Multilevel Decomposition Framework for Reliability Assessment of Assembled Stochastic Linear Structural Systems

Tanmoy Chatterjee[1]; Sondipon Adhikari[2]; and Michael I. Friswell[3]

Abstract: To reduce the computational cost of assembled stochastic linear structural dynamic systems, a three-staged reduced order model-based framework for forward uncertainty propagation was developed. First, the physical domain was decomposed by constructing an equivalent reduced order numerical model that limited the cost of a single deterministic simulation. This was done in two phases: (1) reducing the system matrices of the subcomponents using component mode synthesis and (2) solving the resulting reduced system with the help of domain decomposition in an efficient manner. Second, functional decomposition was carried out in the stochastic space by employing a multioutput machine learning model that reduced the number of eigenvalue analyses to be performed. Thus, a multilevel framework was developed that propagated the dynamic response from the subcomponent level to the assembled global system level efficiently. Subsequently, reliability analysis was performed to assess the safety level and failure probability of linear stochastic dynamic systems. The results achieved by solving a two-dimensional (2D) building frame and a three-dimensional (3D) transmission tower model illustrated good performance of the proposed methodology, highlighting its potential for complex problems. **DOI: 10.1061/AJRUA6.0001119.** © *2021 American Society of Civil Engineers.*

Author keywords: Domain decomposition; Component mode synthesis; Reliability analysis; Neural network; Multioutput.

Introduction

Significant effort has been made to improve the computational framework of numerical modeling and simulation, facilitated by high performance computing (HPC). Accordingly, numerical algorithms are required to scale with the number of processors in order to realize the maximum capability of the HPC platform. The domain decomposition (DD) method is an example of such an algorithm that exhibits scalable features (Badia et al. 2019). In general, the finite-element (FE) domain is decomposed into a set of subdomains, and each of these are assigned to an individual processor. In a discrete model, representative of a continuous system, the resulting linear system is recast as a set of smaller linear subsystems to be solved independently. In doing so, the critical aspect is to ensure that the compatibility and equilibrium conditions are established at the interface degrees of freedom (DOFs). Hence, such partitioning of the domain by DD through scalable and parallel computing allows zooming in the model resolution while minimizing the computational cost. The increase in model resolution is reasonable, because it can significantly reduce the discretization errors in the numerical simulation.

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Note. This manuscript was submitted on June 24, 2020; approved on November 5, 2020; published online on January 12, 2021. Discussion period open until June 12, 2021; separate discussions must be submitted for individual papers. This paper is part of the ***ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering***, © ASCE, ISSN 2376-7642.

In contrast to the foregoing efficient numerical solution techniques, component mode synthesis (CMS) is more of a physics-based model reduction approach (Boo et al. 2018). CMS has been immensely useful for structural dynamics applications. It is particularly effective in large-scale structural models, because in most cases it has been observed that the dynamics can be captured by a relatively low number of linear modes (Hinke et al. 2009). The literature related to CMS methods is well developed, and an extensive review can be found in De Klerk et al. (2008).

Although the foregoing approaches (CMS and DD) employ different strategies, both of them have the same utility, which is to accelerate the solution process of FE approximations of partial differential equations. In this regard, the relevant conceptual similarities and differences of these classes of methods were discussed by Rixen (2006). It has also been noted that these methods have not found a common application arena; for example, CMS is popular in the structural dynamics community, whereas DD is more prevalent in problems such as wave propagation (Sarkar et al. 2009) and flow through porous media (Subber and Sarkar 2014). However, there has been recent interest in DD solvers in nonlinear dynamics (Subber and Sarkar 2018). In this work, CMS and DD are combined to solve linear structural dynamic problems.

The underlying assumption that the material, geometric, and load parameters are precisely known—and, therefore, a deterministic FE response analysis may be performed—is not necessarily valid (Mace et al. 2005). Thus, in addition, to improve the accuracy of a deterministic model, uncertainty quantification of the response due to perturbations in the input parameters is equally important. Another critical aspect of uncertainty quantification is the assessment of the safety levels of a structure, which experiences fluctuation in its response depending on the random heterogenous properties and varying loads. Safety and reliability assessment of structural systems involves identification of the failure conditions based on the exceedance of some response parameters of interest from a prescribed threshold tolerance. The theory of and recent contributions on reliability analysis of dynamic systems is discussed subsequently (see "Structural Reliability Analysis" section).

However, uncertainty quantification of dynamical systems can be computationally expensive. To address this issue, various approaches have been exploited, and some of these are discussed here. CMS combined with perturbation methods for quantifying the uncertainty in the structural dynamic response was studied by Hinke et al. (2009) and Sarsri and Azrar (2016). In the presence of a high level of input uncertainty (when the perturbation methods do not generally work well), efficient stochastic reduced basis projection schemes have been utilized with CMS for the modeling and propagation of spatially distributed joint uncertainties (Dohnal et al. 2009). A CMS technique based on dominant fixed-interface normal modes and the static contribution of higher order modes was developed in Jensen et al. (2017) and employed in stochastic dynamic analysis (González et al. 2019). Sarkar et al. (2009) proposed a theoretical framework for the nonoverlapping DD of stochastic systems. The intrusive polynomial chaos expansion (PCE)-based nonoverlapping DD methods with preconditioned conjugate gradient techniques have demonstrated excellent scalability for problems with high mesh resolution (Subber and Sarkar 2014). More recently, scalable sparse iterative solvers with efficient preconditioners have been proposed in order to deal with high-dimensional problems (Desai et al. 2018). A frequency transformation strategy with a principal component analysis technique was proposed to overcome the limitations of PCE for frequency response function (FRF) simulation in Yaghoubi et al. (2017). Modal analysis was performed to investigate the stochastic dynamic properties of FRFs in Pichler et al. (2009), Chatterjee et al. (2016), and Pryse et al. (2018). Recently, Gaussian process modeling was employed in Lu et al. (2019) and Chatterjee et al. (2020, 2021) to estimate the FRFs of stochastic dynamic systems by a reduced subspace projection technique.

