Supplementary material - The eigenbuckling analysis of hexagonal lattices: Closed-form solutions

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ABSTRACT

This document explains how the equivalent elastic coefficients appearing in the elasticity matrix of the 2D hexagonal lattices are obtained.

Contents

1 Introduction

The elements of the lattice structure are comprised of beams undergoing axial and bending deformations. This document provides detailed mathematical explanations on how the equivalent elastic coefficients appearing in the elasticity matrix of the 2D hexagonal lattices are obtained. This is important as the buckling analysis proposed in the main paper depends on the spectral decomposition of the elasticity matrix. The outline of this document is as follows. The general expressions of the equivalent elastic coefficients are discussed in section 2. These general expressions are applied to the case of small deformation of lattices comprised of Euler-Bernoulli beams in section 3. In section 4 the exact stiffness matrix of axially loaded Euler-Bernoulli beams is derived using a transcendental displacement function. Using this stiffness matrix, the equivalent nonlinear elastic moduli arising due to compressive stress are obtained in section 5. The new analytical results are validated using an independent finite element situation in section 6. In section 7 the works presented in this document are summarised.

2 Elastic moduli of 2D hexagonal lattices

In this document, we express equivalent in-plane elastic properties of the hexagonal lattice in terms of the stiffness matrix elements of the constituent beams. For the case of equivalent properties of the lattice without the axial force, we refer to well-known references [1, 2]. In Figure 1 a representative example of a hexagonal lattice and its corresponding unit cell are shown. The equivalent elastic properties of a lattice can be obtained by exploiting the mechanics of the unit cell. The unit cell is selected such that it represents the whole lattice under tessellation in both directions. Each of the cell walls will bend and compress when subjected to in-plane compressive stresses. When the applied stress is uniform along the out-of-plane direction, each element of the unit cell in Figure 1(b) can be modelled as a beam. In Figure 1(c) a general beam element with six degrees of freedom and two nodes is shown. The stiffness matrix of the beam element can be expressed by a 6×6 matrix with degrees of freedom in each node corresponding to the axial, transverse and rotational deformation.

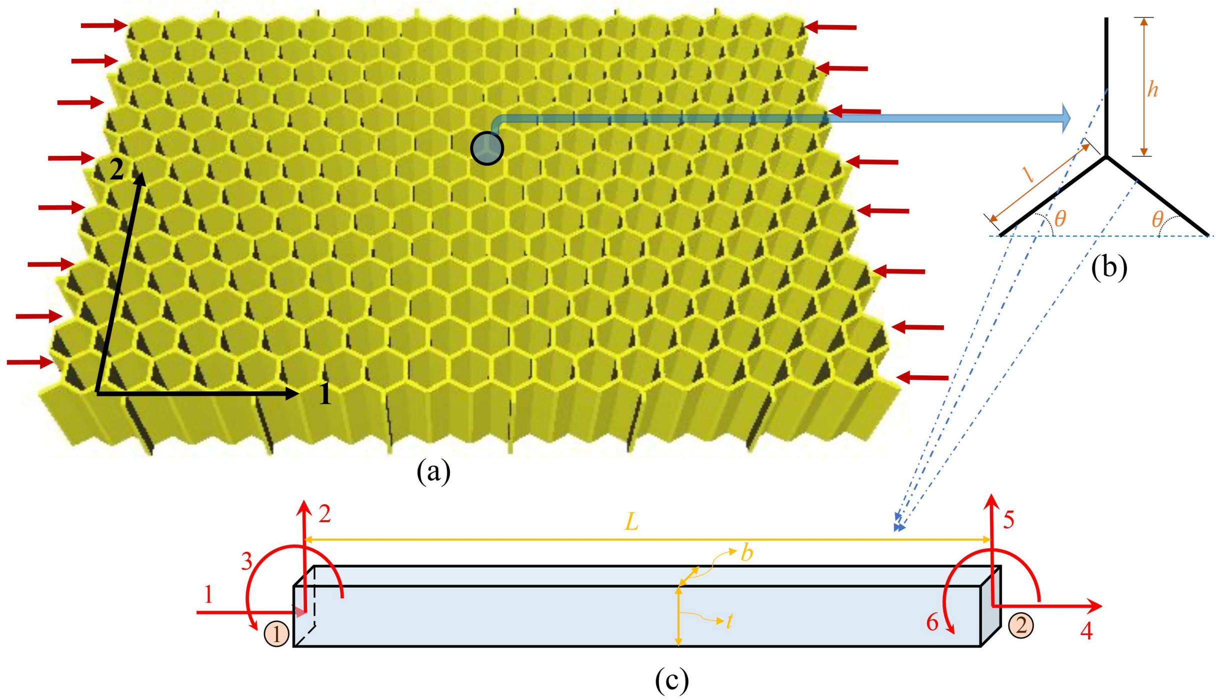


Figure 1. (a) Illustration of a hexagonal lattice subjected to compressive stress in direction-1, (b) The unit cell used to analyse the mechanics of the lattice. It comprised of three beams connected at one point. (c) A beam element (representing each of the three beams in the unit cell model) with six degrees of freedom and two nodes. The degrees of freedom in each node corresponds to the axial, transverse and rotational deformation.

