Supplementary material
Anisotropy tailoring in geometrically isotropic multi-material lattices

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To present necessary insights concerning the anisotropy tailoring in geometrically isotropic multi-material lattices, efficient computational models for Young's moduli of such lattice are required. On the basis of a unit cell (consisting of three beam-like members connected at a single point) based approach, closed-form analytical expressions for the effective Young's moduli in two orthogonal directions are derived here as a function of the intrinsic material properties and structural geometry. Using the closed-form formulae of in-plane elastic moduli, three different cases are formulated, leading to the contours of: I. isotropy (i.e. minimum anisotropy), II. maximum anisotropy and III. a fixed value of anisotropy. Only the bending deformation is considered in the analytical derivation, which is most predominant for thin-walled lattices with axially rigid members. As a consequence, only the intrinsic material properties of the slant members contribute under such assumption and that of the vertical members does not have any effect on the global elastic behaviour of the multi-material lattice.

Contents

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1. Effective elastic properties of multi-material lattices

A unit cell based approach is adopted for deriving the effective mechanical properties of the lattice structures at macro-scale (refer to figure 1 of the main paper). The unit cells of mono- and multi-material lattices along with auxetic and non-auxetic configurations are shown in figure 1.

1.1. Closed-form expression of effective Young's modulus $\bar{E}_1$

The closed-form expression of effective Young's modulus $\bar{E}_1$ is obtained when the stress is applied in direction 1 (note the directions indicated in figure 2). From figure 2A, it can be noted that $C = 0$ and $M = \dfrac{Pl\sin\theta}{2}$. The bending deflections of the two slant members perpendicular to their respective longitudinal directions, as shown in figure 2A, can be obtained as

$$\delta_1 = \frac{Pl^3\sin\theta}{12E_1 I} \tag{1}$$

$$\delta_2 = \frac{Pl^3\sin\theta}{12E_2 I} \tag{2}$$

where E_1 and E_2 represent the intrinsic Young's modulus of the two slant members. I is the second moment of area of the beam cross section i.e. $I = \dfrac{bt^3}{12}$. Here b and t are the out-of-plane thickness of the honeycomb sheet and cell-wall thickness, respectively. The subscripts 1 and 2 with δ are indicated to represent the two slant members of figure 2. The lumped forces acting at the joints can be expressed as $P = \bar{\sigma_1}(h + l\sin\theta)b$.

Strain along the direction 1 can be obtained as

$$\begin{aligned}\bar{\epsilon_1} &= \frac{(\delta_1+\delta_2)\sin\theta}{2l\cos\theta}\\ &= \frac{Pl^3\sin^2\theta}{12\left(\dfrac{bt^3}{12}\right)2l\cos\theta}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\end{aligned} \tag{3}$$

Substituting the expression of P in the above equation,

$$\begin{aligned}\bar{\epsilon_1} &= \frac{\bar{\sigma_1}b(h+l\sin\theta)l^2\sin^2\theta}{bt^3 2\cos\theta}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\\ &= \bar{\sigma_1}\frac{(h+l\sin\theta)l^2\sin^2\theta}{2t^3\cos\theta}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\end{aligned} \tag{4}$$

Noting that the effective Young's modulus in direction 1 can be expressed as $\bar{E}_1 = \dfrac{\sigma_1}{\epsilon_1}$, we get

$$\bar{E}_1 = 2\left(\frac{t}{l}\right)^3 \frac{\cos\theta}{\left(\dfrac{h}{l}+\sin\theta\right)\sin^2\theta\left(\dfrac{1}{E_1}+\dfrac{1}{E_2}\right)} \tag{5}$$

