Supplementary material: The in-plane mechanical properties of highly compressible and stretchable 2D lattices

S. Adhikari[1,*]

[1]Future Manufacturing Research Institute, College of Engineering, Swansea University, Swansea, UK
[*]s.adhikari@swansea.ac.uk

ABSTRACT

The details of the derivation of the equivalent elastic properties of the hexagonal lattice are given.

Contents

1 Introduction

Compressive and tensile behaviour of the overall lattice structure depends on the deformation characteristics of the constituent individual beams. In Section 2 of the main paper, the stiffness matrix of a beam element was derived in closed-form considering compressive and tensile axial forces within the beam. In this document, we express equivalent in-plane elastic moduli of the lattice in terms of the stiffness matrix elements of the beams using the unit cell approach. For the case of equivalent properties of the lattice without the axial force, we refer to well-known references [1, 2]. A key focus in the derivation proposed below is the direct exploitation of the coefficients of the stiffness matrix derived in the main paper. This will enable us to link the axial force-dependent deformation behaviour of an elemental beam with the whole lattice. A beam element of length L is shown in Figure 1 with two nodes and three degrees of freedom per node. The degrees of freedom in each node corresponds to the axial, transverse and rotational deformation. This beam element can be represented by a 6×6 stiffness matrix.

2 Elastic moduli of compressed and stretched lattices

The elements of the 6-degree-of-freedom stiffness matrix are employed in the derivation of the equivalent elastic properties of the lattice in this section. A generic notation K_{ij} is used here. It should be recalled from Section 3 of the main paper that the stiffness coefficients are functions of the internal axial force parameter μ. This functional dependence is omitted here for notational brevity.

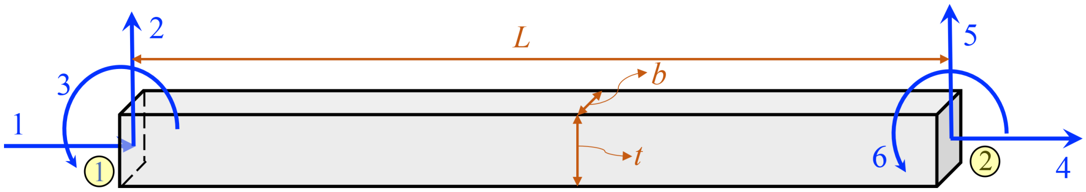


Figure 1. A beam element with six degrees of freedom and two nodes. The degrees of freedom in each node corresponds to the axial, transverse and rotational deformation.

2.1 The longitudinal Young's modulus E_1

A uniform stress field σ_1 is applied to the unit cell in direction-1 as shown in Figure 2 for deriving the expression of the longitudinal Young's modulus. This results in a force P being applied at point A (and B) on the unit cell. The

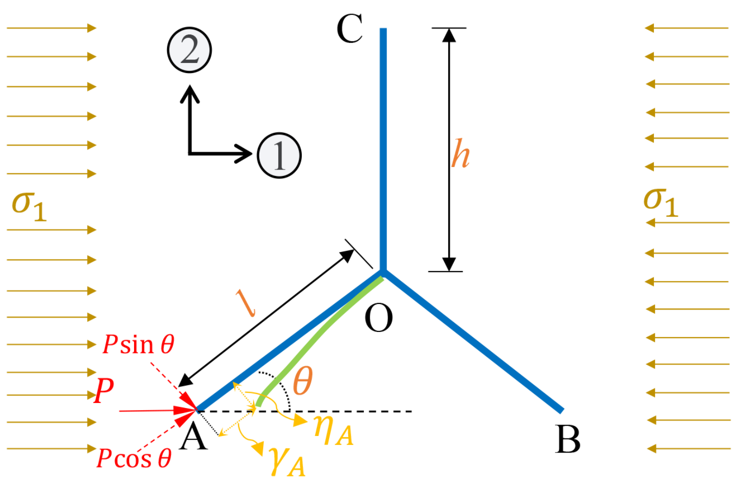


Figure 2. Internal forces and deformation patterns of the unit cell under the application of a stress field σ_1 applied in the 1-direction. This configuration is used for the derivation of the longitudinal Young's modulus E_1.

