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A global two-layer meta-model for response statistics in robust design optimization

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ABSTRACT

Robust design optimization (RDO) of large-scale engineering systems is computationally intensive and requires significant CPU time. Considerable computational effort is still required within conventional meta-model assisted RDO frameworks. The primary objective of this article is to minimize further the computational requirements of meta-model assisted RDO by developing a global two-layered approximation based RDO technique. The meta-model in the inner layer approximates the response quantity and the meta-model in the outer layer approximates the response statistics computed from the response meta-model. This approach eliminates both model building and Monte Carlo simulation from the optimization cycle, and requires considerably fewer actual response evaluations than a single-layered approximation. To demonstrate the approach, two recently developed compressive sensing enabled globally refined Kriging models have been utilized. The proposed framework is applied to one test example and two real-life applications to illustrate clearly its potential to yield robust optimal solutions with minimal computational cost.

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RDO; Kriging; HDMR; compressive sensing; adaptive sparse

1. Introduction

Robust design optimization (RDO) is one of the most popular approaches for dealing with optimization under uncertainty (Du and Chen 2002). RDO has been applied in various applications to enhance the quality of products by yielding the least sensitive design (Roy and Chakraborty 2015; Chakraborty, Goswami, and Rabczuk 2019). RDO constitutes a mathematical framework to minimize the effect of random input parameters on the output response quantities of interest (Zhu, Zhang, and Chen 2015; Vu-Bac *et al.* 2016), thereby resulting in a more robust solution.

The RDO problem involves enhancing the system performance by reducing the performance variation. In most cases, the variation in the performance functions as defined by the objective and/or the constraint functions is expressed in terms of the response statistics. Consequently, the solution of the RDO problem will require repeated computationally expensive simulations integrated within the optimization routine.

In order to reduce the computational requirements, various efficient meta-modelling schemes have emerged. The literature in this field is well developed and an overview of the mathematical framework underpinning the different methods was presented by Chatterjee, Chakraborty, and Chowdhury (2019). Meta-models have been implemented for design optimization under uncertainty

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(Saijal, Ganguli, and Viswamurthy 2011; Murugan, Ganguli, and Harursampath 2012), although the following issues still remain to be addressed.

- The accuracy of the meta-models must be sufficient to enable a successful RDO implementation. Some popular meta-models are susceptible to deviations due to uncertainties within the optimization framework and may yield incorrect optima (Chatterjee, Chakraborty, and Chowdhury 2019).
- It is well known that extensive computations have to be performed within the optimization cycles in the RDO framework. This restricts the applicability to low and medium scaled problems.

The primary objective of the present work is to address the above issues in meta-model assisted RDO frameworks. Note that the first issue regarding the accuracy of the meta-model has been addressed recently by Chatterjee, Chowdhury, and Ramu (2019), and therefore the same meta-models are adopted here. Thus, the focus here is more towards the improvement of the computational efficiency of meta-model assisted RDO.

In a meta-model assisted RDO framework, response statistics within the objective and constraint functions are estimated with the help of a meta-model. In general, the above setting involves a two-step solution: (i) the response quantity of interest is approximated by the meta-model, which involves a relatively small number of actual function evaluations for training the meta-model; and (ii) the response statistics are obtained by performing Monte Carlo simulations (MCSs) on the meta-model. In this regard, a simple yet effective concept has been devised to decouple (i) and (ii) from the optimization cycle in RDO. Thus, a global two-layered meta-model assisted response statistics approximation based RDO framework has been developed, where the response statistics involved in the objective and constraint problems are approximated before the optimization commences. As a result, no model building or MCS (points (i) and (ii)) have to be performed within the optimization routine. This yields optimal solutions in fewer actual response evaluations and results in improvement in terms of computational effort over recent approaches (Chatterjee, Chowdhury, and Ramu 2019) so that it can readily be applied to large-scale problems.

The rest of the article is organized as follows. The conventional meta-assisted RDO framework is briefly outlined in Section 2. In Section 3, the proposed approximation based RDO framework is developed. The formulation of the two meta-models used is briefly summarised in Section 4. The performance of the proposed framework is assessed by solving an analytical test example in Section 5. Section 6 considers two computationally expensive finite element (FE) based practical engineering problems. Finally, the study is concluded in Section 7.

2. Meta-model assisted robust design optimization

This section describes the framework of conventional meta-model assisted RDO to demonstrate the enhancement provided by the proposed techniques. A common approach to define the RDO problem is based on the statistics (mean and standard deviation) of the response quantities of interest in the objective $\hat{y}_0$ and constraint functions $\hat{y}_c$. The optimization problem may be written in the following form (Chen *et al.* 1996):

$$\begin{aligned}
\underset{\mathbf{d}\subset D\in\mathbb{R}^N}{\text{minimize}} \quad & h_f(\mathbf{d}) := \alpha_w \mathbb{E}(\hat{y}_0(\mathbf{x},\mathbf{d})) + (1-\alpha_w)\sqrt{\operatorname{var}(\hat{y}_0(\mathbf{x},\mathbf{d}))} \\
\text{subject to} \quad & h_{g_c}(\mathbf{d}) := \mathbb{E}(\hat{y}_c(\mathbf{x},\mathbf{d})) + k_c\sqrt{\operatorname{var}(\hat{y}_c(\mathbf{x},\mathbf{d}))} \le 0, \quad c = 1,\ldots,n_c \\
& d_{i,l} \le d_i \le d_{i,u}, \quad i = 1,\ldots,n_v,
\end{aligned} \tag{1}$$

where h_f and h_{g_c} are the objective and constraint functions of the RDO problem, respectively; $\mathbf{x}$ and $\mathbf{d}$ are the random variables and design variables (DVs), respectively; and $\mathbb{E}$ and var represent the mean and variance, respectively. Note that $\hat{y}_0$ and $\hat{y}_c$ are the objective and constraint functions of the deterministic optimization problem. Weights $\alpha_w \in [0, 1]$ and k_c are assigned to the mean and variance of the response quantities in Problem (1); this approach is known as the weighted sum method (WSM) and the objective and constraints may be interpreted in terms of confidence intervals.

Remark 2.1: To distinguish between the DVs **d** (controllable parameters in the optimization) and other random variables **x** defined in the probability space, this unique notation is used. To illustrate this further, the following potential instances are discussed: (i) if all the DVs are random and there are no other random variables, either notation **d** or **x** can be used—this is encountered for the problem presented in Section 5 and the notation **d** has been used to denote the random DVs; (ii) if all the DVs are random and there are other random variables as in the problems of Section 6, the notation **d** and **x** has been used for the (random) DVs and other random parameters, respectively.

