Supplementary material
Probing the frequency-dependent elastic moduli of lattice materials

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Abstract

An analytical framework is developed for analysing the frequency dependence of in-plane elastic moduli of lattice metamaterials under vibrating environment. On the basis of a unit cell based approach, closed-form expressions for the complex elastic moduli are derived as a function of frequency by employing the dynamic stiffness matrix of a damped beam element. In this supplementary material, we have provided the detail derivation of the dynamic stiffness matrix for a single beam element first. Thereby, the derivation of closed-form expressions of the frequency-dependent elastic moduli of lattice metamaterials are presented.

Contents

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1. Dynamic stiffness approach

1.1. Equation of motion

Dynamic motion of the overall cellular structure corresponds to vibration of individual beams which constitute each hexagonal unit cells. A pictorial depiction of the beam is shown in figure 1(c) of the main manuscript. One honeycomb unit cell under dynamic environment is shown in figure 1(b) of the main manuscript, wherein a vibrating mode of each constituent members is symbolically shown. If external forces are applied to such vibrating honeycomb, the members will deform following a different rule. Thus the effective elastic moduli of the entire lattice will be different from conventional static elastic moduli. Aim of the present article is to capture the effect of vibration in the effective elastic moduli of hexagonal lattices based on dynamic stiffness method [9, 16]. The dynamic stiffness matrix of a single beam element is derived first (section 1); thereby the expressions of frequency dependent elastic moduli of the lattice metamaterial are developed based on the elements of the dynamic stiffness matrix of a single beam (section 2).

The equation of motion of free vibration of a damped beam (refer to figure 1) can be expressed as

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} + c_1\frac{\partial^5 V(x,t)}{\partial x^4 \partial t} + m\frac{\partial^2 V(x,t)}{\partial t^2} + c_2\frac{\partial V(x,t)}{\partial t} = 0 \tag{1}$$

It is assumed that the behaviour of the beam follows the Euler-Bernoulli hypotheses. In the above equation EI is the bending rigidity, m is mass per unit length, c_1 is the strain-rate-dependent viscous damping coefficient, c_2 is the velocity-dependent viscous damping coefficient and $V(x,t)$ is the transverse displacement. The length of the beam is assumed to be L. Considering a harmonic motion with frequency ω we have

$$V(x,t) = v(x)\exp\left[\mathrm{i}\omega t\right] \tag{2}$$

where $\mathrm{i} = \sqrt{-1}$. Substituting this in the beam equation (1) one obtains

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + \mathrm{i}\omega c_1\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - m\omega^2 v + \mathrm{i}\omega c_2 v = 0 \tag{3}$$

$$\text{or} \quad \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - b^4 v = 0 \tag{4}$$

where

$$b^4 = \frac{m\omega^2 - \mathrm{i}\omega c_2}{EI + \mathrm{i}\omega c_1} \tag{5}$$

Following the damping convention in dynamic analysis as in [18], we consider stiffness and mass proportional damping. Therefore, we express the damping constants as

$$c_1 = \zeta_k(EI) \quad \text{and} \quad c_2 = \zeta_m(m) \tag{6}$$

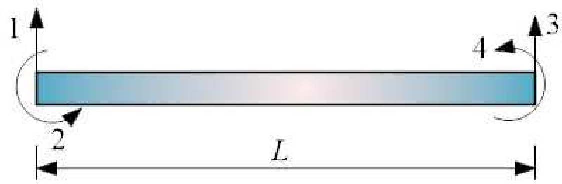

Figure 1: A finite element representing dynamics of an Euler-Bernoulli beam. This element has four degrees of freedom consisting of the transverse deformation and the rotation at the two nodes.

where ζ_k and ζ_m are stiffness and mass proportional damping factors. Substituting these, from Eq. (5) we have

$$b^4 = \frac{m\omega^2 \left(1 - \mathrm{i}\zeta_m/\omega\right)}{EI\left(1 + \mathrm{i}\omega\zeta_k\right)} \tag{7}$$

The constant b is in general a complex number for any physically realistic damping values. The effect of mass proportional damping factor ζ_m linearly decreases with higher frequency whereas the effect of stiffness proportional damping factor ζ_k linearly increases with higher frequency. To obtain the characteristic equation, we consider

$$v(x) = \exp\left[\lambda x\right] \tag{8}$$

Substituting this in Eq. (4) one obtains

$$\lambda^4 - b^4 = 0 \tag{9}$$

$$\text{or} \quad \lambda = \mathrm{i}b, -\mathrm{i}b,\ b, -b \tag{10}$$

Next we use these solutions to obtain the dynamic shape functions of the beam.