deformation of the unit cell is symmetric about the OC line. The magnitude of the force P acting on point A is given by

$$P = \sigma_1 b(h + l\sin\theta) \tag{1}$$

Considering η_A and γ_A as deformations transverse and along the inclined member AO, we have

$$\eta_A = \frac{P\sin\theta}{K_{55}} \quad \text{and} \quad \gamma_A = \frac{P\cos\theta}{K_{44}} \tag{2}$$

Here K_{55} and K_{44} are elements of the stiffness matrix of the inclined member AO of length l. The total deflection in the 1-direction is therefore

$$\begin{aligned} \delta_1 &= \eta_A\sin\theta + \gamma_A\cos\theta = P\left(\frac{\sin^2\theta}{K_{55}} + \frac{\cos^2\theta}{K_{44}}\right) \\ &= \frac{P\sin^2\theta}{K_{55}}\left(1 + \cot^2\theta\frac{K_{55}}{K_{44}}\right) \end{aligned} \tag{3}$$

The strain the 1-direction is obtained as

$$\varepsilon_1 = \frac{\delta_1}{l\cos\theta} = \frac{\sigma_1 b(h/l + \sin\theta)\sin^2\theta}{K_{55}\cos\theta}\left(1 + \cot^2\theta\frac{K_{55}}{K_{44}}\right) \tag{4}$$

Using this, the Young's modulus in 1-direction is obtained in terms of the elements of the stiffness matrix as

$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{K_{55}\cos\theta}{b(h/l+\sin\theta)\sin^2\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)} \tag{5}$$

From equation (5), it can be observed that only two coefficients of the 6×6 element stiffness matrix of the inclined member, namely, K_{55} and K_{44}, contribute towards the value of E_1. The Poisson's ratio corresponding to this stress field, namely ν_{12}, is derived in 2.2.

2.2 The Poisson's ratio ν_{12}

To obtain the Poisson's ratio ν_{12}, we need to obtain the strain in the direction 2 for applied stress in the 1-direction from Figure 2. Using the expressions of the deformations in equation (2), we obtain total deflection in the 2-direction as

$$-\delta_2 = \eta_A\cos\theta - \gamma_A\sin\theta = P\left(\frac{\sin\theta\cos\theta}{K_{55}} - \frac{\sin\theta\cos\theta}{K_{44}}\right) = \frac{P\sin\theta\cos\theta}{K_{55}}\left(1-\frac{K_{55}}{K_{44}}\right) \tag{6}$$

The total strain in the 2-direction is

$$-\varepsilon_2 = \frac{\delta_2}{h+l\sin\theta} = \frac{\sigma_1 b\sin\theta\cos\theta}{K_{55}}\left(1-\frac{K_{55}}{K_{44}}\right) \tag{7}$$

Using the expressions of the strains in directions 1 and 2 given by Eqs. (4) and (7), we obtain the Poisson's ratio ν_{12}

$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = \frac{\cos^2\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{(h/l+\sin\theta)\sin\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)} \tag{8}$$

From equation (8), it can be observed that only two coefficients of the 6×6 element stiffness matrix of the inclined member, namely, K_{55} and K_{44}, contribute towards the value of ν_{12}.