Several other improved RDO methods are given in Messac (1996) and Chen *et al.* (2000). The WSM defined in Problem (1) can be viewed as a multi-objective optimization and the RDO problem may be solved using multi-objective optimization techniques (Srinivas and Deb 1994; Zhang and Li 2007). Other variants of the RDO formulation define unique metrics of robustness (Du, Sudjianto, and Chen 2004; Huang and Du 2007), use sensitivity based approaches (Han and Kwak 2004; Chakraborty, Bhattacharjya, and Haldar 2012) or use adaptive strategies (Mortazavi, Azarm, and Gabriel 2013; Cheng *et al.* 2014).

Broadly speaking, for approximation purposes, meta-models are integrated into the RDO formulation in one of two ways. The first approach is to construct the meta-model outside of the optimization cycle to approximate the response quantities involved in the objective function and constraints (*i.e.* $\hat{y}_0$ and $\hat{y}_c$ in Problem (1)) and perform MCS on the meta-model to evaluate the response statistics (*i.e.* $\mathbb{E}[\hat{y}_0]$, $\mathbb{E}[\hat{y}_c]$, var$[\hat{y}_0]$ and var$[\hat{y}_c]$) at every optimization iteration (Bhattacharjya and Chakraborty 2011). The computational cost using this approach is minimal as no actual response evaluations are required within the optimization cycle. However, the approach can often have difficulty capturing the nonlinear response trends across the functional space. In contrast, the second type of approach involves building a meta-model to approximate the response quantities within each optimization iteration (Chakraborty *et al.* 2017b). This means that the stochastic computations lie within the optimization cycle and make the computation required prohibitive for large-scale problems. An adaptive form of these two approaches has been proposed by Chakraborty *et al.* (2017a) to maintain a balance between required accuracy level and affordable computational cost. An analytical response moment approximation scheme has been developed by Chatterjee, Chakraborty, and Chowdhury (2018) so as to form an MCS-free RDO strategy. However, in the above techniques, the computationally expensive evaluations in the form of actual response calculations and meta-model building still remained within the optimization cycle.

Model building and MCS were removed from the optimization cycle successfully by Chatterjee, Chowdhury, and Ramu (2019). However, only a single approximation layer was devised, which still required a considerable number of actual response evaluations to estimate the statistics. Thus, this approach did not prove to be particularly suited to solving real-world problems as it turned out to be computationally prohibitive. For other improvements in meta-model assisted RDO frameworks, the reader is referred to Ren and Rahman (2013), Zhou *et al.* (2018) and Shimoyama *et al.* (2009).

After assessing the current state of the art of meta-model assisted RDO approaches, it was concluded by the present authors that either the approximation accuracy or the computational effort dominates depending upon the problem complexity. Therefore, in this article, the objective is to develop a framework that reduces the computational cost by completely decoupling the stochastic computations from the optimization cycle and at the same maintaining a desirable level of prediction accuracy to solve real-world problems.

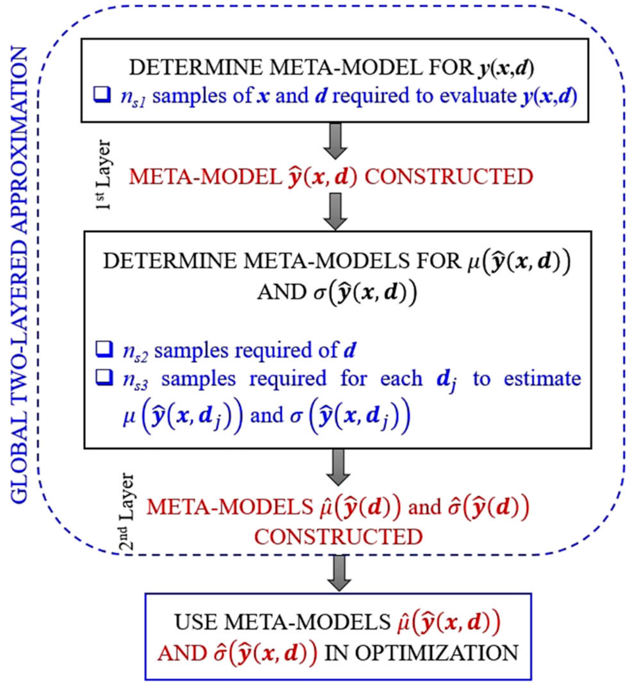

Figure 1. Schematic of the proposed RDO framework.

3. Proposed meta-model assisted robust design optimization framework

The basic idea underlying the proposed meta-model assisted RDO framework is to develop a global approximation of the response statistics, instead of the response quantities. The term 'global approximation' means that the meta-model is built at the start, *i.e.* before the optimization commences. The advantage of this approach is that the expensive response evaluations, meta-model building and MCS to evaluate the response statistics are removed from the optimization cycle. Thus, once the meta-model has been constructed to predict the response statistics, the RDO framework may be viewed as a deterministic optimization. The proposed RDO framework is illustrated schematically in Figure 1.

Initially, n_{s1} input sample points are generated using a suitable experimental design scheme (in this case, Latin-hypercube sampling has been employed) for each design variable $\mathbf{d}$. In case there are additional random variables $\mathbf{x}$, n_{s1} samples of $\mathbf{x}$ are generated. Corresponding to these samples of $\mathbf{x}$ and $\mathbf{d}$, n_{s1} actual responses (denoted by $\mathbf{y}$) involved in the objective and constraint functions are computed. These samples then allow the meta-model to be built that approximates the relationship between the DVs ($\mathbf{d}$) and the output response, $\mathbf{y}(\mathbf{x}, \mathbf{d})$. This is the first-layer meta-model constructed to predict the expensive actual response, $\mathbf{y}$, cheaply with the help of the approximate response, $\hat{\mathbf{y}}$.