1.2. Frequency dependent shape functions

For classical (static) finite element analysis of beams, cubic polynomials are used as shape functions (see for example [20]). Here we aim to incorporate frequency dependent dynamic shape functions, as used with the framework of the dynamic finite element method. The dynamic finite element method belongs to the general class of spectral methods for linear dynamical systems [9]. This approach, or approaches very similar to this, is known by various names such as the dynamic stiffness method [1–7, 10, 11, 17, 19], spectral finite element method [9, 13] and dynamic finite element method [14, 15].

The dynamic shape functions are obtained such that the equation of dynamic equilibrium is satisfied exactly at all points within the element. Similar to the classical finite element method, assume that the frequency-dependent displacement within an element is interpolated from the nodal displacements as

$$v(x, \omega) = \mathbf{N}^T(x, \omega)\widehat{\mathbf{v}}(\omega) \tag{11}$$

Here $\widehat{\mathbf{v}}(\omega) \in \mathbb{C}^n$ is the nodal displacement vector $\mathbf{N}(x, \omega) \in \mathbb{C}^n$ is the vector of frequency-dependent shape functions and $n = 4$ is the number of the nodal degrees-of-freedom. Suppose the $s_j(x, \omega) \in \mathbb{C}, j = 1, \cdots, 4$ are the basis functions which exactly satisfy Eq. (4). It can be shown that the shape function vector can be expressed as

$$\mathbf{N}(x, \omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(x, \omega) \tag{12}$$

where the vector $\mathbf{s}(x, \omega) = \{s_j(x, \omega)\}^T, \forall j = 1, \cdots, 4$ and the complex matrix $\mathbf{\Gamma}(\omega) \in \mathbb{C}^{4\times 4}$ depends on the boundary conditions. The elements of $\mathbf{s}(x, \omega)$ constitutes $\exp[\lambda_j x]$ where the values of λ_j are obtained from the solution of the characteristics equation as given in Eq. (10). An element for the damped beam under bending vibration is shown in figure 1. The degrees-of-freedom for each nodal point include a vertical and a rotational degrees-of-freedom.

In view of the solutions in Eq. (10), the displacement field with the element can be expressed by a linear combination of the basic functions $e^{-\mathrm{i}bx}, e^{\mathrm{i}bx}, e^{bx}$ and e^{-bx} so that in our notations $\mathbf{s}(x, \omega) = \left\{e^{-\mathrm{i}bx}, e^{\mathrm{i}bx}, e^{bx}, e^{-bx}\right\}^T$. We can also express $\mathbf{s}(x, \omega)$ in terms of trigonometric functions. Considering $e^{\pm \mathrm{i}bx} = \cos(bx) \pm \mathrm{i}\sin(bx)$ and $e^{\pm bx} = \cosh(bx) \pm \mathrm{i}\sinh(bx)$, the vector $\mathbf{s}(x, \omega)$ can be alternatively expressed as

$$\mathbf{s}(x, \omega) = \begin{Bmatrix} \sin(bx) \\ \cos(bx) \\ \sinh(bx) \\ \cosh(bx) \end{Bmatrix} \in \mathbb{C}^4 \tag{13}$$

For steady-state dynamic response, the displacement field within the element can be expressed as

$$v(x) = \mathbf{s}(x, \omega)^T \mathbf{v} \tag{14}$$

where $\mathbf{v} \in \mathbb{C}^4$ is the vector of constants to be determined from the boundary conditions.

The relationship between the shape functions and the boundary conditions can be represented as in Table 1, where boundary conditions in each column give rise to the corresponding shape function. Writing Eq. (14) for the above four sets of boundary conditions, one obtains

$$[\mathbf{R}] \left[\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4\right] = \mathbf{I} \tag{15}$$

Table 1: The relationship between the boundary conditions and the shape functions for the bending vibration of beams.

