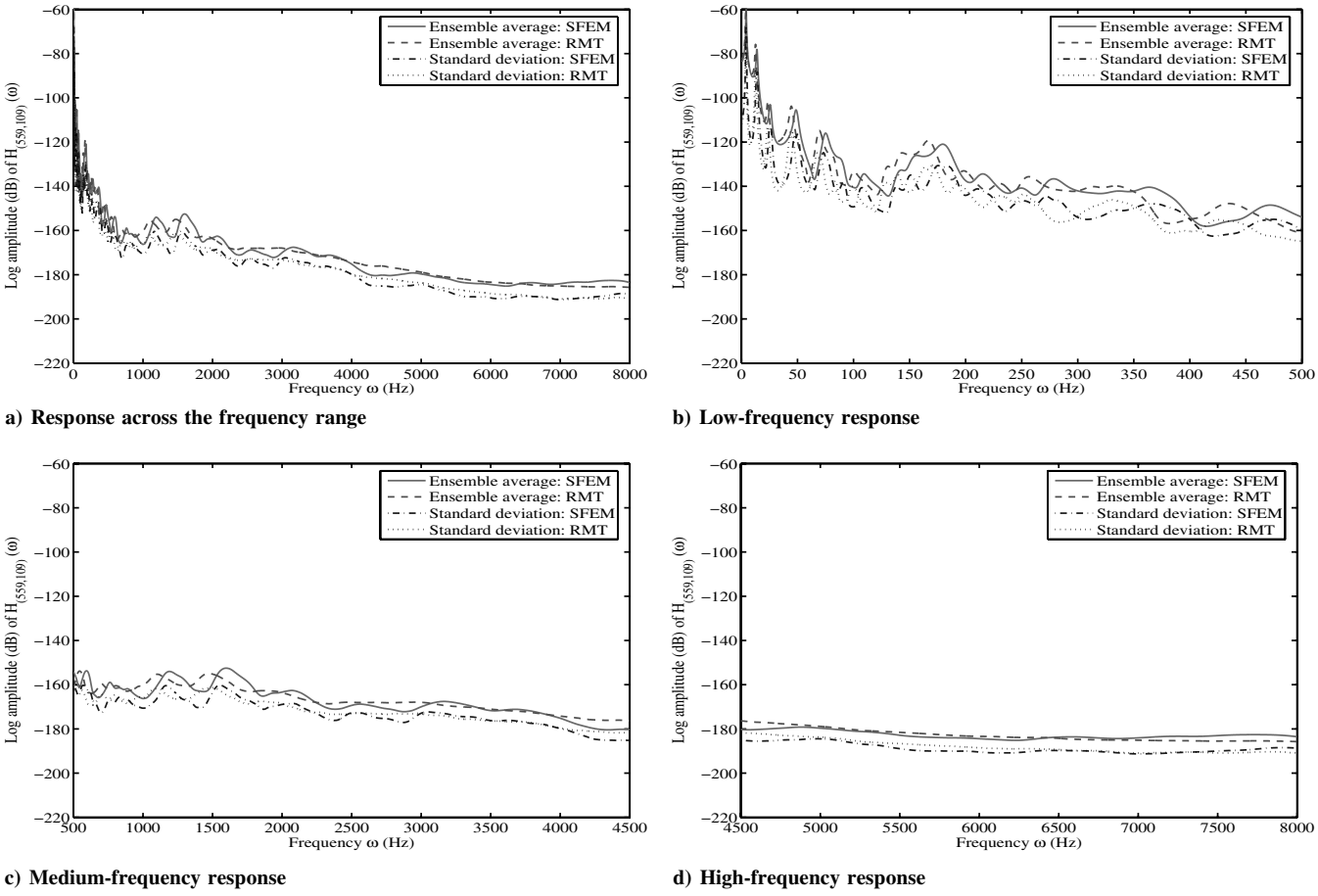
$$\Gamma_n(a) = 2^n \int\!\!\int_{\substack{-\infty<t_{ij}<\infty \\ t_{ii}>0}} \cdots \int_{n(n+1)\,\text{terms}} \exp\left(-\sum_{j\le i}^{n} t_{ij}^2\right) \prod_{i=1}^{n} \left(t_{ii}^2\right)^{a-\frac{1}{2}(n+1)} \times \prod_{i=1}^{n} t_{ii}^{n-i+1}\,\text{d}t_{ij} = 2^n \int\!\!\int_{\substack{-\infty<t_{ij}<\infty \\ t_{ii}>0}} \cdots \int_{n(n+1)\,\text{terms}} \prod_{j\le i}^{n} \exp\left(-t_{ij}^2\right) \times \prod_{i=1}^{n} \left(t_{ii}^2\right)^{a-\frac{1}{2}(i)}\,\text{d}t_{ij} \tag{A6}$$

Separating the integrals involving the diagonal and off-diagonal terms and breaking 2^n into products of n twos, Eq. (A6) can be rewritten as

$$\Gamma_n(a) = \left[\prod_{j\le i}^{n} \int_{-\infty<t_{ij}<\infty} \cdots \int_{n(n-1)\ \text{terms}} \exp\left(-t_{ij}^2\right) dt_{ij}\right] \times \left[\prod_{i=1}^{n} 2 \int_{t_{ii}>0} \cdots \int_{n\ \text{terms}} \exp\left(-t_{ii}^2\right)\left(t_{ii}^2\right)^{a-\frac{1}{2}(i)} dt_{ii}\right] \tag{A7}$$

Equation (A7) is now products of simple one-dimensional integrals which can be evaluated easily [34] to obtain

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{i=1}^{n} \Gamma\left[a - \frac{1}{2}(i-1)\right] \tag{A8}$$



a) Response across the frequency range

b) Low-frequency response

c) Medium-frequency response

d) High-frequency response

Fig. 6 Comparison of mean and standard deviation of amplitude of cross-FRF obtained using direct stochastic finite element simulation and proposed random matrix method.

