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A FRACTIONAL CALCULUS APPROACH TO METADAMPING IN PHONONIC CRYSTALS AND ACOUSTIC METAMATERIALS

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Abstract. Research on phononic and acoustic materials and structures emerged in the recent decade as a result of switching from theoretical physics to applications in various engineering fields. Periodicity is the main characteristic of the phononic medium stemming from periodic material phases, geometry or the boundary condition with wave propagation properties analysed through frequency band structure. To obtain these characteristics, the generalized Bloch theorem is usually applied to obtain the dispersion relations of viscously damped resonant metamaterials. Here we develop a novel analytical approach to analyse the fractionally damped model of phononic crystals and acoustic metamaterials introduced through the fractional-order Kelvin–Voigt and Maxwell damping models. In the numerical study, the results obtained using the proposed models are compared against the elastic cases of the phononic crystal and locally resonant acoustic metamaterial, where significant differences in dispersion curves are identified. We show that the fractional-order Maxwell model is more suitable for describing the dissipation effect throughout the spectrum due to the possibility of fitting both, the order of fractional derivative and the damping parameter.

1. Introduction

Wave propagation analysis of different materials can provide us with important information about their mechanical and dynamic properties or even existence of defects and inclusions. Waves can be of different nature, such as mechanical, electronic or electromagnetic, and they can encompass several length scales, from atomic to macro scales, such as seismic waves. The propagation of elastic waves in periodic heterogeneous materials, also known as phononic crystals, belongs to a special class of problems [1], which has received considerable attention in the scientific community in recent years [2–4]. From the physical viewpoint, periodicity in material causes destructive interference of elastic waves by forbidding them to propagate

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within the structure in certain directions and frequency ranges called band gaps [5]. In the case of locally resonant acoustic metamaterials, internal resonators play a major role in generating band gaps with associated wavelengths that are orders of magnitude longer than the periodicity of the metamaterial. This enables one to design small structures to absorb low frequency vibrations or sound. The concept of band structure, providing the relationship between frequency and the wavenumber, is well known in physics, but in mechanics it represents a dispersion relation describing the free wave propagation in some medium. Therefore, examining the band structure properties of periodic structures or metamaterials is significant from the viewpoint of their application in acoustic wave attenuation, filters for mechanical waves, noise and vibration isolation [6]. In addition, phononic crystals can be used in waveguides to open band gaps for undesired excitation modes in ultrasonic experiments [7].

The problem of wave propagation in viscoelastic materials was analysed in a number of studies [8–10]. It was demonstrated experimentally and theoretically that the presence of dissipation in PCs at lower frequencies widens the bandgaps and decreases the initial forbidden frequency [11]. Since the modulus of viscoelastic materials is frequency dependent, it is shown that the dispersion effect becomes more evident at relatively higher frequencies. Periodic materials with low-frequency bandgaps are usually made of polymers [12], which requires introduction of viscoelastic models to describe a strong damping effect in such materials. Hussein and Fraizer [13] developed the methodology to analyse general damping in phononic crystals and acoustic metamaterials by employing the generalized form of Bloch's theorem [14, 15]. One of the aims was to design an acoustic metamaterial in such a manner to achieve a high level of dissipation while keeping the high stiffness properties. The authors' main idea was to use the feature that damping is most profound at resonance frequency, which is a well known fact from linear vibration analysis in structural mechanics. They observed the frequency band structure to reveal the effect of metadamping not only near resonant frequencies but for the entire frequency-wavenumber spectrum. Chen et al. [16] applied a similar methodology to examine metadamping in a dissipative mass-in-mass lattice system with multiple resonators. In [17] the authors demonstrated that the generalized linear Maxwell model allows an accurate description of nonlinear frequency dependent elastic properties as opposed to the classic Kelvin–Voigt model, and it is widely applied to model the behaviour of many polymeric materials in a realistic manner. Recently, Alamri et al. [18] introduced a Maxwell-type damper to study wave attenuation in dissipative elastic metamaterials. Due to the band gaps merging effect induced by the Maxwell-type damper, a significant improvement of stress wave mitigation in the proposed dissipative metamaterial was demonstrated in both time and frequency domains. Application of the linear viscoelastic models and frequency domain analysis to observe the dispersion characteristics implies linearity of the system, which was justified in [17] based on the generalized Maxwell model. Moreover, in [19] the authors studied wave propagation in a two-dimensional phononic crystal with viscoelasticity based on the fractional-order Kelvin–Voigt model and finite-difference-time-domain method. However, as discussed in [20], frequency domain analysis of the fractional-order Kelvin–Voigt model reveals unusual behaviour of increased frequency for an increase in damping in the system, which cannot be seen by the time domain analysis only. Thus, frequency domain analysis of linear fractional-order derivative viscoelastic models is also required in order to achieve reliable results for their dispersion characteristics.

Besides choosing the appropriate damping model, it is important to choose whether the frequency or the wavenumber is set to be real or complex values. Based on this, Frazier and Hussein [21] defined two classes of problems dealing with damped phononic materials. If the frequencies are assumed to be real, the damping effect is manifested in the form of complex wavenumbers defining one class of problems. This case could be physically represented by the wave propagation in a medium due to a sustained driving frequency and dissipation only in the form of spatial attenuation, which results in a linear or a quadratic eigenvalue problem. If the frequencies are permitted to be complex and wavenumbers are real, the dissipation effect is represented in the form of temporal attenuation [22]. For example, this class of problems can be physically represented by a free dissipative wave motion in a medium due to impulse loading [21]. The problem considered in this study belongs to the second class of the aforementioned problems with a specified real wavenumber and complex frequency as a solution, where the real part represents the damping factor and the imaginary part is the damped frequency. Moreover, the solution to the fractional-order differential equations sought in the frequency domain yields a characteristic polynomial with noninteger exponents, which can be solved by using the methodology from [23].

The aim of this work is to provide the framework to study the complex-eigenfrequency band structure of phononic crystals (PCs) and acoustic metamaterials (AMs) using the fractional-order general damping models. In the developed methodology, simple two-mass unit cell systems are employed for two different fractional-order viscoelastic models. The first model is the fractional Kelvin–Voigt model, and the second one is the fractional Maxwell model, both of which are the fractional-order derivative counterparts of the well known models from the linear theory of viscoelasticity with classical derivatives. Such models can exhibit a wide range of damping behaviours depending on the value of the considered retardation/relaxation times and fractional-order derivative parameters. The solution procedure for obtaining the corresponding dispersion relations is based on the temporal Fourier transform and Bloch theory. The obtained results yield new insights into the wave propagation behaviour of the aforementioned periodic structures described via fractional-order damping models.