	$N_1(x,\omega)$	$N_2(x,\omega)$	$N_3(x,\omega)$	$N_4(x,\omega)$
$y(0)$	1	0	0	0
$\frac{\mathrm{d}y}{\mathrm{d}x}(0)$	0	1	0	0
$y(L)$	0	0	1	0
$\frac{\mathrm{d}y}{\mathrm{d}x}(L)$	0	0	0	1

where

$$\mathbf{R} = \begin{bmatrix} s_1(0) & s_2(0) & s_3(0) & s_4(0) \\ \frac{\mathrm{d}s_1}{\mathrm{d}x}(0) & \frac{\mathrm{d}s_2}{\mathrm{d}x}(0) & \frac{\mathrm{d}s_3}{\mathrm{d}x}(0) & \frac{\mathrm{d}s_4}{\mathrm{d}x}(0) \\ s_1(L) & s_2(L) & s_3(L) & s_4(L) \\ \frac{\mathrm{d}s_1}{\mathrm{d}x}(L) & \frac{\mathrm{d}s_2}{\mathrm{d}x}(L) & \frac{\mathrm{d}s_3}{\mathrm{d}x}(L) & \frac{\mathrm{d}s_4}{\mathrm{d}x}(L) \end{bmatrix} \tag{16}$$

and $\mathbf{y}^k$ is the vector of constants giving rise to the kth shape function. In view of the boundary conditions represented in Table 1 and equation (15), the shape functions for bending vibration can be shown to be given by Eq. (12) where

$$\boldsymbol{\Gamma}(\omega) = \left[\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4\right]^T = \left[\mathbf{R}^{-1}\right]^T \tag{17}$$

By obtaining the matrix $\boldsymbol{\Gamma}(\omega)$ from the above equation, the shape function vector can be obtained from Eq. (12). After some algebraic simplifications, we have represented the frequency dependent complex shape functions as

$$\begin{Bmatrix} N_1(x,\omega) \\ N_2(x,\omega) \\ N_3(x,\omega) \\ N_4(x,\omega) \end{Bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{cS+Cs}{cC-1} & -\frac{1}{2}\frac{1+sS-cC}{cC-1} & -\frac{1}{2}\frac{cS+Cs}{cC-1} & \frac{1}{2}\frac{cC+sS-1}{cC-1} \\ \frac{1}{2}\frac{cC+sS-1}{b(cC-1)} & \frac{1}{2}\frac{-Cs+cS}{b(cC-1)} & -\frac{1}{2}\frac{1+sS-cC}{b(cC-1)} & -\frac{1}{2}\frac{-Cs+cS}{b(cC-1)} \\ -\frac{1}{2}\frac{S+s}{cC-1} & \frac{1}{2}\frac{C-c}{cC-1} & \frac{1}{2}\frac{S+s}{cC-1} & -\frac{1}{2}\frac{C-c}{cC-1} \\ \frac{1}{2}\frac{C-c}{b(cC-1)} & -\frac{1}{2}\frac{S-s}{b(cC-1)} & -\frac{1}{2}\frac{C-c}{b(cC-1)} & -\frac{1}{2}\frac{S-s}{b(cC-1)} \end{bmatrix} \begin{Bmatrix} \sin bx \\ \cos bx \\ \sinh bx \\ \cosh bx \end{Bmatrix} \tag{18}$$

where

$$C = \cosh(bL), \quad c = \cos(bL), \quad S = \sinh(bL) \quad \text{and} \quad s = \sin(bL) \tag{19}$$

and b is defined in (7).

1.3. Element dynamic stiffness matrix

The stiffness and mass matrices can be obtained following the conventional variational formulation [8]. The only difference is instead of classical cubic polynomials as the shape functions, frequency dependent shape functions in (18) should be used. It is convenient to define the dynamic stiffness matrix as

$$\mathbf{D}(\omega) = \mathbf{K}(\omega) - \omega^2 \mathbf{M}(\omega) \tag{20}$$

so that the equation of dynamic equilibrium is

$$\mathbf{D}(\omega)\widehat{\mathbf{v}}(\omega) = \widehat{\mathbf{f}}(\omega) \tag{21}$$

In Eq. (20), the frequency-dependent stiffness and mass matrices can be obtained from

$$\mathbf{K}(\omega) = EI \int_0^L \frac{\mathrm{d}^2\mathbf{N}(x,\omega)}{\mathrm{d}x^2} \frac{\mathrm{d}^2\mathbf{N}^T(x,\omega)}{\mathrm{d}x^2} \mathrm{d}x \tag{22}$$

$$\text{and} \quad \mathbf{M}(\omega) = m \int_0^L \mathbf{N}(x,\omega)\mathbf{N}^T(x,\omega)\mathrm{d}x \tag{23}$$