The next step starts by generating n_{s2} input sample points (comprising samples of $\mathbf{d}$) to train the response statistics corresponding to the mean of each DV by a suitable experimental design scheme (in this work, Latin-hypercube sampling). Subsequently, n_{s3} random points are to be generated according to the probability distribution of the DVs. Then n_{s3} approximate responses $\hat{\mathbf{y}}$ are predicted by the first-layer meta-model (built in the second step) corresponding to the n_{s3} random points generated previously. Next, n_{s2} estimates of the mean, $\mu(\hat{\mathbf{y}})$, and standard deviation (SD), $\sigma(\hat{\mathbf{y}})$, of the response are computed from each set of approximate response vectors of size n_{s3} obtained previously. As the final step of the proposed global approximation framework, the second-layer meta-models are built to capture the trend between the mean of the DVs and the output response statistics (mean, $\mu(\hat{\mathbf{y}})$, and SD, $\sigma(\hat{\mathbf{y}})$, of the approximate response) obtained earlier. These second-layer meta-models

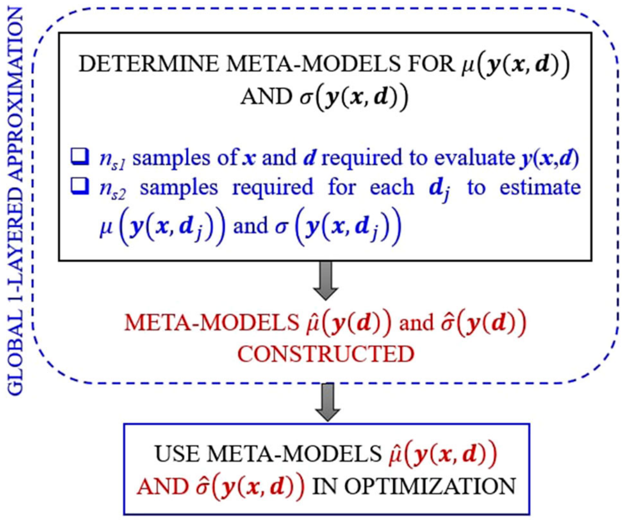

Figure 2. Schematic of the global single layered approximation based RDO framework.

may be readily utilized to predict the response statistics within the objective and constraint functions of the optimization cycle corresponding to each design point. Hence, the optimization cycle is free from any stochastic computations. The kernel of the proposed framework relies on the assumption of a parametric probabilistic description of the stochastic input and/or, DVs. The treatment of spatially varying uncertainties within the proposed framework will be possible with random field modelling. However, this will require further investigation and is not within the scope of the present work.

To implement the above framework, the number of samples n_{s1}, n_{s2} and n_{s3} have to be defined, and their optimal values are determined via various convergence studies. Specifically, n_{s1} is determined corresponding to the minimum approximation error in estimating the response quantity using the first-layer meta-model based on leave-one-out cross validation (LOOCV) (Efron and Tibshirani 1997). After selecting an optimal value of n_{s1}, the variation of $\hat{\mathbf{y}}$ is studied in order to select an appropriate value of n_{s3}. Once n_{s1} and n_{s3} are chosen, n_{s2} is determined corresponding to the minimum LOOCV error in approximating the response statistics using the second-layer meta-model.

Remark 3.1: Although the recently proposed approach by the same authors (Chatterjee, Chowdhury, and Ramu 2019) may seem to be similar, the proposed approach is much more computationally efficient, especially in solving expensive problems. The difference is relatively subtle but very effective. To illustrate the difference of the proposed framework, a flowchart of Chatterjee, Chowdhury, and Ramu (2019) has been illustrated in Figure 2. It can be observed from Figure 2 that the response quantities $\mathbf{y}$ are generated and the response statistics $\mu(\mathbf{y})$ and $\sigma(\mathbf{y})$ are evaluated directly. These statistics are used to train the meta-models and the approximate response statistics $\mu(\mathbf{y})$ and $\sigma(\mathbf{y})$, which are predicted by the models within the optimization cycle. In contrast to Figure 2, the proposed method limits the number of actual function evaluations (which may be computationally expensive in the case of large-scale systems). As illustrated in Figure 1, the response quantities $\mathbf{y}$ are generated and then a meta-model is constructed to approximate $\mathbf{y}$. Now $\hat{\mathbf{y}}$, as predicted by the above surrogate model, is utilized to generate samples of the response statistics $\mu(\hat{\mathbf{y}})$ and $\sigma(\hat{\mathbf{y}})$. These response statistics are used to construct another layer of meta-models. This second layer of meta-models predicts $\mu(\hat{\mathbf{y}})$ and $\sigma(\hat{\mathbf{y}})$ at each optimization iteration.

The dual layer of approximation in the proposed method reduces the number of evaluations of $\mathbf{y}$ to n_{s_1} compared to $(n_{s_1} \times n_{s_2})$ in Chatterjee, Chowdhury, and Ramu (2019), which allows the former to solve comparatively larger-scale problems. It may be noted that, despite the useful framework of

Chatterjee, Chowdhury, and Ramu (2019), only toy problems could be solved, and no real engineering application problem could be addressed due to the dimensionality issue.

Figure 1 shows the general RDO approach, and any meta-model and/or optimization scheme can be employed. However, the meta-models should be robust enough to capture the variation of the response statistics using only a single approximation, before the optimization cycle commences, to predict the statistics accurately at the design points of subsequent iterations. As previously illustrated, a slight deviation in the approximation accuracy of the meta-models in any of the intermediate iterations may easily lead to false optima. Thus, in order to ensure this requirement is satisfied, adaptive sparse refined Kriging models have been employed for the bi-layered approximation of response quantities and moments. These meta-models are briefly outlined in the next section.

4. Proposed meta-modelling techniques

The details of the meta-models introduced here can be found in a recent article by Chatterjee and Chowdhury (2018) where the accuracy of the meta-models has been assessed for the global sensitivity analysis of nonlinear and high-dimensional stochastic systems. The key characteristics of the proposed meta-model for this application are as follows.

- To enhance the global approximation characteristics of Kriging, the term related to the trend is replaced by a high-dimensional model representation (HDMR) (Li, Rosenthal, and Rabitz 2001; Chatterjee, Chakraborty, and Chowdhury 2016).
- Two compressive sensing strategies have been employed that impart adaptive sparsity to the meta-models, and as a result, the computational complexity in dealing with high-dimensional systems is reduced (Doostan and Owhadi 2011).

