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Joint statistics of natural frequencies of stochastic dynamic systems

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Abstract The description of real-life engineering structural systems is usually associated with some amount of uncertainty in specifying material properties, geometric parameters and boundary conditions. In the context of structural dynamics it is necessary to consider joint probability distribution of the natural frequencies in order to account for these uncertainties. Current methods to deal with such problems are dominated by approximate perturbation methods. In this paper a new approach based on an asymptotic approximation of multidimensional integrals is proposed. A closed-form expression for general order joint moments of arbitrary number of natural frequencies of linear stochastic systems is derived. The proposed method does not employ the small-randomness and Gaussian random variable assumption usually used in the perturbation based methods. Joint distributions of the natural frequencies are investigated using numerical examples and the results are compared with Monte Carlo simulation.

Keywords Random eigenvalue problem · Asymptotic method · Stochastic dynamical systems · Probabilistic structural mechanics

Nomenclature

$\kappa_{jk}^{(r_1,r_2)}$	(r_1, r_2)th order cumulant of jth and kth natural frequencies
$\mathbf{D}_{(\bullet)}(\mathbf{x})$	Hessian matrix of $(\bullet)$ at $\mathbf{x}$
$\mathbf{d}_{(\bullet)}(\mathbf{x})$	gradient vector of $(\bullet)$ at $\mathbf{x}$
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\boldsymbol{\mu}$	mean of parameter vector $\mathbf{x}$
$\mathbf{O}$	null matrix
$\boldsymbol{\phi}_j$	jth mode shape of the system
$\boldsymbol{\Sigma}$	covariance matrix of parameter vectors $\mathbf{x}$
$\boldsymbol{\Sigma}_{\omega_{jk}}$	covariance matrix of jth and kth natural frequencies
$\boldsymbol{\theta}$	optimal point
$\mathbf{x}$	basic random variables
ϵ_m, ϵ_k	strength parameters associated with mass and stiffness coefficients
$\mu_{jk}^{(r_1,r_2)}$	(r_1, r_2)th order joint moment of jth and kth natural frequencies
ω_j	jth natural frequencies of the system
$\widetilde{\mathbf{s}}$	vector of complex Laplace parameters s_1 and s_2
$L(\mathbf{x})$	negative of the log-likelihood funtion
m	number of basic random variables
$M_{\omega_j,\omega_k}(s_1, s_2)$	joint moment generating function of ω_j and ω_k
N	degrees-of-freedom of the system
$p_{(\bullet)}$	probability density function of $(\bullet)$
$(\bullet)^{\mathrm{T}}$	matrix transpose
$\mathbb{C}$	space of complex numbers
$\mathbb{R}$	space of real numbers
$\mathrm{Cov}(\bullet, \bullet)$	covariance of random quantities
$\lVert\bullet\rVert$	determinant of a matrix
exp	exponential function

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E [•]	expectation operator
∈	belongs to
↦	maps into
| • |	l_2 norm of (•)
$\overline{(\bullet)}$	deterministic value corresponding to (•)
dof	degrees-of-freedom
jpdf	joint probability density function
pdf	probability density function

1 Introduction

Random eigenvalue problems arise in the dynamic analysis of linear stochastic systems – for example in the problem of dynamic response prediction of cars rolling out from a production chain, cabin noise prediction of nominally identical helicopters across the fleet and vibrations across the ensemble of Micro Electro Mechanical Systems (MEMS) divides. The study of probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics [3,4,6,16,20,36]. In this paper we aim to obtain a closed-form expression of arbitrary order joint moments of the natural frequencies of discrete linear systems or discretized continuous systems. The random eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \omega_j^2\mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j \tag{1}$$

Here ω_j and $\boldsymbol{\phi}_j$ are the natural frequencies (square root of the eigenvalues) and mode shapes (eigenvectors) of the dynamic system. $\mathbf{M}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$ and $\mathbf{K}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$, the mass and stiffness matrices, are assumed to be smooth, continuous and at least twice differentiable functions of a random parameter vector $\mathbf{x} \in \mathbb{R}^m$. The vector $\mathbf{x}$ may consist of material properties, e.g., mass density, Poisson's ratio, Young's modulus; geometric properties, e.g., length, thickness, and boundary conditions. These quantities, that is the elements of the vector $\mathbf{x}$, are considered to be uncertain in this study. Therefore, the statistical properties of the system are completely described by the joint probability density function (jpdf) $p_{\mathbf{x}}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$. For mathematical convenience we express

$$p_{\mathbf{x}}(\mathbf{x}) = \exp\{-L(\mathbf{x})\} \tag{2}$$

where $-L(\mathbf{x})$ is often known as the log-likelihood function. For example, if $\mathbf{x}$ is a m-dimensional multivariate Gaussian random vector with mean $\boldsymbol{\mu} \in \mathbb{R}^m$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times m}$ then

$$L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\|\boldsymbol{\Sigma}\| + \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \tag{3}$$

We consider that the random parameters are non-Gaussian and correlated, i.e., $L(\mathbf{x})$ can have any general form provided it is a smooth, continuous and at least twice differentiable function. It is further assumed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices so that all the eigenvalues are real and positive. In this paper we consider parametric uncertainty approach. Random eigenvalue problems associated with nonparametric approach has been discussed by Soize [37–39].

The aim of studying random eigenvalue problems is to obtain the jpdf of the eigenvalues and the eigenvectors. The current literature on random eigenvalue problems arising in engineering systems is dominated by the mean-centered perturbation methods [7,14,15,26,31–35,40]. These methods work well if the uncertainties are small and the parameters follow a Gaussian distribution. In order to overcome the problems associated with the perturbation method, several authors [2,8,11–13,19,22,25,29,30,41,42] have also proposed non-perturbative methods. Majority of these works consider probabilistic characteristics of *single* eigenvalues. For dynamic response analysis, failure analysis through buckling and in many other practical problems the *joint* distribution of the eigenvalues are required. Under very special circumstances, for example when the system matrix is Gaussian unitary ensemble (GUE) or Gaussian orthogonal ensemble (GOE), an exact closed-form expression can be obtained for the jpdf of the eigenvalues [23,24]. However, the system matrices of real structures may not always follow such distributions and consequently some kind of approximate analysis is required for structural engineering problems. While several papers are available on the distribution of individual eigenvalues, joint distributions of the eigenvalues seem to have received little attention in literature. To the best of authors knowledge only mean-centered first-order perturbation based results are available for the *joint* pdf of the eigenvalues. In this paper a new asymptotic method is proposed to obtain the joint distribution of the natural frequencies of discrete linear systems.

2 Joint natural frequency statistics using the perturbation method

In 1969 using the first-order perturbation method Collins and Thomson [7] derived the jpdf of the eigenvalues and eigenvectors of linear systems with uncertain parameters following Gaussian distribution.

Over the past three decades, several authors [14,15,26, 31–35,40] have applied first-order perturbation method in various problems of practical interest. Adhikari [1] used the first-order perturbation method in complex random eigenvalue problems arising in non-proportionally damped systems. The widespread application of the first-order perturbation method is primarily due its computational efficiency compared to other methods. The limitations of this method are well understood – if uncertainties in the system parameters are 'large' then the method do not produce very accurate results. The accuracy and the range of applicability of the first-order perturbation method can be extended if higher-order terms are included (see the classic book by Scheidt and Purkert [36]). Here we briefly review the joint statistics of the natural frequencies using the second-order perturbation method.

2.1 Perturbation expansion

The mass and the stiffness matrices are in general non-linear functions of the random vector $\mathbf{x}$. We denote the mean of $\mathbf{x}$ as $\boldsymbol{\mu} \in \mathbb{R}^m$, and consider that

$$\mathbf{M}(\boldsymbol{\mu}) = \overline{\mathbf{M}}, \quad \text{and} \quad \mathbf{K}(\boldsymbol{\mu}) = \overline{\mathbf{K}} \tag{4}$$

are the 'deterministic parts' of the mass and stiffness matrices, respectively. In general $\overline{\mathbf{M}}$ and $\overline{\mathbf{K}}$ are different from the mean matrices. The deterministic part of the natural frequencies

$$\overline{\omega}_j = \omega_j(\boldsymbol{\mu}) \tag{5}$$

is obtained from the deterministic eigenvalue problem:

$$\overline{\mathbf{K}}\,\overline{\boldsymbol{\phi}}_j = \overline{\omega}_j^2\,\overline{\mathbf{M}}\,\overline{\boldsymbol{\phi}}_j \tag{6}$$

The natural frequencies, $\omega_j(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ are non-linear functions of the parameter vector $\mathbf{x}$. If the natural frequencies are not repeated, then each $\omega_j(\mathbf{x})$ is expected to be a smooth and twice differentiable function since the mass and stiffness matrices are smooth and twice differentiable functions of the random parameter vector. In the perturbation approach the function $\omega_j(\mathbf{x})$ is expanded by its Taylor series about the point $\mathbf{x} = \boldsymbol{\mu}$ as

$$\omega_j(\mathbf{x}) \approx \omega_j(\boldsymbol{\mu}) + \mathbf{d}_{\omega_j}^{\mathrm{T}}(\boldsymbol{\mu})\,(\mathbf{x}-\boldsymbol{\mu}) + \frac{1}{2}\,(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\,\mathbf{D}_{\omega_j}(\boldsymbol{\mu})\,(\mathbf{x}-\boldsymbol{\mu}) \tag{7}$$

Here $\mathbf{d}_{\omega_j}(\boldsymbol{\mu}) \in \mathbb{R}^m$ and $\mathbf{D}_{\omega_j}(\boldsymbol{\mu}) \in \mathbb{R}^{m\times m}$ are respectively the gradient vector and the Hessian matrix of $\omega_j(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\mu}$, that is

$$\left\{\mathbf{d}_{\omega_j}(\boldsymbol{\mu})\right\}_k = \frac{\partial \omega_j(\mathbf{x})}{\partial x_k}|_{\mathbf{x}=\boldsymbol{\mu}} \quad \text{and} \tag{8}$$

$$\left\{\mathbf{D}_{\omega_j}(\boldsymbol{\mu})\right\}_{kl} = \frac{\partial^2 \omega_j(\mathbf{x})}{\partial x_k\,\partial x_l}|_{\mathbf{x}=\boldsymbol{\mu}} \tag{9}$$

The expressions of the elements of the gradient vector and the Hessian matrix are given in Appendix A. Equation (7) implies that the natural frequencies are effectively expanded about their corresponding deterministic values $\overline{\omega}_j$.