After some algebraic simplifications it can be shown that the dynamic stiffness matrix is given by the following closed-form expression

$$\mathbf{D}(\omega) = \frac{EIb}{(cC-1)} \begin{bmatrix} -b^2(cS+Cs) & -sbS & b^2(S+s) & -b(C-c) \\ -sbS & -Cs+cS & b(C-c) & -S+s \\ b^2(S+s) & b(C-c) & -b^2(cS+Cs) & sbS \\ -b(C-c) & -S+s & sbS & -Cs+cS \end{bmatrix} \tag{24}$$

The elements of this matrix are frequency dependent complex quantities because b is a function of ω and the damping factors.

2. The derivation of frequency-dependent elastic moduli

Considering only the static deformation of a unit cell, the equivalent elastic moduli of the hexagonal cellular materials can be obtained as [12]

$$E_{1GA} = E\left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l}+\sin\theta)\sin^2\theta} \tag{25}$$

$$E_{2GA} = E\left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l}+\sin\theta)}{\cos^3\theta} \tag{26}$$

$$\nu_{12GA} = \frac{\cos^2\theta}{(\frac{h}{l}+\sin\theta)\sin\theta} \tag{27}$$

$$\nu_{21GA} = \frac{(\frac{h}{l}+\sin\theta)\sin\theta}{\cos^2\theta} \tag{28}$$

$$\text{and} \quad G_{12GA} = E\left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l}+\sin\theta\right)}{\left(\frac{h}{l}\right)^2(1+2\frac{h}{l})\cos\theta} \tag{29}$$

where $(.)_{GA}$ represents the expressions of elastic moduli of regular hexagonal honeycombs. The cell walls are treated as beams of thickness t and Young's modulus E. The quantities l and h are the lengths of inclined cell walls having inclination angle θ and the vertical cell walls respectively. A key interest in this section is to obtain equivalent expressions when harmonic forcing is considered. The central idea behind the proposed derivation is to exploit the physical interpretation of the elements of the dynamic stiffness matrix obtained in the previous section.

Using equation (24), the analytical expressions of the frequency dependent in-plane elastic moduli will be obtained. For the purpose of deriving the expressions, the dynamic stiffness matrix is written in the following form for notational convenience

$$\mathbf{D}(\omega) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \tag{30}$$

where D_{ij} $(i,j = 1,2,3,4)$ has the expressions corresponding to the terms of equation (24).

2.1. Derivation of Young's modulus E_1

One cell wall is considered for deriving the expression of the Young's modulus E_1 under the application of stress in direction - 1 as shown in figure 2(a) [12]. In the free body diagram of the slant member in figure 2(a), the rotational displacements of both ends and the bending displacement of one end is considered as zero. To satisfy the equilibrium of forces in direction -2, the force C is needed to be zero. Thus from the dynamic stiffness matrix presented in equation (30), the bending deflection of one end of the slant member with respect to the other end can be written as

$$\delta = \frac{P \sin\theta}{D_{33}} \tag{31}$$

where $P = \sigma_1(h + l\sin\theta)\bar{b}$ (geometric dimensions of a single honeycomb cell is shown in figure 1(b) of the main manuscript. $\bar{b}$ is the width of the beam i.e. thickness of the honeycomb sheet). The component of δ in direction - 1 is $\delta\sin\theta$. Thus the strain component in direction - 1 due to applied stress in the same direction can be expressed as

$$\begin{aligned} \epsilon_{11} &= \frac{\delta \sin\theta}{l\cos\theta} \\ &= \frac{\sigma_1(h + l\sin\theta)\bar{b}\sin^2\theta}{D_{33} l\cos\theta} \end{aligned} \tag{32}$$