4.1. Improving the approximation potential of Kriging

The functional form of the globally refined Kriging model is obtained by substituting HDMR for its trend term (Chatterjee, Adhikari, and Friswell 2020), and is given by

$$\hat{\mathbf{y}} = \left(\mathbf{y}_0 + \sum_{k=1}^{M}\left\{\sum_{i_1=1}^{N-k+1}\cdots\sum_{i_k=i_k-1}^{N}\sum_{r=1}^{k}\left[\sum_{m_1=1}^{s}\sum_{m_2=1}^{s}\cdots\sum_{m_r=1}^{s}\alpha_{m_1 m_2\ldots m_r}^{(i_1 i_2\ldots i_k)i_r}\Psi_{m_1}^{i_1}\cdots\Psi_{m_r}^{i_r}\right]\right\}\right) + \sigma^2\mathbf{Z}(\mathbf{x},\boldsymbol{\omega}), \tag{2}$$

where $\mathbf{y}_0$ represents the zeroth-order component function, implying the mean of response $\mathbf{y}(\mathbf{x})$. $\mathbf{Z}(\mathbf{x},\boldsymbol{\omega})$ denotes a zero mean, unit variance Gaussian process and σ^2 represents the process variance. The maximum order of the component functions and bases Ψ are M and s, respectively. N represents the number of variables. In the next sub-section, the determination of the unknown coefficients $\boldsymbol{\alpha}$ is discussed.

4.2. Sparse recovery of the unknown coefficients

In high-dimensional systems, the resulting system of equations is often underdetermined. In such cases, to ensure a well-posed solution, sparse solutions are recovered by using compressive sensing strategies as shown in Equation (3):

$$P_{\nu,\delta} : \underset{\boldsymbol{\alpha}}{\text{minimize}} \ \|\mathbf{W}\boldsymbol{\alpha}\|_{\nu} \text{ subject to } \|\boldsymbol{\Psi}\boldsymbol{\alpha} - \mathbf{y}\|_2 \leq \varepsilon, \tag{3}$$

where $\nu = 0, 1$, the positive diagonal weight matrix $\mathbf{W}$ is a measure of the sparsity of $\boldsymbol{\alpha}$ and $\| \cdot \|_2$ denotes the ℓ_2-norm quantifying the accuracy of the approximation in constructing the sparse set; ε is the error tolerance.

To solve the optimization problem in Equation (3), orthogonal matching pursuit (OMP), which is a compressive sensing algorithm based on the ℓ_0-norm, and basis pursuit (BP), which is based on the ℓ_1-norm, have been utilized to determine the sparse set of unknown coefficients (Chatterjee and Chowdhury 2018). Hereafter, for notational clarity, the globally refined Kriging model integrated with OMP and BP will be referred to as proposed meta-model 1 (PM1) and proposed meta-model 2 (PM2), respectively.

5. Verification on an analytical test problem

The proposed RDO framework (Section 3), combined with the bi-layered approximation using PM1 and PM2 (Section 4), are used for the numerical verification in this section. An analytical test problem is solved in order to access the performance of the proposed RDO frameworks (PM1-RDO and PM2-RDO). A speed reducer design problem has been selected for investigation, as it consists of nonlinear performance functions and multiple constraints. The deterministic optimization problem statement is presented in the online Supplemental Data, which can be accessed at https://doi.org/10.1080/0305215X.2020.1861262. Other details of the RDO problem formulation can be found in Chatterjee, Chakraborty, and Chowdhury (2019). For the RDO formulation of this problem, each of the DVs are considered to be normally distributed with 5% variation. The RDO problem may be stated as $\mu_{\mathbf{d}} = \operatorname{argmin}[F := \mu(f(\mathbf{d})) + \sigma(f(\mathbf{d}))]$, where $\mu_{\mathbf{d}}$ denotes the mean of DVs at optimum F and $f(\mathbf{d})$ is defined in the Supplemental Data. The primary motive is to access the accuracy of the proposed RDO framework in approximating the nonlinear response trends and to evaluate the computational cost in comparison to several existing meta-model assisted RDO frameworks. The interior-point optimization algorithm has been used, as implemented by the command *fmincon* in MATLAB®.

To access the accuracy of the meta-model assisted RDO frameworks, MCS-based RDO solutions are presented for validation. For comparison, the above surrogate models are implemented in a conventional high-fidelity RDO (HF-RDO) framework (*i.e.* the meta-model building and MCS are performed within the optimization routine). The purpose of this comparison is to illustrate the improvement in computational effort achieved with a slight compromise in the accuracy by using the proposed RDO framework as compared to HF-RDO. The RDO methodology in Chatterjee, Chowdhury, and Ramu (2019) using the same meta-models has also been compared. Results achieved with other meta-models can be found in Chatterjee, Chowdhury, and Ramu (2019).

The details regarding the selection of parameters n_{s_1}, n_{s_2} and n_{s_3} are provided in the Supplemental Data. Table 1 gives the robust optimal solutions of the speed reducer design problem. The computational effort required for each of the meta-models to yield the optimal solutions for the speed reducer design problem are compared in Table 2.

In Table 2, n_a denotes the number of actual response evaluations per optimization iteration and selected based on the LOOCV error; n_{itr} is the number of optimization iterations required for convergence; TNARE represents the total number of actual response evaluations. In Table 2, the computational effort of the proposed RDO framework in terms of total actual response evaluations (TNARE) = n_{s1} whereas for the HF-RDO framework and Chatterjee, Chowdhury, and Ramu (2019), TNARE = $(n_a \times n_{\text{itr}})$ and $(n_{s_1} \times n_{s_2})$, respectively. The reason for expressing the overall computational cost in terms of actual response evaluations is that the use of meta-models is justified in real systems where an actual response evaluation takes significant time compared to the response evaluation using the meta-model. As illustrated in Table 1, the results obtained using the proposed RDO framework are close to those obtained with MCS, which illustrates good approximation accuracy of the former. Tables 1 and 2 also illustrate that considerable savings in computational cost can be achieved by the proposed RDO strategy with a slight compromise in the accuracy as compared to the

Table 1. Robust optimal solutions for the speed reducer design problem.

	MCS	PM1 -RDO	PM2 -RDO	PM1[a]	PM2[a]	HF-RDO (PM1)	HF-RDO (PM2)
μ_{d_1}	3.4863	3.5177	3.5201	3.5177	3.5201	3.4863	3.4863
μ_{d_2}	0.7	0.7	0.7	0.7	0.7	0.7	0.7
μ_{d_3}	17	17	17	17	17	17	17
μ_{d_4}	7.3	7.5524	7.5539	7.5524	7.5539	7.3	7.3
μ_{d_5}	7.7158	7.7028	7.7084	7.7028	7.7084	7.7158	7.7158
μ_{d_6}	3.3515	3.3519	3.3516	3.3519	3.3516	3.3515	3.3515
μ_{d_7}	5.2867	5.2642	5.2647	5.2642	5.2647	5.2866	5.2866
$y^* \times 10^3$	3.2325	3.2297	3.2311	3.2297	3.2311	3.2287	3.2303
$\mu(y^*) \times 10^3$	3.0025	2.9983	2.9996	2.9983	2.9996	3.0025	3.0025
$\sigma(y^*) \times 10^2$	2.3003	2.3138	2.3153	2.3138	2.3153	2.3003	2.3003

[a]Chatterjee, Chowdhury, and Ramu (2019).