2. Analytical models

2.1. Fractional-order dissipative models of lattice systems. A one-dimensional model of the fractionally damped diatomic *phononic crystal* (PC) can be formulated by using the lumped masses connected through elements represented by force-displacement relationships of the fractional-order Kelvin–Voigt or Maxwell

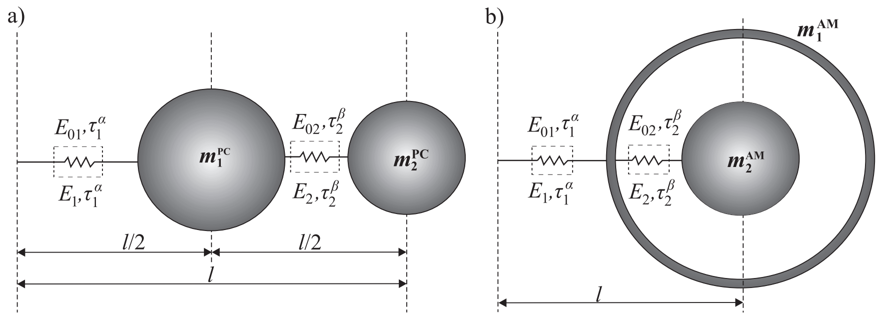

FIGURE 1. Unit cells of periodic chains consisting of masses connected with fractional-order elements a) phononic crystal b) acoustic metamaterial.

type phenomenological models (see Figure 1), which are analogous to the corresponding fractional-order stress-strain constitutive equations [20, 24]. Similar to this, fractional viscoelastic relationships can be used for the *acoustic metamaterial* (AM). Both PC and AM models are represented by a unit cell that is periodically repeated in both directions towards infinity. The equation of motion for the PC unit cell can be written as:

$$m_1\ddot{u}_1^j + f_{j,j-1}^{PC} - f_j^{PC} = 0, \tag{2.1}$$

$$m_2\ddot{u}_2^j + f_j^{PC} - f_{j,j+1}^{PC} = 0, \tag{2.2}$$

and for the AM unit cell as:

$$m_1\ddot{u}_1^j + f_{j,j-1}^{AM} - f_j^{AM} + f_{j,j+1}^{AM} = 0, \tag{2.3}$$

$$m_2\ddot{u}_2^j + f_j^{AM} = 0, \tag{2.4}$$

with u_γ^j denoting the displacement of mass m_γ, $\gamma = 1, 2$, in an arbitrary unit-cell. Forces due to viscoelastic coupling between masses and adjacent cells for the fractional Kelvin–Voigt PC and AM models are defined as:

$$f_{j,j+1}^{PC} = E_{01}[1 + \tau^\alpha D^\alpha](u_1^{j+1} - u_2^j), \qquad f_j^{PC} = E_{02}[1 + \tau^\beta D^\beta](u_2^j - u_1^j),$$
$$f_{j,j-1}^{PC} = E_{01}[1 + \tau^\alpha D^\alpha](u_1^j - u_2^{j-1}), \qquad f_{j,j\pm1}^{AM} = \pm E_{01}[1 + \tau^\alpha D^\alpha](u_1^{j\pm1} - u_1^j),$$
$$f_j^{AM} = E_{02}[1 + \tau^\beta D^\beta](u_2^j - u_1^j),$$

and for the fractional Maxwell PC and AM models as:

$$\begin{aligned}
f_{j,j+1}^{PC} + \tau^\alpha D^\alpha f_{j,j+1}^{PC} &= E_1[1 + \tau^\alpha D^\alpha](u_1^{j+1} - u_2^j),\\
f_j^{PC} + \tau^\beta D^\beta f_j^{PC} &= E_2[1 + \tau^\beta D^\beta](u_2^j - u_1^j),\\
f_{j,j-1}^{PC} + \tau^\alpha D^\alpha f_{j,j-1}^{PC} &= E_1[1 + \tau^\alpha D^\alpha](u_1^j - u_2^{j-1}),\\
f_{j,j\pm1}^{AM} + \tau^\alpha D^\alpha f_{j,j\pm1}^{AM} &= \pm E_1[1 + \tau^\alpha D^\alpha](u_1^{j\pm1} - u_1^j),\\
f_j^{AM} + \tau^\beta D^\beta f_j^{AM} &= E_2[1 + \tau^\beta D^\beta](u_2^j - u_1^j),
\end{aligned}$$

where D^α, D^β are the operators of the Riemann–Liouville derivative (see [25]), τ is the retardation/relaxation time and E_{0i} and E_i for $i = 1, 2$ are prolonged and instantaneous stiffness moduli of the fractional-order Kelvin–Voigt and Maxwell models, respectively. Bearing in mind that the Fourier transform of a fractional derivative is defined as $F[D^\alpha(u(t))] = (i\omega)^\alpha U(i\omega)$ for $i = \sqrt{-1}$, one first needs to perform the temporal Fourier transform [26] over force-displacement relationships of the fractional Kelvin–Voigt model and Eqs. (2.1)–(2.4). After taking that $p = i\omega$, one obtains the following equation for the fractional Kelvin–Voigt PC model:

$$p^2\bar{u}_1^j + \frac{E_{01}}{m_1}[1+\tau^\alpha p^\alpha](\bar{u}_1^j - \bar{u}_2^{j-1}) - \frac{E_{02}}{m_1}[1+\tau^\beta p^\beta](\bar{u}_2^j - \bar{u}_1^j) = 0, \tag{2.5}$$

$$p^2\bar{u}_2^j + \frac{E_{02}}{m_2}[1+\tau^\beta p^\beta](\bar{u}_2^j - \bar{u}_1^j) - \frac{E_{01}}{m_2}[1+\tau^\alpha p^\alpha](\bar{u}_1^{j+1} - \bar{u}_2^j) = 0, \tag{2.6}$$

and for the fractional Kelvin–Voigt AM model:

$$p^2\bar{u}_1^j + \frac{E_{01}}{m_1}[1+\tau^\alpha p^\alpha](2\bar{u}_1^j - \bar{u}_1^{j-1} - \bar{u}_1^{j+1}) + \frac{E_{02}}{m_1}[1+\tau^\beta p^\beta](\bar{u}_1^j - \bar{u}_2^j) = 0, \tag{2.7}$$

$$p^2\bar{u}_2^j + \frac{E_{02}}{m_2}[1+\tau^\beta p^\beta](\bar{u}_2^j - \bar{u}_1^j) = 0. \tag{2.8}$$

In [18], the authors introduced some additional degrees of freedom (and therefore equations) and the principle of virtual work to derive the system's equations and obtain corresponding dispersion relations of the acoustic metamaterial. In this study, there are no additional degrees of freedom since fractional force-displacement relations can be used directly in motion equations after moving to the frequency domain. After performing the temporal Fourier transform over the corresponding force-displacement relations, they are substituted into Eqs. (2.1)–(2.4) for AM and PC models. Therefore, the equation for the fractional Maxwell PC model in the frequency domain is given as:

$$p^2\bar{u}_1^j + \frac{E_1\tau^\alpha p^\alpha(\bar{u}_1^j - \bar{u}_2^{j-1})}{m_1(1+\tau^\alpha p^\alpha)} - \frac{E_2\tau^\beta p^\beta(\bar{u}_2^j - \bar{u}_1^j)}{m_1(1+\tau^\beta p^\beta)} = 0, \tag{2.9}$$

$$p^2\bar{u}_2^j + \frac{E_2\tau^\beta p^\beta(\bar{u}_2^j - \bar{u}_1^j)}{m_2(1+\tau^\beta p^\beta)} - \frac{E_1\tau^\alpha p^\alpha(\bar{u}_1^{j+1} - \bar{u}_2^j)}{m_2(1+\tau^\alpha p^\alpha)} = 0, \tag{2.10}$$

and for the fractional Maxwell AM model as:

$$p^2\bar{u}_1^j + \frac{E_1\tau^\alpha p^\alpha}{m_1(1+\tau^\alpha p^\alpha)}(2\bar{u}_1^j - \bar{u}_1^{j-1} - \bar{u}_1^{j+1}) + \frac{E_2\tau^\beta p^\beta(\bar{u}_1^j - \bar{u}_2^j)}{m_1(1+\tau^\beta p^\beta)} = 0, \tag{2.11}$$

$$p^2\bar{u}_2^j + \frac{E_2\tau^\beta p^\beta(\bar{u}_2^j - \bar{u}_1^j)}{m_2(1+\tau^\beta p^\beta)} = 0. \tag{2.12}$$