Following the aforementioned discussion on model-order reduction, domain decomposition, and uncertainty quantification in dynamical systems, a reduced-order predictive emulator for forward uncertainty propagation and reliability analysis in linear structural dynamic systems is proposed. To the authors' knowledge, the proposed decomposition framework is the first of its kind, resulting from a unique combination of three existing approaches, that is, CMS, DD, and artificial neural network (ANN). The following integral features form the core of the proposed framework:

- CMS was implemented in conjunction with DD solvers to achieve higher computational efficiency compared to the individual methods. The strategy recently proposed by the same authors in Chatterjee et al. (2020) was followed (see "Proposed Deterministic Methodology" section); and
- For efficient uncertainty quantification and reliability analysis, a multioutput neural network was devised in the modal space to propagate the input uncertainty to the frequency response and determine the failure probability in an efficient manner. This is illustrated in the section "Proposed Framework for Stochastic Systems." This is a point of improvement of the present work compared to Chatterjee et al. (2020).

The remainder of the paper is organized in the following sequence. The section "Proposed Deterministic Methodology" presents the proposed deterministic model-order reduced DD method. The stochastic version of the proposed deterministic method is illustrated in the section "Proposed Framework for Stochastic Systems." The section "Numerical Study" demonstrates the proposed approach on two structural engineering applications. Last, the key aspects and contribution of the present work are summarized in the section "Summary and Conclusions."

Proposed Deterministic Methodology

Integrating Model-Reduction in the Framework of Domain Decomposition

Domain decomposition solvers partition the original problem into various subproblems in a parallel manner using different processors. Several domain decomposition solvers are available depending on whether the subdomains are overlapping or nonoverlapping (Bjorstad et al. 1996). Here, we employed the generalized framework of DD for nonoverlapping subdomains as proposed by Sarkar et al. (2009) to illustrate the improvements.

Considering the whole domain Ω of the FE model of an arbitrary system in an n-dimensional subspace partitioned into two nonoverlapping subdomains, the dynamic equation of motion of the system in the frequency domain is

$$\begin{bmatrix} [\mathbf{D}_{II}^{\alpha}]_{n_1\times n_1} & \mathbf{0} & [\mathbf{D}_{IB}^{\alpha}]_{n_1\times n_B} \\ \mathbf{0} & [\mathbf{D}_{II}^{\beta}]_{n_2\times n_2} & [\mathbf{D}_{IB}^{\beta}]_{n_2\times n_B} \\ [\mathbf{D}_{BI}^{\alpha}]_{n_B\times n_1} & [\mathbf{D}_{BI}^{\beta}]_{n_B\times n_2} & [\mathbf{D}_{BB}^{\alpha}+\mathbf{D}_{BB}^{\beta}]_{n_B\times n_B} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_I^{\alpha} \\ \mathbf{x}_I^{\beta} \\ \mathbf{x}_B \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_I^{\alpha} \\ \mathbf{f}_I^{\beta} \\ \mathbf{f}_B^{\alpha}+\mathbf{f}_B^{\beta} \end{Bmatrix} \tag{1}$$

where $\mathbf{D}_{ij}^{s}=(-\omega^2\mathbf{M}_{ij}^{s}+i\omega\mathbf{C}_{ij}^{s}+\mathbf{K}_{ij}^{s})$ denotes the dynamic stiffness matrix; the superscript s represents the subsystems α or β; and the subscripts $ij=II, BB$, and IB refer to the internal DOFs, interface DOFs, and coupling DOFs, respectively.

To capture complex geometries, mesh refinement often leads to a significant increase in the size of the resulting system [represented by Eq. (1)] from the FE model. However, as previously mentioned, in most structural applications, it has been found that the dynamics can be captured by a relatively low number of linear modes (Hinke et al. 2009). Taking advantage of this useful feature, we have integrated model-reduction (specifically the Craig-Bampton approach) into the framework of domain decomposition so as to avoid solving the full system in Eq. (1) and efficiently evaluate the dynamic response.

Applying the Craig-Bampton approach (Bampton and Craig 1968) on the individual subsystems α and β leads to a reduced assembled system ($\mathbf{D}'\mathbf{x}'=\mathbf{f}'$):

$$\begin{bmatrix} [\mathbf{D}_{II}^{\alpha'}]_{n_1'\times n_1'} & \mathbf{0} & [\mathbf{D}_{IB}^{\alpha'}]_{n_1'\times n_B} \\ \mathbf{0} & [\mathbf{D}_{II}^{\beta'}]_{n_2'\times n_2'} & [\mathbf{D}_{IB}^{\beta'}]_{n_2'\times n_B} \\ [\mathbf{D}_{BI}^{\alpha'}]_{n_B\times n_1'} & [\mathbf{D}_{BI}^{\beta'}]_{n_B\times n_2'} & [\mathbf{D}_{BB}^{\alpha}+\mathbf{D}_{BB}^{\beta}]_{n_B\times n_B} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_I^{\alpha'} \\ \mathbf{x}_I^{\beta'} \\ \mathbf{x}_B \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_I^{\alpha'} \\ \mathbf{f}_I^{\beta'} \\ \mathbf{f}_B^{\alpha}+\mathbf{f}_B^{\beta} \end{Bmatrix} \tag{2}$$

where $\mathbf{D}'=\mathbf{G}^{\mathbf{T}}\mathbf{D}\mathbf{G}$ and $\mathbf{G}$ represent the global modal matrix (component modal matrices in assembled form) given by

$$\mathbf{G}=\begin{bmatrix} [\boldsymbol{\phi}_{Ip}^{\alpha}] & \mathbf{0} & -[\mathbf{K}_{II}^{\alpha}]^{-1}[\mathbf{K}_{IB}^{\alpha}] \\ \mathbf{0} & [\boldsymbol{\phi}_{Iq}^{\beta}] & -[\mathbf{K}_{II}^{\beta}]^{-1}[\mathbf{K}_{IB}^{\beta}] \\ \mathbf{0} & \mathbf{0} & [\mathbf{I}] \end{bmatrix} \tag{3}$$

where $\mathbf{\phi}_{Ij}$ are the eigenvectors corresponding to each subcomponent. Only the first p and q modes are retained in subcomponents α and β, respectively; thus, the dimension of the model is reduced. Out of the three columns of $\mathbf{G}$ illustrated in Eq. (3), the first two denote the retained fixed-interface normal modes, and the third denotes the static constraint modes. More details on the Craig-Bampton approach can be found in Krattiger et al. (2019).

