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Short Communication

Stochastic mechanics of metamaterials

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ABSTRACT

The effect of stochasticity in mechanical behaviour of metamaterials is quantified in a probabilistic framework. The stochasticity has been accounted in the form of random material distribution and structural irregularity, which are often encountered due to manufacturing and operational uncertainties. An analytical framework has been developed for analysing the effective stochastic in-plane elastic properties of irregular hexagonal structural forms with spatially random variations of cell angles and intrinsic material properties. Probabilistic distributions of the in-plane elastic moduli have been presented considering both randomly homogeneous and randomly inhomogeneous stochasticity in the system, followed by an insightful comparative discussion. The ergodic behaviour in spatially irregular lattices is investigated as a part of this study. It is found that the effect of random micro-structural variability in structural and material distribution has considerable influence on mechanical behaviour of metamaterials.

1. Introduction

Mechanical metamaterials are artificial microstructures with mechanical properties defined by their structural configuration rather than intrinsic material properties of the constituent members. The global mechanical properties can be engineered by intelligently identifying the material microstructures of metamaterials. This novel class of structural materials with tailorable global mechanical properties (like equivalent elastic moduli, buckling, vibration and wave propagation characteristics) have tremendous potential applications for future aerospace, civil and mechanical structures. Development of such application-specific engineered materials have received immense attention from the concerned scientific community in last few years after the recent advancement in 3D printing technology [1–8]. Fascinating properties such as extremely lightweight, negative elastic moduli, negative mass density, pentamode material characteristic (meta-fluids) can be obtained by cognitively identifying the material microstructure. Considering hexagonal lattices, the structural configurations for obtaining negative and zero Poisson's ratios are depicted in Fig. 1 as an illustration. The lattice in Fig. 1(a) has conventional positive Poisson's ratio, while by changing the microstructural configuration of the hexagonal lattice intuitively, negative (refer to Fig. 1)) and zero Poisson's ratio (refer to Fig. 1(c–e)) can be obtained at a global scale of the material. In case of a material with negative Poisson's ratio (auxetic), it thickens in the dimensions perpendicular to the direction of stretching and vice versa, while the dimensions perpendicular to the direction of stretching/compressing do not change for a material with zero Poisson's ratio. Natural materials can not exhibit such unusual properties that can have various favourable applications in wide range of structural systems.

Metamaterials are consisted of periodic structural forms in two and three dimesional spaces. The most prominent approach of analysing metamaterials is to consider an appropriate unit cell that can represent the entire material micro-structure. However, a significant limitation of the unit cell based approach is that it cannot account for the effect of spatial irregularity in material distribution and structural geometry, which is practically inevitable. Random irregularities in metamaterials can occur due to manufacturing uncertainty, variation in temperature, structural defects, pre-stressing and micro-structural variability in intrinsic material properties. Aim of the present article is to quantify the effect of such random irregularities (material and structural) in a comprehensive probabilistic framework.

To quantify the effect of stochasticity in material and structural properties of metamaterials, a two dimensional hexagonal lattice structural form is considered in this article. Two dimensional hexagonal metamaterials with tailorable elastic moduli have been widely investigated without considering any form of irregularity [9–14]. Such hexagonal lattices of natural and artificial nature can be identified across different length-scales (nano to macro) in auxetic and non-auxetic forms. The basic mechanics of deformation for the lattices being scale-independent, the formulations developed in this context are normally applicable for wide range of materials and structural forms. Experimental and finite element

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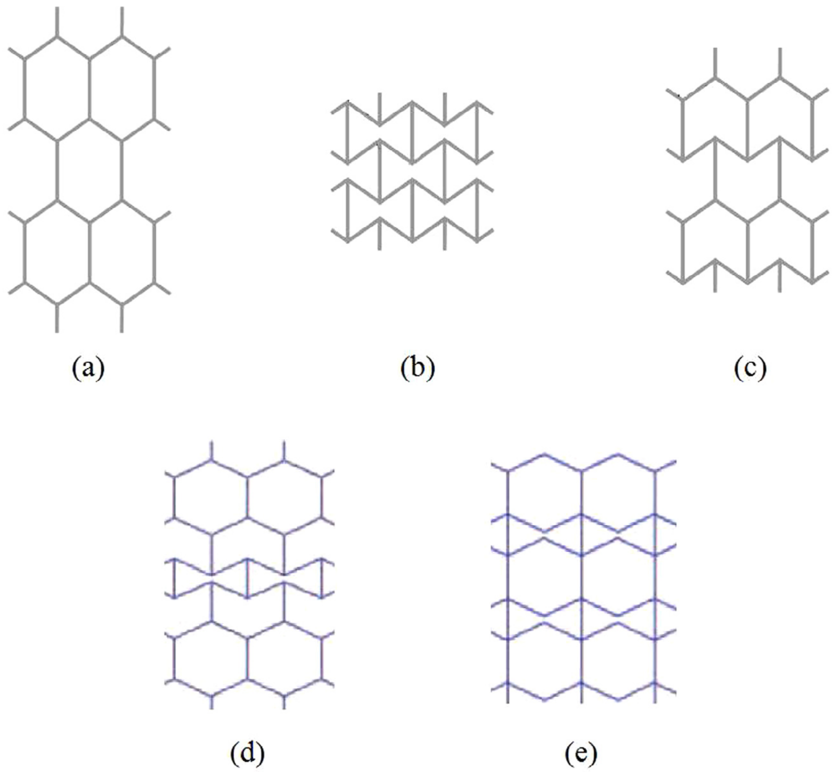