Table 2. Computational cost estimated in terms of response evaluations to solve the speed reducer design problem.

	MCS	PM1-RDO	PM2-RDO	PM1[a]	PM2[a]	HF-RDO (PM1)	HF-RDO (PM2)
n_{s1}	–	700	700	700	700	–	–
n_{s2}	–	1000	1000	700	700	–	–
n_{s3}	–	1000	1000	–	–	–	–
n_a	1×10^5	–	–	–	–	128	128
n_{itr}	10	16	21	16	21	11	11
TNARE	1×10^6	700	700	4.9×10^5	4.9×10^5	1408	1408

[a]Chatterjee, Chowdhury, and Ramu (2019).

conventional HF-RDO setup. Both the proposed framework and the approach in Chatterjee, Chowdhury, and Ramu (2019) yield the same set of optimal solutions. This can be explained by the fact that they both stem from a similar global approximation concept and employ the same meta-models. However, the proposed RDO approach is superior to that of Chatterjee, Chowdhury, and Ramu (2019) as it achieves the same level of accuracy with much smaller computational cost in terms of TNARE (Tables 1 and 2).

6. Application to industrial problems

Two offshore structural problems, representing practical real-world structural problems, are considered in this section. The first problem involves an FE model of a fixed-base offshore jacket platform (OJP), while the second deals with an OJP supported with piles and considers pile–soil interaction. Initially, probabilistic modelling was performed, followed by the formulation of the robust sizing optimization problem statement. The formulated RDO problems are solved using the proposed framework owing to its efficiency, and thus obtaining the optimal solutions is computationally realizable with such realistic and detailed models.

6.1. Fixed offshore jacket platform

A four-legged offshore jacket platform with fixed supports as shown in Figure 3 has been considered in this example. The other details of the FE model of the OJP, offshore sea-states, various loading combinations and random parameters can be found in Section 5 of Chatterjee and Chowdhury (2017). Probabilistic modelling has been performed by considering uncertainties in the material and geometric and loading parameters. In total, 39 random variables were considered. The structural response analysis considered ten load cases and eight load combinations. The stochastic static response analysis of the OJP model with fixed supports was performed in terms of the maximum displacement and von Mises stress. The performance of PM1 and PM2 to approximate the response statistics of the above model in Tables 3 and 4 are assessed by comparison to the results from MCS.

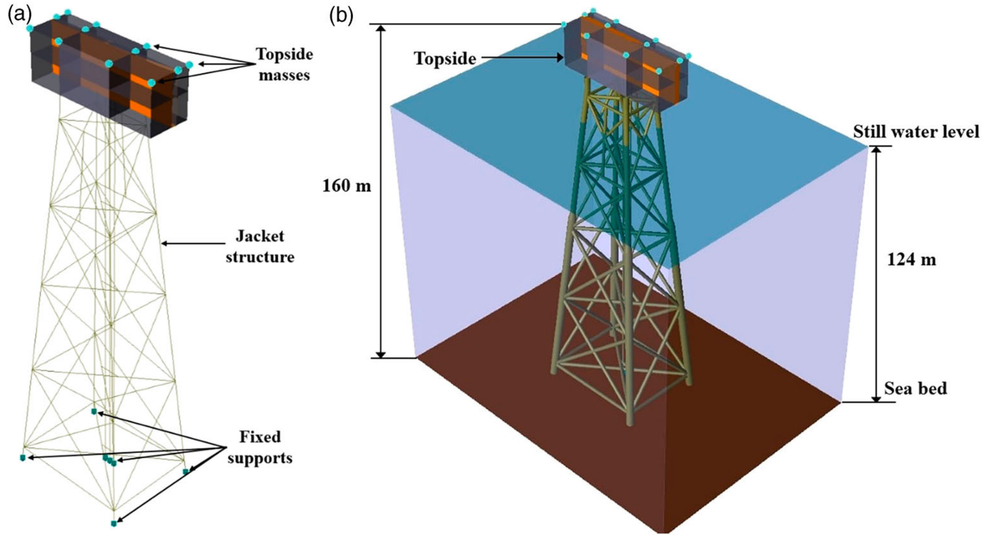

Figure 3. Four-legged offshore jacket platform with fixed supports. (a) Jacket structural model and (b) Height of structure and water level.

Table 3. Comparison of the mean of the response quantities corresponding to the OJP model with fixed supports.

Response	Load combination	MCS ($n_a = 1 \times 10^4$)	PM1 ($n_a = 700$)	PM2 ($n_a = 700$)
Maximum displacement (m)	LC1	0.0600	0.0601	0.0601
	LC2	0.0592	0.0594	0.0592
	LC3	0.2893	0.2893	0.2893
	LC4	0.2834	0.2835	0.2834
	LC5	0.3497	0.3498	0.3498
	LC6	0.3462	0.3461	0.3462
	LC7	0.1834	0.1834	0.1834
	LC8	0.1835	0.1835	0.1835
Maximum von Mises stress (kPa $\times 10^4$)	LC1	9.0153	9.0339	9.0327
	LC2	8.9974	9.0050	9.0103
	LC3	15.826	15.847	15.843
	LC4	15.705	15.726	15.719
	LC5	18.042	18.051	18.047
	LC6	17.871	17.881	17.885
	LC7	16.249	16.249	16.249
	LC8	15.714	15.713	15.722

In Tables 3 and 4, n_a is the number of actual response evaluations. The load combinations are described in Chatterjee and Chowdhury (2017). After probabilistic modelling of the OJP model with fixed supports and performance assessment of the proposed meta-models in the stochastic response analysis, an RDO problem statement is now formulated. The objective is to minimize the total mass of the structural model (W) subject to displacement and stress constraints, and the DVs are the optimal geometrical parameters of the member sections. The RDO problem may be stated as

$$\begin{aligned} \mu^*_{\mathbf{d_v}} = \text{minimize} \quad & f(\mathbf{x}, \mathbf{d_v}) = \alpha_w \mu(W) + (1 - \alpha_w)\sigma(W) \\ \text{subjectto} \quad & g_{d_{\mathrm{LC}}}(\mathbf{x}, \mathbf{d_v}) = [\mu(d_{\mathrm{LC}}) + 3\sigma(d_{\mathrm{LC}})] - d_{\max} \leq 0; \\ & g_{\sigma_{\mathrm{LC}}}(\mathbf{x}, \mathbf{d_v}) = [\mu(\sigma_{\mathrm{LC}}) + 3\sigma(\sigma_{\mathrm{LC}})] - \sigma_{\max} \leq 0. \end{aligned} \tag{4}$$

In Problem (4), μ and σ denote the mean and standard deviation, the DVs are considered as random and $\mu^*_{\mathbf{d_v}}$ denotes the mean of the DVs at the optimum $f(\mathbf{x}, \mathbf{d_v})$, $\mathbf{x}$ represents random variables

Table 4. Comparison of the standard deviation of the response quantities corresponding to the OJP model with fixed supports.