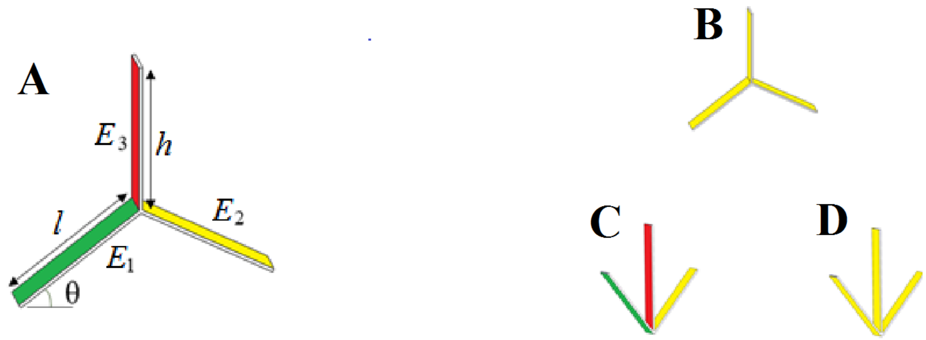

Figure 1: **(A)** Unit cell of a general multi-material hexagonal lattice with dimensions (refer to figure 1 of the main paper). Here the three constituting beam elements have different material properties, as indicated by different colours. **(B)** Unit cell of a typical mono-material hexagonal lattice where material properties are same for all the three constituting materials **(C - D)** Unit cells for re-entrant multi-material and mono-material lattices where the cell angle θ is negative (refer to figure 1 of the main paper).

As a special case, when $E_1 = E_2 = E_s$, the above expression can be reduced to the traditional equations for mono-material lattice [1]

$$\bar{E_1} = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{\left(\frac{h}{l} + \sin\theta\right)\sin^2\theta} \tag{6}$$

1.2. Closed-form expression of effective Young's modulus $\bar{E_2}$

The closed-form expression of effective Young's modulus $\bar{E_2}$ is obtained when the stress is applied in direction 2. From figure 2B, it can be noted that $W_1 + W_2 = W$ and $W = 2\bar{\sigma_2} lb\cos\theta$. The bending deflections of the two slant members perpendicular to their respective longitudinal directions, as shown in figure 2B, can be obtained as

$$\delta_1 = \frac{W_1 l^3 \cos\theta}{12 E_1 I} \tag{7}$$

$$\delta_2 = \frac{W_2 l^3 \cos\theta}{12 E_2 I} \tag{8}$$

From deformation compatibility condition we get

$$\delta_1 \cos\theta = \delta_2 \cos\theta \tag{9}$$

Substituting the expression of δ_1 and δ_2 in the above expression and noting that $W_1 + W_2 = W$, we obtain