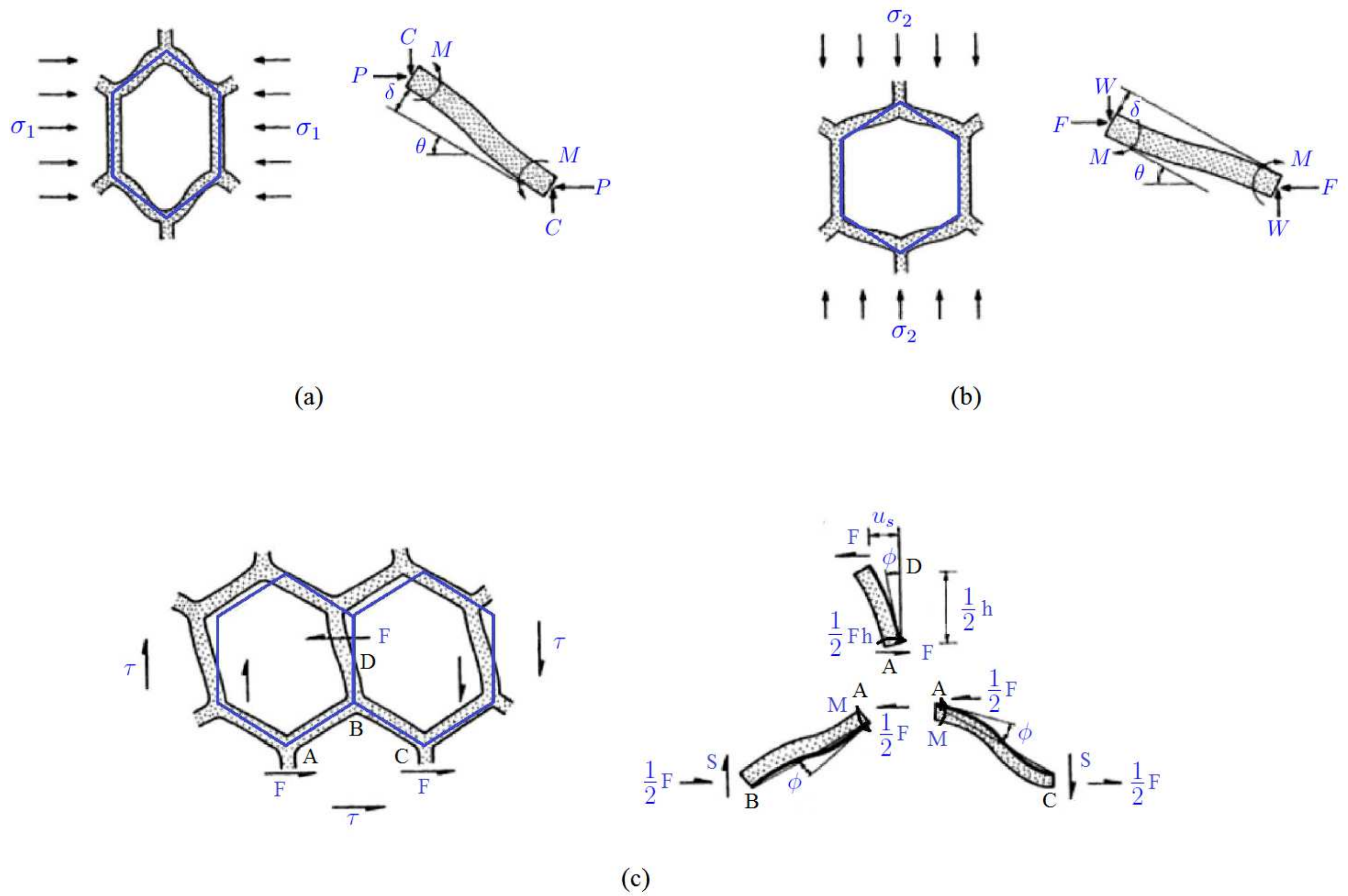

Figure 2: Deformed shapes and free body diagrams under the application of direct stresses and shear stress. The undeformed shapes of the hexagonal cell are indicated using blue colour for each of the loading conditions.

The expression of D_{33} is given in equation (24) and (30). Replacing the expression for D_{33} and $I = \dfrac{\bar{b}t^3}{12}$, the Young's modulus E_1 can be obtained as

$$\begin{aligned} E_1(\omega) = \frac{\sigma_1}{\epsilon_{11}} &= \frac{D_{33} l \cos\theta}{(h + l\sin\theta)\bar{b}\sin^2\theta} \\ &= \frac{E t^3 l \cos\theta b^3 (\cos(bl)\sinh(bl) + \cosh(bl)\sin(bl))}{12(h + l\sin\theta)\sin^2\theta(1 - \cos(bl)\cosh(bl))} \end{aligned} \tag{33}$$

The expression of b is provided in equation (7). E is the intrinsic elastic modulus of the honeycomb material and t is the thickness of honeycomb wall.