2.3 The transverse Young's modulus E_2

For deriving the expression of transverse Young's modulus, a uniform stress field σ_2 is applied to the unit cell in direction-2 as shown in Figure 3. From the free-body diagram depicting the equilibrium, we deduce that the the deformation of the unit cell is symmetric about the OC line. It addition, the point O has no deflection in the 1-direction. Therefore, it is sufficient to consider the deflection of point A or B with respect to point C under the applied stress. Considering point A, the stress results in a vertical force W. The magnitude of this vertical force is given by

$$W = \sigma_2 bl\cos\theta \tag{9}$$

Considering η_A and γ_A as deformations transverse and along the inclined member AO, we have

$$\eta_A = \frac{W\cos\theta}{K_{55}} \quad \text{and} \quad \gamma_A = \frac{W\sin\theta}{K_{44}} \tag{10}$$

Here K_{55} and K_{44} are elements of the stiffness matrix of the member AO. The deflection in the 2-direction is therefore

$$\begin{aligned}\delta_{2_{AO}} &= \eta_A\cos\theta + \gamma_A\sin\theta = W\left(\frac{\cos^2\theta}{K_{55}} + \frac{\sin^2\theta}{K_{44}}\right)\\ &= \frac{W\cos^2\theta}{K_{55}}\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}\right)\end{aligned} \tag{11}$$

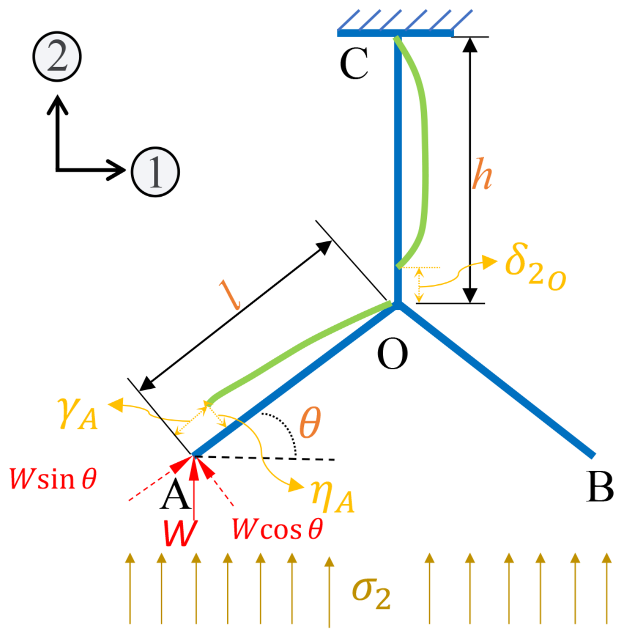


Figure 3. Internal forces and deformation patterns of the unit cell under application of a stress field σ_2 applied in the 2-direction. This configuration is used for the derivation of the transverse Young's modulus E_2.

The total force acting in the 2-direction at point O is $2W$. Therefore, the displacement of point O in the 2-direction arising from the axial deformation of the vertical member OC is

$$\delta_{2_O} = \frac{2W}{K_{44}^{(h)}} \tag{12}$$

Here $(\bullet)^{(h)}$ corresponds to the properties arising from the vertical member OC of length h. The total deflection in the 2-direction is therefore

$$\delta_2 = \delta_{2_{AO}} + \delta_{2_O} = \frac{W \cos^2\theta}{K_{55}} \left(1 + \tan^2\theta \frac{K_{55}}{K_{44}} + 2\sec^2\theta \frac{K_{55}}{K_{44}^{(h)}} \right) \tag{13}$$

The strain the 2-direction is obtained as

$$\varepsilon_2 = \frac{\delta_2}{h + l\sin\theta} = \frac{\sigma_2 b \cos^3\theta}{K_{55}(h/l + \sin\theta)} \left(1 + \tan^2\theta \frac{K_{55}}{K_{44}} + 2\sec^2\theta \frac{K_{55}}{K_{44}^{(h)}} \right) \tag{14}$$

Using this, the Young's modulus in 1-direction is obtained in terms of the elements of the stiffness matrix as

$$E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{K_{55}(h/l + \sin\theta)}{b\cos^3\theta \left(1 + \tan^2\theta \frac{K_{55}}{K_{44}} + 2\sec^2\theta \frac{K_{55}}{K_{44}^{(h)}} \right)} \tag{15}$$

From equation (15), it can be observed that only two coefficients of the 6×6 element stiffness matrix of the inclined member and one coefficients of the 6×6 element stiffness matrix of vertical member, namely, K_{55}, K_{44} and $K_{44}^{(h)}$, contribute towards the value of E_2. The Poisson's ratio corresponding to this stress field, namely ν_{21}, is derived in 2.4.