The second term directly follows from the definition of the univariate gamma function [35]. Next, we introduce the concept of the matrix variate Laplace transform [24].

Definition 5. Matrix variate Laplace transform: Let $f(\mathbf{X})$ be a function of an $n \times n$ symmetric positive-definite matrix $\mathbf{X}$ and $\mathbf{Z} = \mathbf{Z}_r + i\mathbf{Z}_i$ be an $n \times n$ symmetric complex matrix. Then the matrix variate Laplace transform $\mathcal{F}(\mathbf{Z})$ of $f(\mathbf{X})$ is defined as

$$\mathcal{F}(\mathbf{Z}) = \mathbb{L}\{f(\mathbf{X})\} = \int_{\mathbf{X}>0} \text{etr}\{-\mathbf{Z}\mathbf{X}\} f(\mathbf{X})\, \text{d}\mathbf{X} \tag{A9}$$

where the integral is assumed to absolutely convergent in the right half plane $\Re(\mathbf{Z}) = \mathbf{Z}_r > 0$.

From the normalization condition in Eqs. (10) and (29), it may be observed that integrals of the form

$$\int_{\mathbf{G}>0} |\mathbf{G}|^{\nu} \text{etr}\{-\mathbf{\Lambda}_1 \mathbf{G}\}\, \text{d}\mathbf{G}$$

needs to be evaluated. This can be obtained by considering the Laplace transform of $f(\mathbf{X}) = |\mathbf{X}|^m$, for some m. In particular, we consider the Laplace transform

$$\mathbb{L}\{|\mathbf{X}|^{a-(n+1)/2}\} = \int_{\mathbf{X}>0} \text{etr}\{-\mathbf{Z}\mathbf{X}\} |\mathbf{X}|^{a-(n+1)/2}\, \text{d}\mathbf{X} \tag{A10}$$

for $\Re(a) > \frac{1}{2}(n-1)$. Suppose $\mathbf{X} = \mathbf{Z}^{-\frac{1}{2}}\mathbf{Y}\mathbf{Z}^{-\frac{1}{2}}$, so that $\text{d}\mathbf{X} = |\mathbf{Z}|^{-\frac{1}{2}(n+1)}\, \text{d}\mathbf{Y}$ (see Chapter 1 in Mathai [28]). Substituting $\mathbf{X}$ into Eq. (A10) we have

$$\begin{aligned}\mathbb{L}\{|\mathbf{X}|^{a-(n+1)/2}\} &= \int_{\mathbf{Y}>0} \text{etr}\{-\mathbf{Y}\} |\mathbf{Z}^{-\frac{1}{2}}\mathbf{Y}\mathbf{Z}^{-\frac{1}{2}}|^{a-(n+1)/2} |\mathbf{Z}|^{-\frac{1}{2}(n+1)}\, \text{d}\mathbf{Y} \\ &= |\mathbf{Z}|^{-a} \int_{\mathbf{Y}>0} \text{etr}\{-\mathbf{Y}\} |\mathbf{Y}|^{a-(n+1)/2}\, \text{d}\mathbf{Y} \end{aligned} \tag{A11}$$

Using the definition of the multivariate gamma function in Eq. (A1), we have

$$\mathbb{L}\{|\mathbf{X}|^{a-(n+1)/2}\} = \int_{\mathbf{X}>0} \text{etr}\{-\mathbf{Z}\mathbf{X}\} |\mathbf{X}|^{a-(n+1)/2}\, \text{d}\mathbf{X} = |\mathbf{Z}|^{-a} \Gamma_n(a) \tag{A12}$$

This expression is simply the matrix generalization of the well-known scalar ($n = 1$) case [36] $\mathbb{L}\{t^{m-1}\} = \Gamma(m)/s^m$. Equation (A12) turns out to be very useful as can be seen in the next two appendices.

Appendix B: Proof of Theorem 1 and Theorem 2

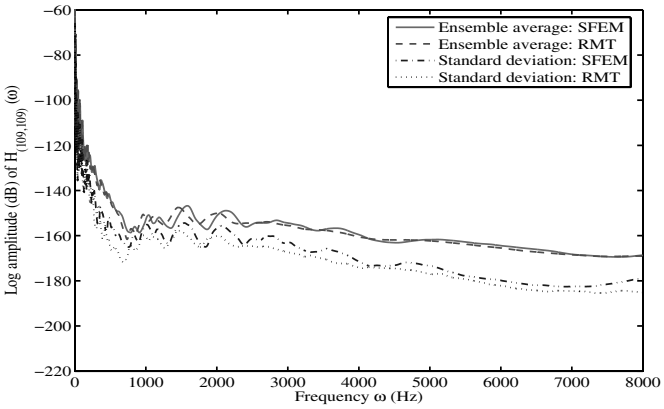
Theorem 1 is a special case of Theorem 2 when $\nu = 0$. Therefore, we will only consider Theorem 2 here. The main task is to obtain the expressions for the Lagrange multipliers $\lambda_0 \in \mathbb{R}$ and $\mathbf{\Lambda}_1 \in \mathbb{R}^n$ appearing in Eq. (29). Substituting $p_{\mathbf{G}}(\mathbf{G})$ from Eq. (29) into the normalization condition in Eq. (10), we have

$$\int_{\mathbf{G}>0} \exp\{-\lambda_0\} \text{etr}\{-\mathbf{\Lambda}_1 \mathbf{G}\} |\mathbf{G}|^{\nu}\, \text{d}\mathbf{G} = 1 \tag{B1}$$