Response	Load combination	MCS ($n_a = 1 \times 10^4$)	PM1 ($n_a = 700$)	PM2 ($n_a = 700$)
Maximum displacement (m)	LC1	0.0023	0.0023	0.0023
	LC2	0.0022	0.0023	0.0022
	LC3	0.0099	0.0099	0.0097
	LC4	0.0097	0.0097	0.0095
	LC5	0.0122	0.0122	0.0122
	LC6	0.0121	0.0120	0.0120
	LC7	0.0122	0.0122	0.0122
	LC8	0.0122	0.0122	0.0122
Maximum von Mises stress (kPa $\times 10^3$)	LC1	2.2548	2.3028	2.1892
	LC2	2.4527	2.4692	2.3479
	LC3	3.6013	3.7153	3.3551
	LC4	3.9068	4.0909	3.6536
	LC5	3.8695	3.8993	3.5960
	LC6	4.0272	4.0517	3.7811
	LC7	4.6985	4.6987	4.6975
	LC8	4.8141	4.8224	4.7025

(other than the DVs). The constraints are defined such that the maximum nodal displacement d_{LC} and maximum element stress σ_{LC} corresponding to the different load combination (LC) should not exceed the allowable displacement d_{max} and stress limit σ_{max}, respectively. Since there are eight load combinations considered for the stochastic response analysis, sixteen constraints are considered in Problem (4). The design variable vector $\mathbf{d_v}$ consists of the stochastic geometric parameters given in Equation (5). The structural member details can be found in the Supplemental Data.

$$\mathbf{d_v} = \left\{ \begin{array}{c} \text{Diameter of non-structural conductors} \\ \text{Diameter of bracings B27–B54} \\ \text{Diameter of bracings B1–B26} \\ \text{Diameter of short stubs} \\ \text{Diameter of legs} \\ \text{Thickness of non-structural conductors} \\ \text{Thickness of bracings B27–B54} \\ \text{Thickness of bracings B1–B26} \\ \text{Thickness of short stubs} \\ \text{Thickness of legs} \\ \text{Thickness of topside deck and walls} \end{array} \right\}. \tag{5}$$

Since an actual response evaluation of the above FE model requires significant computational cost, the proposed efficient framework illustrated in Sections 3 and 4 has been employed to solve the above formulated RDO problem (Problem (4)). It should be noted that MCS-based RDO evaluation could not be performed owing to its prohibitive computational cost. In this context, comparison of the response statistics obtained by PM1, PM2 and MCS in Tables 3 and 4 have been presented to verify the performance assessment of PM1 and PM2 in terms of accuracy.

In solving the RDO problem by PM1 and PM2, n_{s1} is selected as 700, n_{s2} is selected as 5000 and n_{s3} is selected as 2000 from the convergence studies undertaken (which have the same form as performed previously). The robust optimal solutions for the OJP model with fixed supports are presented in Table 5 for $\alpha_w = 0$, 0.5 and 1. In this example, the mean and standard deviation objectives are not strongly conflicting, which means that the optimal solutions are similar for all values of α_w. For

Table 5. Robust optimal solutions for the OJP model with fixed supports.

α_w	Response quantities/statistics	PM1-RDO	PM2-RDO
0	$f^*(\mathbf{x}, \mathbf{d_v})$	130.0833	130.8703
	$\mu(W) \times 10^4$	1.2506	1.2605
	$\sigma(W)$	130.0883	130.8703
	No. of iterations	90	54
0.5	$f^*(\mathbf{x}, \mathbf{d_v}) \times 10^3$	6.3260	6.5503
	$\mu(W) \times 10^4$	1.2521	1.2963
	$\sigma(W)$	130.9474	137.4053
	No. of iterations	26	102
1	$f^*(\mathbf{x}, \mathbf{d_v}) \times 10^4$	1.2503	1.2963
	$\mu(W) \times 10^4$	1.2503	1.2963
	$\sigma(W)$	130.0978	137.4127
	No. of iterations	23	24

Table 6. Total steel mass corresponding to the optimal configuration of the OJP model with fixed supports.

α_w	Total steel quantity/savings	PM1-RDO	PM2-RDO
0	Quantity (t/10^3)	5.9079	6.0069
	Savings (%)	30.4195	29.2540
0.5	Quantity (t/10^3)	5.9233	6.3656
	Savings (%)	30.2388	25.0294
1	Quantity (t/10^3)	5.9047	6.3653
	Savings (%)	30.4571	29.0331

problems with strong conflicting objectives, the proposed methodology can yield optimal solutions corresponding to multiple α_w values, and hence a Pareto front can be obtained with almost no additional computational effort. The total CPU time required to obtain the robust optimal solutions by PM1 and PM2 is 4.97×10^4 s and 4.92×10^4 s, respectively, using an Intel® Xeon® CPU E5645 processor @ 2.4 GHz.

Furthermore, the total saving in the mass of steel for the jacket structure achieved by solving the RDO problem is illustrated in Table 6. The material saving in Table 6 has been evaluated corresponding to the structural configuration with the optimized geometric parameters (DVs) over their mean values. In doing so, the other stochastic parameters of the OJP model have been set to their respective mean values. The total mass of the structural model is 8.4908×10^3 t, which corresponds to the mean of the geometric parameters. Table 6 shows that a 25–30% saving in steel mass has been achieved by the proposed RDO frameworks.