Considering only the bending deformation and ignoring any stretching/shortening deformations, the equivalent elastic moduli of hexagonal cellular materials are obtained by Gibson and

Ashby [1] as

$$E_{1_{\mathrm{GA}}} = E\alpha^3 \frac{\cos\theta}{(\beta+\sin\theta)\sin^2\theta} \tag{1}$$

$$E_{2_{\mathrm{GA}}} = E\alpha^3 \frac{(\beta+\sin\theta)}{\cos^3\theta} \tag{2}$$

$$\nu_{12_{\mathrm{GA}}} = \frac{\cos^2\theta}{(\beta+\sin\theta)\sin\theta} \tag{3}$$

$$\nu_{21_{\mathrm{GA}}} = \frac{(\beta+\sin\theta)\sin\theta}{\cos^2\theta} \tag{4}$$

$$\text{and} \quad G_{12_{\mathrm{GA}}} = E\alpha^3 \frac{(\beta+\sin\theta)}{\beta^2(1+2\beta)\cos\theta} \tag{5}$$

Here E is the elastic modulus of the base material, θ is the cell angle as shown in Figure 1(b), α and β are geometric non-dimensional ratios given by

$$\alpha = \frac{t}{l} \quad \text{(thickness ratio)} \tag{6}$$

$$\text{and} \quad \beta = \frac{h}{l} \quad \text{(height ratio)} \tag{7}$$

When external stress is applied to a cellular material as depicted in Figure 1(a), it results in forces and moments on the unit cell shown in Figure 1(b). The deformation of the unit cell due to the applied stress can be obtained using the coefficients of the stiffness matrix of the typical element shown in Figure 1(c). A general derivation of the equivalent in-plane elastic properties of 2D lattices is presented in [3, 4] in terms of the element of the stiffness matrix $\mathbf{K} \in \mathbb{R}^{6\times 6}$ of the constituent beams. Following the analytical derivation in [3, 4], the exact expressions of the five elastic constants are given by the following closed-form formulae:

$$E_1 = \frac{K_{55}\cos\theta}{b(\beta+\sin\theta)\sin^2\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)} \tag{8}$$

$$E_2 = \frac{K_{55}(\beta+\sin\theta)}{b\cos^3\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)} \tag{9}$$

$$\nu_{12} = \frac{\cos^2\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{(\beta+\sin\theta)\sin\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)} \tag{10}$$

$$\nu_{21} = \frac{(\beta+\sin\theta)\sin\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{\cos^2\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)} \tag{11}$$

and

$$G_{12} = \frac{(\beta + \sin\theta)}{b\cos\theta} \frac{1}{\left(-\frac{h^2}{2lK_{65}} + \frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)} - \left(K_{56}^{(h/2)}\right)^2\right)} + \frac{(\cos\theta + (\beta+\sin\theta)\tan\theta)^2}{K_{44}} \right)} \quad (12)$$

In the above equations, b is the depth of the lattice and K_{ij} are the ij-th element of the stiffness matrix of the beam element corresponding to the inclined member in the unit cell shown in Figure 1(b). The notation $(\bullet)^{h/2}$ denotes stiffness matrix coefficients of the beam element corresponding to the vertical member in the unit cell shown in Figure 1(b) with half the length (that is $h/2$). In the rest of this document, these general expressions will be employed to obtain explicit expressions of the equivalent elastic properties of the hexagonal lattice.

3 The special case of small deformation

In the previous section, the expressions of five quantities characterising the effective in-plane elastic properties of 2D cellular materials have been expressed in terms of the stiffness elements of a beam. The stiffness matrix of an Euler-Bernoulli beam element [5, 6] is expressed by

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (13)$$

We are considering beam elements with rectangular cross section as shown in Figure 1. The moment of inertia and the cross section area appearing in the stiffness matrix in equation (13) are therefore given by

$$I = \frac{1}{12}bt^3 \quad \text{and} \quad A = bt \quad (14)$$

From the expressions in the previous section, it can be observed that two coefficients of the 6×6 element stiffness matrix of the inclined member and one coefficients of the 6×6 element stiffness matrix of vertical member, namely, K_{55}, K_{44} and $K_{44}^{(h)}$, are necessary to obtain E_1, E_2 ν_{12} and ν_{21}. Using the expressions of the moment of inertia and the cross-sectional area in equation (14), the stiffness coefficients are given by

$$K_{55} = \frac{12EI}{l^3} = Eb\alpha^3, \quad K_{44} = \frac{EA}{l} = Eb\alpha \quad \text{and} \quad K_{44}^{(h)} = \frac{EA}{h} = \frac{Ebt}{h} = \frac{Eb\alpha}{\beta} \quad (15)$$