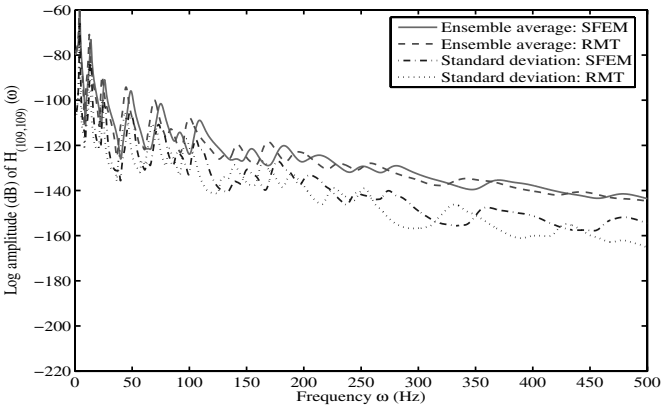
or

$$\exp\{\lambda_0\} = \int_{G>0} \text{etr}\{-\mathbf{\Lambda}_1 \mathbf{G}\} |\mathbf{G}|^{\nu}\, \text{d}\mathbf{G} \tag{B2}$$

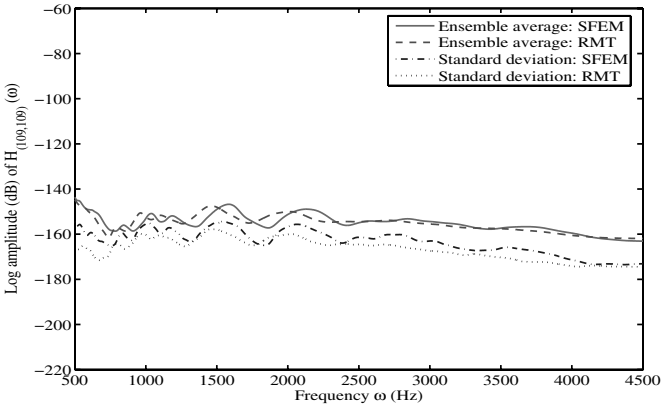
The last integral can be evaluated exactly in closed form using the



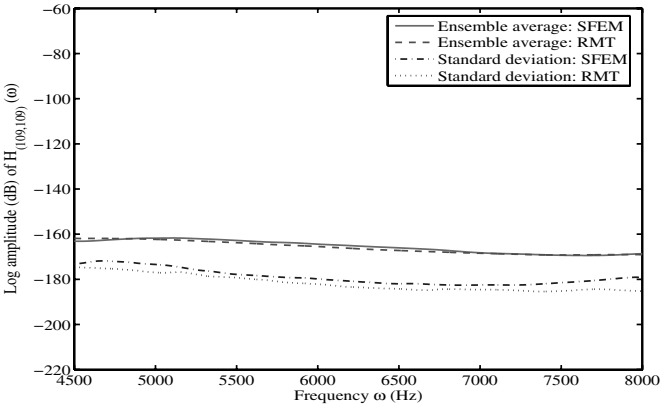
a) Response across the frequency range



b) Low-frequency response



c) Medium-frequency response



d) High-frequency response

Fig. 7 Comparison of mean and standard deviation of amplitude of driving-point FRF obtained using the direct stochastic finite element simulation and proposed random matrix method.

Laplace transform Eq. (A12) by substituting $a = \nu + \frac{1}{2}(n+1)$ as

$$\exp\{\lambda_0\} = |\mathbf{\Lambda}_1|^{-[\nu+(n+1)/2]}\Gamma_n\left(\nu + \frac{1}{2}(n+1)\right) = |\mathbf{\Lambda}_1|^{-r}\Gamma_n(r) \tag{B3}$$

where $r = [\nu + (n+1)/2]$ as defined in Eq. (30). Substituting $\exp\{\lambda_0\}$ from Eq. (B3) in the expressions of the pdf in Eq. (29), we have

$$p_{\mathbf{G}}(\mathbf{G}) = \{\Gamma_n(r)\}^{-1}|\mathbf{\Lambda}_1|^r|\mathbf{G}|^{r-\frac{1}{2}(n+1)}\text{etr}\{-\mathbf{\Lambda}_1\mathbf{G}\} \tag{B4}$$

Now we need to obtain the matrix $\mathbf{\Lambda}_1$ from the second constraint Eq. (11). To avoid the direct evaluation of this integral, we will obtain the mean corresponding to the distribution in Eq. (B4) using the characteristic function. The matrix variate characteristic function of $\mathbf{G}$ can be defined as

$$\phi_{\mathbf{G}}(\mathbf{\Omega}) = E[\text{etr}\{i\mathbf{\Omega}\mathbf{G}\}] = \int_{\mathbf{G}>0}\text{etr}\{i\mathbf{\Omega}\mathbf{G}\}p_{\mathbf{G}}(\mathbf{G})\,\mathrm{d}\mathbf{G} \tag{B5}$$

where $\mathbf{\Omega}$ is a symmetric matrix. Substituting the expression of the pdf from Eq. (B4) into the preceding equation, we have

$$\phi_{\mathbf{G}}(\mathbf{\Omega}) = \{\Gamma_n(r)\}^{-1}|\mathbf{\Lambda}_1|^r\int_{\mathbf{G}>0}\text{etr}\{-(\mathbf{\Lambda}_1 - i\mathbf{\Omega})\mathbf{G}\}|\mathbf{G}|^{r-\frac{1}{2}(n+1)}\,\mathrm{d}\mathbf{G} \tag{B6}$$