2.2. Derivation of Young's modulus E_2

Similar to the derivation of E_1, the bending deformation of one end of the slant beam under the application of σ_2 (as shown in figure 2(b)) can be expressed as

$$\delta = \frac{W\cos\theta}{D_{33}} \tag{34}$$

where $W = \sigma_2 l\bar{b}\cos\theta$. The expression for strain component in direction - 2 due to application of stress in the same direction can be obtained as

$$\begin{aligned} \epsilon_{22} &= \frac{\delta\cos\theta}{(h + l\sin\theta)} \\ &= \frac{\sigma_2 l\bar{b}\cos^3\theta}{D_{33}(h + l\sin\theta)} \end{aligned} \tag{35}$$

Replacing the expression for D_{33} and $I = \dfrac{\bar{b}t^3}{12}$, the Young's modulus E_2 can be obtained as

$$\begin{aligned} E_2(\omega) &= \frac{\sigma_2}{\epsilon_{22}} = \frac{D_{33}(h + l\sin\theta)}{l\bar{b}\cos^3\theta} \\ &= \frac{Et^3(h + l\sin\theta)b^3\left(\cos(bl)\sinh(bl) + \cosh(bl)\sin(bl)\right)}{12l\cos^3\theta(1 - \cos(bl)\cosh(bl))} \end{aligned} \tag{36}$$

The expression of b is provided in equation (7), E is the intrinsic elastic modulus of the honeycomb material and t is the thickness of honeycomb wall as before.

2.3. Derivation of shear modulus G_{12}

For deriving the expression of G_{12}, two members of the honeycomb cell are needed to be considered (vertical member with length $\dfrac{h}{2}$ and a slant member with length l) as shown in figure 2(c). The points A, B and C will not have any relative movement due to symmetrical structure. The total shear deflection u_s consists of two components, bending deflection of the member BD and its deflection due to rotation of joint B.

It can be noted here that the elements of the dynamic stiffness matrix (refer to equation(30)) will be different for the vertical member and the slant member due to their different lengths. Using the stiffness components of the dynamic stiffness matrix (refer to equation (30)), the bending deformation of point D with respect to point B in direction - 1 can be obtained as

$$\delta_b = \frac{F}{\left(D^v_{33} - \dfrac{D^v_{34}D^v_{43}}{D^v_{44}}\right)} = \frac{F}{\left(D^v_{33} - \dfrac{(D^v_{34})^2}{D^v_{44}}\right)} \tag{37}$$

Here $F = 2\tau l\bar{b}\cos\theta$ and we make use of the symmetry of the elements of the dynamic stiffness matrix. The superscript v in the elements of the dynamic stiffness matrix is used to indicate the stiffness element corresponding to the vertical member.

From the free body diagram presented in figure 2(c),

$$M = \frac{Fh}{4} \tag{38}$$

On the basis of equation (30), deflection of the end B with respect to the end C due to application of moment M at the end B is given as

$$\delta_r = \frac{M}{-D_{43}^s} \tag{39}$$

Here the superscript s in D_{43} is used to indicate the stiffness element corresponding to the slant member and the negative sign arises due to the direction of the rotation as given in figure 1. Thus the rotation of joint B can be expressed as

$$\begin{aligned}\phi &= \frac{\delta_r}{l} \\ &= -\frac{Fh}{4lD_{43}^s}\end{aligned} \tag{40}$$

Total shear deformation under the application of shear stress τ can be expressed as

$$\begin{aligned}u_s &= \frac{1}{2}\phi h + \delta_b \\ &= -\frac{Fh^2}{8lD_{43}^s} + \frac{F}{\left(D_{33}^v - \frac{(D_{34}^v)^2}{D_{44}^v}\right)}\end{aligned} \tag{41}$$

The shear strain is given by

$$\begin{aligned}\gamma &= \frac{2u_s}{(h + l\sin\theta)} \\ &= \frac{F}{(h + l\sin\theta)}\left(-\frac{h^2}{4lD_{43}^s} + \frac{2}{\left(D_{33}^v - \frac{(D_{34}^v)^2}{D_{44}^v}\right)}\right) \\ &= \frac{2\tau l\bar{b}\cos\theta}{(h + l\sin\theta)}\left(-\frac{h^2}{4lD_{43}^s} + \frac{2}{\left(D_{33}^v - \frac{(D_{34}^v)^2}{D_{44}^v}\right)}\right)\end{aligned} \tag{42}$$