In general Eq. (7) represents a quadratic form in basic non-Gaussian random variables. The first-order perturbation can be obtained from Eq. (7) by neglecting the Hessian matrix. In this case the natural frequencies are simple linear functions of the basic random variables. This formulation is expected to produce acceptable results when the random variation in $\mathbf{x}$ is small. If the basic random variables are Gaussian then the first-order perturbation results in a joint Gaussian distribution of the natural frequencies [7]. In this case a closed-form expression for their jpdf can be obtained easily. When the second-order terms are retained in Eq. (7), each $\omega_j(\mathbf{x})$ results in a quadratic form in $\mathbf{x}$. In this case the resulting distribution of $\omega_j(\mathbf{x})$ is however not Gaussian. Joint statistics of the natural frequencies are discussed in the next subsection using the theory of quadratic forms.

2.2 Joint statistics of the natural frequencies

Discussions on quadratic forms in Gaussian random variables can be found in the books by Johnson and Kotz [17] (Chapt. 29) and Mathai and Provost [21]. Using the methods outlined in these references joint moments/cumulants of the natural frequencies are obtained in this section.

Considering $\mathbf{x}$ as multivariate Gaussian random vector with mean $\boldsymbol{\mu} \in \mathbb{R}^m$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times m}$, the joint moment generating function of $\omega_j(\mathbf{x})$ and $\omega_k(\mathbf{x})$, for any $s_1, s_2 \in \mathbb{C}$, can be obtained as

$$\begin{aligned} M_{\omega_j,\omega_k}(s_1,s_2) &= \mathrm{E}\left[\exp\left\{s_1\omega_j(\mathbf{x}) + s_2\omega_k(\mathbf{x})\right\}\right] \\ &= \int_{\mathbb{R}^m} \exp\left\{s_1\omega_j(\mathbf{x}) + s_2\omega_k(\mathbf{x}) - L(\mathbf{x})\right\} \mathrm{d}\mathbf{x} \\ &= (2\pi)^{-m/2}\|\boldsymbol{\Sigma}\|^{-1/2} \int_{\mathbb{R}^m} \exp\{s_1\omega_j(\mathbf{x}) \\ &\quad + s_2\omega_k(\mathbf{x}) - \frac{1}{2}\,(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x}-\boldsymbol{\mu})\}\mathrm{d}\mathbf{x} \end{aligned} \tag{10}$$

Applying the quadratic expansion in Eq. (7) for j and k, and rearranging the terms within the exponent we have

$$s_1\overline{\omega}_j + s_2\overline{\omega}_k + \left(s_1\mathbf{d}_{\omega_j} + s_2\mathbf{d}_{\omega_k}\right)^{\mathrm{T}}(\mathbf{x}-\boldsymbol{\mu}) - \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \times \left[\boldsymbol{\Sigma}^{-1} - s_1\mathbf{D}_{\omega_j} - s_2\mathbf{D}_{\omega_k}\right](\mathbf{x}-\boldsymbol{\mu}) \tag{11}$$

The dependence on $\boldsymbol{\mu}$ has been omitted for notational convenience. Using the transformation $\mathbf{y} = (\mathbf{x}-\boldsymbol{\mu})$ the integral in (10) can be evaluated exactly as

$$M_{\omega_j,\omega_k}(s_1,s_2) = \left\|\mathbf{I} - s_1\mathbf{D}_{\omega_j}\boldsymbol{\Sigma} - s_2\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right\|^{-1/2} \times \exp\left\{\left(s_1\overline{\omega}_j + s_2\overline{\omega}_k\right) + \frac{1}{2}\left(s_1\mathbf{d}_{\omega_j} + s_2\mathbf{d}_{\omega_k}\right)^{\mathrm{T}} \times \boldsymbol{\Sigma}\left[\mathbf{I} - s_1\mathbf{D}_{\omega_j}\boldsymbol{\Sigma} - s_2\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right]^{-1}\left(s_1\mathbf{d}_{\omega_j} + s_2\mathbf{d}_{\omega_k}\right)\right\} \tag{12}$$

To obtain the joint pdf of ω_j and ω_k, the two-dimensional inverse Laplace transform of (12) is required. An exact closed-form expressions for the general case is not possible. Therefore, we calculate the joint cumulants of the natural frequencies.

If first-order perturbation is used then $\mathbf{D}_{\omega_j} = \mathbf{D}_{\omega_k} = \mathbf{O}$ and from Eq. (12) we obtain

$$M_{\omega_j,\omega_k}(s_1,s_2) = \exp\left\{\boldsymbol{\mu}_{\omega_{jk}}^{\mathrm{T}}\widetilde{\mathbf{s}} + \frac{1}{2}\widetilde{\mathbf{s}}^{\mathrm{T}}\boldsymbol{\Sigma}_{\omega_{jk}}\widetilde{\mathbf{s}}\right\} \tag{13}$$

where

$$\widetilde{\mathbf{s}} = \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix}, \quad \boldsymbol{\mu}_{\omega_{jk}} = \begin{Bmatrix} \overline{\omega}_j \\ \overline{\omega}_k \end{Bmatrix}, \quad \text{and} \quad \boldsymbol{\Sigma}_{\omega_{jk}} = \begin{bmatrix} \mathbf{d}_{\omega_j}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{d}_{\omega_j} & \mathbf{d}_{\omega_j}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{d}_{\omega_k} \\ \mathbf{d}_{\omega_j}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{d}_{\omega_k} & \mathbf{d}_{\omega_k}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{d}_{\omega_k} \end{bmatrix} \tag{14}$$

This implies that ω_j and ω_k are jointly Gaussian distributed with mean $\boldsymbol{\mu}_{\omega_{jk}}$ and covariance matrix $\boldsymbol{\Sigma}_{\omega_{jk}}$.

For the second-order perturbation, the joint cumulants of ω_j and ω_k can be obtained by taking the logarithm of the joint moment generating function (also known as the cumulant generating function). A general (r_1,r_2)th order cumulant of the jth and kth natural frequencies can be obtained from

$$\kappa_{jk}^{(r_1,r_2)} = \frac{\partial^{r_1+r_2}}{\partial s_1^{r_1}\partial s_2^{r_2}} \ln M_{\omega_j,\omega_k}(s_1,s_2)|_{s_1=0,s_2=0} \tag{15}$$

After some simplifications the following results can be derived

$$\kappa_{jk}^{(1,0)} = \mathrm{E}\left[\omega_j\right] = \overline{\omega}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right), \tag{16}$$

$$\kappa_{jk}^{(0,1)} = \mathrm{E}\left[\omega_k\right] = \overline{\omega}_k + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right), \tag{17}$$

$$\kappa_{jk}^{(1,1)} = \mathrm{Cov}\left(\omega_j,\omega_k\right) = \frac{1}{2}\mathrm{Trace}\left(\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right)\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right)\right) + \mathbf{d}_{\omega_j}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{d}_{\omega_k} \tag{18}$$

The general case can be obtained as

$$\begin{aligned}\kappa_{jk}^{(r_1,r_2)} = \frac{1}{2}(r_1+r_2-2)!\Big\{&(r_1+r_2-1) \times \mathrm{Trace}\left(\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right)^{r_1}\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right)^{r_2}\right) \\ &+ r_1(r_1-1)\,\mathbf{d}_{\omega_j}^{\mathrm{T}}\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right)^{r_1-2}\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right)^{r_2}\boldsymbol{\Sigma}\,\mathbf{d}_{\omega_j} \\ &+ r_2(r_2-1)\mathbf{d}_{\omega_k}^{\mathrm{T}}\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right)^{r_1}\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right)^{r_2-2}\boldsymbol{\Sigma}\,\mathbf{d}_{\omega_k} \\ &+ 2r_1r_2\,\mathbf{d}_{\omega_j}^{\mathrm{T}}\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right)^{r_1-1}\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right)^{r_2-1}\boldsymbol{\Sigma}\,\mathbf{d}_{\omega_k}\Big\}, \\ &\text{for} \quad r_1 \geq 1, \; r_2 \geq 1\end{aligned} \tag{19}$$

Because all the cumulants are known from the preceding expressions, the jpdf of the natural frequencies can be calculated, for example using Edgeworth expansion, up to any desired accuracy. Recall that the limitation of this approach arises from the second-order Taylor expansion of the natural frequencies around the deterministic values. Therefore, the inclusion of higher order cumulants in the expression of the joint pdf will not overcome this fundamental limitation.