2.4 The Poisson's ratio ν_{21}

To obtain the Poisson's ratio ν_{21}, we need to obtain the strain in the direction 1 due to the applied stress in the 2-direction from Figure 3. Using the expressions of the deformations in equation (10), we obtain total deflection in the 1-direction as

$$\begin{aligned} \delta_1 &= \gamma_A \cos\theta - \eta_A \sin\theta = -W\left(\frac{\sin\theta\cos\theta}{K_{55}} - \frac{\sin\theta\cos\theta}{K_{44}}\right) \\ &= -\frac{W\sin\theta\cos\theta}{K_{55}}\left(1-\frac{K_{55}}{K_{44}}\right) \end{aligned} \tag{16}$$

The total strain in the 1-direction is

$$\varepsilon_1 = \frac{\delta_1}{l\cos\theta} = -\frac{\sigma_2 b\sin\theta}{lK_{55}}\left(1-\frac{K_{55}}{K_{44}}\right) \tag{17}$$

Using the expressions of the strains in directions 1 and 2 given by Eqs. (4) and (7), we obtain the Poisson's ratio ν_{21}

$$\nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2} = \frac{(h/l+\sin\theta)\sin\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{\cos^2\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)} \tag{18}$$

The proposed expressions of the elastic moduli and Poisson's ratio conform to the reciprocal theorem

$$\begin{aligned} &E_1\nu_{21} = E_2\nu_{12} = \\ &\frac{K_{55}}{b\sin\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)}\frac{\left(1-\frac{K_{55}}{K_{44}}\right)}{\cos\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)} \end{aligned} \tag{19}$$

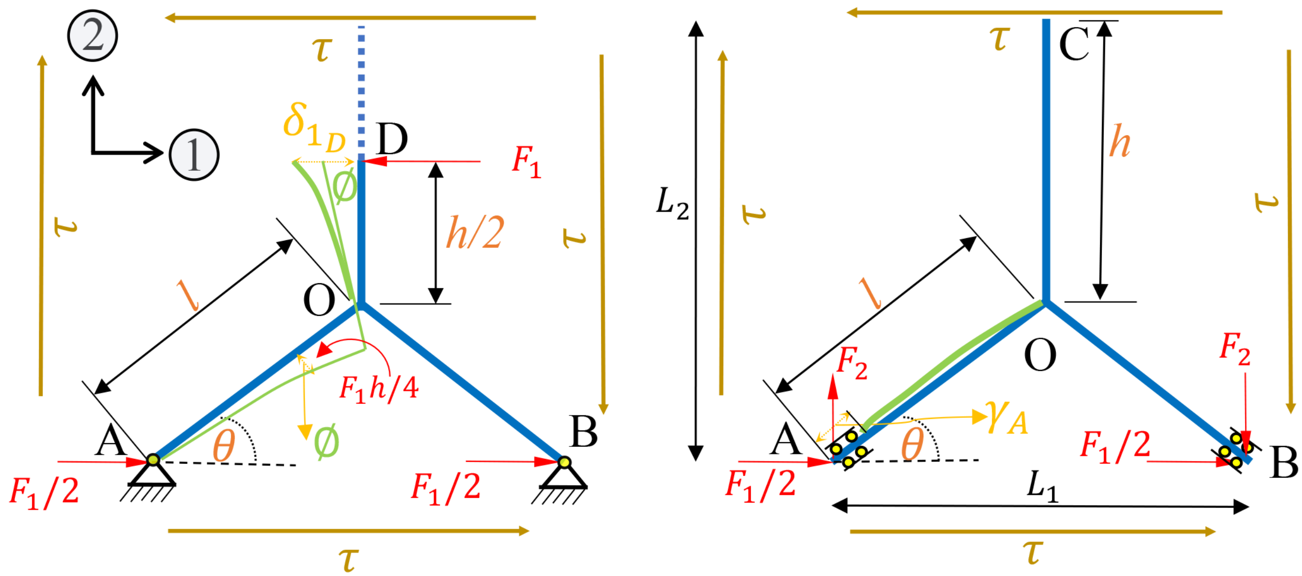
From Eq. (8), it can be observed that only two coefficients of the 6×6 element stiffness matrix of the inclined member and one coefficients of the 6×6 element stiffness matrix of vertical member, namely, K_{55}, K_{44} and $K_{44}^{(h)}$, contribute towards the value ν_{21}.