Efficient Solution Scheme by Schur Complement

The reduced system of equations represented by Eq. (2) can be partitioned and rearranged in the following form:

$$\left(\underbrace{[\mathbf{D}_{BB}^{\alpha}]-[\mathbf{D}_{BI}'][\mathbf{D}_{II}^{\alpha'}]^{-1}[\mathbf{D}_{IB}^{\alpha'}]}_{\mathbf{S}_1}+\underbrace{[\mathbf{D}_{BB}^{\beta}]-[\mathbf{D}_{BI}^{\beta'}][\mathbf{D}_{II}^{\beta'}]^{-1}[\mathbf{D}_{IB}^{\beta'}]}_{\mathbf{S}_2}\right)\{\mathbf{x}_B\}$$
$$=\left[\underbrace{\{\mathbf{f}_B^{\alpha}\}-[\mathbf{D}_{BI}^{\alpha'}][\mathbf{D}_{II}^{\alpha'}]^{-1}\{\mathbf{f}_I^{\alpha'}\}}_{\mathbf{F}_1}\right]+\left[\underbrace{\{\mathbf{f}_B^{\beta}\}-[\mathbf{D}_{BI}^{\beta'}][\mathbf{D}_{II}^{\beta'}]^{-1}\{\mathbf{f}_I^{\beta'}\}}_{\mathbf{F}_2}\right] \quad (4)$$

$$[\mathbf{D}_{II}^{\alpha'}]\{\mathbf{x}_I^{\alpha'}\}=\mathbf{f}_I^{\alpha'}-[\mathbf{D}_{IB}^{\alpha'}]\{\mathbf{x}_B\} \quad (5)$$

$$[\mathbf{D}_{II}^{\beta'}]\{\mathbf{x}_I^{\beta'}\}=\mathbf{f}_I^{\beta'}-[\mathbf{D}_{IB}^{\beta'}]\{\mathbf{x}_B\} \quad (6)$$

The interface DOFs can be obtained by solving the Schur complement matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ in parallel using Eq. (4). Then, $\mathbf{x}_B$ is substituted into Eqs. (5) and (6) to obtain the response at the internal DOFs $\mathbf{x}_I^{\alpha'}$ and $\mathbf{x}_I^{\beta'}$. This partitioning of the domain into interior and interface DOFs results in a more efficient solution of the global system compared to the direct inversion of the coefficient matrix in Eq. (2). This seamlessly provides the general treatment for solving an assembled system comprising multiple subdomains in a parallel manner. For the generalized formulation for multiple subdomains, refer to Chatterjee et al. (2020). In the next section, the implementation of the proposed approach in stochastic systems is illustrated.

Proposed Framework for Stochastic Systems

This section has been divided into two subsections. The first subsection briefly discusses the theory of and recent contributions to reliability analysis. The second subsection illustrates functional decomposition in the stochastic space and efficient implementation of a neural network in the presence of parametric uncertainties.

Structural Reliability Analysis

Structural reliability analysis (SRA) can be viewed as the postprocessing of the stochastic response analysis. The aim of SRA is to quantify the probability of system failure considering the effects of variation in the system parameters and/or loads. Intuitively, it involves assessing if the resistance of the structure exceeds the effect of applied load. This is considered as a safe instance; otherwise, the case corresponds to failure. The simulations are performed for combinations of random realizations of the stochastic parameters (Zhu et al. 2020). This classification is based upon a performance function $g(\mathbf{X})$, referred to as the limit state function, where $\mathbf{X}$ denotes the vector of random input parameters, $g(\mathbf{X}) = 0$ refers to the hyperplane, which represents the boundary of the safe $\{\mathbf{X}|g(\mathbf{X})<0\}$ and failure $\{\mathbf{X}|g(\mathbf{X})>0\}$ regions. Thus, the failure probability p_f can be evaluated by solving the integral problem given by

$$p_f=\int_{\{\mathbf{X}|g(\mathbf{X})>0\}} f_{\mathbf{X}}(\mathbf{X})d\mathbf{X} \quad (7)$$

where $f_{\mathbf{X}}(\mathbf{X})$ is the joint probability density function of $\mathbf{X}$.

Solving the foregoing integral problem is often computationally expensive, because, in most cases, closed form solutions do not exist. Monte Carlo simulation (MCS) is one of the most straightforward techniques for SRA (Zhang et al. 2010). However, it requires large numbers of samples for convergence. Several improvements to address this issue have been proposed (Misraji et al. 2020). These approaches fall under the category of sampling-based SRA; nonsampling-based SRA approaches are well documented in Haldar and Mahadevan (2000). The latter category of approaches is more efficient than the former; however, they are only effective for solving linear or weakly nonlinear problems (Bagheri et al. 2020). Metamodel-based techniques, which are a hybrid combination of the aforementioned approaches, are used in this work. Important recent applications of metamodels in SRA are briefly discussed in the following.

A wavelet support vector machine—(SVM-) based neural network metamodel was developed for reliability analysis in Dai et al. (2015). Time dependent reliability assessment was performed for welded joints with surface cracks in Dong et al. (2020) by employing adaptive metamodels. An online learning–based metamodel framework coupled with importance sampling was proposed to estimate multiple rare failure events (Razaaly and Congedo 2018). Adaptive kriging was used in conjunction with stratified importance sampling for SRA in Xiao et al. (2020). Sequential adaptive sampling-based support vector regression (SVR) was employed for SRA in Roy and Chakraborty (2020). Reliability analysis of corroded pipelines was performed using tree regression in Keshtegar and Kisi (2017). A polynomial chaos-based kriging metamodel was developed for determining small failure probabilities in Schobi et al. (2017). Compressive sensing with an adaptive wavelet basis was utilized for SRA under missing data in Comerford et al. (2017). An adaptive parametric metamodel was used to measure the first excursion probability of structural systems with friction-based devices under stochastic excitation in Jensen et al. (2020). An adaptive approach was developed for reliability analysis by ensemble learning of multiple competitive metamodels, including kriging, polynomial chaos expansion, and SVR in Cheng and Lu (2020). A multicomponent dynamic reliability analysis of an aeroengine high-pressure turbine blisk with blade and disk was performed using a decomposition and coordination strategy, genetic algorithm, and kriging in Fei et al. (2019) and Lu et al. (2020). A review of applications of ANN in the reliability analysis of steel structures can be found in Chojaczyk et al. (2015).