Replacing the expressions for the stiffness components from equation (24) and (30), the shear modulus can be obtained as

$$\begin{aligned}G_{12}(\omega) = \frac{\tau}{\gamma} &= \frac{(h + l\sin\theta)}{2l\bar{b}\cos\theta}\frac{1}{\left(-\frac{h^2}{4lD_{43}^s} + \frac{2}{\left(D_{33}^v - \frac{(D_{34}^v)^2}{D_{44}^v}\right)}\right)} \\ &= \frac{(h + l\sin\theta)}{2l\bar{b}\cos\theta}\frac{4EIb^3\sin(bl)\sinh(bl)\left(1 + \cos(bh/2)\cosh(bh/2)\right)}{\begin{array}{l}h^2 b\left(1 - \cos(bl)\cosh(bl)\right)\left(1 + \cos(bh/2)\cosh(bh/2)\right) \\ \quad + 8l\sin(bl)\sinh(bl)\left(\cosh(bh/2)\sin(bh/2) - \sinh(bh/2)\cos(bh/2)\right)\end{array}} \\ &= \frac{Et^3(h + l\sin\theta)b^3\sin(bl)\sinh(bl)\left(1 + \cos(bh/2)\cosh(bh/2)\right)}{\begin{array}{l}6l\cos\theta\,[h^2 b\left(1 - \cos(bl)\cosh(bl)\right)\left(1 + \cos(bh/2)\cosh(bh/2)\right) \\ \quad +8l\sin(bl)\sinh(bl)\left(\cosh(bh/2)\sin(bh/2) - \sinh(bh/2)\cos(bh/2)\right)]\end{array}}\end{aligned} \tag{43}$$

The expression of the complex variable b is provided in equation (7).

2.4. Derivation of Poisson's ratios ν_{12} and ν_{21}

The strain components in direction - 1 and direction - 2 under the application of stress σ_1 are given by (refer to figure 2(a))

$$\epsilon_{11} = \frac{\delta \sin\theta}{l \cos\theta} \tag{44}$$

$$\epsilon_{21} = \frac{-\delta \cos\theta}{h + l \sin\theta} \tag{45}$$

Thus the Poisson's ratio for loading direction - 1 can be obtained as

$$\begin{aligned} \nu_{12} &= -\frac{\epsilon_{21}}{\epsilon_{11}} \\ &= \frac{l \cos^2\theta}{(h + l\sin\theta)\sin\theta} \end{aligned} \tag{46}$$

Similarly the Poisson's ratio for loading direction - 2 can be obtained as

$$\begin{aligned} \nu_{21} &= -\frac{\epsilon_{12}}{\epsilon_{22}} \\ &= \frac{(h + l\sin\theta)\sin\theta}{l \cos^2\theta} \end{aligned} \tag{47}$$

For the convenience of readers, the derived expressions for the frequency-dependent elastic moduli of hexagonal lattices are listed in Table 2. It can be noted that the in-plane Poisson's ratios (Equation 46 and 47) are not dependent on frequency and the expressions are same as the case of static deformation provided in literature [12].

Table 2: Summary of the newly-derived formulae for effective frequency-dependent elastic moduli (E_1, E_2 and G_{12}) of hexagonal lattices (the in-plane Poisson's ratios are not dependent on frequency)

Expressions of the frequency-dependent elastic moduli
$E_1(\omega) = \frac{Et^3 l \cos\theta b^3(\cos(bl)\sinh(bl) + \cosh(bl)\sin(bl))}{12(h + l\sin\theta)\sin^2\theta(1 - \cos(bl)\cosh(bl))}$
$E_2(\omega) = \frac{Et^3(h + l\sin\theta)b^3\left(\cos(bl)\sinh(bl) + \cosh(bl)\sin(bl)\right)}{12l\cos^3\theta(1 - \cos(bl)\cosh(bl))}$
$G_{12}(\omega) = \frac{Et^3(h + l\sin\theta)b^3\sin(bl)\sinh(bl)\left(1 + \cos(bh/2)\cosh(bh/2)\right)}{6l\cos\theta\left[h^2 b\left(1 - \cos(bl)\cosh(bl)\right)\left(1 + \cos(bh/2)\cosh(bh/2)\right) + 8l\sin(bl)\sinh(bl)\left(\cosh(bh/2)\sin(bh/2) - \sinh(bh/2)\cos(bh/2)\right)\right]}$

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