The method described in this section is only useful when the basic random variables are Gaussian. If the elements of $\mathbf{x}$ are non-Gaussian then neither the first-order perturbation nor the second-order perturbation methods are helpful because there is no general method to obtain the resulting statistics in a simple manner. In such cases the method proposed in the next section might be useful.

3 Asymptotic integral based method

Consider a function $f(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ which is smooth and at least twice differentiable. Suppose we want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} \exp\left\{-f(\mathbf{x})\right\} \mathrm{d}\mathbf{x} \tag{20}$$

This is an m-dimensional integral over the unbounded domain $\mathbb{R}^m$. The maximum contribution to this integral comes from the neighborhood where $f(\mathbf{x})$ reaches its global minimum. Suppose that $f(\mathbf{x})$ reaches its global minimum at an *unique* point $\boldsymbol{\theta} \in \mathbb{R}^m$. Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \quad \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0} \tag{21}$$

Using this, the integral in (20) can be approximated using the asymptotic method [5,27,43] as

$$\mathcal{J} \approx \exp\{-f(\boldsymbol{\theta})\} \int_{\mathbb{R}^m} \left\| \mathbf{D}_f(\boldsymbol{\theta}) \right\|^{-1/2} \exp\left\{ -\frac{1}{2}(\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}) \right\} \mathrm{d}\boldsymbol{\xi} \tag{22}$$

$$\text{or} \quad \mathcal{J} \approx (2\pi)^{m/2} \exp\{-f(\boldsymbol{\theta})\} \left\| \mathbf{D}_f(\boldsymbol{\theta}) \right\|^{-1/2} \tag{23}$$

Here $\mathbf{D}_f(\boldsymbol{\theta})$ is the Hessian matrix of $f(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\theta}$. The approximation in Eq. (22) is also known as the saddle point approximation.

3.1 Joint moments of two natural frequencies

A general (r_1, r_2)th order joint moment of two natural frequencies ω_j and ω_k can be expressed as

$$\begin{aligned}\mu_{jk}^{(r_1,r_2)} &= \mathrm{E}\left[\omega_j^{r_1}(\mathbf{x})\omega_k^{r_2}(\mathbf{x})\right] = \int_{\mathbb{R}^m} \omega_j^{r_1}(\mathbf{x})\omega_k^{r_2}(\mathbf{x}) p_{\mathbf{x}}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^m} \exp\left\{-\left(L(\mathbf{x}) - r_1 \ln \omega_j(\mathbf{x}) - r_2 \ln \omega_k(\mathbf{x})\right)\right\} \mathrm{d}\mathbf{x}, \\ &\quad r_1, r_2 = 1, 2, 3, \ldots\end{aligned} \tag{24}$$

This equation can be expressed in the form of Eq. (20) by choosing

$$f(\mathbf{x}) = L(\mathbf{x}) - r_1 \ln \omega_j(\mathbf{x}) - r_2 \ln \omega_k(\mathbf{x}) \tag{25}$$

Differentiating the above equation with respect to x_i we obtain

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{\partial L(\mathbf{x})}{\partial x_i} - \frac{r_1}{\omega_j(\mathbf{x})}\frac{\partial \omega_j(\mathbf{x})}{\partial x_i} - \frac{r_2}{\omega_k(\mathbf{x})}\frac{\partial \omega_k(\mathbf{x})}{\partial x_i} \tag{26}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from (21) by equating the above expression to zero. Therefore at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = 0, \quad \forall i \quad \text{or} \tag{27}$$

$$\frac{r_1}{\omega_j(\mathbf{x})}\frac{\partial \omega_j(\mathbf{x})}{\partial x_i} + \frac{r_2}{\omega_k(\mathbf{x})}\frac{\partial \omega_k(\mathbf{x})}{\partial x_i} = \frac{\partial L(\boldsymbol{\theta})}{\partial x_i} \quad \text{or} \tag{28}$$

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r_1}{\omega_j(\boldsymbol{\theta})}\mathbf{d}_{\omega_j}(\boldsymbol{\theta}) + \frac{r_2}{\omega_k(\boldsymbol{\theta})}\mathbf{d}_{\omega_k}(\boldsymbol{\theta}) \tag{29}$$

The elements of the Hessian matrix $\mathbf{D}_f(\boldsymbol{\theta})$ can be obtained by differentiating Eq. (26) with respect to x_l as

$$\begin{aligned}\frac{\partial^2 f(\mathbf{x})}{\partial x_i\,\partial x_l} &= \frac{\partial^2 L(\mathbf{x})}{\partial x_i\,\partial x_l} + \frac{r_1}{\omega_j^2(\mathbf{x})}\frac{\partial \omega_j(\mathbf{x})}{\partial x_l}\frac{\partial \omega_j(\mathbf{x})}{\partial x_i} \\ &\quad - \frac{r_1}{\omega_j(\mathbf{x})}\frac{\partial^2 \omega_j(\mathbf{x})}{\partial x_i\,\partial x_l} + \frac{r_2}{\omega_k^2(\mathbf{x})}\frac{\partial \omega_k(\mathbf{x})}{\partial x_l}\frac{\partial \omega_k(\mathbf{x})}{\partial x_i} \\ &\quad - \frac{r_2}{\omega_k(\mathbf{x})}\frac{\partial^2 \omega_k(\mathbf{x})}{\partial x_i\,\partial x_l}\end{aligned} \tag{30}$$

Combining this equation for all i and l we have

$$\begin{aligned}\mathbf{D}_f(\boldsymbol{\theta}) &= \mathbf{D}_L(\boldsymbol{\theta}) + \frac{r_1}{\omega_j^2(\boldsymbol{\theta})}\mathbf{d}_{\omega_j}(\boldsymbol{\theta})\mathbf{d}_{\omega_j}^{\mathrm{T}}(\boldsymbol{\theta}) - \frac{r_1}{\omega_j(\boldsymbol{\theta})}\mathbf{D}_{\omega_j}(\boldsymbol{\theta}) \\ &\quad + \frac{r_2}{\omega_k^2(\boldsymbol{\theta})}\mathbf{d}_{\omega_k}(\boldsymbol{\theta})\mathbf{d}_{\omega_k}^{\mathrm{T}}(\boldsymbol{\theta}) - \frac{r_2}{\omega_k(\boldsymbol{\theta})}\mathbf{D}_{\omega_k}(\boldsymbol{\theta})\end{aligned} \tag{31}$$

Using the asymptotic approximation (22), the joint moment of two natural frequencies can be obtained as

$$\begin{aligned}\mu_{jk}^{(r_1,r_2)} &\approx (2\pi)^{m/2}\omega_j^{r_1}(\boldsymbol{\theta})\omega_k^{r_2}(\boldsymbol{\theta}) \\ &\quad \times \exp\{-L(\boldsymbol{\theta})\} \left\| \mathbf{D}_f(\boldsymbol{\theta}) \right\|^{-1/2}\end{aligned} \tag{32}$$

The mean of the natural frequencies can be obtained by substituting $r_1 = 1$ and $r_2 = 0$ in Eq. (32) as

$$\begin{aligned}\mathrm{E}\left[\omega_j\right] = \mu_{jk}^{(1,0)} &\approx (2\pi)^{m/2}\omega_j(\boldsymbol{\theta}) \\ &\quad \times \exp\{-L(\boldsymbol{\theta})\} \left\| \mathbf{D}_f(\boldsymbol{\theta}) \right\|^{-1/2}\end{aligned} \tag{33}$$

where $\boldsymbol{\theta}$ is obtained from

$$\mathbf{d}_L(\boldsymbol{\theta}) = \mathbf{d}_{\omega_j}(\boldsymbol{\theta}) / \omega_j(\boldsymbol{\theta}) \quad \text{and} \tag{34}$$

$$\begin{aligned}\mathbf{D}_f(\boldsymbol{\theta}) &= \mathbf{D}_L(\boldsymbol{\theta}) + \left(\mathbf{d}_{\omega_j}(\boldsymbol{\theta})\mathbf{d}_{\omega_j}^{\mathrm{T}}(\boldsymbol{\theta}) / \omega_j(\boldsymbol{\theta}) \right. \\ &\quad \left. - \mathbf{D}_{\omega_j}(\boldsymbol{\theta})\right) / \omega_j(\boldsymbol{\theta})\end{aligned} \tag{35}$$

The elements of the covariance matrix of the natural frequencies can obtained as

$$\begin{aligned}\mathrm{Cov}(\omega_j, \omega_k) &= \mathrm{E}\left[\left(\omega_j - \mathrm{E}\left[\omega_j\right]\right)\left(\omega_k - \mathrm{E}\left[\omega_k\right]\right)\right] \\ &= \mu_{jk}^{(1,1)} - \mu_{jk}^{(1,0)}\mu_{jk}^{(0,1)}\end{aligned} \tag{36}$$