6.2. Offshore jacket platform with pile supports considering soil–structure interaction

The previous example in Section 6.1 assumed that the OJP model was fixed at the bottom. However, it has been observed that flexibility of the foundation, and compressibility of the soil underneath, may affect the structural responses significantly (Ajamy *et al.* 2014; Nour El-Din and Kim 2014). Thus, consideration of the soil–structure interaction in the response analysis may prove to be critical in terms of structural safety and prevention of damage. Moreover, uncertainty related to key geotechnical parameters such as local soil conditions with varying depths should be accounted for in the realistic modelling of their interaction with the structural foundation.

In this example, the same jacket structure has been employed, additionally accounting for pile–soil interaction (Abdel Raheem, Ahmed, and Alazrak 2015). Among various simplified models for capturing the complex phenomenon of soil–structure interaction, a pseudo-static beam on a nonlinear Winkler foundation (BNWF), also referred to as the p–y method, has been utilized in this study (Matlock 1970). The p–y method is recommended by API-RP-2A (American Petroleum Institute 2000) and is widely used among researchers and practitioners. The various soil types and their location are

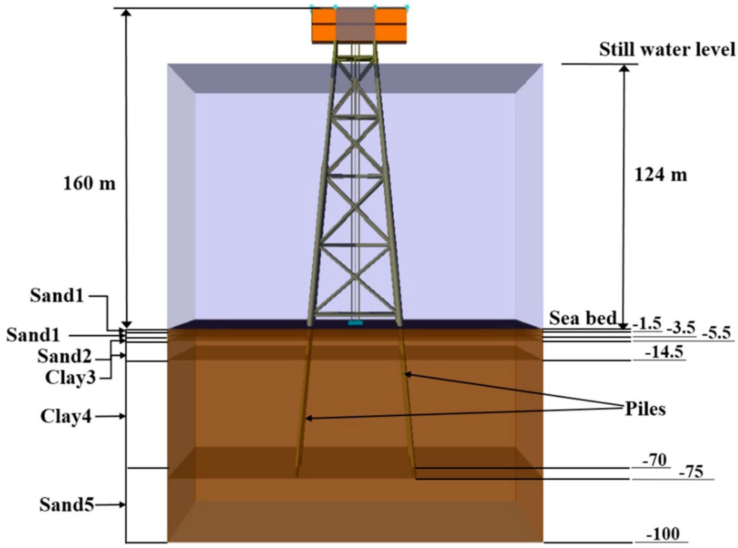

Figure 4. The jacket structural model with specifications of soil types.

Table 7. Comparison of the mean of the response quantities corresponding to the OJP model, also considering pile–soil interaction.

Response	Load combination	MCS ($n_a = 1 \times 10^4$)	PM1 ($n_a = 1000$)	PM2 ($n_a = 1000$)
Maximum displacement (m)	LC1	0.1198	0.1199	0.1199
	LC2	0.1177	0.1178	0.1178
	LC3	0.5998	0.5997	0.5995
	LC4	0.5937	0.5935	0.5934
	LC5	0.7367	0.7363	0.7361
	LC6	0.7314	0.7305	0.7308
	LC7	0.2743	0.2742	0.2742
	LC8	0.2707	0.2706	0.2706
Maximum von Mises stress (kPa $\times 10^5$)	LC1	1.3896	1.3896	1.3894
	LC2	1.3879	1.3881	1.3878
	LC3	3.2297	3.2277	3.2282
	LC4	3.2216	3.2204	3.2201
	LC5	3.6397	3.6370	3.6374
	LC6	3.6408	3.6361	3.6396
	LC7	2.6021	2.5996	2.6013
	LC8	2.5973	2.5956	2.5961

presented in Figure 4 and the details can be found in Section 6 of Chatterjee and Chowdhury (2018). In total, 65 stochastic variables are considered for this problem, accounting for uncertainties in the material, geometric, loading and geotechnical parameters. The rest of the data, including load combinations, are the same as those of the previous example.

The performance of the proposed meta-models in approximating the response statistics of the piled OJP model is given in Tables 7 and 8. The results achieved by using PM1 and PM2 are close to those of MCS, and illustrate good performance of the meta-models in terms of approximation accuracy.

Subsequently, the same RDO formulation of the OJP model considering the pile–soil interaction given by Problem (4) is investigated. The design variable vector $\mathbf{d}_v$ consists of the stochastic geometric

Table 8. Comparison of the standard deviation of the response quantities corresponding to the OJP model, also considering pile–soil interaction.

Response	Load combination	MCS ($n_a = 1 \times 10^4$)	PM1 ($n_a =$ 1000)	PM2 ($n_a =$ 1000)
Maximum displacement (m)	LC1	0.0048	0.0048	0.0048
	LC2	0.0049	0.0049	0.0048
	LC3	0.0198	0.0099	0.0097
	LC4	0.0195	0.0195	0.0194
	LC5	0.0246	0.0248	0.0245
	LC6	0.0257	0.0256	0.0249
	LC7	0.0088	0.0088	0.0087
	LC8	0.0087	0.0086	0.0086
Maximum von Mises stress (kPa $\times 10^3$)	LC1	4.0268	3.9904	3.9795
	LC2	4.0326	4.0168	3.9711
	LC3	9.3677	9.3123	9.2873
	LC4	9.3418	9.3003	9.2594
	LC5	1.0604	1.0610	1.0476
	LC6	1.0630	1.0567	1.0464
	LC7	7.3077	7.2823	7.2054
	LC8	7.2941	7.2387	7.1986

parameters given in Equation (6).

$$\mathbf{d_v} = \begin{Bmatrix} \text{Diameter of non-structural conductors} \\ \text{Diameter of bracings B27–B54} \\ \text{Diameter of bracings B1–B26} \\ \text{Diameter of piles} \\ \text{Diameter of short stubs} \\ \text{Diameter of legs} \\ \text{Thickness of non-structural conductors} \\ \text{Thickness of bracings B27–B54} \\ \text{Thickness of bracings B1–B26} \\ \text{Thickness of piles} \\ \text{Thickness of short stubs} \\ \text{Thickness of legs} \\ \text{Thickness of topside deck and walls} \end{Bmatrix}. \tag{6}$$

The proposed RDO framework illustrated in Sections 3 and 4 has been employed to solve this sizing optimization problem. From the convergence studies undertaken, n_{s1}, n_{s2} and n_{s3} are selected as 1000, 5000 and 2000, respectively. The robust optimal solutions for the above OJP model are given in Table 9 for $\alpha_w = 0$, 0.5 and 1. The total CPU time required to yield the robust optimal solutions by PM1 and PM2 is 7.42×10^4 s and 7.34×10^4 s, respectively, using an Intel® Xeon® CPU E5645 processor @ 2.4 GHz. Further, the total savings in the mass of the steel of the jacket structure achieved by solving the RDO problem are given in Table 10. As for the previous example, the material saving in Table 10 has been evaluated with respect to the total mass of the structural model corresponding to the means of the geometric parameters, which is 9.6944×10^3 t. Table 10 shows that a 27.5–30% saving in steel mass has been achieved by the proposed RDO frameworks.