2.2. Dispersion relations using the Bloch solution. Let us assume the solution to Eqs. (2.5)–(2.12) using the Bloch's theorem [13] as:

$$\bar{u}_{\gamma}^{j+n} = U_{\gamma} e^{i(\kappa x_{\gamma}^{j} + n\kappa L)}, \quad \gamma = 1, 2, \tag{2.13}$$

with $\bar{u}_{\gamma}^{j+n}$ representing the displacement of mass γ in the $(j+n)$-th unit-cell of the periodic chain (here $n = -1, 0, 1$ denote the previous, present and subsequent unit cells), κ is the wavenumber, U_{γ} is the complex wave amplitude, L is the length of the unit-cell and $x_{\gamma}^{j} = j + L$ for the AM and $x_{\gamma}^{j} = j + \gamma L/2$ for the PC model. By substituting Eq. (2.13) in each of Eqs. (2.5)–(2.12) one obtains the system of equations in the matrix form:

$$\begin{bmatrix} S_{11}^{PC,AM} & S_{12}^{PC,AM} \\ S_{21}^{PC,AM} & S_{22}^{PC,AM} \end{bmatrix} \begin{Bmatrix} U_{1}^{PC,AM} \\ U_{2}^{PC,AM} \end{Bmatrix} = 0, \tag{2.14}$$

for both fractional Kelvin–Voigt and Maxwell models, where elements of matrices $S_{i,j}^{PC,AM}$ are given in the Appendix A. Here we prescribe the real wavenumbers $0 \leqslant \kappa L \leqslant \pi$ spanning the first irreducible Brillouin zone, which yields wavenumber-dependent complex roots. These roots can be obtained from the characteristic equation found by solving the determinant of matrix S_{ij}, which for the fractional Kelvin–Voigt PC model is given as:

$$\begin{aligned} p^4 + p^2\big[(\omega_{11}^2 + \omega_{12}^2)(1 + \tau^{\alpha} p^{\alpha}) + (\omega_{21}^2 + \omega_{22}^2)(1 + \tau^{\beta} p^{\beta})\big] \\ + 2\omega_{11}^2 \omega_{22}^2 (1 - \cos \kappa L)(1 + \tau^{\beta} p^{\beta})(1 + \tau^{\alpha} p^{\alpha}) = 0, \end{aligned} \tag{2.15}$$

and for the fractional Kelvin–Voigt AM as

$$\begin{aligned} p^4 + p^2\big[2\omega_{11}^2(1 - \cos \kappa L)(1 + \tau^{\alpha} p^{\alpha}) + (\omega_{21}^2 + \omega_{22}^2)(1 + \tau^{\beta} p^{\beta})\big] \\ + 2\omega_{11}^2 \omega_{22}^2 (1 - \cos \kappa L)(1 + \tau^{\beta} p^{\beta})(1 + \tau^{\alpha} p^{\alpha}) = 0. \end{aligned} \tag{2.16}$$

Further, for the fractional Maxwell PC model we have

$$\begin{aligned} p^4 + p^2\Big[\frac{(\tilde{\omega}_{11}^2 + \tilde{\omega}_{12}^2)\tau^{\alpha} p^{\alpha}}{(1 + \tau^{\alpha} p^{\alpha})} + \frac{(\tilde{\omega}_{21}^2 + \tilde{\omega}_{22}^2)\tau^{\beta} p^{\beta}}{(1 + \tau^{\beta} p^{\beta})}\Big] \\ + \frac{2\tilde{\omega}_{11}^2 \tilde{\omega}_{22}^2 (1 - \cos \kappa L)(\tau p)^{\alpha+\beta}}{(1 + \tau^{\alpha} p^{\alpha})(1 + \tau^{\beta} p^{\beta})} = 0, \end{aligned} \tag{2.17}$$

and for the fractional Maxwell AM model as

$$\begin{aligned} p^4 + p^2\Big[\frac{2\tilde{\omega}_{11}^2 \tau^{\alpha} p^{\alpha}(1 - \cos \kappa L)}{(1 + \tau^{\alpha} p^{\alpha})} + \frac{(\tilde{\omega}_{21}^2 + \tilde{\omega}_{22}^2)\tau^{\beta} p^{\beta}}{(1 + \tau^{\beta} p^{\beta})}\Big] \\ + \frac{2\tilde{\omega}_{11}^2 \tilde{\omega}_{22}^2 (1 - \cos \kappa L)(\tau p)^{\alpha+\beta}}{(1 + \tau^{\alpha} p^{\alpha})(1 + \tau^{\beta} p^{\beta})} = 0, \end{aligned} \tag{2.18}$$

where $\omega_{11}^2 = E_{01}/m_1$, $\omega_{12}^2 = E_{01}/m_2$, $\omega_{21}^2 = E_{02}/m_1$, $\omega_{22}^2 = E_{02}/m_2$, for the fractional Kelvin–Voigt model while for the Maxwell it is given as $\tilde{\omega}_{11}^2 = E_1/m_1$, $\tilde{\omega}_{12}^2 = E_1/m_2$, $\tilde{\omega}_{21}^2 = E_2/m_1$, $\tilde{\omega}_{22}^2 = E_2/m_2$. In comparison to the work by Hussein and Frazier [13], where characteristic equations are fourth-order polynomials, one can notice that in this work the characteristic equations have noninteger exponents due to the introduced fractional-order damping terms. The solutions cannot be obtained in the classical manner and some other techniques from the literature need to be adopted. The above characteristic equations have two pairs of complex conjugate roots that can be obtained in a manner similar to [23]. A detailed procedure of obtaining the roots of the characteristic equation is given in the Appendix B, where one first introduces the replacement $p = re^{i\psi}$ and separates the real and imaginary parts. After some algebra one can obtain corresponding transcendental equations having two zeros for some fixed value of angle ψ given in the range $\pi/2 < \psi < \pi$, while there are no stable roots in the range $0 \leqslant \psi \leqslant \pi/2$. After finding the unknown r, two pairs of complex conjugate roots of the characteristic equation can be obtained as

$$p_s = r_s e^{\pm i\psi} = \delta_s \pm \Omega_s i, \quad s = 1, 2, \tag{2.19}$$

representing the complex frequency function of the presented fraction-order system, thus permitting the wave attenuation in time. In the above equation, the imaginary part of the complex conjugate roots Ω_s represents the damped frequency and the real part δ_s is the damping factor. Therefore, the imaginary part of the complex root obtained in terms of the wavenumber represents the dispersion relation of the observed PC or AM structure with fractional-order damping. The limiting case with no damping, where the frequency of the fractionally damped system is approaching the one corresponding to the equivalent elastic system, occurs for the values $\psi \to \pi/2$. One should state that fractional derivative viscoelastic models undergo certain limitations in the values of fractional-order parameters coming from the second law of thermodynamics. However, these limitations do not apply to the fractional Maxwell and Kelvin–Voigt models since they belong to groups of diffusion-wave and diffusion type rheological models, respectively.