Using these, we obtain the ratios

$$\frac{K_{55}}{K_{44}} = \alpha^2 \quad \text{and} \quad \frac{K_{55}}{K_{44}^{(h)}} = \alpha^2\beta \quad (16)$$

Therefore, when the Euler-Bernoulli beam stiffness elements are used, from equations (8), (9), (10) and (11) we have

$$E_1 = \frac{K_{55}\cos\theta}{b(\beta+\sin\theta)\sin^2\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)} = \frac{E\alpha^3\cos\theta}{(\beta+\sin\theta)\left(\sin^2\theta+\alpha^2\cos^2\theta\right)} \tag{17}$$

$$E_2 = \frac{K_{55}(\beta+\sin\theta)}{b\cos^3\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)} = \frac{E\alpha^3(\beta+\sin\theta)}{(1-\alpha^2)\cos^3\theta+\alpha^2(2\beta+1)\cos\theta} \tag{18}$$

$$\nu_{12} = \frac{\cos^2\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{(\beta+\sin\theta)\sin\theta\left(1+\cot^2\theta\frac{K_{55}}{K_{44}}\right)} = \frac{\cos^2\theta\left(1-\alpha^2\right)}{(\beta+\sin\theta)\sin\theta\left(1+\alpha^2\cot^2\theta\right)} \tag{19}$$

$$\nu_{21} = \frac{(\beta+\sin\theta)\sin\theta\left(1-\frac{K_{55}}{K_{44}}\right)}{\cos^2\theta\left(1+\tan^2\theta\frac{K_{55}}{K_{44}}+2\sec^2\theta\frac{K_{55}}{K_{44}^{(h)}}\right)} = \frac{(\beta+\sin\theta)\sin\theta\left(1-\alpha^2\right)}{(1-\alpha^2)\cos^2\theta+\alpha^2(2\beta+1)} \tag{20}$$

For the shear modulus, five elements from two different stiffness matrices are necessary. They are two coefficients of the 6×6 element stiffness matrix of the inclined member, namely, K_{65}, K_{44} as in equation (15) with $K_{65} = -6\frac{EI}{l^2} = -1/2\,\frac{Ebt^3}{l^2}$. We also need three elements of the stiffness matrix of the vertical member with half the length given by

$$K_{55}^{(h/2)} = \frac{12EI}{(h/2)^3} = \frac{8Ebt^3}{h^3}, \quad K_{56}^{(h/2)} = -\frac{6EI}{(h/2)^2} = -\frac{2Ebt^3}{h^2} \quad \text{and} \quad K_{66}^{(h/2)} = \frac{4EI}{(h/2)} = \frac{2Ebt^3}{3h} \tag{21}$$

Using these expressions we obtain

$$\begin{aligned} G_{12} &= \frac{(\beta+\sin\theta)}{b\cos\theta}\frac{1}{\left(-\frac{h^2}{2lK_{65}}+\frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)}-\left(K_{56}^{(h/2)}\right)^2\right)}+\frac{(\cos\theta+(\beta+\sin\theta)\tan\theta)^2}{K_{44}}\right)} \\ &= \frac{E\alpha^3(\beta+\sin\theta)}{\left(\beta^2(1+2\beta)+\alpha^2\left(\cos\theta+(\beta+\sin\theta)\tan\theta\right)^2\right)\cos\theta} \end{aligned} \tag{22}$$

For a lattice with very thin constituent beams $\alpha^2 << 1$. Therefore, substituting $\alpha^2 \to 0$ in the equations derived here, it can be observed that they exactly reduce to the corresponding classical expressions by Gibson and Ashby [1] given in equations (1) – (5). This provides an independent analytical validation of the general expressions given here for the special case of small deformation showing a linear stress-strain relationship.

4 The exact stiffness matrix of axially loaded beams

When the entire lattice is subjected to compressive stress, the constitutive beam members undergo a compressive force. If the axial forces are small, they do not have a significant impact on the bending of the beam. However, if such forces are large, their effect cannot be ignored. We use the Euler-Bernoulli beam theory to characterise the underlying deformation of the beams. A beam with a compressive axial force N is shown in Figure 2. The equation governing the

Figure 2. An Euler-Bernoulli beam subjected to an axial force N. The beam element has 2 nodes and four degrees of freedom as shown.

transverse deflection of a beam modelled using the Euler-Bernoulli beam theory [see for example, 5] subjected to a compressive axial force N is given by the following fourth-order ordinary differential equation

$$EI\frac{\mathrm{d}^4W(x)}{\mathrm{d}x^4}+N\frac{\mathrm{d}^2W(x)}{\mathrm{d}x^2}=F(x) \tag{23}$$