To illustrate the computational benefits of the proposed RDO framework, the time required by MCS to solve the offshore structural problems was estimated assuming that it would require the same number of iterations to converge as that by PM1 and PM2. The total time using MCS was estimated as 1×10^8 s with no parallel processing and 9.2×10^6 s with parallel processing on a 12-core processor. The proposed RDO framework required only 0.8–1% of the CPU time to yield the optimal solutions compared to conventional MCS-based RDO with parallel processing. This indicates that the proposed

Table 9. Robust optimal solutions for the OJP model considering pile–soil interaction.

α_W	Response quantities/statistics	PM1-RDO	PM2-RDO
0	$f^*(\mathbf{x}, \mathbf{d_v})$	132.5560	132.5486
	$\mu(W) \times 10^4$	1.3623	1.3618
	$\sigma(W)$	132.5560	132.5486
	No. of iterations	84	146
0.5	$f^*(\mathbf{x}, \mathbf{d_v}) \times 10^3$	6.7883	6.7594
	$\mu(W) \times 10^4$	1.3444	1.3387
	$\sigma(W)$	132.4876	132.1259
	No. of iterations	59	88
1	$f^*(\mathbf{x}, \mathbf{d_v}) \times 10^4$	1.3444	1.3387
	$\mu(W) \times 10^4$	1.3444	1.3387
	$\sigma(W)$	132.4876	132.1259
	No. of iterations	39	37

Table 10. Total steel mass corresponding to the optimal configuration of the OJP model considering pile–soil interaction.

α_W	Total Steel quantity/savings	PM1-RDO	PM2-RDO
0	Quantity (t/10^3)	7.0256	7.0212
	Savings (%)	27.5291	27.5749
0.5	Quantity (t/10^3)	6.8472	6.7899
	Savings (%)	29.3698	29.9610
1	Quantity (t/10^3)	6.8472	6.7899
	Savings (%)	29.3698	29.9610

framework is useful for dealing with such computationally intensive problems, the solution of which would otherwise be rendered prohibitive.

Tables 3 and 7 show that the same structural model subjected to same set of loading conditions, yield responses of higher magnitude by accounting for the soil–structure interaction, compared to the fixed base. This can be explained by the fact that the structure becomes more flexible and exhibits responses of higher magnitude because of the flexibility of the foundation and the compressibility of the soil underneath compared to the fixed-base assumption. In order to illustrate this fact further, the response ratio, defined as the ratio of the response considering the pile–soil interaction to the response of the fixed model, has been presented in Table 11. The mean response

Table 11. Comparison of response ratios of the OJP model.

Response	Load combination	MCS	PM1	PM2
Maximum displacement	LC1	1.9967	1.9950	1.9950
	LC2	1.9882	1.9832	1.9899
	LC3	2.0733	2.0729	2.0722
	LC4	2.0949	2.0935	2.0939
	LC5	2.1067	2.1049	2.1043
	LC6	2.1127	2.1107	2.1109
	LC7	1.4956	1.4951	1.4951
	LC8	1.4752	1.4747	1.4747
Maximum von Mises stress	LC1	1.5414	1.5382	1.5382
	LC2	1.5426	1.5415	1.5402
	LC3	2.0408	2.0368	2.0376
	LC4	2.0513	2.0478	2.0485
	LC5	2.0173	2.0148	2.0155
	LC6	2.0373	2.0335	2.0350
	LC7	1.6014	1.5998	1.6009
	LC8	1.6529	1.6519	1.6513

quantities obtained in Tables 3 and 7 have been utilized to obtain the response ratios in Table 11. Table 11 shows that the response ratios vary within a range of 1.47 to 2.11. The notable increase in responses illustrate the fact that ignoring the soil–structure interaction may prove to be detrimental to the structural integrity. This is even more important for offshore structures, considering the high level of uncertainties present in the form of marine loading and the geotechnical environment.

7. Summary and conclusions

The novelty and contribution of this work lies in the fact that a global bi-layered response statistics approximation based RDO framework has been developed. The proposed RDO framework is completely free from any stochastic computations within the optimization cycle and thus may be considered as an equivalent deterministic optimization strategy. This work can be perceived as a double-layered extension of the recently proposed single-layered framework (Chatterjee, Chowdhury, and Ramu 2019), which results in improvement of the overall computational cost. To limit the number of computationally expensive function evaluations, two adaptive sparse refined Kriging-based models (Chatterjee and Chowdhury 2018; Chatterjee, Chowdhury, and Ramu 2019) are utilized for the two-layered approximation of the response functions and moments. These models have been employed here because they were capable of capturing the global response for the full range of design parameters. However, the proposed RDO framework is general, so that any meta-modelling approach may be implemented, as long as it can capture the variation in the response accurately.

Owing to the efficiency of the proposed RDO framework, two computationally intractable offshore structural problems were solved by the proposed methods. In this regard, it is worth mentioning that the proposed framework can yield optimal solutions corresponding to multiple weighing factors α_w without any increase in computational effort, as opposed to MCS-based RDO or conventional meta-model assisted RDO frameworks. Thus, the study provides the rationale for considering the proposed framework to solve complex real-world applications.

In spite of the above advantages, there are scenarios as listed below where the potential of the proposed framework may not be realized and the advantage might be less than for the problems investigated.

(i) It may prove difficult to achieve an adequate approximation by the proposed framework when the DVs vary within wide intervals. However, it is reasonable to assume that the decision makers will possess sufficient knowledge of the variability of the DVs, as these include controllable parameters that are to be optimized.
(ii) Quite understandably, the proposed methodology may not be preferable for addressing problems in which the DVs are deterministic and/or the random DVs are fewer than the other stochastic parameters, in comparison to the conventional meta-model assisted RDO approaches. The latter scenario is often encountered in topology optimization under uncertainty.

A few potential aspects that could be improved include adaptive sampling schemes and error minimization using Kriging prediction variance. Additionally, considering spatial uncertainty would enhance the present version of the proposed methodology. Further considerations to predict quantities such as reliability or the probability of failure would allow its extension to reliability-based design optimization.

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Disclosure statement

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