3.2 Arbitrary joint moments of multiple natural frequencies

The formulation presented in the previous subsection can be readily generalized to obtain arbitrary order joint moments of multiple natural frequencies. We want to obtain

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1, r_2, \ldots, r_n)} = \int_{\mathbb{R}^m} \left\{ \omega_{j_1}^{r_1}(\mathbf{x})\omega_{j_2}^{r_2}(\mathbf{x}) \cdots \omega_{j_n}^{r_n}(\mathbf{x}) \right\} p_{\mathbf{x}}(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{37}$$

By choosing the function

$$f(\mathbf{x}) = L(\mathbf{x}) - r_1 \ln \omega_{j_1}(\mathbf{x}) \\ -r_2 \ln \omega_{j_2}(\mathbf{x}) - \cdots - r_n \ln \omega_{j_n}(\mathbf{x}) \tag{38}$$

and applying the asymptotic approximation in Eq. (22) it can be shown that

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1, r_2, \ldots, r_n)} \approx (2\pi)^{m/2} \left\{ \omega_{j_1}^{r_1}(\boldsymbol{\theta})\, \omega_{j_2}^{r_2}(\boldsymbol{\theta}) \cdots \omega_{j_n}^{r_n}(\boldsymbol{\theta}) \right\} \\ \times \exp\{-L(\boldsymbol{\theta})\} \left\| \mathbf{D}_f(\boldsymbol{\theta}) \right\|^{-1/2} \tag{39}$$

where $\boldsymbol{\theta}$ is obtained by solving

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r_1}{\omega_{j_1}(\boldsymbol{\theta})} \mathbf{d}_{\omega_{j_1}}(\boldsymbol{\theta}) + \frac{r_2}{\omega_{j_2}(\boldsymbol{\theta})} \mathbf{d}_{\omega_{j_2}}(\boldsymbol{\theta}) \\ + \cdots + \frac{r_n}{\omega_{j_n}(\boldsymbol{\theta})} \mathbf{d}_{\omega_{j_n}}(\boldsymbol{\theta}) \tag{40}$$

and the Hessian matrix is given by

$$\mathbf{D}_f(\boldsymbol{\theta}) = \mathbf{D}_L(\boldsymbol{\theta}) + \sum_{\substack{j=j_1, j_2, \ldots \\ r=r_1, r_2, \ldots}}^{j_n, r_n} \frac{r}{\omega_j^2(\boldsymbol{\theta})} \mathbf{d}_{\omega_j}(\boldsymbol{\theta}) \mathbf{d}_{\omega_j}^{\mathrm{T}}(\boldsymbol{\theta}) \\ - \frac{r}{\omega_j(\boldsymbol{\theta})} \mathbf{D}_{\omega_j}(\boldsymbol{\theta}) \tag{41}$$

Equation (39) is perhaps the most general formula to obtain the moments of the natural frequencies of linear stochastic dynamic systems. Once the joint moments are known, the jpdfs of the natural frequencies can be obtained, for example, using the maximum entropy principle [18]. It should be recalled that whether the system parameters are Gaussian or not, the jpdf of the eigenvalues must expected to be non-Gaussian in general.

4 Computational approaches for the asymptotic method

The computational efficiency of the asymptotic method crucially depends on the computation of the optimal point $\boldsymbol{\theta} \in \mathbb{R}^m$. The elements of the vector $\boldsymbol{\theta}$ should be calculated by solving the coupled non-linear set of Eq. (29). Because the explicit analytical expression of $\mathbf{d}_{\omega_j}$ in terms of the derivative of the mass and stiffness matrices is available (see Appendix A), expensive numerical differentiation of $\omega_j(\mathbf{x})$ at each step is not needed. Since $p_{\mathbf{x}}(\mathbf{x})$ is available in closed-form, the expression of $\mathbf{d}_L(\mathbf{x})$ can be obtained easily by differentiating it successively. We illustrate the proposed method to systems with multivariate Gaussian random variables.

In this case $L(\mathbf{x})$ is given by Eq. (3) and by differentiating Eq. (3) successively we obtain

$$\mathbf{d}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \quad \text{and} \tag{42}$$

$$\mathbf{D}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1} \tag{43}$$

We are interested in the mean and covariance of the natural frequencies. For the elements of the covariance matrix of the natural frequencies, the optimal point $\boldsymbol{\theta}$ should be obtained by substituting $r_1 = 1$ and $r_2 = 1$ in Eq. (29) as

$$\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta} - \boldsymbol{\mu}) = \frac{\mathbf{d}_{\omega_j}(\boldsymbol{\theta})}{\omega_j(\boldsymbol{\theta})} + \frac{\mathbf{d}_{\omega_k}(\boldsymbol{\theta})}{\omega_k(\boldsymbol{\theta})} \quad \text{or} \tag{44}$$

$$\boldsymbol{\theta} = \boldsymbol{\mu} + \boldsymbol{\Sigma}\left[\frac{\mathbf{d}_{\omega_j}(\boldsymbol{\theta})}{\omega_j(\boldsymbol{\theta})} + \frac{\mathbf{d}_{\omega_k}(\boldsymbol{\theta})}{\omega_k(\boldsymbol{\theta})}\right] \tag{45}$$

This equation can be used to obtain $\boldsymbol{\theta}$ in an iterative manner by following these steps:

1. Select the initial guess as $\boldsymbol{\theta} = \boldsymbol{\mu}$ and an error tolerance ϵ.
2. Obtain an updated value of $\boldsymbol{\theta}$ from Eq. (45) as

$$\boldsymbol{\theta}^{(\text{new})} = \boldsymbol{\mu} + \boldsymbol{\Sigma}\left[\frac{\mathbf{d}_{\omega_j}(\boldsymbol{\theta})}{\omega_j(\boldsymbol{\theta})} + \frac{\mathbf{d}_{\omega_k}(\boldsymbol{\theta})}{\omega_k(\boldsymbol{\theta})}\right] \tag{46}$$

3. If $\mid \boldsymbol{\theta}^{(\text{new})} - \boldsymbol{\theta} \mid < \epsilon$ then select $\boldsymbol{\theta} = \boldsymbol{\theta}^{(\text{new})}$ and leave the iteration process.
4. Otherwise select $\boldsymbol{\theta} = \boldsymbol{\theta}^{(\text{new})}$ and continue from step 2.

The convergence of the above iteration scheme can neither be guaranteed nor be proved analytically. For every iteration step, the solution of a deterministic eigenvalue problem is required. If n_r is the number of iteration, then $n_r N(N+1)/2$ number of eigenvalue problems needs to be solved to obtain the complete covariance matrix.

For the problems considered in the numerical examples later in the paper, only the first few steps in the iterative procedure produced an acceptable results.

Note that the mean-centered perturbation (for which $\boldsymbol{\theta} = \boldsymbol{\mu}$) can be viewed as the zeroth order case in this iterative procedure. In the numerical examples in Sect. 5, only one step in the iteration procedure is used. For this case only one additional eigenvalue problem needs to be solved compared to the perturbation method. Substituting $L(\mathbf{x})$ from Eq. (3), the joint moment of the natural frequencies can now be obtained from Eq. (32) as

$$\mu_{jk}^{(1,1)} \approx \omega_j(\boldsymbol{\theta})\omega_k(\boldsymbol{\theta}) \exp\left\{-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\boldsymbol{\theta} - \boldsymbol{\mu})\right\} \\ \times \left\| \mathbf{I} + \widetilde{\mathbf{D}}_f(\boldsymbol{\theta}) \right\|^{-1/2} \tag{47}$$

where

$$\widetilde{\mathbf{D}}_f(\boldsymbol{\theta}) = \boldsymbol{\Sigma}^{-1}\left[\frac{\mathbf{d}_{\omega_j}(\boldsymbol{\theta})\mathbf{d}^{\mathrm{T}}_{\omega_j}(\boldsymbol{\theta})}{\omega_j^2(\boldsymbol{\theta})} - \frac{\mathbf{D}_{\omega_j}(\boldsymbol{\theta})}{\omega_j(\boldsymbol{\theta})} + \frac{\mathbf{d}_{\omega_k}(\boldsymbol{\theta})\mathbf{d}^{\mathrm{T}}_{\omega_k}(\boldsymbol{\theta})}{\omega_k^2(\boldsymbol{\theta})} - \frac{\mathbf{D}_{\omega_k}(\boldsymbol{\theta})}{\omega_k(\boldsymbol{\theta})}\right] \tag{48}$$

The optimal point for the mean of the natural frequencies can be obtained using a similar iterative procedure with

$$\boldsymbol{\theta}_1^{(\mathrm{new})} = \boldsymbol{\mu} + \frac{\boldsymbol{\Sigma}\mathbf{d}_{\omega_j}(\boldsymbol{\theta}_1)}{\omega_j(\boldsymbol{\theta}_1)} \tag{49}$$

Once $\boldsymbol{\theta}_1$ is obtained, the mean values can be calculated from

$$\mathrm{E}\left[\omega_j\right] = \omega_j(\boldsymbol{\theta}_1)\exp\left\{-\frac{1}{2}(\boldsymbol{\theta}_1-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}_1-\boldsymbol{\mu})\right\} \times \left\|\mathbf{I} + \frac{\boldsymbol{\Sigma}^{-1}\mathbf{d}_{\omega_j}(\boldsymbol{\theta}_1)\mathbf{d}^{\mathrm{T}}_{\omega_j}(\boldsymbol{\theta}_1)}{\omega_j^2(\boldsymbol{\theta}_1)} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{D}_{\omega_j}(\boldsymbol{\theta}_1)}{\omega_j(\boldsymbol{\theta}_1)}\right\|^{-1/2} \tag{50}$$

In the next section the proposed asymptotic method is applied to a 20 degree-of-freedom (dof) system with uncertain properties and a 3 dof random system with closely spaced natural frequencies.