$$W_1 = \frac{E_1}{E_1 + E_2} W \tag{10}$$

$$W_2 = \frac{E_2}{E_1 + E_2} W \tag{11}$$

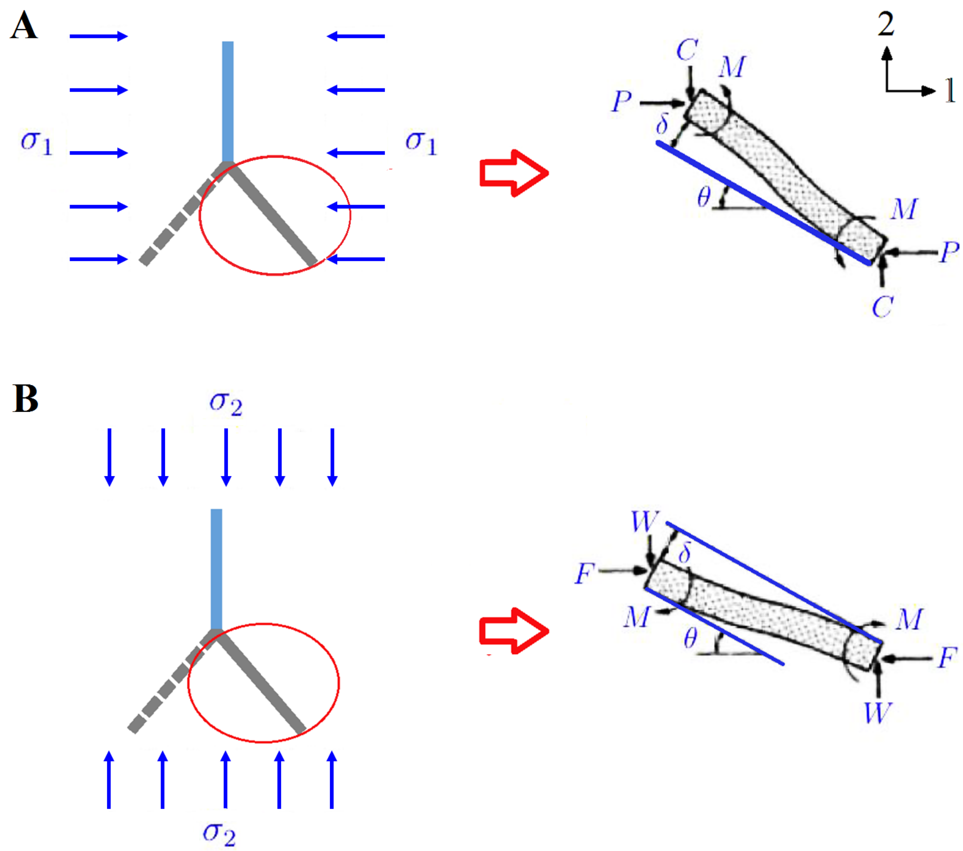

Figure 2: **(A)** Unit cell of a multi-material hexagonal lattice subjected to compressive stress in direction 1 (Here three different members are made of three different materials and indicated using different colours and line styles.) **(B)** Unit cell of a multi-material hexagonal lattice subjected to compressive stress in direction 2.

Considering a linear strain field, strain along the direction 2 can be written using Equations 7, 8, 10 and 11 as

$$\begin{aligned}
\bar{\epsilon_2} &= \frac{\left(\frac{(\delta_1+\delta_2)}{2}\cos\theta\right)}{(h+l\sin\theta)} \\
&= \frac{l^3\cos^2\theta}{12I(2(h+l\sin\theta))}\left(\frac{W_1}{E_1}+\frac{W_2}{E_2}\right) \\
&= \frac{l^3\cos^2\theta}{12\left(\frac{bt^3}{12}\right)2(h+l\sin\theta)}\left(\frac{W}{E_1+E_2}+\frac{W}{E_1+E_2}\right)
\end{aligned} \tag{12}$$

Replacing $W = 2\bar{\sigma_2} lb\cos\theta$ and noting that the effective Young's modulus in direction 2 can be expressed as $\bar{E_2} = \frac{\sigma_2}{\epsilon_2}$, we can obtain

$$\bar{E_2} = \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l}+\sin\theta\right)(E_1+E_2)}{2\cos^3\theta} \tag{13}$$

As a special case, when $E_1 = E_2 = E_s$, the above expression can be reduced to the traditional equation for mono-material lattice [1]

$$\bar{E_2} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\cos^3\theta} \tag{14}$$

1.3. Closed-form expression of effective Poisson's ratio $\bar{\nu_{12}}$

The closed-form expression of effective Poisson's ratio $\bar{\nu_{12}}$ is obtained when the stress is applied in direction 1. From Equation 3 we can write the strain component in direction 1 due to applied stress in direction 1 as

$$\bar{\epsilon_1} = \frac{(\delta_1 + \delta_2)\sin\theta}{2l\cos\theta} \tag{15}$$

Considering a linear strain field, total strain in direction 2 due to applied stress in direction 1 is

$$\bar{\epsilon_2} = -\frac{(\delta_1 + \delta_2)\cos\theta}{2(h + l\sin\theta)} \tag{16}$$

The closed-form expression of effective Poisson's ratio $\bar{\nu_{12}}$ is given by

$$\bar{\nu_{12}} = -\frac{\bar{\epsilon_2}}{\bar{\epsilon_1}} = \frac{\cos^2\theta}{(h/l + \sin\theta)\sin\theta} \tag{17}$$

From the above expression, we note that the Poisson's ratio $\bar{\nu_{12}}$ does not depend on the intrinsic material properties of the beam members. It only depends on the structural geometry of the lattice. The expression of $\bar{\nu_{12}}$ exactly matches with the standard formulae given in literature [1]. Thus there is no effect of multiple materials in the lattice on Poisson's ratio.