Again, this integral can be evaluated exactly in closed form using the Laplace transform Eq. (A12) by substituting $\mathbf{Z} = \mathbf{\Lambda}_1 - i\mathbf{\Omega}$ and $a = r$ as

$$\phi_{\mathbf{G}}(\mathbf{\Omega}) = |\mathbf{\Lambda}_1|^r|\mathbf{\Lambda}_1 - i\mathbf{\Omega}|^{-r} = |\mathbf{I} - i\mathbf{\Omega}\mathbf{\Lambda}_1^{-1}|^{-r} \tag{B7}$$

Therefore, the cumulant generating function

$$\ell n\,\phi_{\mathbf{G}}(\mathbf{\Omega}) = -r\,\ell n\,|\mathbf{I} - i\mathbf{\Omega}\mathbf{\Lambda}_1^{-1}| = r\left[i\mathbf{\Omega}\mathbf{\Lambda}_1^{-1} + \left(i\mathbf{\Omega}\mathbf{\Lambda}_1^{-1}\right) + \cdots\right] \tag{B8}$$

The mean of $\mathbf{G}$ can be obtained as

$$E[\mathbf{G}] = \frac{\partial\,\ell n\,\phi_{\mathbf{G}}}{\partial(i\mathbf{\Omega})}\bigg|_{\mathbf{\Omega}=O} = r\mathbf{\Lambda}_1^{-1} \tag{B9}$$

Comparing this with Eq. (11), we have

$$r\mathbf{\Lambda}_1^{-1} = \bar{\mathbf{G}} \quad \text{or} \quad \mathbf{\Lambda}_1 = r\bar{\mathbf{G}}^{-1} \tag{B10}$$

Equations (B3) and (B10) define both the unknown constants in the pdf of $\mathbf{G}$. Substituting $\mathbf{\Lambda}_1$ in Eq. (B4), we have

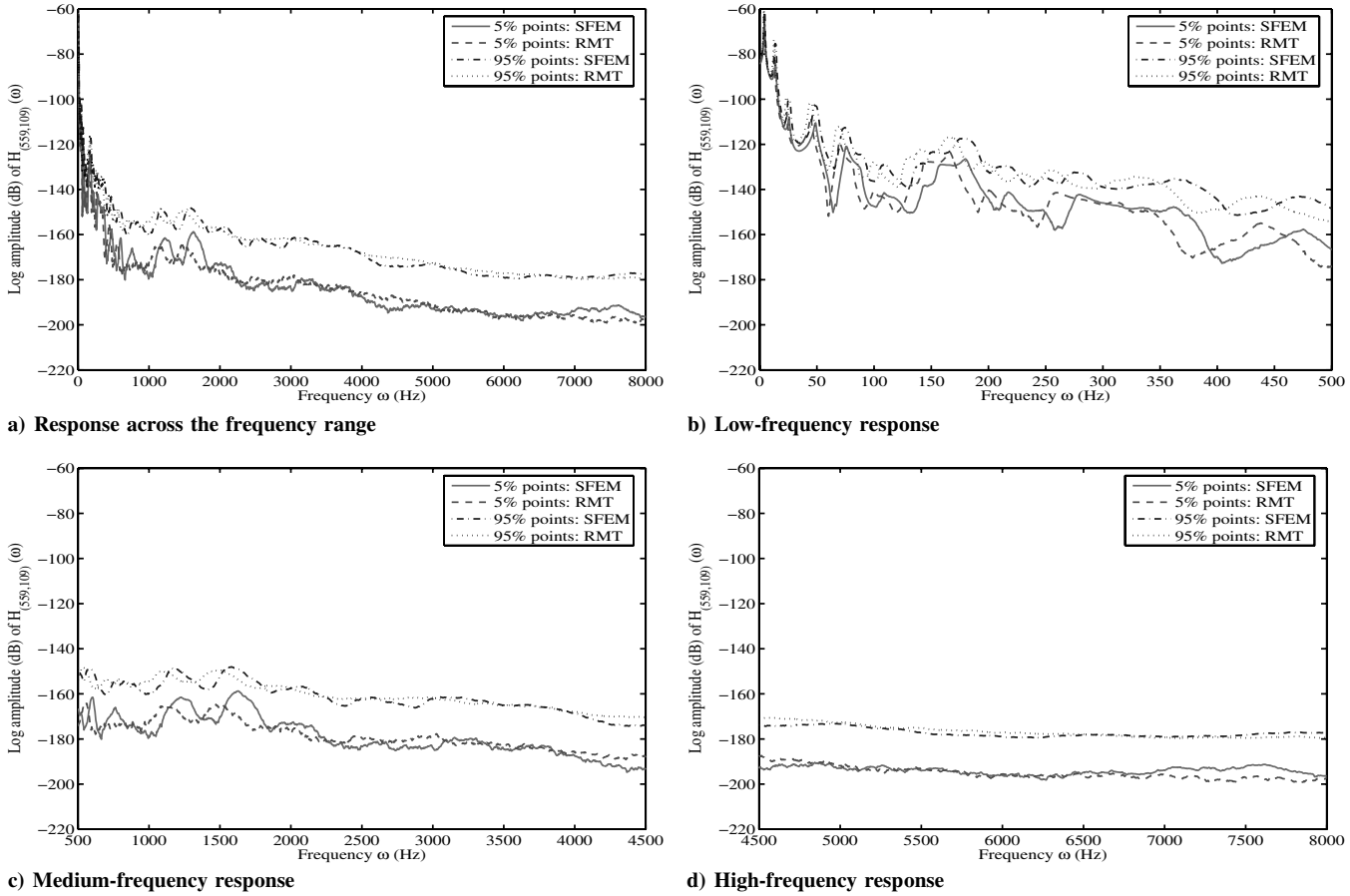
$$\begin{aligned} p_{\mathbf{G}}(\mathbf{G}) &= \{\Gamma_n(r)\}^{-1}|r\bar{\mathbf{G}}^{-1}|^r|\mathbf{G}|^{r-\frac{1}{2}(n+1)}\text{etr}\{-r\bar{\mathbf{G}}^{-1}\mathbf{G}\} \\ &= r^{nr}\{\Gamma_n(r)\}^{-1}|\bar{\mathbf{G}}|^{-r}|\mathbf{G}|^{\nu}\text{etr}\{-r\bar{\mathbf{G}}^{-1}\mathbf{G}\} \end{aligned} \tag{B11}$$

which proves the theorem.