Fig. 1. (a) Hexagonal lattice with positive Poisson's ratio (b) hexagonal lattice with negative Poisson's ratio (auxetic) (c – e) hexagonal lattice with zero Poisson's ratio.

studies have been reported for impact, crushing, elastic moduli and other mechanical responses considering manufacturing irregularities and defects [15–20]. However all the above mentioned studies considering irregularities in hexagonal lattices are based on limited number of samples. Both experimental investigation and finite element analysis being very expensive and time consuming, its not feasible to quantify the effect of random irregularities by considering reasonably high number of samples. The problem aggravates in case of uncertainty quantification for the responses of irregular lattices, where large number of samples ($\sim 10^3$) are required for Monte Carlo simulation [21–24]. An analytical framework could be a simple, insightful, yet an efficient way to quantify the responses in a probabilistic paradigm. Recently analytical approaches have been presented [25–27] for elastic properties of irregular honeycombs considering only structural irregularities (random over expansion and under expansion of cells [17]). However, there exists a strong rationale to extend the analytical approaches to a probabilistic framework including the effect of random material property distribution and cell wall thickness to comprehensively analyse such structural forms.

In the present paper, an analytical formulation is developed to account for the effect of spatially random variation of material properties along with structural irregularities for hexagonal lattices. Results are presented for randomly homogeneous and randomly inhomogeneous stochasticity in the system originated from random material property distribution and structural geometry. As mechanics based closed-form formulae are developed in this article, the stochastic responses of hexagonal metamaterials can be quantified in a computationally efficient, yet physically insightful manner. Similar approaches can be adopted to quantify the effect of stochasticity for other metamaterials in future. This article is organized hereafter as follows: the stochastic mechanics for in-plane elastic moduli of hexagonal metamaterials is derived in Section 2; results are presented in Section 3 following a probabilistic framework for stochasticity in material and geometric parameters; and finally, Section 4 provides concluding remarks and outlook of the present article.

2. In-plane elastic properties of irregular lattices

The underlying concept to obtain the equivalent in-plane elastic moduli of the entire stochastic metamaterial/lattice structure is that the irregular quasi-periodic lattice is assumed to be consisted of several constituent representative unit cell elements (RUCEs) having different individual elastic properties depending on their respective structural geometry and material property. Each of the RUCEs have a common pattern in their structural configuration, but they are randomly varied along a two dimensional space (refer to Fig. 2(a)). In the proposed bottom-up approach, the irregularity in structural geometry and random material property distribution are accounted implicitly by means of the RUCEs. The expressions for the strain components for a generalized RUCE in different directions are derived first, and thereby the effective in-plane elastic moduli of the entire stochastic lattice are derived based on force equilibrium and deformation compatibility conditions. Only bending deformation has been accounted in the present analysis as the effect due to axial and shear deformation becomes negligible for very high axial rigidity and small value of the cell wall thickness compared to the other dimensions. The proposed formulae for in-plane elastic moduli of stochastic hexagonal lattices are applicable for both tensile as well as compressive stresses. In this context it can be noted that effectively three directions of applied stresses are needed to be considered for analysing five in-plane elastic moduli as depicted in Fig. 2(b–d).