1.4. Closed-form expression of effective Poisson's ratio $\bar{\nu_{21}}$

The closed-form expression of effective Poisson's ratio $\bar{\nu_{21}}$ is obtained when the stress is applied in direction 2. From Equation 12 we can write the strain component in direction 2 due to applied stress in direction 2 as

$$\bar{\epsilon_2} = \frac{(\delta_1 + \delta_2)\cos\theta}{2(h + l\sin\theta)} \tag{18}$$

Considering a linear strain field, total strain in direction 1 due to applied stress in direction 2 is

$$\bar{\epsilon_1} = -\frac{(\delta_1 + \delta_2)\sin\theta}{2l\cos\theta} \tag{19}$$

The closed-form expression of effective Poisson's ratio $\bar{\nu_{21}}$ is given by

$$\bar{\nu_{21}} = -\frac{\bar{\epsilon_1}}{\bar{\epsilon_2}} = \frac{(h/l + \sin\theta)\sin\theta}{cos^2\theta} \tag{20}$$

From the above expression, we note that the Poisson's ratio $\bar{\nu_{21}}$ does not depend on the intrinsic material properties of the beam members. It only depends on the structural geometry of the lattice. The expression of $\bar{\nu_{21}}$ exactly matches with the standard formulae given in literature [1]. Thus there is no effect of multiple materials in the lattice on Poisson's ratio.

2. Critical remarks

2.1. Remark 1: Special cases

Case 1: For a structurally regular (/geometrically isotropic) lattice ($\theta = 30^o; h = l$) when $E_1 = E_2 = E_s$

$$\bar{E_1} = \bar{E_2} = 2.3 E_s \left(\frac{t}{l}\right)^3 \tag{21}$$

$$\bar{\nu_{12}} = \bar{\nu_{21}} = 1 \tag{22}$$

The above expressions exactly match with the literature [1, 2].

Case 2: For a structurally regular (/geometrically isotropic) lattice ($\theta = 30^o; h = l$) when $E_1 \neq E_2$ (i.e. structurally regular multi-material lattice)

$$\bar{E_1} = \frac{8}{\sqrt{3}} \frac{E_1 E_2}{E_1 + E_2} \left(\frac{t}{l}\right)^3 \tag{23}$$

$$\bar{E_2} = \frac{2}{\sqrt{3}} (E_1 + E_2) \left(\frac{t}{l}\right)^3 \tag{24}$$

$$\bar{\nu_{12}} = \bar{\nu_{21}} = 1 \tag{25}$$

The expressions show that $\bar{E_1} \neq \bar{E_2}$ in structurally regular multi-materiel lattices ($\theta = 30^o$ and $h = l$).

2.2. Remark 2: Non-dimensionalization

Since effective mechanical properties of lattice metamaterials made of same intrinsic material(s) vary with the microstructural geometry, such effective properties are often presented as non-dimensional ratios of $\frac{\bar{E_1}}{E_s}$ and $\frac{\bar{E_2}}{E_s}$. Based on Equation 5 and 13, the effective Young's moduli in non-dimensional form can be written as

$$\frac{\bar{E_1}}{E_s} = 2\,(t/l)^3 \frac{\cos\theta}{(h/l + \sin\theta)\sin^2\theta(1+\alpha)} \tag{26}$$

$$\frac{\bar{E_2}}{E_s} = (t/l)^3 \frac{(h/l + \sin\theta)(1+\alpha)}{2\alpha\cos^3\theta} \tag{27}$$

where, $\frac{E_1}{E_2} = \alpha$ and $E_1 = \alpha E_2 = E_s$. Here we define degree of anisotropy as $q = \frac{\bar{E_1}}{\bar{E_2}}$. Each material and geometric terms in the above equations are presented in non-dimensional form. It is interesting to notice that the in-plane elastic moduli (Young's moduli and Poisson's ratios) for a hexagonal multi-material lattice do not depend up on the intrinsic material property of the vertical member when bending deformation is predominant. In fact, the Poisson's ratios are found to be only dependent upon the microstructural geometric properties. However, if we have to consider the effect of axial deformation (such as the case of structural members where axial rigidity is not very high), the intrinsic material properties of the vertical members are expected to contribute significantly.

2.3. Remark 3: Exact analytical validation and numerical validation

For $\alpha = 1$, the expressions of elastic moduli for multi-material lattice (refer to equations 26 and 27) reduce to the traditional expressions of mono-material lattice [1] as a special case. This observation provides an exact analytical validation of the derived formulae. To validate the proposed expressions of $\bar{E}_1$ and $\bar{E}_2$ for different other values of α, we adopt a finite element based approach as discussed in the following paragraph.