2.5 The shear modulus G_{21}

The derivation of the shear modulus G_{12} requires the superpositions strain contributions arising from bending and axial deformations. In Figure 4, the consideration of both the cases is depicted. For deriving the bending contributions, considering the deformation of the adjacent cells, it can be deduced that the midpoint of the vertical member will only have a deformation in the 1-direction due to shear. Therefore, in Figure 4(a) we consider the unit cell with the vertical member with length $h/2$ and a slant member with the usual length l. The points A and O will not have any relative movement due to the symmetrical structure. The shear deflection γ_D due to bending consists of two components, namely, bending deflection of the member OD and its deflection due to rotation of joint O arising from the bending of the slant members.

It can be noted here that the elements of the stiffness matrix (refer to equation (40) for example) will be different for the vertical member and the slant member due to their different lengths. Using the stiffness elements of the stiffness matrix with length $h/2$, the bending deformation of point D with respect to point O in direction the 1 can be obtained as

$$\eta_D = \frac{F_1}{\left(K_{55}^{(h/2)} - \frac{K_{56}^{(h/2)}K_{65}^{(h/2)}}{K_{66}^{(h/2)}}\right)} = \frac{F_1 K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)} - \left(K_{56}^{(h/2)}\right)^2\right)} \tag{20}$$



(a) Shear strain due to bending

(b) Shear strain due to axial deformation

Figure 4. Internal forces and deformation patterns of the unit cell under the application of the shear stress field τ. These configurations are used for the derivation of the shear modulus G_{12}.

Here

$$F_1 = 2\tau lb\cos\theta \tag{21}$$

and we make use of the symmetry of the elements of the stiffness matrix. Here $(\bullet)^{(h/2)}$ corresponds to the properties arising from the vertical member OD of length $h/2$ as shown in Figure 4(a).

From the diagram in Figure 4(a), the moment acting on point O is obtained as

$$M = \frac{F_1}{2} \times \frac{h}{2} = \frac{F_1 h}{4} \tag{22}$$

On the basis of the degrees of freedom as denoted in Figure 1, deflection of the end O with respect to the end A due to application of moment M at the end O is given as

$$\delta_r = \frac{M}{-K_{65}} \tag{23}$$

Here K_{65} is the stiffness element corresponding to the slant member and the negative arise due to the direction of the rotation as given in Figure 1. Thus the rotation of joint O can be expressed as

$$\begin{aligned} \phi &= \frac{\delta_r}{l} \\ &= -\frac{F_1 h}{4lK_{65}} \end{aligned} \tag{24}$$

Shear deformation in the 1-direction due to bending at point D under the application of shear stress τ can be expressed as

$$\begin{aligned} \delta_{1_D} &= 2\left(\phi\frac{h}{2} + \eta_D\right) \\ &= -\frac{F_1 h^2}{4lK_{65}} + \frac{2F_1 K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)} - \left(K_{56}^{(h/2)}\right)^2\right)} \end{aligned} \tag{25}$$

The factor 2 in the above expression arises due to the consideration of two units shown in Figure 4(a) to capture the total shear deformation by representing a complete unit cell that can create the entire lattice structure on tessellation.