On the basis of standard principles of structural mechanics, the total strain in direction-1 (ϵ_{11}) and direction-2 (ϵ_{21}) due to application of stress in direction-1 (refer to Fig. 2(b)) for a RUCE can be expressed as

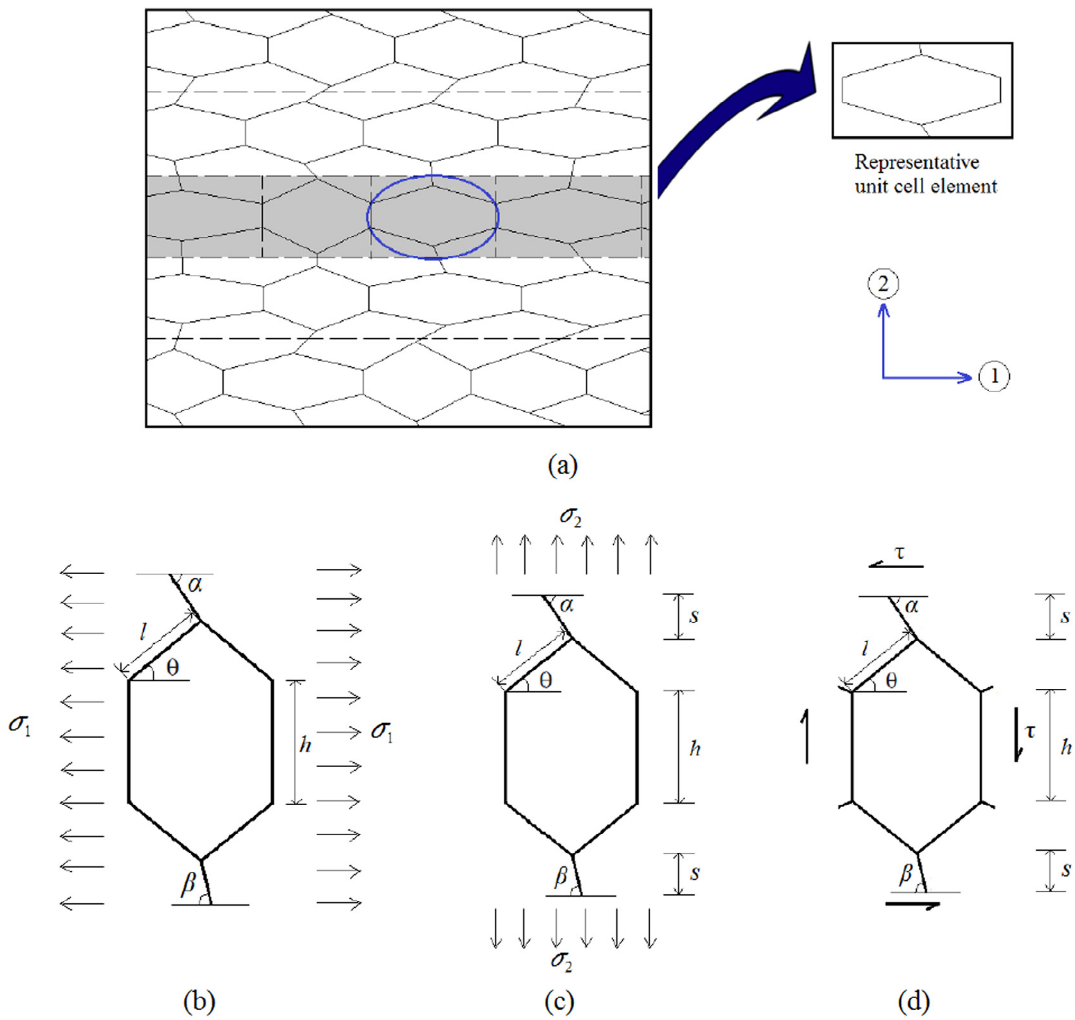


Fig. 2. (a) Bottom-up approach for analysing irregular lattices (b) considered RUCE for the analysis of E_1 and ν_{12} (c) considered RUCE for the analysis of E_2 and ν_{21} (d) considered RUCE for the analysis of G_{12}.

$$\epsilon_{11}(\bar{\omega}) = \frac{\sigma_1(\bar{\omega})\left(\frac{h(\bar{\omega})}{l(\bar{\omega})} + \sin\theta(\bar{\omega})\right)\sin^2\theta(\bar{\omega})}{E_s(\bar{\omega})\left(\frac{t(\bar{\omega})}{l(\bar{\omega})}\right)^3 \cos\theta(\bar{\omega})} \quad (1)$$

$$\epsilon_{21}(\bar{\omega}) = \frac{2\sigma_1(\bar{\omega})\sin\theta(\bar{\omega})\cos\theta(\bar{\omega})\left(\frac{h(\bar{\omega})}{l(\bar{\omega})} + \sin\theta(\bar{\omega})\right)}{E_s(\bar{\omega})\left(\frac{t(\bar{\omega})}{l(\bar{\omega})}\right)^3\left(\frac{h(\bar{\omega})}{l(\bar{\omega})} + 2\frac{s(\bar{\omega})}{l(\bar{\omega})} + 2\sin\theta(\bar{\omega})\right)} \quad (2)$$