Here $W(x)$ and $F(x)$ are the transverse displacement and applied transverse forcing on the beam. The quantity EI is the bending stiffness of the beam, I is the inertia moment of the beam cross section and E is the Young's modulus of the beam material. The natural boundary conditions of the beam are expressed in terms of the displacement, rotation ($\Theta(x)$), bending moment ($M(x)$) and shear force ($V(x)$). They are given by

$$\Theta(x)=\frac{\mathrm{d}W}{\mathrm{d}x}, M(x)=EI\frac{\mathrm{d}^2W}{\mathrm{d}x^2} \quad \text{and} \quad -V(x)=EI\frac{\mathrm{d}^3W}{\mathrm{d}x^3}+N\frac{\mathrm{d}W}{\mathrm{d}x} \tag{24}$$

Introducing the normalised length

$$\xi=x/L \tag{25}$$

where L is the length of the beam, the governing equation without the forcing term can be expressed as

$$\frac{\mathrm{d}^4w(\xi)}{\mathrm{d}\xi^4}+\mu^2\frac{\mathrm{d}^2w(\xi)}{\mathrm{d}\xi^2}=0 \tag{26}$$

Here $w(\xi)\equiv W(x)$ and the non-dimensional axial force is given by

$$\mu^2=\frac{NL^2}{EI} \tag{27}$$

Assuming a solution of the form

$$w(\xi)=\exp\{\lambda\xi\} \tag{28}$$

and substituting in the governing equation (26) results

$$\lambda^2(\lambda^2+\mu^2)=0 \quad \text{or} \quad \lambda^2=0, \lambda=\pm \mathrm{i}\mu \tag{29}$$

In view of the roots in equation (29), the general solution can be expressed as

$$\begin{aligned} & w(\xi)=c_1+c_2\xi+c_3\sin\mu\xi+c_4\cos\mu\xi \\ \text{or} \quad & w(\xi)=\mathbf{s}^T(\xi)\mathbf{c} \end{aligned} \tag{30}$$

where the vectors

$$\mathbf{s}(\xi)=\{1,\xi,\sin\mu\xi,\cos\mu\xi\}^T \tag{31}$$

$$\text{and} \quad \mathbf{c}=\{c_1,c_2,c_3,c_4\}^T \tag{32}$$

Applying the force and displacement boundary conditions at the two ends of the beam (denoted by '1' and '2' in Figure 2) and eliminating the unknown constant vector **c**, the stiffness matrix of the beam can be derived. After some algebraic simplifications we obtain

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} d_1 & d_2L & -d_1 & d_2L \\ & d_3L^2 & -d_2L & d_4L^2 \\ & & d_1 & -d_2L \\ sym & & & d_3L^2 \end{bmatrix} \begin{bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{bmatrix} \tag{33}$$

The non dimensional coefficients in the above equation are given by

$$\begin{aligned} & d_1=-\mu^3\sin(\mu)/\Delta, d_2=\mu^2(\cos(\mu)-1)/\Delta \\ & d_3=\mu(\mu\cos(\mu)-\sin(\mu))/\Delta, d_4=\mu(\sin(\mu)-\mu)/\Delta \\ \text{and} \quad & \Delta=\mu\sin(\mu)-2(1-\cos(\mu)) \end{aligned} \tag{34}$$

The four unique non-dimensional coefficients are functions of the axial force parameter μ only. Expanding them in a Taylor series about $\mu=0$ results

$$\begin{aligned} d_1 &= 12-\frac{6}{5}\mu^2-\frac{1}{700}\mu^4-\frac{1}{63000}\mu^6-\frac{37}{194040000}\mu^8-\frac{59}{25225200000}\mu^{10}+O\left(\mu^{12}\right) \\ d_2 &= 6-\frac{1}{10}\mu^2-\frac{1}{1400}\mu^4-\frac{1}{126000}\mu^6-\frac{37}{388080000}\mu^8-\frac{59}{50450400000}\mu^{10}+O\left(\mu^{12}\right) \\ d_3 &= 4-\frac{2}{15}\mu^2-\frac{11}{6300}\mu^4-\frac{1}{27000}\mu^6-\frac{509}{582120000}\mu^8-\frac{14617}{681080400000}\mu^{10}+O\left(\mu^{12}\right) \\ d_4 &= 2+\frac{1}{30}\mu^2+\frac{13}{12600}\mu^4+\frac{11}{378000}\mu^6+\frac{907}{1164240000}\mu^8+\frac{27641}{1362160800000}b^{10}+O\left(\mu^{12}\right) \end{aligned} \tag{35}$$

Considering only the first term in the above expansion, it can be confirmed that the stiffness matrix in equation (33) reduces to the classical stiffness matrix of the Euler-Bernoulli beam [5]. If the second term of this expansion is considered, then we obtain the classical tangent stiffness matrix of Euler-Bernoulli beams. The higher-order terms, therefore, quantify the extended effect of the axial force on the transverse deflection of the beam. From (34) we observe that the coefficients of the stiffness matrix are nonlinear functions of the in-plane force. As the stiffness coefficients are exact, they are valid for any values of the axial force parameter μ, including the case of buckling. By applying different boundary conditions, it can be shown that the classical Euler buckling loads can be obtained exactly by setting the determinant of the coefficient matrix to zero.