The lattice structure is modelled in the ANSYS environment (commercial finite element code) considering different values of α. Each of the members are modelled as beam elements. The boundary conditions are given on the joints of the lattice and depends on the Young's modulus that is calculated. For explaining the boundary conditions, we represent two mutually perpendicular in-plane directions as i and j, where $i, j = 1, 2$ and $i \neq j$. For calculating the Young's modulus in direction i, concentrated forces are applied on the joints of one of the edges that is perpendicular to the direction i. Note that the stress in direction i can be calculated from the applied concentrated loads on the joints by assuming an uniform stress distribution along the edge. In the opposite edge of the applied loads, the translational deflections of the joints in the loading direction (i.e direction i) are restrained while the translational deflections in direction j and rotations of the joints are allowed. In the joints of one of the edges parallel to the loading direction i, the translational deformations in direction j are restrained while the deflections in direction i and rotational deformations are allowed. In the remaining edge, all deformations are allowed at all the joints. The effective Young's moduli are calculated based on global strain and applied stress in longitudinal and transverse directions. The finite element analysis results are presented in figure 3 along with the corresponding analytical results (obtained based on equations 26 and 27) for different values of α. A good agreement between the results of two forms of analysis can be noticed. It is also interesting to note the increasing level of deviation between $\bar{E}_1$ and $\bar{E}_2$ for higher values of α, essentially corroborating

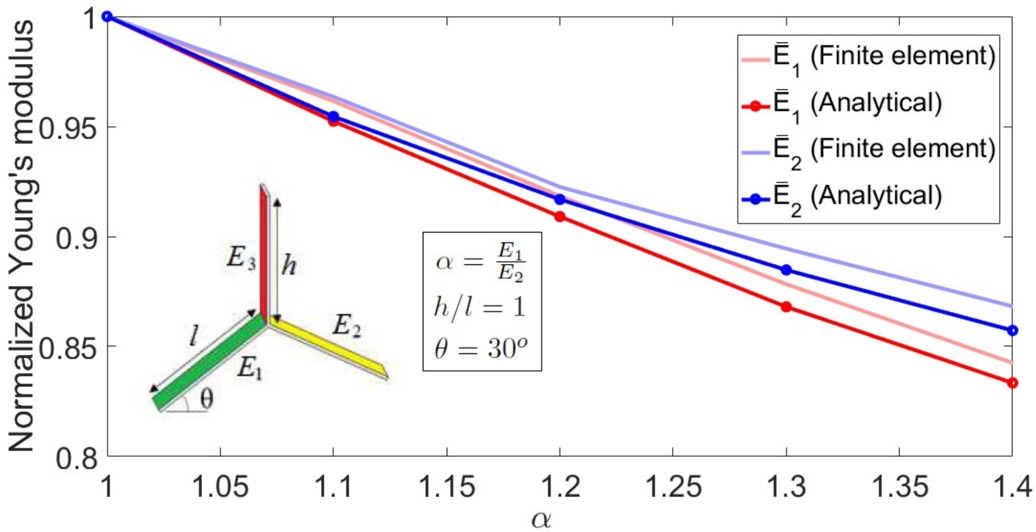

Figure 3: Numerical validation of the proposed analytical formulae of $\bar{E}_1$ and $\bar{E}_2$ (effective longitudinal and transverse Young's moduli of multi-material lattices) with finite element model. The normalized Young's moduli obtained using the proposed analytical model and finite element model are presented for different values of α $(= \dfrac{E_1}{E_2})$. The normalization of the Young's moduli is carried out with respect to the corresponding Young's moduli of conventional monomaterial lattices (i.e. $\alpha = 1$). The microstructural details of the lattice considered for obtaining the results are shown in inset of the figure.

the preliminary evidence of the possibility of anisotropy tailoring based on multi-material parameters. In general, the exact analytical validation for $\alpha = 1$ and finite element based numerical validation for different values of α generate necessary confidence to utilize the proposed analytical formulae of $\bar{E}_1$ and $\bar{E}_2$ further for demonstrating the aspect of anisotropy tailoring.

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