where $\bar{\omega}$ is used to represent the stochastic character of the parameters. The total strain in direction-1 (ϵ_{12}) and direction-2 (ϵ_{22}) due to application of stress in direction-2 (refer to Fig. 2(c)) for a RUCE can be expressed as

$$\epsilon_{12}(\bar{\omega}) = -\frac{\sigma_2(\bar{\omega})\sin\theta(\bar{\omega})\cos\theta(\bar{\omega})}{E_s(\bar{\omega})\left(\frac{t(\bar{\omega})}{l(\bar{\omega})}\right)^3} \quad (3)$$

The total shear strain in the direction parallel to direction-1 (γ_{12}) due to application of shear stress in the same direction (refer to Fig. 2(d)) for a RUCE can be expressed as

$$\gamma_{12}(\bar{\omega}) = \frac{2\tau(\bar{\omega})\cos\theta(\bar{\omega})\left(2\left(\frac{s(\bar{\omega})}{l(\bar{\omega})}\right)^2 + 4\left(\frac{s(\bar{\omega})}{l(\bar{\omega})}\right)^3\left(\frac{1}{\sin\alpha(\bar{\omega})} + \frac{1}{\sin\beta(\bar{\omega})}\right) + \left(\frac{h(\bar{\omega})}{l(\bar{\omega})}\right)^3 + \frac{1}{2}\left(\frac{h(\bar{\omega})}{l(\bar{\omega})}\right)^2\right)}{E_s(\bar{\omega})\left(\frac{t(\bar{\omega})}{l(\bar{\omega})}\right)^3\left(\frac{h(\bar{\omega})}{l(\bar{\omega})} + 2\frac{s(\bar{\omega})}{l(\bar{\omega})} + 2\sin\theta(\bar{\omega})\right)} \quad (5)$$

The detail derivation of the above expressions for strain components are provided as Supplementary Material with the paper.

On the basis of the expressions for strain components for a RUCE (Eqs. (1)–(5)), the final expressions for five in-plane elastic moduli of an entire irregular hexagonal lattice are obtained following a bottom-up approach as described in Fig. 3. The in-plane

$$\epsilon_{22}(\bar{\omega}) = \frac{\sigma_2(\bar{\omega})\cos\theta(\bar{\omega})\left(2\cos^2\theta(\bar{\omega}) + 8\left(\frac{s(\bar{\omega})}{l(\bar{\omega})}\right)^3\left(\frac{\cos^2\alpha(\bar{\omega})}{\sin^3\alpha(\bar{\omega})} + \frac{\cos^2\beta(\bar{\omega})}{\sin^3\beta(\bar{\omega})}\right) + 2\left(\frac{s(\bar{\omega})}{l(\bar{\omega})}\right)^2\left(cot^2\alpha(\bar{\omega}) + cot^2\beta(\bar{\omega})\right)\right)}{E_s(\bar{\omega})\left(\frac{t(\bar{\omega})}{l(\bar{\omega})}\right)^3\left(\frac{h(\bar{\omega})}{l(\bar{\omega})} + 2\frac{s(\bar{\omega})}{l(\bar{\omega})} + 2\sin\theta(\bar{\omega})\right)} \quad (4)$$