3. Numerical results

In the numerical study, we perform the parametric study and compare the results obtained by the herein presented fractional viscoelastic models of phononic crystal and acoustic metamaterial against those for the elastic case. For this reason, in the following numerical experiment we adopted the values of ratios of elastic coefficients and masses the same as in the paper by Hussein et al. [13] i.e. we adopted the ratios of prolonged elasticity coefficients given in the fractional derivative model and masses as $\omega_{11}^2 = (250000)^{1/2}$, $\omega_{12}^2 = E_{01}/m_2 = (50000)^{1/2}$, $\omega_{21}^2 = (50000)^{1/2}$, $\omega_{22}^2 = 100$, and similar for the Maxwell model as $\tilde{\omega}_{11}^2 = (250000)^{1/2}$, $\tilde{\omega}_{12}^2 = (50000)^{1/2}$, $\tilde{\omega}_{21}^2 = (50000)^{1/2}$, $\tilde{\omega}_{22}^2 = 100$, with the length of unit cells $L = 1$. We should note that the parameters in this work can be read in any consistent system of physical units. For the numerical study we adopted the values of fractional parameters in the range $0 < \alpha, \beta < 1$ while instead of choosing the retardation/relaxation times we used fixed values of angle $\psi > \pi/2$ (since no stable roots are available for $\psi < \pi/2$), which, after solving the transcendent equation numerically, yields two roots X_1 and X_2. Introducing these two roots into the equations for r and Eq. (2.19) gives two pairs of complex conjugate poles of

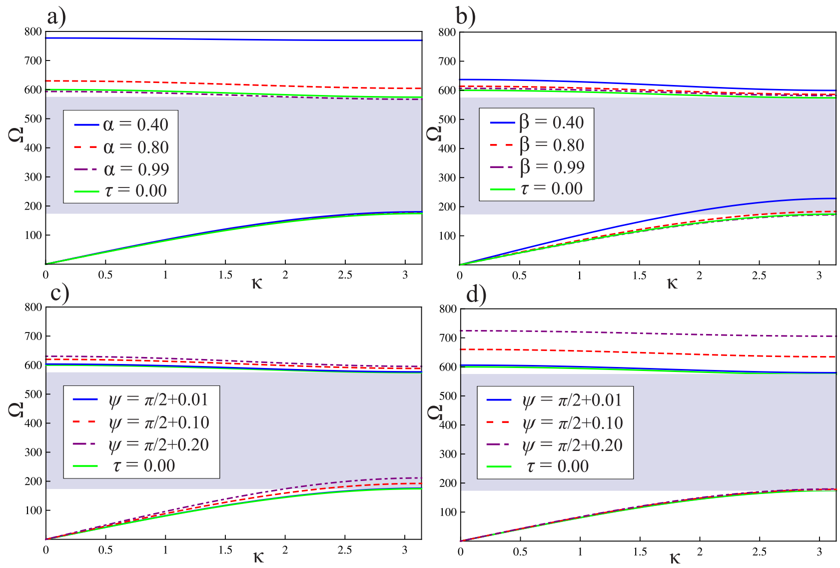

Figure 2. Dispersion curves of the elastic system (green lines) with the corresponding gap (gray colored square) and fractional-order viscoelastic Kelvin–Voigt phononic crystal for varying model parameters and fixed values of a) $\psi = \pi/2 + 0.2$ and $\beta = 0.9$, b)$\psi = \pi/2 + 0.2$ and $\alpha = 0.9$, c) $\alpha = 0.9$ and $\beta = 0.5$,d) $\alpha = 0.5$ and $\beta = 0.9$

the system as roots of the characteristic equation i.e. damped frequency Ω_s and damping factor δ_s.

Figure 2 shows the dispersion curves of the fractional PC Kelvin–Voigt model for different values of fractional-order derivative parameters, and damping parameters corresponding to angle ψ. In all figures one can notice two sets of dispersion curves where the upper ones belong to optical and the lower ones to acoustic branches. One can notice an obvious frequency shift due to the change of damping parameter and orders of fractional derivatives, which is more obvious in the optical branch. The dispersion curves of the equivalent elastic phononic structure are represented by the green lines while the corresponding band gaps are marked by squares coloured in grey. One can notice significantly higher frequencies in the optical branches for lower values of the fractional derivative order. This effect is more pronounced for changes of fractional parameter α. This is also noticed for the changes of the damping parameter (by changes of angle ψ), which yields higher frequencies and band gaps for an increase in damping and fixed values of fractional-order derivatives. Therefore, elastic waves cannot propagate through the system within the obtained frequency band gap region. The results for the fractional-order Kelvin–Voigt model lack physical interpretation since frequency increases with an increase in the damping parameter at lower values of the fractional derivative order.

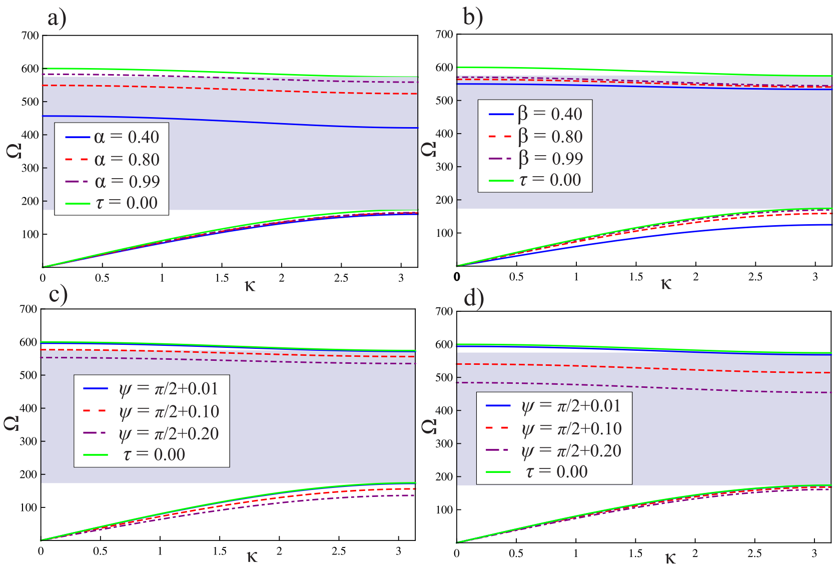

FIGURE 3. Dispersion curves of the elastic system (green lines) with the corresponding band gap (gray colored square) and fractional-order viscoelastic Maxwell phononic crystal for varying model parameters and fixed values of a) $\psi = \pi/2+0.2$ and $\beta = 0.9$, b)$\psi = \pi/2 + 0.2$ and $\alpha = 0.9$, c) $\alpha = 0.9$ and $\beta = 0.5$,d) $\alpha = 0.5$ and $\beta = 0.9$

This feature of the fractional Kelvin–Voigt model is attributed to its diffusion type nature and instability and sensitivity to small variations in the damping parameter as discussed in [20].