5 Numerical examples

5.1 A 20 dof random system

5.1.1 System model and computational methodology

A linear system consisting of an array of spring-mass oscillators is considered to illustrate a possible use of the expressions developed so far. Figure 1 shows the system together with the numerical values assumed for the system parameters. Twenty masses, each of with nominal mass $\overline{m}_u$, are connected by springs of nominal stiffness value $\overline{k}_u$. The mass matrix of the system has the form $\mathbf{M} = \mathrm{diag}\left[m_{u_j}\right]$ and the stiffness matrix of the system is given by

$$\mathbf{K} = \begin{bmatrix} k_{u_1}+k_{u_2} & -k_{u_2} & & & \\ -k_{u_2} & k_{u_2}+k_{u_3} & -k_{u_3} & & \\ & \ddots & \ddots & \ddots & \\ & & & k_{u_{19}}+k_{u_{20}} & -k_{u_{20}} \\ & & & -k_{u_{20}} & k_{u_{20}} \end{bmatrix} \in \mathbb{R}^{20\times 20} \tag{51}$$

It is assumed that the mass and stiffness associated with all the units are random. Randomness associated with each unit has the following form

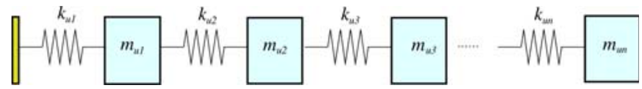


Fig. 1 Linear array of random spring-mass oscillators; $n = 20$, $m_{u_j} = \overline{m}_u\left(1+\epsilon_m x_j\right)$, $k_{u_j} = \overline{k}_u\left(1+\epsilon_k x_{20+j}\right)$, $\overline{m}_u = 1\,\mathrm{kg}$, $\overline{k}_u = 2{,}500\,\mathrm{N/m}$

$$\begin{aligned} m_{u_j} &= \overline{m}_u\left(1+\epsilon_m x_j\right) \\ k_{u_j} &= \overline{k}_u\left(1+\epsilon_k x_{20+j}\right) \end{aligned} \tag{52}$$

Here $\mathbf{x} = \{x_1, x_2, \ldots, x_{40}\}^{\mathrm{T}} \in \mathbb{R}^{40}$ is the vector of random variables. It is assumed that all random variables are Gaussian and uncorrelated with zero mean and unit standard deviation, that is $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$. Therefore, the mean values of m_{u_j} and k_{u_j} are given by $\overline{m}_u = 1\,\mathrm{kg}$ and $\overline{k}_u = 2{,}500\,\mathrm{N/m}$. The numerical values of the 'strength parameters' are assumed to be $\epsilon_m = 0.15$ and $\epsilon_k = 0.20$, that is the randomness associated with mass and stiffness values are 15 and 20%, respectively.

In order to obtain the statistics of the natural frequencies using the methods developed in this paper, the gradient vector and the Hessian matrix of the natural frequencies are required. As shown in Appendix A, this in turn requires the derivative of the system matrices with respect to the entries of $\mathbf{x}$. For most practical problems, which usually involve Finite Element modeling, these derivatives need to be determined numerically. However, for this relatively simple example the derivatives can be obtained in closed-form. For the mass matrix we have

$$\frac{\partial \mathbf{M}}{\partial x_i} = \overline{m}_u\epsilon_m \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1^{\{i\text{th diagonal}\}} & 0 \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{20\times 20} \tag{53}$$

and $\frac{\partial \mathbf{M}}{\partial x_i} = \mathbf{O}$ when $i > 20$. For the stiffness matrix, $\frac{\partial \mathbf{K}}{\partial x_i} = \mathbf{O}$ when $i \le 20$,

$$\frac{\partial \mathbf{K}}{\partial x_{21}} = \overline{k}_u\epsilon_k \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{20\times 20} \quad \text{and} \tag{54}$$

$$\frac{\partial \mathbf{K}}{\partial x_{20+i}} = \overline{k}_u\epsilon_k \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & 1^{\{(i-1)\text{th row}\}} & -1^{\{i\text{th column}\}} & 0 \\ \vdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{20\times 20}, \quad 2 \le i \le 20 \tag{55}$$

We calculate the joint moments and jpdfs of the natural frequencies of the system. Attention is restricted

Table 1 Deterministic and mean values of the first ten natural frequencies

Deterministic	First-order perturbation	Second-order perturbation	Proposed method	MCS (15 k samples)
3.8303	3.8303 (2.1130)	3.7588 (0.2072)	3.7598 (0.2335)	3.7510
11.4683	11.4683 (2.0845)	11.2544 (0.1805)	11.2573 (0.2060)	11.2342
19.0391	19.0391 (2.0718)	18.6845 (0.1709)	18.6894 (0.1971)	18.6527
26.4982	26.4982 (2.0522)	26.0058 (0.1561)	26.0126 (0.1822)	25.9653
33.8017	33.8017 (2.0398)	33.1757 (0.1501)	33.1843 (0.1761)	33.1260
40.9069	40.9069 (1.9997)	40.1527 (0.1192)	40.1630 (0.1450)	40.1049
47.7720	47.7720 (1.9829)	46.8964 (0.1137)	46.9082 (0.1389)	46.8431
54.3568	54.3568 (1.9073)	53.3682 (0.0539)	53.3818 (0.0794)	53.3394
60.6225	60.6225 (1.8299)	59.5312 (−0.0032)	59.5460 (0.0216)	59.5331
66.5326	66.5326 (1.7557)	65.3511 (−0.0513)	65.3670 (−0.0270)	65.3846

The numbers in the parenthesis correspond to the percentage error with respect to the MCS with 15 k samples

Table 2 Standard deviation of the first ten natural frequencies

First-order perturbation	Second-order perturbation	Proposed method	MCS (15 k samples)
0.1295 (−7.3038)	0.1327 (−5.0327)	0.1436 (2.8147)	0.1397
0.3878 (−8.0871)	0.3982 (−5.6108)	0.4088 (−3.1070)	0.4219
0.6438 (−6.9767)	0.6643 (−4.0072)	0.6784 (−1.9777)	0.6920
0.8960 (−6.1678)	0.9315 (−2.4480)	0.9418 (−1.3643)	0.9549
1.1429 (−6.3319)	1.2006 (−1.6091)	1.1995 (−1.6991)	1.2202
1.3832 (−6.4917)	1.4727 (−0.4423)	1.4511 (−1.8993)	1.4792
1.6153 (−5.1973)	1.7487 (2.6306)	1.6933 (−0.6200)	1.7038
1.8379 (−5.5455)	2.0362 (4.6426)	1.9321 (−0.7054)	1.9459
2.0498 (−4.5226)	2.3324 (8.6411)	2.1534 (0.3039)	2.1469
2.2496 (−4.0835)	2.6462 (12.8241)	2.3583 (0.5508)	2.3454

The numbers in the parenthesis correspond to the percentage error with respect to the MCS with 15 k samples

up to the second-order joint statistics of two natural frequencies. Following four methods are used to obtain the joint moments and the jpdfs:

1. *First-order perturbation*: For this case the mean and covariance matrix of the natural frequencies are calculated using Eqs. (16) and (18) by substituting the Hessian matrices $\mathbf{D}_{\omega_j} = \mathbf{O}$ and $\mathbf{D}_{\omega_k} = \mathbf{O}$. Recalling that for this problem $\boldsymbol{\Sigma} = \mathbf{I}$, the resulting statistics for this special case can be obtained as

$$\mathrm{E}\left[\omega\right]_j = \overline{\omega}_j \quad \text{and} \tag{56}$$

$$\mathrm{Cov}\left(\omega_j, \omega_k\right) = \mathbf{d}_{\omega_j}^{\mathrm{T}} \mathbf{d}_{\omega_k} \tag{57}$$

The gradient vector $\mathbf{d}_{\omega_j}$ can be obtained from Eq. (73) using the system derivative matrices given by Eqs. (53)–(55).

2. *Second-order perturbation*: In this case the Hessian matrices $\mathbf{D}_{\omega_j}$ and $\mathbf{D}_{\omega_k}$ are used in calculating the joint statistics of the natural frequencies using Eqs. (16) and (18). The elements of the Hessian matrices $\mathbf{D}_{\omega_j}$ and $\mathbf{D}_{\omega_k}$ can be calculated using Eq. (75). The resulting statistics for this special case can be obtained as

$$\mathrm{E}\left[\omega\right]_j = \overline{\omega}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\omega_j}\right) \quad \text{and} \tag{58}$$

$$\mathrm{Cov}\left(\omega_j, \omega_k\right) = \mathbf{d}_{\omega_j}^{\mathrm{T}} \mathbf{d}_{\omega_k} + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\omega_j}\mathbf{D}_{\omega_k}\right) \tag{59}$$

Comparing these results with Eqs. (56) and (57), the contributions of the Hessian matrices can be regarded as the corrections to the first-order perturbation results.