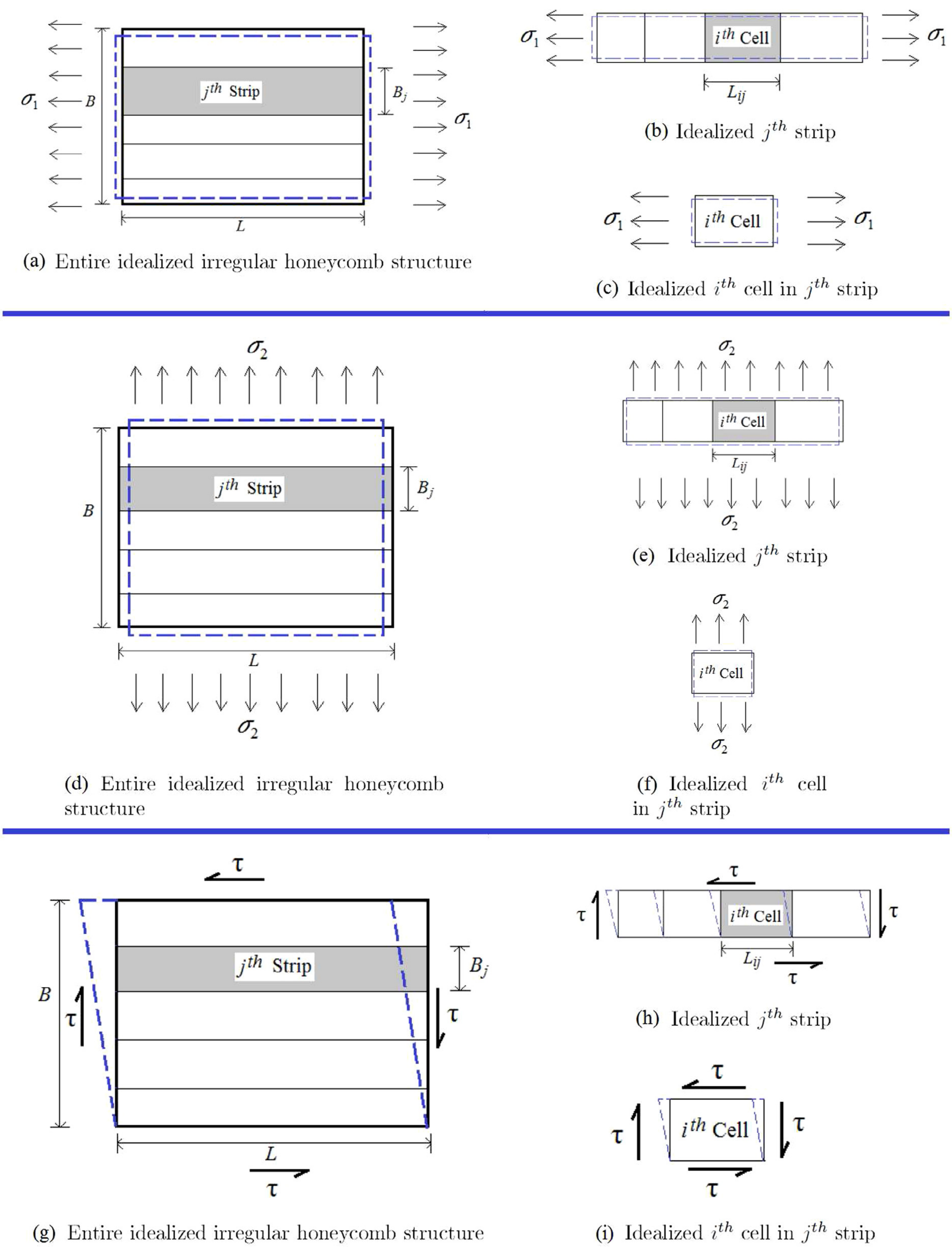


Fig. 3. (a – c) Bottom-up analysis for E_1 and ν_{12} of the entire irregular lattice (d – f) bottom-up analysis for E_2 and ν_{21} of the entire irregular lattice (g – i) bottom-up analysis for G_{12} of the entire irregular lattice.

elastic moduli of a RUCE (Z_{Uij}, where Z represents a particular elastic modulus) can be obtained from the expressions of strain components as: $E_{1Uij} = \frac{\sigma_1}{\epsilon_{11}}$; $E_{2Uij} = \frac{\sigma_2}{\epsilon_{22}}$; $\nu_{12ij} = -\frac{\epsilon_{21}}{\epsilon_{11}}$; $\nu_{21ij} = -\frac{\epsilon_{12}}{\epsilon_{22}}$ and $G_{12ij} = \frac{\tau}{\gamma_{12}}$. Here the entire irregular lattice is assumed to be consisted of m and n number of RUCEs in direction-1 and direction-2, respectively. A particular RUCE having position at ith column and jth row is denoted as (i,j), where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In the proposed bottom-up multi-step approach, expression of each of the elastic moduli are obtained at the strip level first and thereby at the global level ($Z_{eq}(\bar{\omega})$) of the entire irregular lattice as shown in Fig. 3. The expressions for in-plane elastic moduli of the entire irregular lattice accounting random variation of cell angle and intrinsic material properties can be obtained as:

$$E_{1eq}(\bar{\omega}) = \frac{L(\bar{\omega})}{B(\bar{\omega})}\sum_{j=1}^{n}\frac{B(\bar{\omega})_j}{\sum_{i=1}^{m}\frac{2l(\bar{\omega})_{ij}^3\left(h(\bar{\omega})_{ij}+l(\bar{\omega})_{ij}\sin\theta(\bar{\omega})_{ij}\right)\sin^2\theta(\bar{\omega})_{ij}}{E_s(\bar{\omega})_{ij}t(\bar{\omega})_{ij}^3}} \tag{6}$$