To compare this pdf with the expression of the Wishart distribution in Eq. (3), substitute the expression of $r = (2\nu + n + 1)/2$ in Eq. (B11) to obtain

$$\begin{aligned} p_{\mathbf{G}} = &\left(\frac{2\nu+n+1}{2}\right)^{n(2\nu+n+1/2)}\left\{\Gamma_n\left(\frac{2\nu+n+1}{2}\right)\right\}^{-1} \\ &\times|\bar{\mathbf{G}}|^{-n(2\nu+n+1/2)}|\mathbf{G}|^{\nu}\text{etr}\left\{-\left(\frac{2\nu+n+1}{2}\right)\bar{\mathbf{G}}^{-1}\mathbf{G}\right\} \end{aligned} \tag{B12}$$

This expression can be rearranged as



a) Response across the frequency range **b) Low-frequency response**

c) Medium-frequency response **d) High-frequency response**

Fig. 8 Comparison of 5 and 95% probability points of amplitude of cross-FRF obtained using direct stochastic finite element simulation and proposed random matrix method.

$$p_{\mathbf{G}} = (2)^{[-n(2\nu+n+1)]/2}\left\{\Gamma_n\left(\frac{2\nu+n+1}{2}\right)\right\}^{-1} \times \left|\left(\frac{\bar{\mathbf{G}}}{2\nu+n+1}\right)\right|^{-[(2\nu+n+1)/2]} |\mathbf{G}|^{(1/2)\{(2\nu+n+1)-(n+1)\}} \text{etr}\left\{-\frac{1}{2} \times \left(\frac{\bar{\mathbf{G}}}{2\nu+n+1}\right)^{-1}\mathbf{G}\right\} \tag{B13}$$

Comparing Eq. (B13) with the Wishart distribution in Eq. (3) it can be observed that $\mathbf{G}$ has the Wishart distribution with parameters $p = 2\nu + n + 1$ and $\mathbf{\Sigma} = \bar{\mathbf{G}}/(2\nu + n + 1)$.

Appendix C: Proof of Theorem 3

Our proof is based on the characteristic function of $\mathbf{G}$. Considering $\mathbf{G} \sim W_n(p, \mathbf{\Sigma})$, the probability density function is given by

$$p_{\mathbf{G}}(\mathbf{G}) = \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}|\mathbf{G}|^{\frac{1}{2}(p-n-1)}\text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{G}\right\} \tag{C1}$$

Using this, the matrix variate characteristic function can be obtained as

$$\phi_{\mathbf{G}}(\mathbf{\Omega}) = E[\text{etr}\{i\mathbf{\Omega}\mathbf{G}\}] = \int_{\mathbf{G}>0}\text{etr}\{i\mathbf{\Omega}\mathbf{G}\}p_{\mathbf{G}}(\mathbf{G})\,\text{d}\mathbf{G} \tag{C2}$$

$$= \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}\int_{\mathbf{G}>0}\text{etr}\left\{i\mathbf{\Omega}\mathbf{G} - \frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{G}\right\} |\mathbf{G}|^{\frac{1}{2}(p-n-1)}\,\text{d}\mathbf{G} \tag{C3}$$

$$= \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}\int_{\mathbf{G}>0}\text{etr}\left\{-\frac{1}{2}(\mathbf{I}_n - 2i\mathbf{\Omega}\mathbf{\Sigma})\mathbf{\Sigma}^{-1}\mathbf{G}\right\} |\mathbf{G}|^{\{p/2-(n+1)/2\}}\,\text{d}\mathbf{G} \tag{C4}$$

The $n(n+1)/2$ dimensional integral appearing in the second part of the preceding equation can be evaluated exactly using the Laplace transform in Eq. (A12) by considering $a = p/2$ and $\mathbf{Z} = -\frac{1}{2}(\mathbf{I}_n - 2i\mathbf{\Omega}\mathbf{\Sigma})\mathbf{\Sigma}^{-1}$ as

$$\int_{\mathbf{G}>0}\text{etr}\left\{-\frac{1}{2}(\mathbf{I}_n - 2i\mathbf{\Omega}\mathbf{\Sigma})\mathbf{\Sigma}^{-1}\mathbf{G}\right\}|\mathbf{G}|^{\{p/2-(n+1)/2\}}\,\text{d}\mathbf{G} = \left|\frac{1}{2}(\mathbf{I}_n - 2i\mathbf{\Omega}\mathbf{\Sigma})\mathbf{\Sigma}^{-1}\right|^{-p/2}\Gamma_n\left(\frac{1}{2}p\right) = 2^{\frac{1}{2}np}|(\mathbf{I}_n - 2i\mathbf{\Omega}\mathbf{\Sigma})|^{-p/2}|\mathbf{\Sigma}|^{\frac{1}{2}p}\Gamma_n\left(\frac{1}{2}p\right) \tag{C5}$$