3. *Method based on the asymptotic integral*: In this case the mean and covariance matrix of the natural frequencies are calculated using Eqs. (33) and (36). The function $L(\mathbf{x})$ can be obtained by substituting $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$ in Eq. (3) as

$$L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x} \tag{60}$$

The gradient vector and the Hessian matrix of $L(\mathbf{x})$, needed to apply this method, are given by

$$\mathbf{d}_L(\mathbf{x}) = \mathbf{x} \qquad \text{and} \qquad \mathbf{D}_L(\mathbf{x}) = \mathbf{I} \tag{61}$$

In order to calculate the optimal points, we have used only one step in the iterative procedure outlined in Sect. 4. Recall that with one step iteration process, only one additional deterministic eigenvalue needs to be solved to obtain the optimal point $\boldsymbol{\theta}$.

4. *Monte Carlo simulation* (*MCS*): The samples of the 40 independent Gaussian random variables $x_i, i = 1, \ldots, 40$ are generated and the natural frequencies are computed directly from Eq. (1). A total of 15,000 samples are used to obtain the statistical moments and histograms for the jpdf of the natural frequencies. The results obtained from MCS are assumed to be the benchmark for the purpose of comparing the analytical methods.

5.1.2 Numerical results

The nominal undamped natural frequencies of the system are uniformly spaced and range from near 3.8 to approximately 100 rad/s. Among the 20 natural frequencies, we consider only the first ten for statistical analysis. Table 1 shows the deterministic values and the mean of the first ten natural frequencies obtained using the first-order perturbation method, second-order perturbation method, proposed asymptotic method and MCS with 15 k samples. Table 2 shows the standard deviation the first ten natural frequencies obtained using the four methods discussed before. Percentage error associated with the computed values are also shown in Tables 1 and 2. For the analytical methods, the percentage error associated with any quantity is calculated as

$$\text{Error}_{i\text{th method}} = \frac{[(\bullet)_{i\text{th method}} - (\bullet)_{\text{MCS}}]}{(\bullet)_{\text{MCS}}} \times 100, \quad i = 1, \ldots, 3 \tag{62}$$

Using the proposed asymptotic method, the covariance matrix of the first ten natural frequencies is obtained as

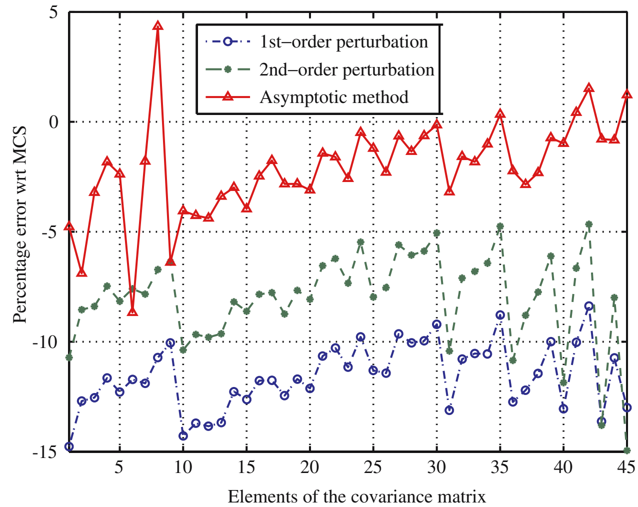


Fig. 2 Percentage error with respect to Monte Carlo simulation (MCS, 15 k samples) in the elements of the covariance matrix. Only the consecutive rows of the *triangular* part above the diagonal corresponding to Eq. (63) are shown

Due to the symmetry of the covariance matrix, only the elements of the upper triangular part are shown above. The square-root of the diagonal elements of the above matrix are the standard deviations of the natural frequencies which are also shown in Table 2. Figure 2 shows the percentage error with respect to MCS in the elements of the upper triangular part of the covariance matrix of the natural frequencies. For the mean and the standard deviation, the first-order perturbation method is the least accurate, followed by the second-order perturbation method. The same fact is also mostly true for the elements of the covariance matrix. For all the calculations, the asymptotic method is clearly the most accurate among the three analytical methods used in this study.

Now we consider the pdf of the natural frequencies. Because the asymptotic method is the most accurate among the three methods discussed here, we will only pursue this method in the remaining discussions.

$$\begin{bmatrix}
0.0206 & 0.0374 & 0.0593 & 0.0856 & 0.1097 & 0.1329 & 0.1443 & 0.1769 & 0.2069 & 0.2021 \\
 & 0.1671 & 0.1863 & 0.2570 & 0.3279 & 0.4002 & 0.4618 & 0.5222 & 0.5858 & 0.6475 \\
 & & 0.4602 & 0.4268 & 0.5398 & 0.6546 & 0.7648 & 0.8652 & 0.9647 & 1.0649 \\
 & & & 0.8871 & 0.7605 & 0.9114 & 1.0609 & 1.2040 & 1.3513 & 1.4779 \\
 & & & & 1.4387 & 1.1743 & 1.3579 & 1.5369 & 1.7286 & 1.8855 \\
 & & & & & 2.1057 & 1.6689 & 1.8753 & 2.0850 & 2.2882 \\
 & & & & & & 2.8672 & 2.2539 & 2.4638 & 2.6840 \\
 & & & & & & & 3.7331 & 2.8848 & 3.0627 \\
 & & & & & & & & 4.6372 & 3.5769 \\
 & & & & & & & & & 5.5617
\end{bmatrix} \tag{63}$$

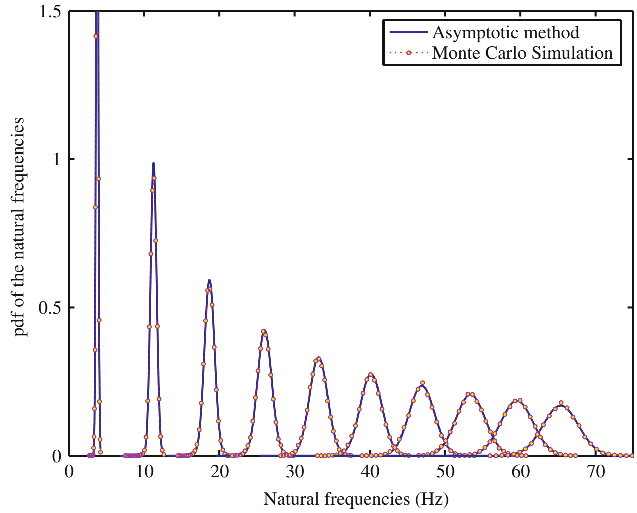


Fig. 3 Probability density function of the first ten natural frequencies of the linear spring-mass system

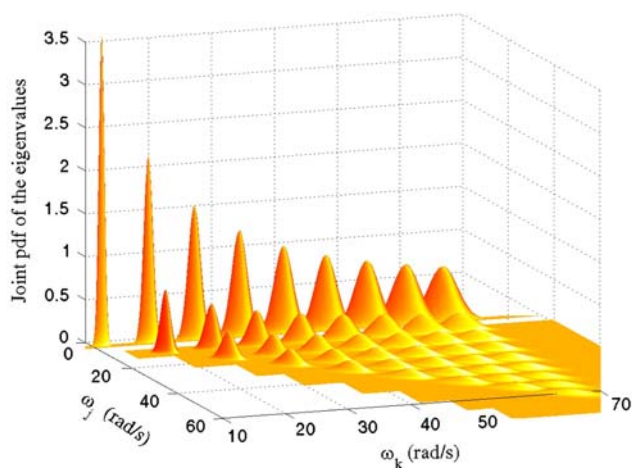


Fig. 4 Fitted joint Gaussian probability density function of the natural frequencies using asymptotic method

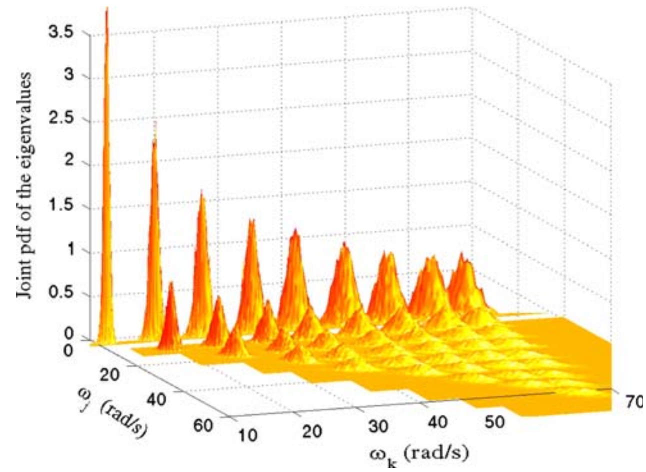


Fig. 5 Joint probability density function of the natural frequencies from MCS

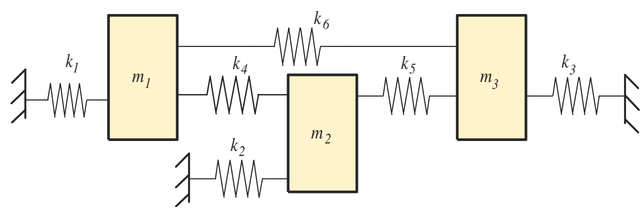


Fig. 6 The three dof random system. $\overline{m}_i = 1.0\,\text{kg}$ for $i = 1, 2, 3$; $\overline{k}_i = 1.0\,\text{N/m}$ for $i = 1, \ldots, 5$ and $\overline{k}_6 = 1.5\,\text{N/m}$

Gaussian distributions are fitted with the mean and standard deviation of the natural frequencies given in Tables 1 and 2 and compared with MCS. The marginal pdf of the first ten natural frequencies obtained from the asymptotic method and MCS are shown in Fig. 3. Each MCS pdf in Fig. 3 is obtained by normalizing the histogram of the samples so that the area under the curve obtained by joining the middle points of the histogram bins is equal to unity. The Gaussian distributions calculated from the asymptotic method fit quite well to the MCS. This result implies that the pdf of the individual natural frequencies may be approximated well using a Gaussian distribution with correct set of parameters.