$$E_{2eq}(\bar{\omega}) = \frac{B(\bar{\omega})}{L(\bar{\omega})}\left(\sum_{j=1}^{n}\frac{B(\bar{\omega})_j}{\sum_{i=1}^{m}\frac{E_s(\bar{\omega})_{ij}t(\bar{\omega})_{ij}^3\left(h(\bar{\omega})_{ij}+2s(\bar{\omega})_{ij}+2l(\bar{\omega})_{ij}\sin\theta(\bar{\omega})_{ij}\right)}{l(\bar{\omega})_{ij}^3\cos^2\theta(\bar{\omega})_{ij}+4s(\bar{\omega})_{ij}^3\left(\frac{\cos^2\alpha(\bar{\omega})_{ij}}{\sin^3\alpha(\bar{\omega})_{ij}}+\frac{\cos^2\beta(\bar{\omega})_{ij}}{\sin^3\beta(\bar{\omega})_{ij}}\right)+s(\bar{\omega})_{ij}^2 l(\bar{\omega})_{ij}(cot^2\alpha(\bar{\omega})_{ij}+cot^2\beta(\bar{\omega})_{ij})}}\right)^{-1} \tag{7}$$

$$\nu_{12eq}(\bar{\omega}) = \frac{L(\bar{\omega})}{B(\bar{\omega})}\sum_{j=1}^{n}\frac{B(\bar{\omega})_j}{\sum_{i=1}^{m}\frac{\left(h(\bar{\omega})_{ij}+2s(\bar{\omega})_{ij}+2l(\bar{\omega})_{ij}\sin\theta(\bar{\omega})_{ij}\right)\sin\theta(\bar{\omega})_{ij}}{\cos\theta(\bar{\omega})_{ij}}} \tag{8}$$

$$\nu_{21eq}(\bar{\omega}) = \frac{B(\bar{\omega})}{L(\bar{\omega})}\times\left(\sum_{j=1}^{n}\frac{B(\bar{\omega})_j}{\sum_{i=1}^{m}\frac{\sin\theta(\bar{\omega})_{ij}\cos\theta(\bar{\omega})_{ij}l(\bar{\omega})_{ij}^3\left(h(\bar{\omega})_{ij}+2s(\bar{\omega})_{ij}+2l(\bar{\omega})_{ij}\sin\theta(\bar{\omega})_{ij}\right)}{l(\bar{\omega})_{ij}^3\cos^2\theta(\bar{\omega})_{ij}+4s(\bar{\omega})_{ij}^3\left(\frac{\cos^2\alpha(\bar{\omega})_{ij}}{\sin^3\alpha(\bar{\omega})_{ij}}+\frac{\cos^2\beta(\bar{\omega})_{ij}}{\sin^3\beta(\bar{\omega})_{ij}}\right)+s(\bar{\omega})_{ij}^2 l(\bar{\omega})_{ij}(cot^2\alpha(\bar{\omega})_{ij}+cot^2\beta(\bar{\omega})_{ij})}}\right)^{-1} \tag{9}$$

$$G_{12eq}(\bar{\omega}) = \frac{B(\bar{\omega})}{L(\bar{\omega})}\times\left(\sum_{j=1}^{n}\frac{B(\bar{\omega})_j}{\sum_{i=1}^{m}\frac{E_s(\bar{\omega})_{ij}t(\bar{\omega})_{ij}^3\left(h(\bar{\omega})_{ij}+2s(\bar{\omega})_{ij}+2l(\bar{\omega})_{ij}\sin\theta(\bar{\omega})_{ij}\right)}{\left(2l(\bar{\omega})_{ij}s(\bar{\omega})_{ij}^2+h(\bar{\omega})_{ij}^2\left(h(\bar{\omega})_{ij}+\frac{l(\bar{\omega})_{ij}}{2}\right)+4s(\bar{\omega})_{ij}^3\left(\frac{1}{sin\alpha(\bar{\omega})_{ij}}+\frac{1}{sin\beta(\bar{\omega})_{ij}}\right)\right)}}\right)^{-1} \tag{10}$$