When the axial deformation is considered, the governing equation is expressed by a second-order ordinary differential equation as

$$EA\frac{\partial^2 U(x)}{\partial x^2} = F_a(b) \tag{36}$$

where $U(x)$ and $F_a(x)$ are the axial displacement and applied axial forcing on the beam. Here EA is the axial stiffness of the beam and A is the area of the beam cross-section. The complete beam element is shown in Figure 1(c) with two nodes and three degrees of freedom per node. The degrees of freedom in each node corresponds to the axial, transverse and rotational deformation. The stiffness matrix of the beam element in Figure 1(c) can be expressed by

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{d_1EI}{L^3} & \frac{d_2EI}{L^2} & 0 & -\frac{d_1EI}{L^3} & \frac{d_2EI}{L^2} \\ 0 & \frac{d_2EI}{L^2} & \frac{d_3EI}{L} & 0 & -\frac{d_2EI}{L^2} & \frac{d_4EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{d_1EI}{L^3} & -\frac{d_2EI}{L^2} & 0 & \frac{d_1EI}{L^3} & -\frac{d_2EI}{L^2} \\ 0 & \frac{d_2EI}{L^2} & \frac{d_4EI}{L} & 0 & -\frac{d_2EI}{L^2} & \frac{d_3EI}{L} \end{bmatrix} \tag{37}$$

The displacements corresponding to degrees of freedom 1 and 4 correspond to the axial deformation governed by equation (36), while the displacements corresponding to degrees of freedom 2, 3, 5 and 6 correspond to the bending deformation governed by equation (23). Next, we utilise this stiffness matrix in the general expressions of the equivalent elastic properties derived in section 2.

5 Equivalent elastic properties of compressed lattices

From the general expression in section 2 (equations (8)–(12)), it can be observed that two coefficients of the 6×6 element stiffness matrix of the inclined member and one coefficient of the 6×6 element stiffness matrix of the vertical member, namely, K_{55}, K_{44} and $K_{44}^{(h)}$, are necessary to obtain E_1, E_2, ν_{21} and ν_{21}. The respective coefficients are given in the element stiffness matrix in equation (37). Using the expressions of moment of inertia and the cross-sectional area in equation (14), the stiffness coefficients are derived as

$$K_{55} = \frac{d_1EI}{l^3} = Eb\alpha^3\frac{d_1}{12}, K_{44} = \frac{EA}{l} = Eb\alpha \quad \text{and} \quad K_{44}^{(h)} = \frac{EA}{h} = \frac{Ebt}{h} = \frac{Eb\alpha}{\beta} \tag{38}$$

Using these, we obtain the ratios

$$\frac{K_{55}}{K_{44}} = \alpha^2 \frac{d_1}{12} \quad \text{and} \quad \frac{K_{55}}{K_{44}^{(h)}} = \alpha^2 \beta \frac{d_1}{12} \tag{39}$$

Substituting these expressions in equations (8) – (11) we obtain the general expressions

$$E_1 = \frac{E\alpha^3 d_1 \cos\theta}{(\beta+\sin\theta)\left(12\sin^2\theta + d_1\alpha^2\cos^2\theta\right)} \tag{40}$$

$$E_2 = \frac{E\alpha^3 d_1(\beta+\sin\theta)}{(12-d_1\alpha^2)\cos^3\theta + d_1\alpha^2(2\beta+1)\cos\theta} \tag{41}$$

$$\nu_{12} = \frac{\cos^2\theta\left(12-d_1\alpha^2\right)}{(\beta+\sin\theta)\sin\theta\,(12+d_1\alpha^2\cot^2\theta)} \tag{42}$$

$$\nu_{21} = \frac{(\beta+\sin\theta)\sin\theta\left(12-d_1\alpha^2\right)}{(12-d_1\alpha^2)\cos^2\theta + d_1\alpha^2(2\beta+1)} \tag{43}$$

For the shear modulus, five elements from two different stiffness matrices are necessary. They are two coefficients of the 6×6 element stiffness matrix of the inclined member, namely, K_{65}, K_{44} and three elements of the stiffness matrix of the vertical member with half the length (see the supplementary document for further details). The vertical member is not subjected to any axial forcing due to the applied shear stress. Therefore, the only term in the expression of the shear modulus in equation (12) affected by axial stress is K_{65} corresponding to the inclined members of the unit cell. Therefore, we have

$$K_{65} = -d_2\frac{EI}{l^2} = -d_2\frac{Ebt^3}{12l^2} \tag{44}$$

We also need three elements of the stiffness matrix of the vertical member with half the length given by

$$K_{55}^{(h/2)} = \frac{12EI}{(h/2)^3} = \frac{8Ebt^3}{h^3}, K_{56}^{(h/2)} = -\frac{6EI}{(h/2)^2} = -\frac{2Ebt^3}{h^2} \quad \text{and} \quad K_{66}^{(h/2)} = \frac{4EI}{(h/2)} = \frac{2Ebt^3}{3h} \tag{45}$$