Figure 3 shows the dispersion curves for the phononic crystal fractional Maxwell model. One can observe lower values of frequency, especially in the optical branch, and narrower band gaps for lower values of the fractional-order parameter compared to the elastic case (green line). This effect is more pronounced for the changes of fractional parameter α while a decrease in β is more visible in the acoustic branch together with the overall shift of the band gap. However, as expected for dissipation models, in the increase of damping parameter (through an increase in the angle ψ) causes a decrease in frequency. This effect is more pronounced for the case when $\alpha < \beta$.

In the following, the dispersion curves for the acoustic metamaterials with the fractional Kelvin–Voigt and Maxwell damping models are given in Figures 4 and 5, respectively. In Figure 4 one can observe much different dispersion curves of acoustic metamaterials compared to the phononic crystals in both acoustic and optical branches. The obtained band gaps are at lower frequencies and narrower than for the phononic crystals. However, for lower values of fractional parameter β one can notice an increase in frequency and band gap widening. Similar behaviour

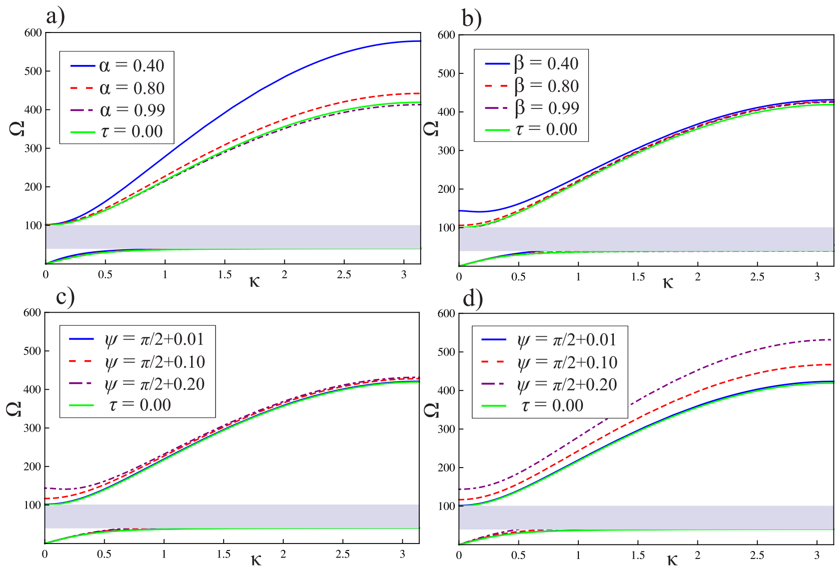

FIGURE 4. Dispersion curves of the elastic system (green lines) with the corresponding band gap (gray colored square) and fractional-order viscoelastic Kelvin–Voigt acoustic metamaterial for varying model parameters and fixed values of a) $\psi = \pi/2 + 0.2$ and $\beta = 0.9$, b)$\psi = \pi/2 + 0.2$ and $\alpha = 0.9$, c) $\alpha = 0.9$ and $\beta = 0.5$,d) $\alpha = 0.5$ and $\beta = 0.9$

can be noticed for an increase in the damping parameter in Figure 4 c) and d). However, this behaviour lacks physical interpretation since in this case we use the diffusion type fractional Kelvin–Voigt model as mentioned before. In addition, Figure 5 shows the dispersion curves for the acoustic metamaterial fractional Maxwell model. It can be noticed that the band gap is narrower at lower values of fractional parameter β. However, changes of dispersion curves are more pronounced in the optical branch where frequency decreases with an increase in the damping parameter (angle ψ). One can observe that the main dissipation effect occurs outside the band gap, where frequency in the optical branch is significantly damped compared to the elastic case. Consequently, dissipation models show that damping can affect the band gap width but the main dissipation is outside this region. Potential application of acoustic metamaterials is in the blast wave mitigation and wave isolation. This was demonstrated in [18] based on the classical Maxwell damping model of acoustic metamaterial, where broadband acoustic wave attenuation was achieved based on the system with several resonators. Since the relationship between the damping ratio, frequency and damping factor cannot be established in the classical sense, we have omitted the analyses of real parts (damping factors) of complex roots for the fractional viscoelastic models. An advantage of fractional viscoelastic models lies in the fact that they have been proven in the literature as superior to the conventional models in fitting the damping behaviour of variety of materials. However, for wave attenuation in a broader frequency spectrum it would be necessary to introduce more resonators or hierarchical design [28] of the metamaterial structure.

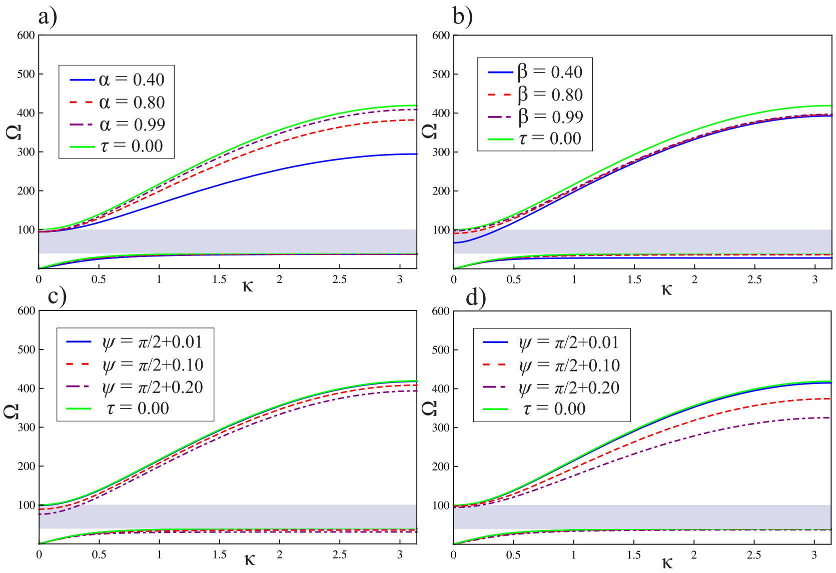

FIGURE 5. Dispersion curves of the elastic system (green lines) with the corresponding band gap (gray colored square) and fractional-order viscoelastic Maxwell acoustic metamaterial for varying model parameters and fixed values of a) $\psi = \pi/2 + 0.2$ and $\beta = 0.9$, b)$\psi = \pi/2 + 0.2$ and $\alpha = 0.9$, c) $\alpha = 0.9$ and $\beta = 0.5$,d) $\alpha = 0.5$ and $\beta = 0.9$

4. Conclusions

In this communication, we studied general fractional-order derivative models of a two-mass phononic crystal and acoustic metamaterial. The fractional Kelvin–Voigt and Maxwell viscoelasticity models are introduced through corresponding force-displacement relationships to account for the dissipation effects due to temporal attenuation of waves. It was demonstrated that the proposed models exhibit different dissipation behaviour from the conventional viscoelastic and elastic models. In the case of the fractional Kelvin–Voigt model, anomalous behaviour of increased frequency for an increase in the damping parameter was revealed in the dispersion curves at lower values of fractional parameters. However, the fractional Maxwell model exhibited larger dissipation for AMs and band gap narrowing for PCs in the entire spectrum. In conclusion, we can state that the fractional-order dissipation models of PCs and AMs can simulate a wide range of behaviours and dissipation in the entire frequency-wavenumber spectrum, which can fit some of the possible behaviours of real-world phononic structures and acoustic metamaterials. The presented methodology can be extended to study metamaterials with a more complex unit cell hierarchy.