Next, the implementation of the metamodel and the proposed framework is illustrated in presence of parametric uncertainties.

Functional Decomposition in the Stochastic Space Using Machine Learning

Repeated simulations were required in order to compute the system response corresponding to the random realizations of the input parameters. MCS performed on the proposed model-reduction based DD required a smaller computational effort compared to DD of the unreduced system. The computational cost was further reduced by limiting the number of actual function evaluations.

The basic idea was to approximate the eigenvectors of the retained DOFs of the model-reduced substructures by using a metamodel and then performing the DD and Schur complement operations based on metamodel predictions of the reduced system. To be specific, the term $\boldsymbol{\phi}_{Ij}$ corresponding to each subcomponent in Eq. (3) was estimated via the metamodel. This leads to the following advantages. First, only n_{samp} eigenvalue analyses have to be performed, where n_{samp} denotes the number of training samples for the metamodel, compared to n_{MCS} eigenvalue analyses, where n_{MCS} is the number of MCS samples. This can lead to time efficiency for solving real-world structural systems in which even a single simulation takes significant time. Second, to some extent, the accuracy aspect of the metamodel may be compromised as a trade-off with computational cost in the case of large-scale systems, because it is well known that the effect of eigenvectors on the dynamic response is not particularly strong.

Therefore, to limit the number of eigenvalue analyses, functional decomposition in the stochastic space was performed using an ANN. ANNs are considered to be complex predictive models due to their ability to handle multidimensional data, nonlinearity, and adaptive learning capability and generalization (Goodfellow et al. 2016). The basic framework of a neural network comprises four atomic elements, namely: (1) nodes, (2) connections/weights, (3) layers, and (4) activation function. In an ANN, the neurons represent the building blocks. The neurons represent the simplest processing units, which return weighted input signals and an output signal using an activation function. The neural network reduces the error by optimization algorithms, such as a back-propagation algorithm (Rumelhart et al. 1986). The optimal weights of each connection between a set of layers are evaluated during each backward pass of a training dataset, which is also used for weight optimization using the derivatives obtained from the input and predicted values of the training data.

A particular variation of neural networks is the feed-forward neural network (FFNN), also known as multilayer perceptron. It is widely used in modeling complex tasks, and the generic architecture is shown in Fig. 1. As the figure shows, the elementary model structure comprises three type of layers—the input, hidden, and output layers. In an FFNN, each individual neuron is interconnected to the output of each unit within the next layer, as shown in Fig. 1. Consequently, it has been proven that an ANN, trained to minimize a loss or cost function between an input and output target variable using sufficient data, can accurately produce an estimate of the posterior probability of the output classes based on the discriminative conditioning of the input vector, which is the applied strategy in this work.

To help understanding, a self-explanatory flowchart of the proposed framework is shown in Fig. 2 in order to illustrate the model reduction-based DD in stochastic systems (see "Proposed Deterministic Methodology" and "Proposed Framework for Stochastic Systems" sections). Fig. 2 shows that the metamodel building block that involves a limited number of high-fidelity simulations (computationally expensive, because they involve actual FE analysis) and the MCS block that involves large number of low-fidelity simulations (cheap to compute, because they are performed on the metamodel) are separately represented. The multioutput ANN model is only trained to approximate the mode shape vectors up to the retained modes of individual subsystems, because the reduced configuration is already selected in a deterministic sense corresponding to the nominal values of input parameters (as indicated by the left-hand block in Fig. 2). To ensure common modal vector shapes and reasonable approximation accuracy by the metamodel, the sign of the modes was kept consistent by using a reference mode shape, and the mode shapes were mass normalized.

MCS is performed on the metamodel to obtain the approximate eigenvectors of the individual subsystems. Using these estimated eigenvectors, the transformation matrix in Eq. (3) is formed, resulting in reduced system matrices of the individual subsystems in Eq. (2). Next, the reduced system matrices of the subsystems are repartitioned and assembled and solved efficiently with the help of Eqs. (4)–(6). The frequency response functions of the assembled system are computed for each MCS realization. The uncertainty propagated in the assembled system response due to the random input parameters of the individual subsystems are quantified by the global FRF statistics. Furthermore, the forced response is evaluated in the frequency domain. The time domain response is obtained by inverse Fourier transformation. All instances of the response magnitude exceeding the prescribed safety threshold are counted as structural system failure. The probability of failure is computed, and the reliability is evaluated as the ratio of the number of failure instances to the total number of (MCS) simulations (as per the definition of the limit state).

By using the proposed methodology, the cost of a single analysis can be reduced due to the model reduction performed within the DD framework, as illustrated in the section "Proposed Deterministic Methodology," and the number of actual simulations can be reduced by the ANN presented in the section "Proposed Framework for Stochastic Systems." Therefore, a two-tier improvement of both aspects of computational cost can be achieved by the proposed framework via bilayered decomposition in the physical and functional space.

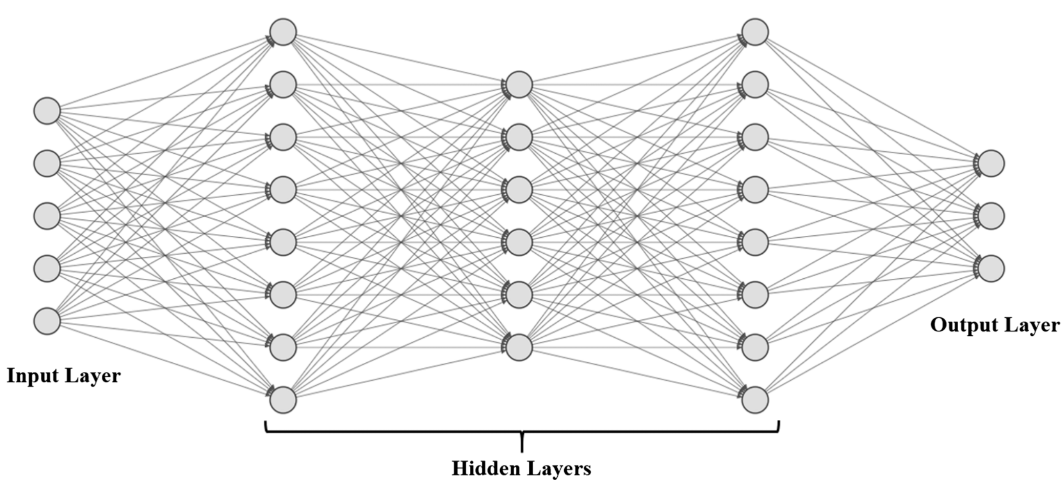

Fig. 1. Schematic of the architecture of a deep neural network. Note that three hidden layers have been shown for illustration.