The stochastic structural dimensions for the RUCEs and the entire irregular lattice, as used in the above expressions, are shown in Figs. 2 and 3. Detail derivation of Eqs. (6)–(10) are provided as Supplementary Material due to paucity of space. It is worthy to note here that the derived closed-form expressions of Young's moduli for the entire irregular lattice Eqs. (6)–(10) reduces to the standard formulae provided by Gibson and Ashby [9] in case of regular honeycombs (i.e. $B(\bar{\omega})_1 = B(\bar{\omega})_2 = \cdots = B(\bar{\omega})_n$; $s(\bar{\omega})_{ij} = \frac{h(\bar{\omega})_{ij}}{2}$; $\alpha(\bar{\omega})_{ij} = \beta(\bar{\omega})_{ij} = 90^\circ$; $l(\bar{\omega})_{ij} = l$ and $\theta(\bar{\omega})_{ij} = \theta$, for all i and j). The above expressions can obtain the equivalent elastic moduli of irregular lattices accounting the spatial variability of structural geometry due to random cell angles (including cell wall thickness (t)) and intrinsic material property (E_s). Thus considering appropriate probabilistic distribution for the spatially random attributes at micro-structural level, the effect of stochasticity in the equivalent material properties at macro-scale can be quantified following a computationally efficient manner.

3. Results and discussion

The developed analytical closed-form expressions for in-plane elastic moduli of irregular hexagonal lattices are validated with a finite element code, while the finite element code is validated for regular hexagonal lattices using results from avaliable literature (regular hexagonal honeycomb) [9]. Comparative results for the five in-plane elastic moduli are provided for multiple random realizations considering spatial variation of structural attributes in Table 1, wherein a good agreement between the elastic moduli obtained using the analytical formulae and finite element method is found. The probabilistic descriptions for the elastic moduli are furnished for different degree of random variation in cell angles ($\Delta\theta = 0^\circ, 1^\circ, 3^\circ, 5^\circ, 7^\circ$) later in this section. For a particular cell angle θ (the cell angle is shown in Fig. 4), the results have been obtained using a set of uniformly distributed 2500 random samples in the range of $[\theta - \Delta\theta, \theta + \Delta\theta]$. Thus the set of input parameters for a particular sample is consisted of N number of cell angles in the specified range ($N(= n \times m)$ is the total number of

Table 1
Results for validation of the proposed closed-form formulae (Eqs. (6)–(10)) with respect to the results obtained from finite element method (FEM) simulation considering multiple random realizations for different degree of irregularities ($\Delta\theta$). The results are presented as a ratio of the elastic modulus for irregular lattice (corresponding to a lattice configuration having cell angle 30° and h/l ratio of 1.5) and that of regular lattice.

$\Delta\theta$	Approach followed	In-plane elastic properties				
		E_1	E_2	ν_{12}	ν_{21}	G_{12}
0	Analytical	1.000	1.000	1.000	1.000	1.000
	FEM	1.002	1.001	1.015	1.003	1.006
1	Analytical	0.990	0.950	0.996	0.953	0.955
	FEM	0.991	0.956	0.989	0.951	0.960
3	Analytical	0.989	0.895	0.991	0.933	0.941
	FEM	0.990	0.906	0.997	0.937	0.938
5	Analytical	0.987	0.789	0.989	0.877	0.921
	FEM	0.981	0.796	0.980	0.867	0.919
7	Analytical	0.979	0.665	0.986	0.779	0.901
	FEM	0.980	0.658	0.982	0.781	0.897

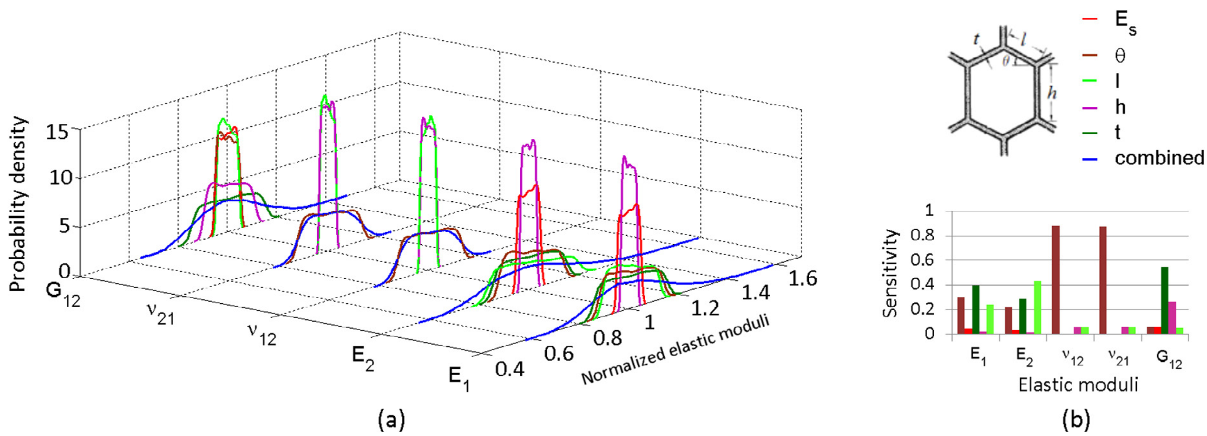


Fig. 4. (a) Probabilistic characterization of five in-plane elastic moduli considering individual and combined stochasticity in the input parameters (randomly homogeneous stochasticity) (b) sensitivity quantification for the variable input parameters.