Appendix A. Matrix elements

Elements of matrix S_{ij}^{PC} for the fractional Kelvin–Voigt PC model are given as

$$S_{11}^{PC} = p^2 + \frac{E_{01}}{m_1}(1+\tau^\alpha p^\alpha) + \frac{E_{02}}{m_1}(1+\tau^\beta p^\beta),$$

$$S_{12}^{PC} = -\frac{E_{01}}{m_1}(1+\tau^\alpha p^\alpha)e^{-i\kappa L} - \frac{E_{02}}{m_1}(1+\tau^\beta p^\beta),$$

$$S_{21}^{PC} = -\frac{E_{01}}{m_2}(1+\tau^\alpha p^\alpha)e^{i\kappa L} - \frac{E_{02}}{m_2}(1+\tau^\beta p^\beta),$$

$$S_{22}^{PC} = p^2 + \frac{E_{01}}{m_2}(1+\tau^\alpha p^\alpha) + \frac{E_{02}}{m_2}(1+\tau^\beta p^\beta).$$

Elements of matrix S_{ij}^{PC} for the fractional Kelvin-Voigt PC model are given as

$$S_{11}^{PC} = p^2 + \frac{E_{01}}{m_1}(1+\tau^\alpha p^\alpha) + \frac{E_{02}}{m_1}(1+\tau^\beta p^\beta),$$

$$S_{12}^{PC} = -\frac{E_{01}}{m_1}(1+\tau^\alpha p^\alpha)e^{-i\kappa L} - \frac{E_{02}}{m_1}(1+\tau^\beta p^\beta),$$

$$S_{21}^{PC} = -\frac{E_{01}}{m_2}(1+\tau^\alpha p^\alpha)e^{i\kappa L} - \frac{E_{02}}{m_2}(1+\tau^\beta p^\beta),$$

$$S_{22}^{PC} = p^2 + \frac{E_{01}}{m_2}(1+\tau^\alpha p^\alpha) + \frac{E_{02}}{m_2}(1+\tau^\beta p^\beta).$$

Similarly, by substituting the Eq. (2.13) in each of the Eqs. (2.7)–(2.12) one can obtain elements of matrix S_{ij}^{PC} for the Maxwell model and elements of matrices S_{ij}^{AM} for both, fractional-order Kelvin–Voigt and Maxwell models.

Appendix B. Roots of the characteristic equations

Here, the procedure for obtaining the roots of polynomials with noninteger exponents in an analytical manner is presented. After using the following replacement $p = re^{i\psi}$ in Eqs. (2.5)–(2.8) and separating the real and imaginary parts, one obtains the following relations for the fractional Kelvin–Voigt PC model:

$$\text{(B.1)}\quad r^4\cos 4\psi + (\omega_{11}^2+\omega_{12}^2)r^2R_\alpha\cos(2\psi+\phi_\alpha) + (\omega_{21}^2+\omega_{22}^2)r^2R_\beta\cos(2\psi+\phi_\beta) + 2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha R_\beta\cos(\phi_\alpha+\phi_\beta) = 0$$

$$\text{(B.2)}\quad r^4\sin 4\psi + (\omega_{11}^2+\omega_{12}^2)r^2R_\alpha\sin(2\psi+\phi_\alpha) + (\omega_{21}^2+\omega_{22}^2)r^2R_\beta\sin(2\psi+\phi_\beta) + 2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha R_\beta\sin(\phi_\alpha+\phi_\beta) = 0,$$

and for the fractional Kelvin–Voigt AM model as:

$$\begin{aligned}(\text{B.3})\quad & r^4\cos 4\psi + 2\omega_{11}^2 r^2 R_\alpha(1-\cos\kappa L)\cos(2\psi+\phi_\alpha) \\ &+ (\omega_{21}^2+\omega_{22}^2) r^2 R_\beta\cos(2\psi+\phi_\beta) \\ &+ 2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha R_\beta\cos(\phi_\alpha+\phi_\beta) = 0,\end{aligned}$$

$$\begin{aligned}(\text{B.4})\quad & r^4\sin 4\psi + 2\omega_{11}^2 r^2 R_\alpha(1-\cos\kappa L)\sin(2\psi+\phi_\alpha) \\ &+ (\omega_{21}^2+\omega_{22}^2) r^2 R_\beta\sin(2\psi+\phi_\beta) \\ &+ 2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha R_\beta\sin(\phi_\alpha+\phi_\beta) = 0,\end{aligned}$$

where R_α, R_β, ϕ_α and ϕ_β are polar coordinates for bracket terms in (2.5)–(2.8) given as:

$$R_\alpha = \sqrt{1+X^\alpha+X^{2\alpha}},\quad R_\beta = \sqrt{1+X^\beta+X^{2\beta}},$$

$$\tan\phi_\alpha = \frac{X^\alpha\sin\alpha\psi}{1+X^\alpha\cos\alpha\psi},\quad \tan\phi_\beta = \frac{X^\beta\sin\beta\psi}{1+X^\beta\cos\beta\psi},$$

where the replacement $X = r\tau$ is used. First, one should multiply Eqs. (B.1) and (B.2) with $\sin 4\psi$ and $\cos 4\psi$, respectively, and subtracting one from another eliminate the term r^4. The same equations should be multiplied again with $\sin\phi_\alpha+\phi_\beta$ and $\cos\phi_\alpha+\phi_\beta$ and then subtracted one from another to eliminate the last terms in equations. After taking into account some trigonometric identities, for the fractional Kelvin–Voigt PC model one can obtain two equations in terms of r^2 as:

$$r^2 = -\frac{2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha R_\beta\sin(\gamma_\alpha+\gamma_\beta)}{(\omega_{11}^2+\omega_{12}^2)R_\alpha\sin\gamma_\alpha+(\omega_{21}^2+\omega_{22}^2)R_\beta\sin\gamma_\beta},$$

$$r^2 = -\frac{(\omega_{11}^2+\omega_{12}^2)R_\alpha\sin\gamma_\beta+(\omega_{21}^2+\omega_{22}^2)R_\beta\sin\gamma_\alpha}{\sin(\gamma_\alpha+\gamma_\beta)},$$

where $\gamma_\alpha = \phi_\alpha - 2\psi$ and $\gamma_\beta = \phi_\beta - 2\psi$. Using the same procedure over Eqs. (B.3) and (B.4) the following equations in terms of r^2 can be obtained for the fractional Kelvin–Voigt AM model:

$$r^2 = -\frac{2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha R_\beta\sin(\gamma_\alpha+\gamma_\beta)}{2\omega_{11}^2 R_\alpha(1-\cos\kappa L)\sin\gamma_\alpha+(\omega_{21}^2+\omega_{22}^2)R_\beta\sin\gamma_\beta},$$

$$r^2 = -\frac{2\omega_{11}^2 R_\alpha(1-\cos\kappa L)\sin\gamma_\beta+(\omega_{21}^2+\omega_{22}^2)R_\beta\sin\gamma_\alpha}{\sin(\gamma_\alpha+\gamma_\beta)}.$$