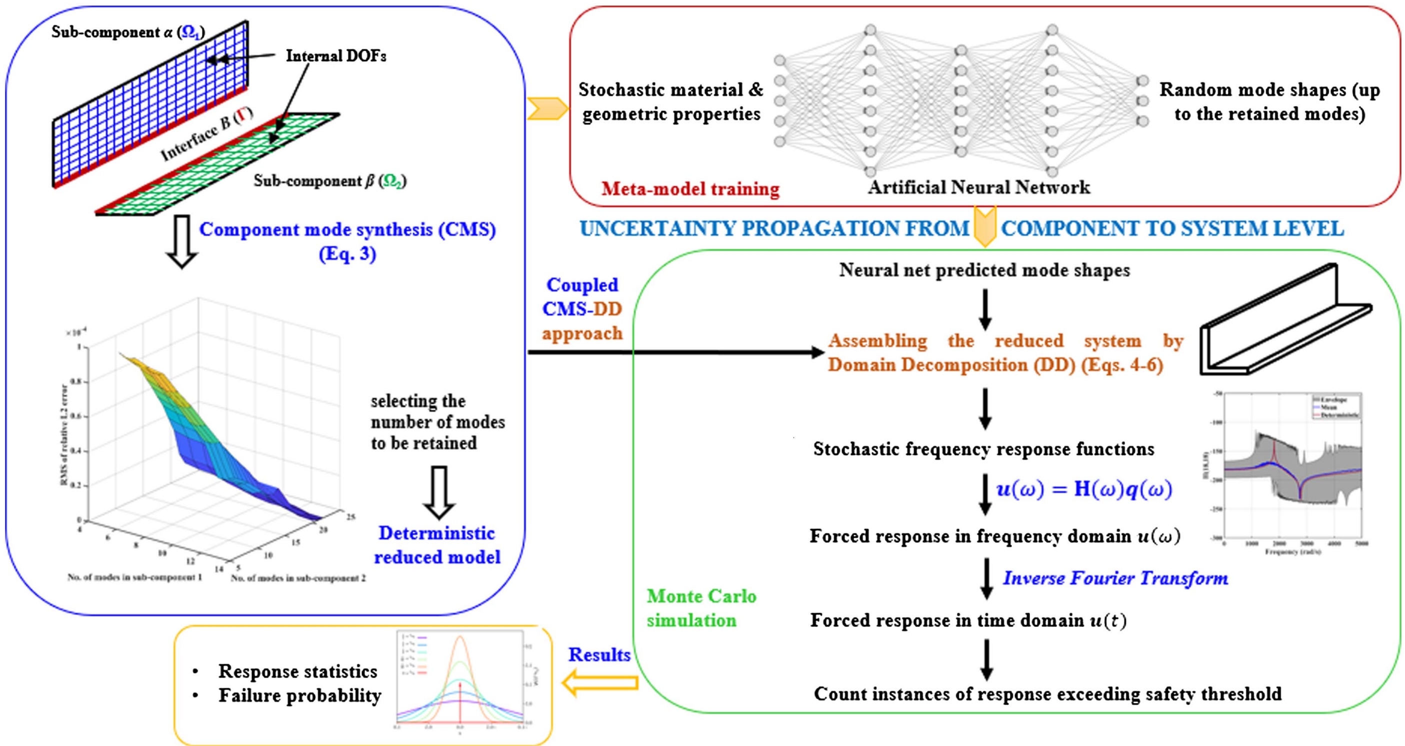

Fig. 2. Schematic representation of the proposed framework for uncertainty propagation from component to system level and structural reliability analysis.

Numerical Study

Eight-Story Building Frame

A two-dimensional building frame is considered in this section, as presented in Fig. 3. The eight-story and three-bay structure was modeled as a plane frame having three DOFs per node and consisting of 140 nodes and 160 elements. The four supports (bottom nodes) were assumed to be fixed in all DOFs. In total, there were 408 DOFs in the plane frame model. The structure was meshed using Gmsh version 4.5.3, an open-source meshing software (Geuzaine and Remacle 2009). The mesh details were exported to MATLAB version 2018b for the finite-element modeling. A snapshot of the mesh with the node locations within the Gmsh environment is presented in Fig. 3.

Nominal values of material density, namely $\rho = 2{,}700\ \text{kg/m}^3$, elastic modulus $E = 200$ GPa, modulus of rigidity $G = 77$ GPa and a square cross section with a width of 0.3 m, were adopted for performing the FE analysis. The damping of the system was assumed to be proportional, with the form $\mathbf{C} = \theta\mathbf{K}$, where the parameter θ is assumed to be 10^{-3}, and $\mathbf{K}$ is the stiffness matrix.

To implement the proposed CMS integrated DD framework, the three-bay frame of each story was considered as a subcomponent. Hence, there were eight subcomponents and seven interfaces. In this context, the multiple subdomain formulation for model-reduced DD can be found in Chatterjee et al. (2020). The number of internal DOFs in subcomponents 1–7 was 39, and the number in subcomponent 8 was 51. Each of the seven interfaces comprised 12 DOFs; hence, the total number of interface DOFs was 84. In this work, our intent was only to reduce the internal DOFs of the subcomponents; all of the interface DOFs were retained in the subsequent analysis. However, when there is a high number of interface DOFs (common in plate structures), they can easily be reduced with the help of characteristic constraint modes.