Now we focus on the joint distribution of the natural frequencies. In line with the univariate Gaussian distributions shown in Fig. 3, we can obtain bivariate Gaussian distribution for each pair of natural frequencies. Joint probability density function of the natural frequencies obtained from the asymptotic method and MCS are shown in Figs. 4 and 5. In total 55 joint distributions, $p_{\omega_j,\omega_k}, j = 1, \ldots, 10, k = j, \ldots, 10$ are shown in Figs. 4 and 5. Each analytical jpdf in Fig. 4 is obtained by fitting a bivariate Gaussian distribution with the mean vector and covariance matrix taken from Table 1 and Eq. (63) for the corresponding set of natural frequencies. The MCS pdf in Fig. 5 is obtained by normalizing the two dimensional histogram of the samples so that the volume under the surface obtained by joining the middle points of the histogram bins is equal to unity. It is difficult to compare two 3D plots directly, however one can see similar trends in Figs. 4 and 5.

5.2 A three DOF system with closely spaced natural frequencies

5.2.1 System model

A three-dof undamped spring-mass system, taken from reference [10], is shown in Fig. 6. The main purpose of this example is to understand how the proposed methods work when some of the system natural frequencies are closely spaced. This is an interesting case because it is well known that closely spaced eigenvalues are parameter sensitive. We will investigate how the parameter uncertainty affects the joint eigenvalue distribution in such cases. This study has particular relevance to the dynamics of nominally symmetric rotating machines, for example, turbine blades with random imperfections. The mass and stiffness matrices of the example system are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and}$$

$$\mathbf{K} = \begin{bmatrix} k_1+k_4+k_6 & -k_4 & -k_6 \\ -k_4 & k_4+k_5+k_2 & -k_5 \\ -k_6 & -k_5 & k_5+k_3+k_6 \end{bmatrix} \tag{64}$$

It is assumed that all mass and stiffness constants are random. The randomness in these parameters are assumed to be of the following form:

$$m_i = \overline{m}_i\,(1+\epsilon_m x_i)\,, \quad i=1,2,3 \tag{65}$$

$$k_i = \overline{k}_i\,(1+\epsilon_k x_{i+3})\,, \quad i=1,\ldots,6 \tag{66}$$

Here $\mathbf{x} = \{x_1,\ldots,x_9\}^{\mathrm{T}} \in \mathbb{R}^9$ is the vector of standard Gaussian random variables, $\epsilon_m = 0.15$, $\epsilon_k = 0.20$ and values of $\overline{m}_i$ and $\overline{k}_i$ are shown in the caption of Fig. 6.

5.2.2 Numerical results

For the given parameter values the natural frequencies (in rad/s) of the corresponding deterministic system is given by

$$\overline{\omega}_1 = 1, \quad \overline{\omega}_2 = 2, \quad \text{and} \quad \overline{\omega}_3 = 2.2361 \tag{67}$$

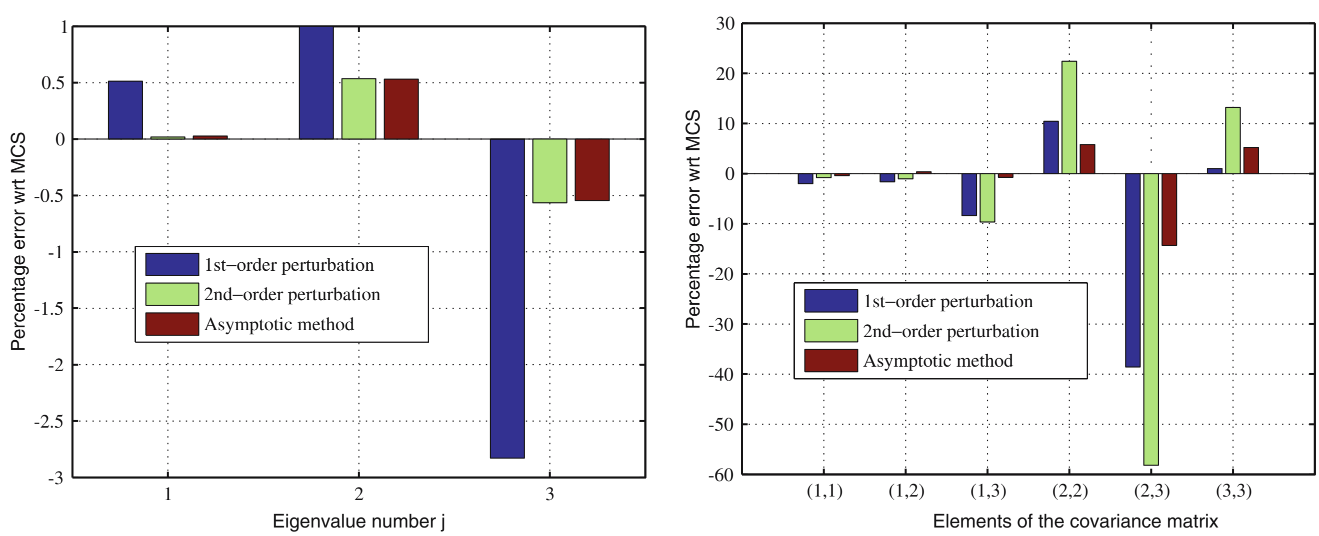
Figure 7 shows percentage error with respect to MCS in the elements of the mean vector and covariance matrix of the natural frequencies. The general trend of these errors are similar to the previous example except that the magnitudes of the errors corresponding to second and third natural frequencies are significantly higher compared to the first one. This is expected because these two natural frequencies are close to each other.

Now consider the pdf of the natural frequencies. Only the asymptotic method will be considered because from Fig. 7 it is clear that this is the most accurate among the three methods discussed here. First we focus on the marginal pdf of the natural frequencies. Using the asymptotic method, the mean and standard deviation of the natural frequencies are obtained as

$$\boldsymbol{\mu}_\omega = \{0.9952,\ 1.9847,\ 2.2886\}^{\mathrm{T}} \quad \text{and} \tag{68}$$

$$\boldsymbol{\sigma}_\omega = \{0.0728,\ 0.1508,\ 0.1884\}^{\mathrm{T}} \tag{69}$$

Gaussian distributions are fitted with these parameters and compared with MCS. The marginal pdf of the natural frequencies obtained from the asymptotic method and MCS are shown in Fig. 8. The Gaussian distributions calculated from the asymptotic method fit quite well to the MCS. This result implies that the pdf of the individual natural frequencies can be approximated reasonably well using a Gaussian distribution even when the natural frequencies are closely spaced.



(a) Error in the mean values.

(b) Error in the elements of the upper triangular part of the covariance matrix.

Fig. 7 Percentage error with respect to MCS (15 k samples)

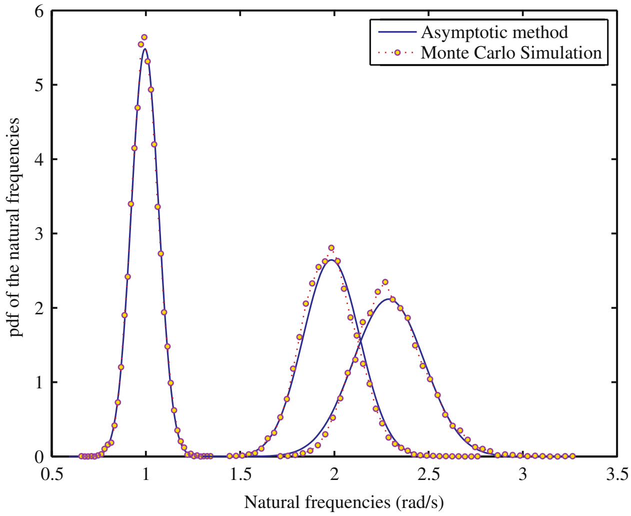


Fig. 8 Probability density function of the natural frequencies of the three dof system