By equating the equations for r^2 obtained for the fractional Kelvin–Voigt PC or AM models one can obtain the corresponding transcendent equation with two zeros defining the unknown X for the fixed values of angle ψ, which are denoted as X_s, $s = 1, 2$. Further, by substituting X_s in any of the equations for r and then in Eq. (2.19), one can find four roots of the characteristic polynomial with fractional exponents.

After using the same replacement $p = re^{i\psi}$ in Eqs. (2.9)–(2.12) and separating real and imaginary parts, for the Maxwell PC model one obtains:

$$\text{(B.5)}\quad \begin{aligned} &r^4\cos 4\psi + (\omega_{11}^2+\omega_{12}^2)r^2R_\alpha^{-1}X^\alpha\cos(2\psi-\phi_\alpha+\alpha\psi)\\ &\quad+2\omega_{11}^2\omega_{22}^2R_\alpha^{-1}R_\beta^{-1}X^{\alpha+\beta}(1-\cos\kappa L)\cos(\psi\alpha+\psi\beta-\phi_\alpha-\phi_\beta)\\ &\qquad\qquad+(\omega_{21}^2+\omega_{22}^2)r^2R_\beta^{-1}X^\beta\cos(2\psi-\phi_\beta+\beta\psi)=0,\end{aligned}$$

$$\text{(B.6)}\quad \begin{aligned} &r^4\sin 4\psi + (\omega_{11}^2+\omega_{12}^2)r^2R_\alpha^{-1}X^\alpha\sin(2\psi-\phi_\alpha+\alpha\psi)\\ &\quad+2\omega_{11}^2\omega_{22}^2R_\alpha^{-1}R_\beta^{-1}X^{\alpha+\beta}(1-\cos\kappa L)\sin(\psi\alpha+\psi\beta-\phi_\alpha-\phi_\beta)\\ &\qquad\qquad+(\omega_{21}^2+\omega_{22}^2)r^2R_\beta^{-1}X^\beta\sin(2\psi-\phi_\beta+\beta\psi)=0,\end{aligned}$$

and for the Maxwell AM model as:

$$\text{(B.7)}\quad \begin{aligned} &r^4\cos 4\psi + 2\omega_{11}^2r^2R_\alpha^{-1}X^\alpha(1-\cos\kappa L)\cos(2\psi-\phi_\alpha+\alpha\psi)\\ &\quad+2\omega_{11}^2\omega_{22}^2R_\alpha^{-1}R_\beta^{-1}X^{\alpha+\beta}(1-\cos\kappa L)\cos(\psi\alpha+\psi\beta-\phi_\alpha-\phi_\beta)\\ &\qquad\qquad+(\omega_{21}^2+\omega_{22}^2)r^2R_\beta^{-1}X^\beta\cos(2\psi-\phi_\beta+\beta\psi)=0,\end{aligned}$$

$$\text{(B.8)}\quad \begin{aligned} &r^4\sin 4\psi + 2\omega_{11}^2r^2R_\alpha^{-1}X^\alpha(1-\cos\kappa L)\sin(2\psi-\phi_\alpha+\alpha\psi)\\ &\quad+2\omega_{11}^2\omega_{22}^2R_\alpha^{-1}R_\beta^{-1}X^{\alpha+\beta}(1-\cos\kappa L)\sin(\psi\alpha+\psi\beta-\phi_\alpha-\phi_\beta)\\ &\qquad\qquad+(\omega_{21}^2+\omega_{22}^2)r^2R_\beta^{-1}X^\beta\sin(2\psi-\phi_\beta+\beta\psi)=0,\end{aligned}$$

where replacements for X, R_α, R_β, ϕ_α and ϕ_β are the same as for the fractional Kelvin–Voigt model. One should now multiply Eqs. (B.5) and (B.6) with $\sin 4\psi$ and $\cos 4\psi$, respectively, and subtract one from another to eliminate the term with r^4. Then the same equations should be multiplied again with $\sin -\phi_\alpha-\phi_\beta+\alpha\psi+\beta\psi$ and $\sin -\phi_\alpha-\phi_\beta+\alpha\psi+\beta\psi$ and subtracted one from another to eliminate the last terms in equations. The same procedure is repeated for Eqs. (B.7) and (B.8). After taking into account some trigonometric identities, one obtains two equations in terms of r^2 for the Maxwell PC model as:

$$r^2=-\frac{2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha^{-1}R_\beta^{-1}X^{\alpha+\beta}\sin(\theta_\alpha+\theta_\beta)}{(\omega_{11}^2+\omega_{12}^2)R_\alpha^{-1}X^\alpha\sin\theta_\alpha+(\omega_{21}^2+\omega_{22}^2)R_\beta^{-1}X^\beta\sin\theta_\beta}$$

$$r^2=-\frac{(\omega_{11}^2+\omega_{12}^2)R_\alpha^{-1}X^\alpha\sin\theta_\beta+(\omega_{21}^2+\omega_{22}^2)R_\beta^{-1}X^\beta\sin\theta_\alpha}{\sin(\theta_\alpha+\theta_\beta)}$$

and for the Maxwell AM model as

$$r^2=-\frac{2\omega_{11}^2\omega_{22}^2(1-\cos\kappa L)R_\alpha^{-1}R_\beta^{-1}X^{\alpha+\beta}\sin(\theta_\alpha+\theta_\beta)}{2\omega_{11}^2R_\alpha^{-1}X^\alpha(1-\cos\kappa L)\sin\theta_\alpha+(\omega_{21}^2+\omega_{22}^2)R_\beta^{-1}X^\beta\sin\theta_\beta},$$

$$r^2=-\frac{2\omega_{11}^2R_\alpha^{-1}X^\alpha(1-\cos\kappa L)\sin\theta_\beta+(\omega_{21}^2+\omega_{22}^2)R_\beta^{-1}X^\beta\sin\theta_\alpha}{\sin(\theta_\alpha+\theta_\beta)},$$

where $\theta_\alpha=\phi_\alpha-\alpha\psi+2\psi$ and $\theta_\beta=\phi_\beta-\beta\psi+2\psi$. In the same manner as above, by equating the equations for r^2 of the fractional Maxwell PC or AM models, one can obtain the corresponding transcendent equation with two zeros defining the unknown X for fixed values of angle ψ, which are denoted as X_s, $s=1,2$.