The relative L2 error of the Frobenius norm of the frequency response of the deterministic assembled structure was studied by varying the number of modes in the individual subcomponents (with regard to the unreduced model). A reduced model

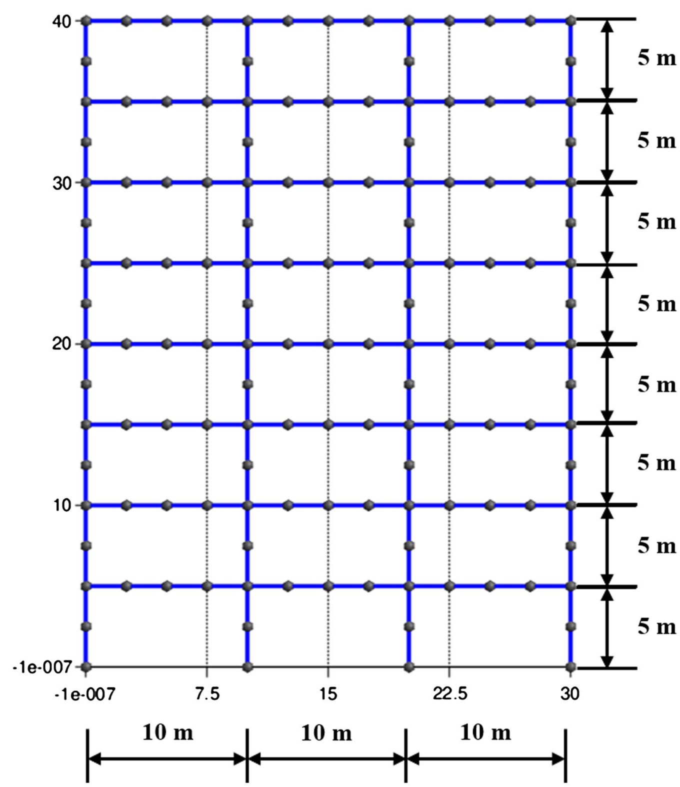

Fig. 3. Schematic diagram of the building frame model showing the node locations.

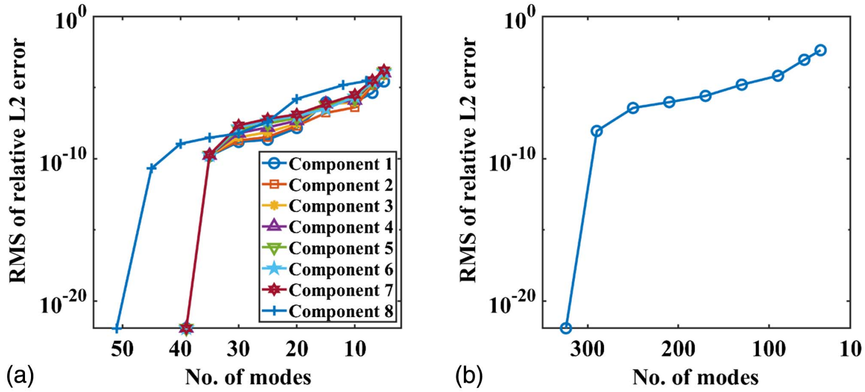

Fig. 4. Convergence of RMS of relative L2 error of the Frobenius norm of the frequency response of the reduced model with respect to the full model with varying numbers of retained modes in the individual subcomponents: (a) model reduction in individual components when full models of the other subcomponents are considered; and (b) model reduction in all the subcomponents considered simultaneously. RMS = root mean square.

configuration was then selected which was capable of capturing the full model-based assembled system response without significant loss of accuracy. The convergence of the error norm with varying number of modes in the subcomponents is shown in Fig. 4. Each of the line plots in Fig. 4(a) corresponds to the error convergence by varying the number of modes in one subcomponent when the full models of the other subcomponents are considered. By contrast, Fig. 4(b) shows the error convergence by varying the number of modes of all the subcomponents simultaneously. Considering the error tolerance to be 10^{-3}, the number of modes retained from each subcomponent for subcomponents 1–7 and from subcomponent 8 were 7 and 12, respectively. This implies that the reduced model utilized 18.8% of the internal DOFs compared to the full model in a deterministic sense. Thus, 81.2% savings in the size of the system matrices were obtained due to the reduced model configuration with regard to the full model corresponding to every stochastic simulation.

To simulate the randomness in the system, the material and geometric properties of each discretized element of the individual subsystems was considered as stochastic. Specifically, the density, elastic modulus, and cross-sectional dimension of each element was considered to be lognormally distributed with 10% variation. The mean of these parameters was the same as their nominal values reported in the foregoing.

Eighty training points were generated by Latin-hypercube sampling to train the network. One ANN model was trained to approximate each mode shape up to the retained internal DOFs of the individual substructures. To do this, a multioutput architecture of the neural network was employed; this is not possible with most conventional metamodels, because they are single-output only. For implementation, the FFNN toolbox in MATLAB was used; trainlm was used as the network training function that updated the weight and biases. This back-propagation algorithm utilizes Levenberg-Marquardt optimization and has been observed to be one of the fastest algorithms. However, several other options are available in the toolbox. The network architecture was selected based on the root-mean square error (RMSE) convergence presented in Table 1. Note that the RMSE values reported represent the mean obtained by approximating the terms of the mode shape matrix. The RMSE was evaluated in comparison to 5,000 samples of MCS. The ANN, composed of three layers and 40 neurons in each layer, was selected based on RMSE = 0.0479 (below the set threshold of 0.05) as indicated in bold in Table 1. The hyperbolic tangent sigmoid transfer function was used for all of the layers.

As stated previously, the model was trained to approximate the eigenvectors corresponding to the retained internal DOFs of the individual subcomponents. Specifically, the dimensions of the approximated quantities (mode shape matrix) were (39×7) and (51×12) for each subcomponent among subcomponents 1–7 and subcomponent 8, respectively. Therefore, only 7 and 12 ANN models were required to be trained (one for each of the retained mode shapes) for each subcomponent among subcomponents 1–7 and subcomponent 8, respectively, because the multioutput feature was utilized here (unlike single-output conventional surrogate models). The stochastic frequency response of the assembled system was obtained by solving the proposed model reduced DD framework as illustrated in Eqs. (4)–(6) in conjunction with the ANN-predicted mode shapes. Sample FRF band plots obtained using MCS (5,000 samples) and ANN (80 samples) are presented in Fig. 5. Close proximity between the predicted (right-hand side) and actual results (left-hand side) demonstrate that a satisfactory level of approximation accuracy was achieved.