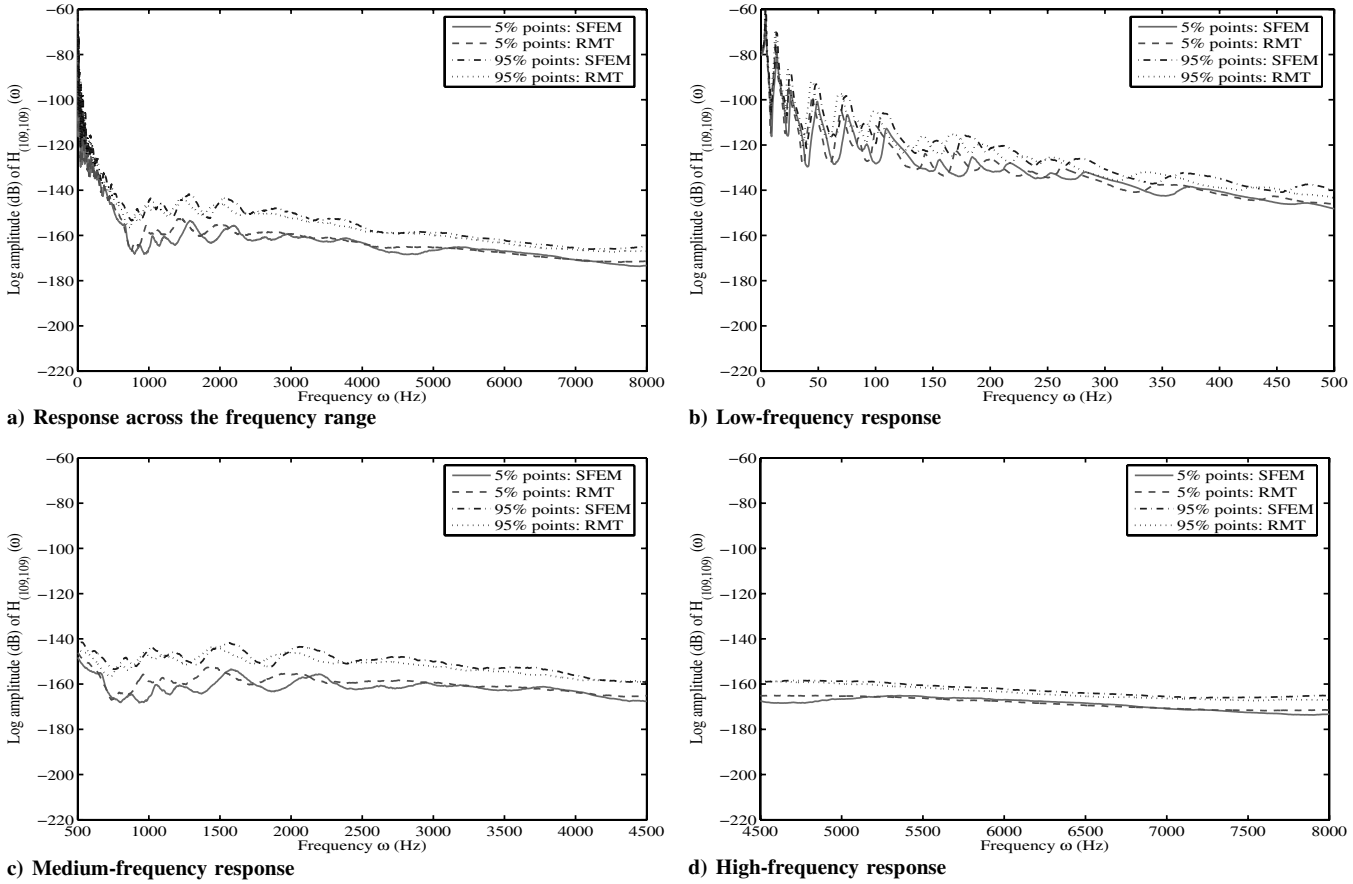
Using this and simplifying Eq. (C4), we have

$$\phi_{\mathbf{G}}(\mathbf{\Omega}) = |\mathbf{I}_n - 2i\mathbf{\Omega}\mathbf{\Sigma}|^{-p/2} \tag{C6}$$

Now, we derive the characteristic function considering $\mathbf{G} = \mathbf{X}\mathbf{X}^T$. If the resulting expression is the same as in Eq. (C6) then the theorem is proved. Because $\mathbf{X} \sim N_{n,p}(\mathbf{O}_{n,p}, \mathbf{\Sigma} \otimes \mathbf{I}_p)$, a rectangular Gaussian random matrix with zero mean and $[\mathbf{\Sigma} \otimes \mathbf{I}_p]$ covariance, the probability density function of $\mathbf{X}$ can be obtained from Eq. (2) as

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2}|\mathbf{\Sigma}|^{-p/2}\text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{X}^{\text{T}}\right\} \tag{C7}$$

The matrix variate characteristic function of $\mathbf{G}$ can be obtained using



a) Response across the frequency range

b) Low-frequency response

c) Medium-frequency response

d) High-frequency response

Fig. 9 Comparison of 5 and 95% probability points of amplitude of driving-point FRF obtained using direct stochastic finite element simulation and proposed random matrix method.

the definition in Eq. (B5) as

$$\phi_{\mathbf{G}}(\boldsymbol{\Omega}) = E[\text{etr}\{i\boldsymbol{\Omega}\mathbf{G}\}] = E[\text{etr}\{i\boldsymbol{\Omega}\mathbf{X}\mathbf{X}^T\}] \tag{C8}$$