To obtain the shear deformation due to axial stretching deformation, we consider the forcing F_2 in the 2-direction as

$$F_2 = \tau b(h + l\sin\theta) \tag{26}$$

Due to the symmetry of the unit cell as depicted in Figure 4(b), the deformation in the 1-direction of member AO and BO will be the same. On the other hand, the amplitude of the deformation in the 2-direction of member AO and BO will be the same, but in the opposite direction. There is no axial deformation in the vertical member OC. It is therefore sufficient to consider only one inclined element in our calculation. The lengths of the unit cell in Figure 4(b) in the 1 and 2 directions are given by

$$L_1 = 2l\cos\theta \tag{27}$$

$$\text{and} \quad L_2 = (h + l\sin\theta) \tag{28}$$

Total force acting in the axial direction of AO is given by

$$F_{\mathrm{AO}} = F_1/2\cos\theta + F_2\sin\theta = \tau lb\left(\cos^2\theta + (h/l + \sin\theta)\sin\theta\right) \tag{29}$$

The axial deformation of point A is therefore

$$\gamma_A = \frac{F_{\mathrm{AO}}}{K_{44}} \tag{30}$$

Using this, the deformation in the 1 and 2 directions are obtained as

$$\delta_{1_A} = \gamma_A\cos\theta = \frac{\tau lb}{K_{44}}\left(\cos^2\theta + (h/l + \sin\theta)\sin\theta\right)\cos\theta \tag{31}$$

$$\delta_{2_A} = \gamma_A\sin\theta = \frac{\tau lb}{K_{44}}\left(\cos^2\theta + (h/l + \sin\theta)\sin\theta\right)\sin\theta \tag{32}$$

The total shear strain arising due to bending and axial deformation is given by

$$\gamma = \frac{\delta_{1_A} + \delta_{1_D}}{L_2} + \frac{2\delta_{2_A}}{L_1} = \frac{\delta_{1_A} + \delta_{1_D}}{h + l\sin\theta} + \frac{2\delta_{2_A}}{2l\cos\theta} \tag{33}$$

$$= \underbrace{\frac{\delta_{1_D}}{h + l\sin\theta}}_{\gamma_b} + \underbrace{\frac{\delta_{1_A}}{h + l\sin\theta} + \frac{\delta_{2_A}}{l\cos\theta}}_{\gamma_s} \tag{34}$$

Here γ_b and γ_s are respectively the bending and stretching components of the total shear strain. Using Eq. (25) we obtain the bending component of the shear strain as

$$
\begin{aligned}
\gamma_b &= \frac{\delta_{1_D}}{(h+l\sin\theta)} \\
&= \frac{F_1}{(h+l\sin\theta)}\left(-\frac{h^2}{4lK_{65}}+\frac{2K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)}-\left(K_{56}^{(h/2)}\right)^2\right)}\right) \\
&= \frac{2\tau lb\cos\theta}{(h+l\sin\theta)}\left(-\frac{h^2}{4lK_{65}}+\frac{2K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)}-\left(K_{56}^{(h/2)}\right)^2\right)}\right) \\
&= \frac{\tau b\cos\theta}{(h/l+\sin\theta)}\left(-\frac{h^2}{2lK_{65}}+\frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)}-\left(K_{56}^{(h/2)}\right)^2\right)}\right)
\end{aligned}
\tag{35}
$$

The stretching component of the shear strain can be simplified as

$$
\gamma_s = \frac{\delta_{1_A}}{h+l\sin\theta}+\frac{\delta_{2_A}}{l\cos\theta} \tag{36}
$$

$$
= \frac{\tau lb}{K_{44}}\left(\cos^2\theta+(h/l+\sin\theta)\sin\theta\right)\left(\frac{\cos\theta}{h+l\sin\theta}+\frac{\sin\theta}{l\cos\theta}\right) \tag{37}
$$

$$
= \frac{\tau b}{K_{44}}\frac{\left(\cos^2\theta+(h/l+\sin\theta)\sin\theta\right)^2}{\cos\theta(h/l+\sin\theta)} \tag{38}
$$