For the reliability analysis, lateral forces acting at every story level of the building frame were considered. Harmonic forces of randomly varying amplitudes (with a lognormal distribution) were assumed to act at the story level. The mean force amplitudes at stories 1–8 were 5, 10, 10, 15, 15, 15, 20, and 20 kN, respectively. A 5% variation was considered in the force amplitudes to simulate loading uncertainty. The analysis was solved as a series system reliability problem in which at least one of the potential failure events can lead to system failure (Mahadevan et al. 2001). Therefore, system failure $F_S = \cup_i E_i$ can be defined as the union of potential failure events E_i. Three limit state functions were considered to simulate three potential failure modes specific to high-rise building designs subjected to lateral forces (for example, seismic and wind forces). The failure events E_i were defined as (1) E_1: the overall drift exceeding 0.1% of the story height; (2) E_2: the interstory drift ratio exceeding 0.2% of the story height; and (3) E_3: the maximum

Table 1. Convergence of RMSE with different network architectures

No. of layers	No. of neurons per layer	RMSE
1	30	0.0668
2	30	0.0533
2	40	0.0533
3	20	0.0531
3	**40**	**0.0479**

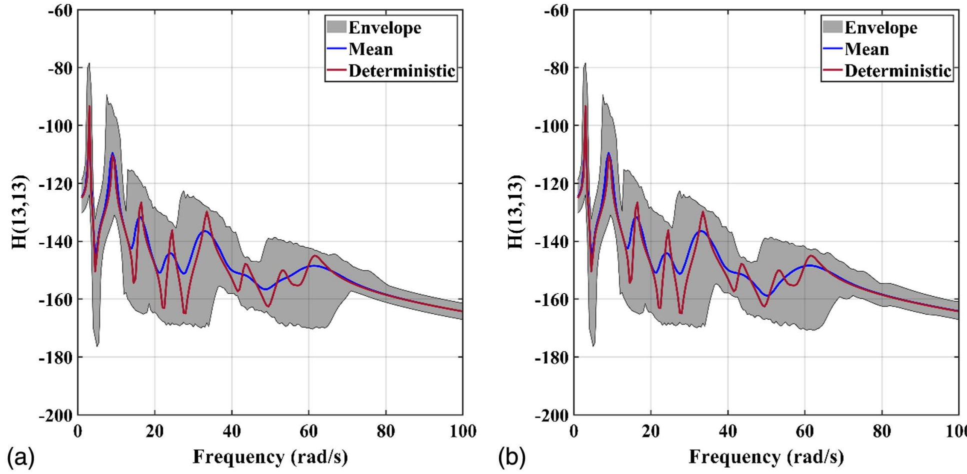

Fig. 5. Displacement FRF band plots (dB) for the building frame example, H(13,13): (a) MCS; and (b) ANN. Note that 5,000 MCS samples were performed. Eighty training points were employed to construct the neural network.

Table 2. Comparison of failure probabilities obtained by MCS and ANN corresponding to different threshold displacement values for the building frame example

Threshold displacement (m)	Probability of failure (MCS; 5,000 samples)	Probability of failure (ANN; 80 samples)	Probability of failure (kriging; 80 samples)	Probability of failure (SVR; 80 samples)
5×10^{-2}	0.1010	0.1020	0.1020	0.0990
8×10^{-2}	0.0530	0.0540	0.0530	0.0510
1×10^{-1}	0.0370	0.0370	0.0370	0.0350

displacement exceeding threshold (defined in Table 2). It was observed that events (1) and (2) were not satisfied at any instance, and, therefore, did not lead to failure. The failure probabilities corresponding to different threshold displacement values obtained by MCS and ANN are reported in Table 2. Other existing metamodels such as kriging and SVR are also compared for validation. From the results in Table 2, ANN and the other metamodels are observed to approximate the failure probability accurately.

Three-Dimensional Transmission Tower Model

A three-dimensional transmission tower model studied in Chatterjee et al. (2020) is presented in this section, and is shown in Fig. 6. The structure was modeled as a space frame having six DOFs per node and consisted of 175 nodes and 246 elements. The four supports (nodes at level $z = 0$) were assumed to be fixed in all DOFs. In total, there were 1,026 DOFs in the space frame model. All other details regarding the FE model, system parameters, subcomponent division, and stochastic modeling can be found in Section 6 of Chatterjee et al. (2020).

The same configuration of the ANN used in the previous example (selected on the basis of the RMSE) was employed here. Specifically, the dimensions of the approximated quantities (mode shape matrices) were (114×30), (600×175) and (192×78) for subcomponents 1, 2, and 3, respectively [see Chatterjee et al. (2020) for details of the substructuring and model reduction]. Therefore, only 30, 175, and 78 ANN models were required to be trained (one for each of the retained mode shapes) for each of the subcomponents, because the multioutput feature was utilized here (unlike single-output conventional surrogate models). Sample FRF band plots obtained using MCS (5,000 samples) and ANN (100 samples) are presented in Fig. 7. Close proximity between the predicted (right-hand side) and actual results (left-hand side)

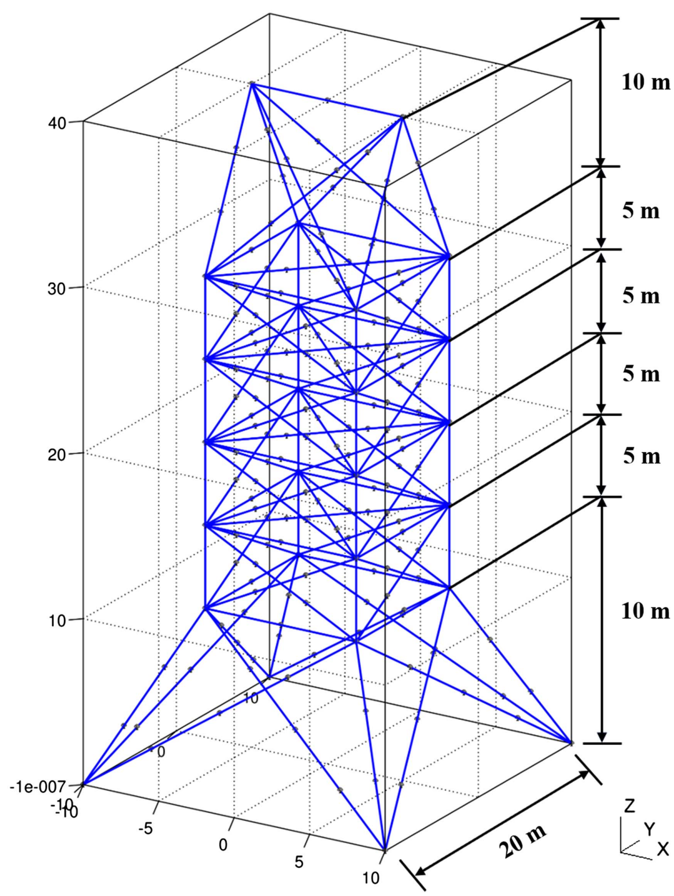

Fig. 6. Schematic diagram of the transmission tower model showing the node locations.