Using the probability density function of $\mathbf{X}$ in Eq. (C7), we have

$$\begin{aligned}\phi_{\mathbf{G}}(\boldsymbol{\Omega}) &= (2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-p/2}\int_{\mathbb{R}_{n,p}} \text{etr}\left\{i\boldsymbol{\Omega}\mathbf{X}\mathbf{X}^T - \frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{X}^T\right\}\mathrm{d}\mathbf{X}\\ &= (2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-p/2}\int_{\mathbb{R}_{n,p}} \text{etr}\left\{-\frac{1}{2}[\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}]\mathbf{X}\mathbf{X}^T\right\}\mathrm{d}\mathbf{X}\end{aligned} \tag{C9}$$

The domain of the preceding integral is the space of all $n \times p$ real matrices. Rearranging the matrix product within the trace function and noting that $[\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}]$ is a symmetric matrix, one has

$$\begin{aligned}\phi_{\mathbf{G}}(\boldsymbol{\Omega}) &= (2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-p/2}\\ &\times \int_{\mathbb{R}_{n,p}} \text{etr}\left\{-1/2[\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}]^{1/2}\mathbf{X}\mathbf{X}^T[\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}]^{1/2}\right\}\mathrm{d}\mathbf{X}\end{aligned} \tag{C10}$$

We use a linear transformation

$$\mathbf{X} = [\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}]^{-1/2}\mathbf{Y} \tag{C11}$$

The Jacobian associated with preceding transformation can be obtained as

$$\mathrm{d}\,\mathbf{X} = |\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}|^{-p/2}\,\mathrm{d}\mathbf{Y} \tag{C12}$$

Substituting $\mathbf{X}$ and $\mathrm{d}\mathbf{X}$ from Eqs. (C11) and (C12) into Eq. (C10), we have

$$\begin{aligned}\phi_{\mathbf{G}}(\boldsymbol{\Omega}) &= (2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-p/2}\int_{\mathbb{R}_{n,p}} \text{etr}\left\{-\tfrac{1}{2}\mathbf{Y}\mathbf{Y}^T\right\}|\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Omega}|^{-p/2}\mathrm{d}\mathbf{Y}\\ &= |\mathbf{I}_n - 2i\boldsymbol{\Omega}\,\boldsymbol{\Sigma}|^{-p/2}\left((2\pi)^{-np/2}\int_{\mathbb{R}_{n,p}} \text{etr}\left\{-\frac{1}{2}\mathbf{Y}\mathbf{Y}^T\right\}\mathrm{d}\mathbf{Y}\right)\end{aligned} \tag{C13}$$

Algorithm 1 MATLAB Code for the Sample-Generation of the System Matrices

Here we show an example-code in MATLAB to generate the samples of Wishart random matrices. The function GetWishartParameters calculates the parameters of the Wishart matrices following the procedure outlined in Sec. V.

```
% An example program to generate the samples of the
% Wishart random matrices in Matlab
% Mbar,delta_M and Kbar,delta_K assumed to be known
% nsamp: number of samples in the Monte Carlo simulation
% M_j, K_j: sample realizations of the mass and stiffness matrices
%-----------------------------------------------------------------
.....
[p_M,Sigma_M]=GetWishartParameters(Mbar,delta_M,n);
[p_K,Sigma_K]=GetWishartParameters(Kbar,delta_K,n);
for j=1: nsamp    % MSC loop over the number of samples
   M_j=wishrnd(Sigma_M,p_M);
   K_j=wishrnd(Sigma_K,p_K);
.....
end
% This function calculates the parameters of the Wishart system matrices
function [p_G,Sigma_G]=GetWishartParameters(Gbar,delta_G,n);
theta_G =( 1+ (trace(Gbar))^2/trace(Gbar*Gbar) )/delta_G^2 -(n+1);
if theta_G < 4
   theta_G=4;
end
alpha_G=sqrt(theta_G*(n+1+theta_G));
p_G=(n+1+theta_G);    % The scalar parameter of the Wishart distribution
Sigma_G=Gbar/alpha_G;    % The matrix parameter of the Wishart
   distribution
```

The second part of the preceding equation is the integration of the probability density function of an $n \times p$ Gaussian random matrix with zero mean and unit covariance [a special case of Eq. (2)]. Therefore,

$$(2\pi)^{-np/2} \int_{\mathbb{R}_{n,p}} \text{etr}\left\{-\frac{1}{2}\mathbf{Y}\mathbf{Y}^T\right\} \text{d}\mathbf{Y} = 1 \tag{C14}$$

and, consequently

$$\phi_{\mathbf{G}}(\boldsymbol{\Omega}) = |\mathbf{I}_n - 2i\boldsymbol{\Omega}\boldsymbol{\Sigma}|^{-p/2} \tag{C15}$$

This is exactly what was obtained in Eq. (C6) and the theorem is proved. For alternative proofs of this theorem see [21,22,24,32].

Acknowledgment

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References

[1] Shinozuka, M., and Yamazaki, F., "Stochastic Finite Element Analysis: An Introduction," *Stochastic Structural Dynamics: Progress in Theory and Applications*, edited by S. T. Ariaratnam, G. I. Schueller, and I. Elishakoff, Elsevier, London, 1998.

[2] Ghanem, R., and Spanos, P., *Stochastic Finite Elements: A Spectral Approach*, Springer–Verlag, New York, 1991.

[3] Kleiber, M., and Hien, T. D., *Stochastic Finite Element Method*, Wiley, Chichester, England, U.K., 1992.

[4] Matthies, H. G., Brenner, C. E., Bucher, C. G., and Soares, C. G., "Uncertainties in Probabilistic Numerical Analysis of Structures and Solids: Stochastic Finite Elements," *Structural Safety*, Vol. 19, No. 3, 1997, pp. 283–336.