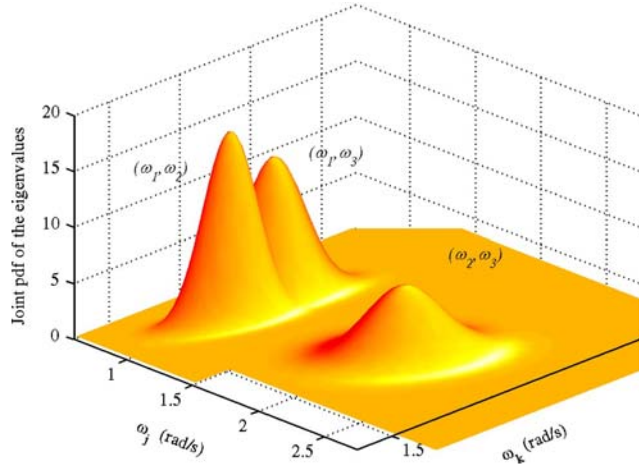


Fig. 9 Fitted joint Gaussian probability density function of the natural frequencies using asymptotic method

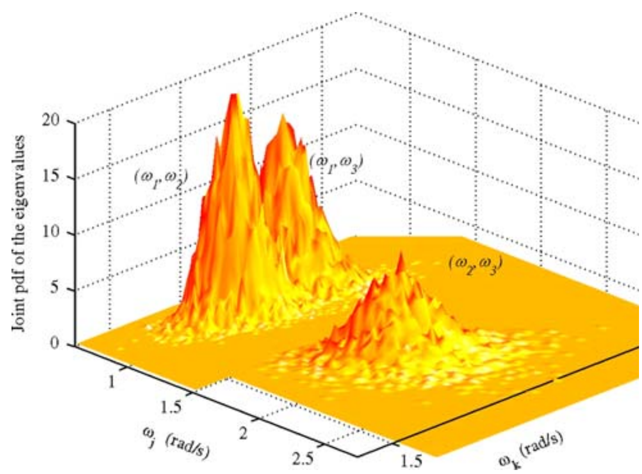


Fig. 10 Joint probability density function of the natural frequencies from MCS

Now we focus on the joint distribution of the natural frequencies. The covariance matrix and the matrix of correlation coefficients were obtained using the asymptotic method as

$$\mathbf{\Sigma}_\omega = \begin{bmatrix} 0.5293 & 0.5528 & 0.6157 \\ 0.5528 & 2.2752 & 1.1049 \\ 0.6157 & 1.1049 & 3.5483 \end{bmatrix} \times 10^{-2} \tag{70}$$

and

$$\boldsymbol{\rho}_\omega = \begin{bmatrix} 1.0000 & 0.5037 & 0.4493 \\ 0.5037 & 1.0000 & 0.3889 \\ 0.4493 & 0.3889 & 1.0000 \end{bmatrix} \tag{71}$$

This indicates that the natural frequencies are moderately correlated. The correlation between ω_1 and ω_2 is more than that between ω_1 and ω_3. This is expected because from ω_1, ω_3 is more distant than ω_2. However, the correlation between ω_1 and ω_3 is more than that between ω_2 and ω_3 in spite of ω_1 being further from ω_3 compared to ω_2.

Joint probability density function of the natural frequencies obtained from the asymptotic method and MCS are shown in Figs. 9 and 10. In total three joint distributions, namely p_{ω_1,ω_2}, p_{ω_1,ω_3} and p_{ω_2,ω_3} are shown in Figs. 9 and 10. Each analytical jpdf in Fig. 9 is obtained by fitting a bivariate Gaussian distribution with the mean vector and covariance matrix taken from Eqs. (68) and (70) for the corresponding set of natural frequencies. At first it may appear that, like the marginal pdfs in Fig. 8, the jpdfs of the natural frequencies are also jointly Gaussian distributed. But a closer inspection reveals that this is not always the case. Figure 11 compares the contours of the analytical jpdf with that obtained from MCS. The adjacent natural frequencies, that is, ω_1 and ω_2 and ω_2 and ω_3 are not jointly Gaussian distributed as the contours of the analytical jpdf is quite different from that obtained using MCS. The jpdf of ω_1 and ω_3 is however close to a bivariate Gaussian density function. The important conclusion that can be drawn from this limited numerical results is that the natural frequencies are in general not *jointly* Gaussian distributed although *individually* they may be. Further research is however required to investigate the generality of this conclusion.

Another factor influences the Gaussian nature of the eigenvalues is the number of random variables in the system. For a system with large number of random variables, the jpdf of the eigenvalues may be close to the Gaussian distribution due to the central limit theorem. In the example in Subsect. 5.1 there are 40 random variables as opposed to only 9 random variables appearing in this example. As a result, the jpdf of the eigenvalues

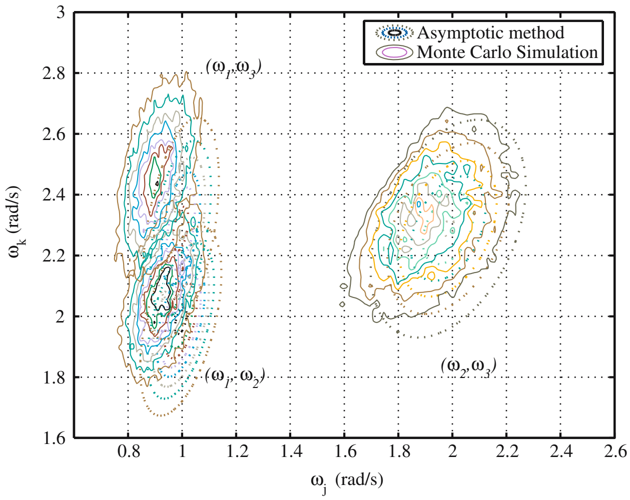


Fig. 11 Contours of the joint probability density function of the natural frequencies

in the previous example is more close to Gaussian than what is obtained here.

6 Conclusions

Joint statistics of the natural frequencies of discrete linear dynamic systems with parameter uncertainties have been considered. It is assumed that the mass and stiffness matrices are smooth and at least twice differentiable functions of the random variables describing the uncertainty of the system. The random variables are in general considered to be non-Gaussian and correlated. The usual assumption of small randomness is not employed in this study. A new approach based on asymptotic evaluation of multidimensional integrals has been proposed to obtain joint statistics of the natural frequencies. A closed-form asymptotically correct expression for the general order joint moments of arbitrary number of natural frequencies of linear stochastic systems with general non-Gaussian distribution has been derived.

The proposed formulae are applied to a 20 dof spring-mass system and a 3 dof system with closely spaced natural frequencies. The mean, covariance and the jpdf of the natural frequencies match well with the corresponding MCS results. However, when some natural frequencies are closely spaced, the proposed methods do not produce very accurate results. It was observed that for the system with closely spaced natural frequencies, the natural frequencies are not jointly Gaussian distributed although their marginal pdfs are Gaussian distributed.

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Appendix

A The gradient vector and the hessian matrix of the natural frequencies

The eigenvectors of symmetric linear systems are orthogonal with respect to the mass and stiffness matrices. We normalize the eigenvectors such that

$$\boldsymbol{\phi}_j^{\mathrm{T}} \mathbf{M} \boldsymbol{\phi}_j = 1 \tag{72}$$

Using this and differentiating Eq. (1) with respect to x_k it can be shown that [9] for any $\mathbf{x}$

$$\frac{\partial \omega_j(\mathbf{x})}{\partial x_k} = \frac{\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x})}{2\omega_j(\mathbf{x})} \tag{73}$$

$$\text{where} \quad \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) = \left[\frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_k} - \omega_j^2(\mathbf{x}) \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k} \right] \tag{74}$$

Differentiating Eq. (1) with respect to x_k and x_l Plaut and Huseyin [28] have shown that, providing the natural frequencies are distinct,

$$\frac{\partial^2 \omega_j(\mathbf{x})}{\partial x_k\, \partial x_l} = \left[\frac{1}{2\omega_j(\mathbf{x})} \frac{\partial^2 \left(\omega_j^2(\mathbf{x})\right)}{\partial x_k\, \partial x_l} - \frac{1}{\omega_j(\mathbf{x})} \frac{\partial \omega_j(\mathbf{x})}{\partial x_l} \frac{\partial \omega_j(\mathbf{x})}{\partial x_k} \right] \tag{75}$$

where

$$\begin{aligned} \frac{\partial^2 \left(\omega_j^2(\mathbf{x})\right)}{\partial x_k\, \partial x_l} &= \boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}} \left[\frac{\partial^2 \mathbf{K}(\mathbf{x})}{\partial x_k\, \partial x_l} - \omega_j^2(\mathbf{x}) \frac{\partial^2 \mathbf{M}(\mathbf{x})}{\partial x_k\, \partial x_l} \right] \boldsymbol{\phi}_j(\mathbf{x}) \\ &- \left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}} \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k} \boldsymbol{\phi}_j(\mathbf{x}) \right) \left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{jl}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right) \\ &- \left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}} \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_l} \boldsymbol{\phi}_j(\mathbf{x}) \right) \left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right) \\ &+ 2 \sum_{r=1}^{N} \frac{\left(\boldsymbol{\phi}_r(\mathbf{x})^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right) \left(\boldsymbol{\phi}_r(\mathbf{x})^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{jl}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right)}{\omega_j^2(\mathbf{x}) - \omega_r^2(\mathbf{x})} \end{aligned} \tag{76}$$

Equations (73) and (75) completely define the elements of the gradient vector and Hessian matrix of the natural frequencies.

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