Substituting the expressions of both the shear strains, the modulus can be obtained as

$$
\begin{aligned}
G_{12} &= \frac{\tau}{\gamma} = \frac{\tau}{\gamma_b+\gamma_s} \\
&= \frac{1}{\frac{b\cos\theta}{(h/l+\sin\theta)}\left(-\frac{h^2}{2lK_{65}}+\frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)}-\left(K_{56}^{(h/2)}\right)^2\right)}\right)+\frac{b}{K_{44}}\frac{\left(\cos^2\theta+(h/l+\sin\theta)\sin\theta\right)^2}{\cos\theta(h/l+\sin\theta)}} \\
&= \frac{(h/l+\sin\theta)}{b\cos\theta}\frac{1}{\left(-\frac{h^2}{2lK_{65}}+\frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)}-\left(K_{56}^{(h/2)}\right)^2\right)}+\frac{\left(\cos\theta+(h/l+\sin\theta)\tan\theta\right)^2}{K_{44}}\right)}
\end{aligned}
\tag{39}
$$

From equation (39) it can be observed that in total five elements of two different stiffness matrices contribute to the shear modulus. They include two coefficients of the 6×6 element stiffness matrix of the inclined member, namely, K_{65}, K_{44}. Additionally three elements of the stiffness matrix of the vertical member with half the length, namely, $K_{55}^{(h/2)}$, $K_{56}^{(h/2)}$ and $K_{66}^{(h/2)}$ contribute to the shear modulus.

3 The special case of small deformation

In the previous section, the expressions of five quantities characterising the effective in-plane elastic properties of 2D cellular materials have been derived in terms of the stiffness element of a beam. The stiffness matrix of an

Euler-Bernoulli beam element [3, 4] is expressed by

$$\mathbf{K}=\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{40}$$

We are considering beam elements with rectangular cross section as shown in Figure 1 of the main paper. The moment of inertia and the cross section area appearing in the stiffness matrix in equation (40) are therefore given by

$$I=\frac{1}{12}bt^3 \quad \text{and} \quad A=bt \tag{41}$$

For notational convenience, the following non-dimensional geometric coefficients are defined

$$\alpha=\frac{t}{l} \quad \text{and} \quad \beta=\frac{h}{l} \tag{42}$$

From the derivations in subsection 2.1 and subsection 2.3, it can be observed that two coefficients of the 6×6 element stiffness matrix of the inclined member and one coefficients of the 6×6 element stiffness matrix of vertical member, namely, K_{55}, K_{44} and $K_{44}^{(h)}$, are necessary to obtain E_1, E_2 ν_{12} and ν_{21}. Using the expressions of moment of inertia and the cross-sectional area in Eq. (41), the stiffness coefficients are given by

$$K_{55}=\frac{12EI}{l^3}=Eb\alpha^3, K_{44}=\frac{EA}{l}=Eb\alpha \quad \text{and} \quad K_{44}^{(h)}=\frac{EA}{h}=\frac{Ebt}{h}=\frac{Eb\alpha}{\beta} \tag{43}$$

Using these, we obtain the ratios

$$\frac{K_{55}}{K_{44}}=\alpha^2 \quad \text{and} \quad \frac{K_{55}}{K_{44}^{(h)}}=\alpha^2\beta \tag{44}$$