[5] Manohar, C. S., and Adhikari, S., "Dynamic Stiffness of Randomly Parametered Beams," *Probabilistic Engineering Mechanics*, Vol. 13, No. 1, Jan. 1998, pp. 39–51.

[6] Adhikari, S., and Manohar, C. S., "Dynamic Analysis of Framed Structures with Statistical Uncertainties," *International Journal for Numerical Methods in Engineering*, Vol. 44, No. 8, 1999, pp. 1157–1178.

[7] Adhikari, S., and Manohar, C. S., "Transient Dynamics of Stochastically Parametered Beams," *Journal of Engineering Mechanics*, Vol. 126, No. 11, Nov. 2000, pp. 1131–1140.

[8] Haldar, A., and Mahadevan, S., *Reliability Assessment Using Stochastic Finite Element Analysis*, Wiley, New York, 2000.

[9] Sudret, B., and Der-Kiureghian, A., "Stochastic Finite Element Methods and Reliability," Dept. of Civil and Environmental Engineering, TR UCB/SEMM-2000/08, Univ. of California, Berkeley, CA, Nov. 2000.

[10] Elishakoff, I., and Ren, Y. J., *Large Variation Finite Element Method for Stochastic Problems*, Oxford Univ. Press, Oxford, England, U.K., 2003.

[11] Soize, C., "Comprehensive Overview of a Non-Parametric Probabilistic Approach of Model Uncertainties for Predictive Models in Structural Dynamics," *Journal of Sound and Vibration*, Vol. 288, No. 3, 2005, pp. 623–652.

[12] Lyon, R. H., and Dejong, R. G., *Theory and Application of Statistical Energy Analysis*, 2nd ed., Butterworth–Heinmann, Boston, 1995.

[13] Langley, R. S., and Bremner, P., "Hybrid Method for the Vibration Analysis of Complex Structural-Acoustic Systems," *Journal of the Acoustical Society of America*, Vol. 105, No. 3, March 1999, pp. 1657–1671.

[14] Sarkar, A., and Ghanem, R., "Mid-Frequency Structural Dynamics with Parameter Uncertainty," *Computer Methods in Applied Mechanics and Engineering*, Vol. 191, Nos. 47–48, 2002, pp. 5499–5513.

[15] Sarkar, A., and Ghanem, R., "Substructure Approach for the Midfrequency Vibration of Stochastic Systems, Part 1," *Journal of the Acoustical Society of America*, Vol. 113, No. 4, 2003, pp. 1922–1934.

[16] Sarkar, A., and Ghanem, R., "Reduced Models for the Medium-Frequency Dynamics of Stochastic Systems," *Journal of the Acoustical Society of America*, Vol. 113, No. 2, 2003, pp. 834–846.

[17] Wishart, J., "Generalized Product Moment Distribution in Samples from a Normal Multivariate Population," *Biometrika*, Vol. 20, A, 1928, pp. 32–52.

[18] Wigner, E. P., "Distribution of the Roots of Certain Symmetric Matrices," *Annals of Mathematics and Artificial Intelligence*, Vol. 67, No. 2, 1958, pp. 325–327.

[19] Mezzadri, F., and Snaith, N. C. (eds.), *Recent Perspectives in Random Matrix Theory and Number Theory*, London Mathematical Society Lecture Note, Cambridge Univ. Press, Cambridge, England, U.K., 2005.

[20] Tulino, A. M., and Verdú, S., *Random Matrix Theory and Wireless Communications*, Now Publishers, Hanover, MA, 2004.

[21] Eaton, M. L., *Multivariate Statistics: A Vector Space Approach*, Wiley, New York, 1983.

[22] Muirhead, R. J., *Aspects of Multivariate Statistical Theory*, Wiley, New York, 1982.

[23] Mehta, M. L., *Random Matrices*, 2nd ed., Academic Press, San Diego, CA, 1991.

[24] Gupta, A., and Nagar, D., *Matrix Variate Distributions*, Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall/CRC, London, 2000.

[25] Kapur, J. N., and Kesavan, H. K., *Entropy Optimization Principles with Applications*, Academic Press, San Diego, CA, 1992.

[26] Udwadia, F. E., "Response of Uncertain Dynamic-Systems: 1," *Applied Mathematics and Computation*, Vol. 22, Nos. 2–3, 1987, pp. 115–150.

[27] Udwadia, F. E., "Response of Uncertain Dynamic-Systems: 2," *Applied Mathematics and Computation*, Vol. 22, Nos. 2–3, 1987, pp. 151–187.

[28] Mathai, A. M., *Jacobians of Matrix Transformation and Functions of Matrix Arguments*, World Scientific, London, 1997.

[29] Steeb, W.-H., *Matrix Calculus and the Kronecker Product with Applications and C++ Programs*, World Scientific, London, 1997.

[30] Harville, D. A., *Matrix Algebra from a Statistician's Perspective*, Springer–Verlag, New York, 1998.

[31] Magnus, J. R., and Neudecker, H., *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Wiley, New York, 1999.

[32] Mathai, A. M., and Provost, S. B., *Quadratic Forms in Random Variables: Theory and Applications*, Marcel Dekker, New York, 1992.

[33] Johnson, N. L., Kotz, S., and Balakrishnan, N., *Continuous Multivariate Distributions, Volume 1: Models and Applications*, Wiley Series in Probability and Mathematical Statistics, Wiley, New York, 2000.

[34] Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series and Products*, 5th ed., Academic Press, Boston, MA, 1994, (translated from Russian by Scripta Technica, Washington, D.C.).

[35] Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.

[36] Kreyszig, E., *Advanced Engineering Mathematics*, 9th ed., Wiley, New York, 2006.

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