Using these expressions we obtain

$$\begin{aligned} G_{12} &= \frac{(\beta+\sin\theta)}{b\cos\theta}\frac{1}{\left(-\frac{h^2}{2lK_{65}} + \frac{4K_{66}^{(h/2)}}{\left(K_{55}^{(h/2)}K_{66}^{(h/2)} - \left(K_{56}^{(h/2)}\right)^2\right)} + \frac{(\cos\theta+(\beta+\sin\theta)\tan\theta)^2}{K_{44}}\right)} \\ &= \frac{E\alpha^3(\beta+\sin\theta)}{\left(\beta^2(6/d_2+2\beta) + \alpha^2\left(\cos\theta+(\beta+\sin\theta)\tan\theta\right)^2\right)\cos\theta} \end{aligned} \tag{46}$$

Substituting $\alpha^2 = 0$ and taking the $\lim_{\mu\to 0}$, the equations derived here exactly reduce to the corresponding classical expressions by Gibson and Ashby [1] in equations (1) – (5) (i.e., the case of considering only the bending deformation without any prestress). For a lattice without the axial compression effect, $d_1 \to 12$ and $d_1 \to 6$. Substituting these in equations (40) – (43) and (46) it can be observed that they exactly reduce to the corresponding expressions for the small deformation case given by equations (17) – (20) and (22). Therefore, equations (40) – (43) and (46) are the most general expressions of the equivalent elastic properties of hexagonal lattices subjected to compressive stress. Note that these expressions are nonlinear functions of the compressive force parameter μ as the coefficients d_1 and d_2 appearing in these equations are nonlinear functions of μ as given by equation (35). In the main paper, these exact closed-form expressions are used for the eigenbuckling analysis of the lattice.

6 Comparison with the finite element analysis

The analytical expressions of the equivalent elastic properties of the lattice were derived exactly considering one beam element for each section of the unit cell shown in Figure 1(c). Although the expressions are exact, it might be insightful to compare the analytical results with independent numerical results. The commercial software ANSYS has been used to obtain the finite element results. In Figure 3 we show finite element models of the entire lattice and the corresponding unit cell models for two geometrical configurations. Two geometrical configurations, namely, when $\theta = 30°$, $\alpha = 0.1$, $\beta = 1$ and $\theta = 45°$, $\alpha = 0.1$, $\beta = 1$ are shown. Solid elements are used in the finite element models. A mesh convergence study has been carried out and the final number of nodes and elements used are as below:

(a) the unit cell model for $\theta = 30°$, $\alpha = 0.1$, $\beta = 1$, 20,000 nodes, 3846 elements,

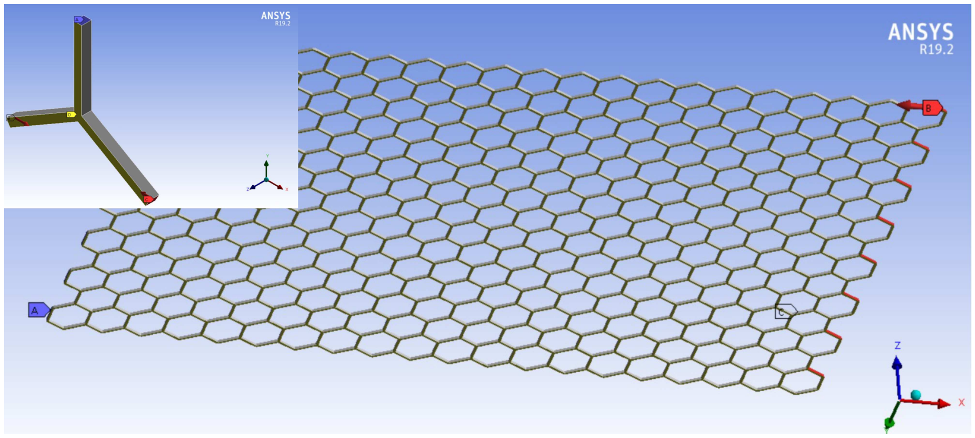
(b) the full lattice model for $\theta = 30°$, $\alpha = 0.1$, $\beta = 1$, 156,320 nodes, 82,390 elements,

(c) the unit cell model for $\theta = 45°$, $\alpha = 0.1$, $\beta = 1$, 20,148 nodes, 3870 elements, and