References

1. M. I. Hussein, M. J. Leamy, M. Ruzzene, *Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook*, App. Mech. Rev. **66** (2014), 040802.
2. D. Krattiger, M. I. Hussein, *Generalized Bloch mode synthesis for accelerated calculation of elastic band structures*, J. Comput. Phys. **357** (2018), 183–205.
3. J. Lou, L. He, J. Yang, S. Kitipornchai, H. Wu, *Wave propagation in viscoelastic phononic crystal rods with internal resonators*, Appl. Acoust. **141** (2018), 382–392.
4. P. G. Martinsson, A. B. Movchan, *Vibrations of lattice structures and phononic band gaps*, Q. J. Mech. Appl. Math. **56** (2003), 45–64.
5. K. H. Matlack, S. Krödel, A. Bauhofer, D. Chiara, *Advanced structured composites as novel phononic crystals and acoustic metamaterials*, in: C. Ralph, M. Silberstein, P. Thakre, R. Singh, (ed.), *Mechanics of Composite and Multi-functional Materials*, Springer, Cham, 2016, 155–162.
6. J. Lee, J. P. Singer, E. L. Thomas, *Micro-/nanostructured mechanical metamaterials*, Adv. Mater. **24** (2012), 4782–4810.
7. P. S. Y. E. Kwon, E. Kwon, Y. Y. Kim, *Wave dispersion tailoring in an elastic waveguide by phononic crystals*, Adv. Mater. **103** (2013), 151901.
8. W. Gang, Y. Liu, J. Wen, D. Yu, *Visco-elastic effects on wave dispersion in three-phase acoustic metamaterials*, J. Mech. Phys. Solids. **96** (2016), 29–47.
9. B. Merheb, P. A. Deymier, M. Jain, M. Aloshyna-Lesuffleur, S. Mohanty, A. Berker, R. W. Greger, *Elastic and viscoelastic effects in rubber/air acoustic band gap structures: A theoretical and experimental study*, J. App. Phys. **104** (2008), 064913.
10. Y-F. Wang, Y-S. Wang, V. Laude, *Wave propagation in two-dimensional viscoelastic metamaterials*, Phys. Rev. B **92** (2015), 104110.
11. X. Zhu, S. Zhong, H. Zhao, *Band gap structures for viscoelastic phononic crystals based on numerical and experimental investigation*, App. Acoust. **106** (2016), 93–104.
12. W. Gang, Y. Liu, J. Wen, D. Yu, *Formation mechanism of the low-frequency locally resonant band gap in the two-dimensional ternary phononic crystals*, Chinese Phys. **15** (2006), 407.
13. M. I. Hussein, M. J. Frazier, *Metadamping: An emergent phenomenon in dissipative metamaterials*, J. Sound. Vib. **332** (2013), 4767–4774.
14. M. I. Hussein, *Theory of damped Bloch waves in elastic media*, Phys. Rev. B **80** (2009), 212301.
15. M. I. Hussein, M. J. Frazier, *Band structure of phononic crystals with general damping*, J. App. Phys. **108** (2010), 093506.
16. Y. Y. Chen, M. V. Barnhart, J. K. Chen, G. K. Hu, C. T. Sun, G. L. Huang, *Dissipative elastic metamaterials for broadband wave mitigation at subwavelength scale*, Compos. Struct. **136** (2016), 358–371.
17. M. A. Lewińska, V. G. Kouznetsova, J. A. W. van Dommelen, A. O. Krushynska, M. G. D. Geers, *The attenuation performance of locally resonant acoustic metamaterials based on generalised viscoelastic modelling*, Int. J. Solids Struct. **126** (2017), 163–174.
18. S. Alamri, B. Li, K. T. Tan, *Dynamic load mitigation using dissipative elastic metamaterials with multiple Maxwell-type oscillators*, J. Appl. Phys. **123** (2018), 095111.
19. Y. Liu, D. Yu, H. Zhao, J. Wen, X. Wen, *Theoretical study of two-dimensional phononic crystals with viscoelasticity based on fractional derivative models*, J. Phys. D Appl. Phys. **41** (2008), 065503.
20. Y. A. Rossikhin, M. V. Shitikova, *Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results*, Appl. Mech. Rev. **63** (2010), 010801.
21. M. J. Frazier, M. I. Hussein, *Generalized Bloch's theorem for viscous metamaterials: Dispersion and effective properties based on frequencies and wavenumbers that are simultaneously complex*, C. R. Phys. **17** (2016), 565–577.
22. D. DePauw, H. Al Ba'ba'a, M. Nouh, *Metadamping and energy dissipation enhancement via hybrid phononic resonators*, Extreme Mech. Lett. **18** (2018), 36–44.

23. Y. A. Rossikhin, M. V. Shitikova, *A new method for solving dynamic problems of fractional derivative viscoelasticity*, Int. J. Eng. Sci. **39** (2001), 149–176.
24. Y. A. Rossikhin, *Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids*, Appl. Mech. Rev. **63** (2010), 010701.
25. T. M. Atanacković, S. Pilipović, B. Stanković, D. Zorica, *Fractional calculus with applications in mechanics: wave propagation, impact and variational principles*, John Wiley & Sons, City, 2014.
26. S. Holm, S. P. Näsholm, *A causal and fractional all-frequency wave equation for lossy media*, J. Acoust. Soc. Am. **130** (2011), 2195–2202.
27. Y. Wang, *A causal and fractional all-frequency wave equation for lossy media*, J. Acoust. Soc. Am. **204** (2015), 1216–1221.
28. C. Liu, C. Reina, *Broadband locally resonant metamaterials with graded hierarchical architecture*, J. Appl. Phys. **123** (2018), 095108.

ПРИСТУП БАЗИРАН НА ФРАКЦИОНОМ РАЧУНУ ЗА ОПИСИВАЊЕ МЕТА-ПРИГУШЕЊА У ФОНОНСКИМ КРИСТАЛИМА И АКУСТИЧНИМ МЕТАМАТЕРИЈАЛИМА

Резиме. У последњој деценији, истраживања на тему фононских и акустичних материјала и структура су у успону као резултат примене резултата истраживања из теоријске физике у разним областима инжењерства. Главна карактеристика фононског медијума је периодичност која произилази из периодичности материјалних фаза, геометрије или граничних услова са особинама пропагације таласа анализираним преко зонске структуре (подручја забрањених и допуштених фреквенција пропагације таласа). За добијање наведених карактеристика и дисперзних релација резонантих метаматеријала са вискозним пригушењем у литертури се најчешће примењује Блохова теорема. У овом раду предложен је нов приступ за теоријску анализу модела фононских кристала и акустичних метаматеријала са фракционим пригушењем уведеним преко фракционог Келвин-Војтовог и Максвеловог модела. У нумеричкој анализи, резултати добијени за предложене моделе са пригушењем су упоређени са резултатима добијеним за еквивалентне еластичне моделе фононских кристала и локално резонантних акустичних метаматеријала и показана је знатна разлика у вредностима дисперзних криви у ова два случаја. Показано је да је фракциони Максвелов модел погоднији за описивање ефекта пригушења због могућности фитовања два параметра, реда фракционог извода и параметра пригушења.

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