When the Euler-Bernoulli beam stiffness elements are used, from Eqs. (5), (15), (8) and (18) we have

$$E_1=\frac{K_{55}\cos\theta}{b(\beta+\sin\theta)\sin^2\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)}=\frac{E\alpha^3\cos\theta}{(\beta+\sin\theta)\left(\sin^2\theta+\alpha^2\cos^2\theta\right)} \tag{45}$$

$$E_2=\frac{K_{55}(\beta+\sin\theta)}{b\cos^3\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)}=\frac{E\alpha^3(\beta+\sin\theta)}{(1-\alpha^2)\cos^3\theta+\alpha^2(2\beta+1)\cos\theta} \tag{46}$$

$$\nu_{12}=\frac{\cos^2\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{(\beta+\sin\theta)\sin\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)}=\frac{\cos^2\theta\left(1-\alpha^2\right)}{(\beta+\sin\theta)\sin\theta\left(1+\alpha^2\cot^2\theta\right)} \tag{47}$$

$$\nu_{21}=\frac{(\beta+\sin\theta)\sin\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{\cos^2\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)}=\frac{(\beta+\sin\theta)\sin\theta\left(1-\alpha^2\right)}{(1-\alpha^2)\cos^2\theta+\alpha^2(2\beta+1)} \tag{48}$$

For the shear modulus, five elements from two different stiffness matrices are necessary. They are two coefficients of the 6×6 element stiffness matrix of the inclined member, namely, K_{65}, K_{44} as in Eq. (43) with $K_{65}=-6\frac{EI}{l^2}=-1/2\,\frac{Ebt^3}{l^2}$. We also need three elements of the stiffness matrix of the vertical member with half the length given by

$$K_{55}^{(h/2)}=\frac{12EI}{(h/2)^3}=\frac{8Ebt^3}{h^3}, K_{56}^{(h/2)}=-\frac{6EI}{(h/2)^2}=-\frac{2Ebt^3}{h^2} \quad \text{and} \quad K_{66}^{(h/2)}=\frac{4EI}{(h/2)}=\frac{2Ebt^3}{3h} \tag{49}$$

Using these expressions we obtain

$$G_{12} = \frac{(\beta + \sin\theta)}{b\cos\theta} \frac{1}{\left(-\frac{h^2}{2lK_{65}} + \frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)} K_{66}^{(h/2)} - \left(K_{56}^{(h/2)} \right)^2 \right)} + \frac{(\cos\theta + (\beta + \sin\theta)\tan\theta)^2}{K_{44}} \right)}$$
$$= \frac{E\alpha^3(\beta + \sin\theta)}{\left(\beta^2(1+2\beta) + \alpha^2 (\cos\theta + (\beta + \sin\theta)\tan\theta)^2 \right) \cos\theta} \tag{50}$$

Substituting $\alpha^2 = 0$, the equations derived here exactly reduce to the corresponding classical expressions [1] (i.e., the case of considering only the bending deformation and ignoring the axial stretching/shortening of the beams).

For a regular lattice $\theta = \frac{\pi}{6}$ and $\beta = \frac{h}{l} = 1$. Substituting these in Eqs. (45)–(48) and (50) we have

$$E_1 = \frac{4E\alpha^3}{\sqrt{3}(3\alpha^2+1)}, E_2 = \frac{4E\alpha^3}{\sqrt{3}(3\alpha^2+1)}, \nu_{12} = \frac{1-\alpha^2}{3\alpha^2+1}, \nu_{21} = \frac{1-\alpha^2}{3\alpha^2+1} \tag{51}$$

$$\text{and} \quad G_{12} = \frac{E\alpha^3}{\sqrt{3}(\alpha^2+1)} \tag{52}$$

References

1. Gibson, L. & Ashby, M. F. *Cellular Solids Structure and Properties* (Cambridge University Press, Cambridge, UK, 1999).

2. Masters, I. & Evans, K. Models for the elastic deformation of honeycombs. *Compos. Struct.* **35**, 403–422 (1996).

3. Dawe, D. *Matrix and Finite Element Displacement Analysis of Structures* (Oxford University Press, Oxford, UK, 1984).

4. Petyt, M. *Introduction to Finite Element Vibration Analysis* (Cambridge University Press, Cambridge, UK, 1990).