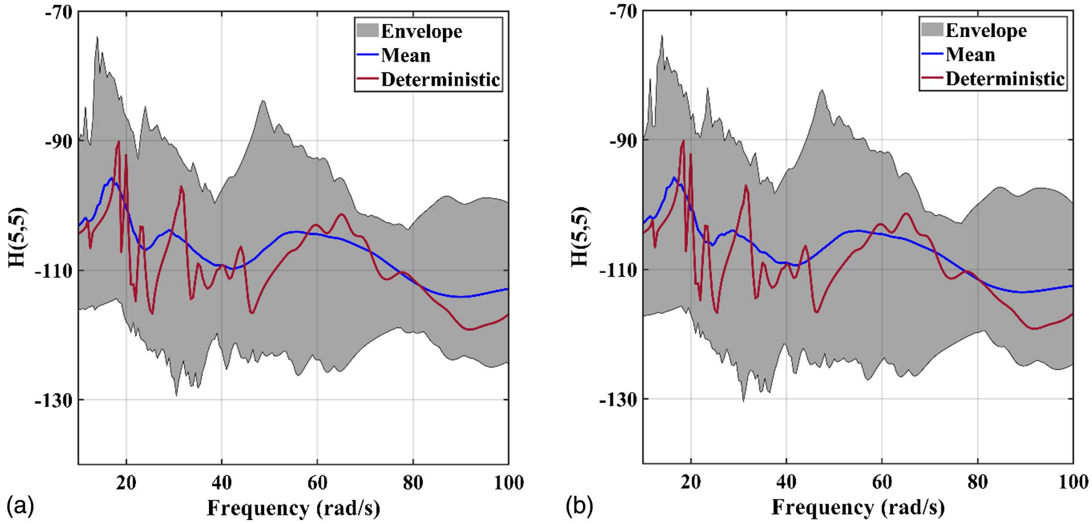

Fig. 7. Displacement FRF band plots (dB) for the transmission tower, H(5,5): (a) MCS; and (b) ANN. Note that 5,000 MCS samples were performed. One hundred training points were employed to construct the neural network.

Table 3. Comparison of failure probabilities obtained by MCS and ANN corresponding to different threshold displacement values for the transmission tower example

Threshold displacement (m)	Probability of failure (MCS; 5,000 samples)	Probability of failure (ANN; 100 samples)
0.22	0.182	0.176
0.25	0.050	0.048
0.28	0.014	0.014

demonstrate that a satisfactory level of approximation accuracy was achieved.

For the reliability analysis, harmonic forces of randomly varying amplitudes (with a lognormal distribution) were assumed to act at different locations. The mean force amplitudes at DOFs 1, 2, 3, 8, 9, 13, 26, 31, 32, 37, 50, 55, 56, 61, 74, 79, 80, 85, 92, 103, 104, 109, 116, 121, and 122 were 10, −20, −20, −20, −20, 10, −20, 10, −20, 7.5, −15, 7.5, −15, 7.5, −15, 7.5, −15, 5, −10, 5, −10, 5, −10, 5, and −10 kN, respectively. The positive directions of the coordinate axes are shown in Fig. 6. A 5% variation was considered in the force amplitudes to simulate loading uncertainty. The limit state function was based on the maximum displacement exceeding threshold (defined in Table 3). The failure probabilities corresponding to different threshold displacement values obtained by MCS and ANN are reported in Table 3. From the results, ANN is observed to approximate the failure probability accurately.

Summary and Conclusions

An efficient computational framework was developed for forward uncertainty propagation from the individual subcomponent level to system level response in built-up structures. In doing so, a theoretical framework of dynamic substructuring combining model reduction with the domain decomposition approach was presented. A multioutput configuration of neural networks was utilized to approximate the mode shapes. Thus, in this context, the present work can be considered as an improvement on the work of Chatterjee et al. (2020). In addition, structural reliability assessment was performed with the help of machine learning.

The proposed framework was explored for application in linear stochastic dynamic systems. Two practical structural engineering application problems were undertaken in which the performance of the proposed approach was observed to be satisfactory. Therefore, the main contribution of this work is that the proposed framework leads to a three-tier improvement in the computational framework of conventional domain decomposition as follows:

- First, the existing computational framework of DD solvers was enhanced by integrating model-order reduction for the local dynamic behavior of substructures together with interface solution problems. This improvement led to a reduction in the computational effort required to solve the actual FE model in a deterministic sense;
- Second, in the presence of random parameters at the subcomponent level, functional decomposition in the stochastic space efficiently captured the uncertainty propagation from the input variables in the individual substructures to the assembled system level dynamic response. The proposed framework was also able to accurately perform reliability assessment of the global structural system; and
- At the stochastic level, further computational leverage was gained by the use of a multioutput framework of neural networks for estimating the retained mode shapes compared to conventional single-output metamodels.

The proposed method will be extended in terms of increasing the complexity of application problems. This can be seamlessly executed without any loss of generality to realize the actual potential of the proposed framework (especially in terms of time efficiency), because it is scalable to the size of the model. Currently, the proposed framework is being extended for multilevel robust and/or reliability-based design optimization with hybrid intelligent machine learning, as recently proposed in Fei et al. (2020). Although this work considers parametric uncertainty modeling, random field modeling will be performed in the future to capture the spatial variability within the subcomponents and the joint/interface using the proposed framework. Another potential area for extending the proposed methodology would be to employ recently developed physics informed deep learning approaches (Raissi et al. 2019) and develop a simulation-free uncertainty quantification framework for structural dynamic systems (Karumuri et al. 2020).

Data Availability Statement

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request. Specifically, (1) the MATLAB data files in (.mat) format for both examples and (2) the geometric details of the second example in Gmsh will be available.

Acknowledgments

The authors gratefully acknowledge the support of the Engineering and Physical Sciences Research Council through the award of a program grant, "Digital Twins for Improved Dynamic Design," Grant No. EP/R006768.

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