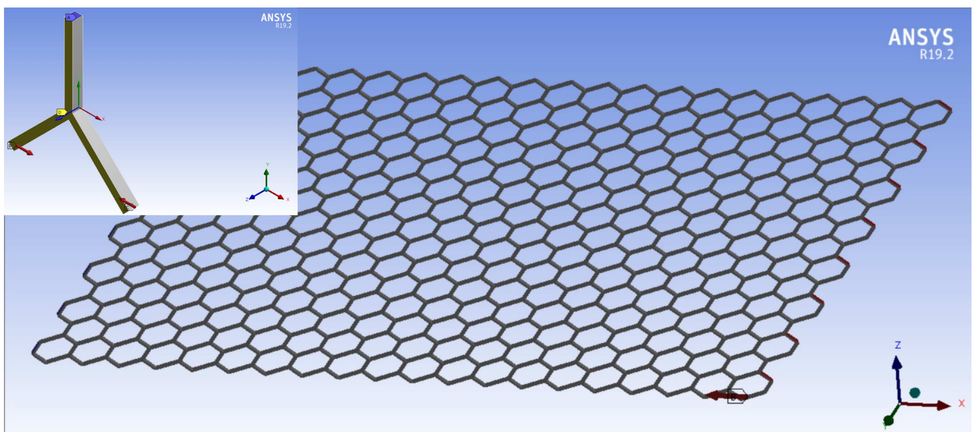
(d) the full lattice model for $\theta = 45°$, $\alpha = 0.1$, $\beta = 1$, 150,757 nodes, 78,323 elements.

For the solution, the nonlinear analysis was used by invoking the 'large deformation' option.

In Figure 4 analytical results are compared with direct nonlinear finite element simulation results. Equivalent normalised Young's modulus $E_1/E\alpha^3$ is plotted as functions of the normalised compressive stress $\sigma_1/E\alpha^3$ in direction 1. Using the displacement responses at the nodes of the right edge of the lattice (where the distributed force is applied) in Figure 3, the effective displacement is calculated by computing the average of nodal displacements over all the edge nodes. The strain is obtained by diving this average displacement with the length of the lattice. The effective stress is derived by diving the total force with the surface area of the edge. Finally, the equivalent Young's modulus is obtained by diving the stress with the effective strain. A similar process is implemented for the unit cell also. In Figure 4, the result obtained using the classical expression by Gibson and Ashby in equation (1) is also shown. These values do not change with increasing axial forces. The finite element modes shown in Figure 3(a) and



(a) $\theta = 30°$, $\alpha = t/l = 0.1$ and $\beta = h/l = 1$.

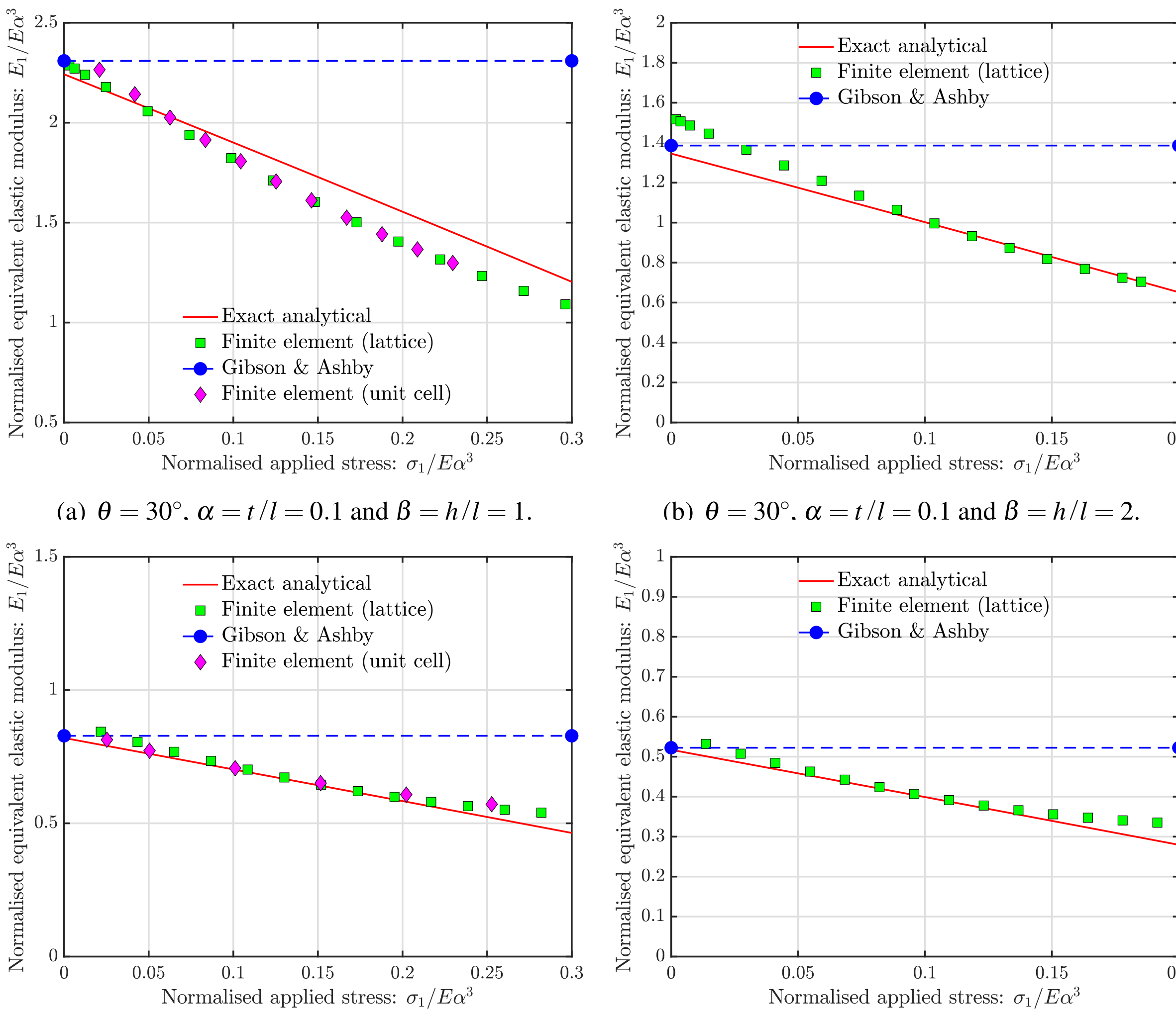


(b) $\theta = 45°$, $\alpha = t/l = 0.1$ and $\beta = h/l = 1$.

Figure 3. Finite element models of the entire lattice and their corresponding unit cell for two geometrical configurations. The left edge of the lattice is fixed and a uniformly distributed force is applied at the right edge for the analysis of in-plane displacement.

Figure 3 (b) correspond to the results shown in Figure 4(a) and Figure 4(c) respectively. It can be seen that the finite element results from the unit cell match well with the full lattice simulation results. Although the finite element results are not identical to the results obtained from the analytical expressions, the trend is similar, and the error is within 10%. It is remarkable that even simple closed-form expressions such as the one in equation (40) produce an excellent agreement with the full scale nonlinear finite element analysis for the four different geometries analysed in Figure 4.

In the main paper, the eigenvalue problem involving the elasticity matrix is solved. It is shown that the eigenvalues can be directly related to the equivalent stiffness coefficients compared here. Similarity with the finite element results reported here, therefore, verifies the subsequent analytical developments on buckling loads as they are mathematically exact. It may also be possible to directly compare the expressions of the critical buckling stresses derived in the main paper



(a) $\theta = 30°$, $\alpha = t/l = 0.1$ and $\beta = h/l = 1$.

(b) $\theta = 30°$, $\alpha = t/l = 0.1$ and $\beta = h/l = 2$.

(c) $\theta = 45°$, $\alpha = t/l = 0.1$ and $\beta = h/l = 1$.

(d) $\theta = 45°$, $\alpha = t/l = 0.1$ and $\beta = h/l = 2$.

Figure 4. Equivalent normalised Young's modulus $E_1/E\alpha^3$ plotted as functions of the normalised compressive stress $\sigma_1/E\alpha^3$ in direction 1. The results from analytical expressions are directly compared with the results from nonlinear finite element analysis (with the full lattice and the unit cell). Results for four different lattice geometries are shown.

with finite element results. In order to achieve this, care must be taken on how the finite element model is constructed. The critical buckling stresses are derived from a 'materials' perspective instead of a 'structures' perspective often adopted in practice. As a result, the expressions are independent of boundary conditions. In contrast, boundary conditions are necessary for a finite element model, and results are in general dependent on such boundary conditions. One way this conceptual discrepancy could be addressed is by considering a very large finite element model. Future research is necessary in this direction.

7 Summary

The in-plane mechanics of compressible lattice materials is considered. A key feature of this problem is the underlying geometric nonlinearly leading to axial softening of the constituent beam elements. A physics-based analytical approach leading to closed-form expressions of in-plane equivalent elastic properties of hexagonal lattices was presented. The route to this analytical derivation has three key steps. Firstly, using the mechanics of a selected unit cell, Young's moduli, Poisson's ratios and the shear modulus of the lattice were presented such that the resulting expressions utilise the stiffness matrix elements of constituent beams. Secondly, noting that the constitutive members of lattices are thin beams, the stiffness matrix of a beam under compression is derived exactly. This is achieved using transcendental displacement functions which are exact solutions of the governing ordinary differential equation with appropriate boundary conditions. Finally, combining these two steps, equivalent elastic properties were obtained for compressed lattices. Equivalent elastic properties are nonlinear functions of applied compressive stresses (through trigonometric functions). This results in a nonlinear homogeneous stress-strain relationship for the cellular material. The closed-form expressions of the equivalent elastic constants were validated for the special case of thin beams with respect to classical expressions for the case of small deformation. For the case of large deformation, the equivalent elastic moduli were validated using independent nonlinear finite element analysis. The expressions of the element of the elasticity matrix derived here are utilised in the main paper for the eigenbuckling analysis.

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