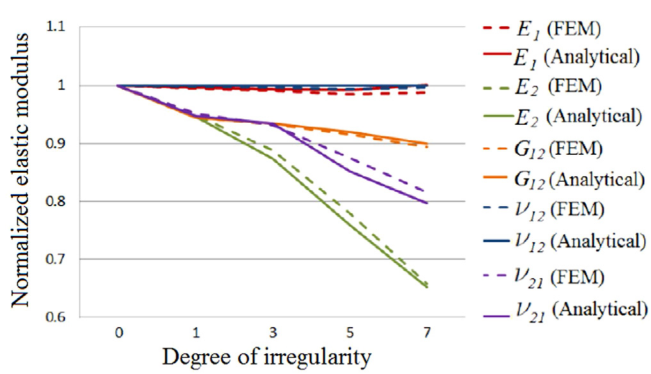


Fig. 5. Mean normalized elastic moduli with different degree of irregularities following the analytical approach and finite element simulation considering randomly inhomogeneous stochasticity (ratio of different elastic moduli for irregular lattices and regular lattices are plotted).

RUCEs in the entire irregular lattice). As the proposed analytical formulae for irregular lattices reduce to the standard formulae of Gibson and Ashby [9] in case of regular lattice configuration, the results in Table 1 corresponding to $\Delta\theta = 0$ represents validation of the finite element code with respect to Gibson and Ashby's [9] results.

The proposed analytical approach is capable of obtaining the equivalent in-plane elastic properties of irregular lattices from known spatial variation of cell angle, thickness of cell walls and intrinsic material property distribution. Probabilistic descriptions for the elastic moduli accounting the effect of structural and material randomness at micro-scale are presented using the efficient analytical formulae (Eqs. (6)–(10)) in this section. The results are furnished for two different classes of stochasticity: randomly homogeneous system and randomly inhomogeneous system. In randomly homogeneous system, no spatial variability is considered. It is assumed that structural and material attributes remain same spatially. However the stochastic parameters vary from sample to sample following a probabilistic distribution (a Monte Carlo simulation based random variable approach). In randomly inhomogeneous system, spatial variability of the stochastic structural attributes are accounted, wherein each sample of the Monte Carlo simulation includes the spatially random distribution of structural and materials attributes. The results for in-plane elastic moduli are presented in this article as a non-dimensional ratio of the elastic modulus for irregular lattice (corresponding to a lattice configuration having cell angle $\theta = 30^\circ$ and h/l ratio of 1.5) and that of regular lattice, unless otherwise mentioned.

Fig. 4(a) shows effect of randomly homogeneous form of stochasticity in the material and structural attributes considering individual and combined effects. The probabilistic descriptions

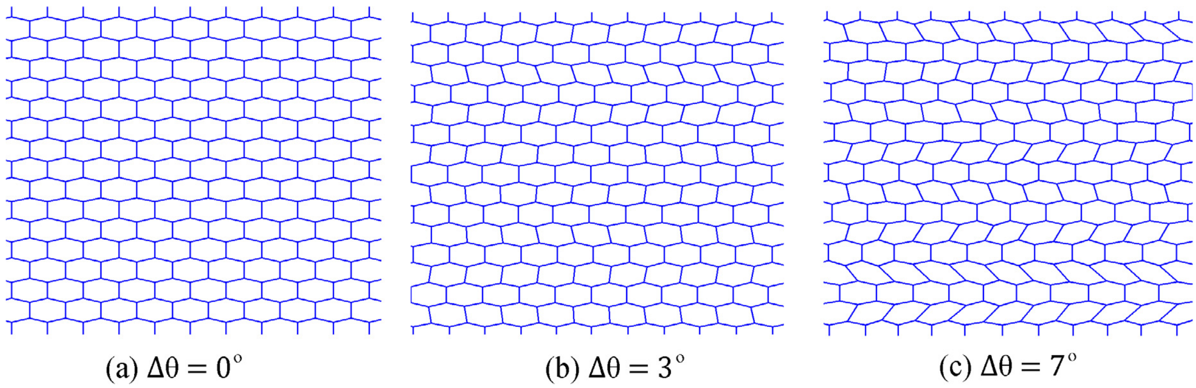


Fig. 6. Irregular undeformed lattice configurations with different degree of spatial irregularities.

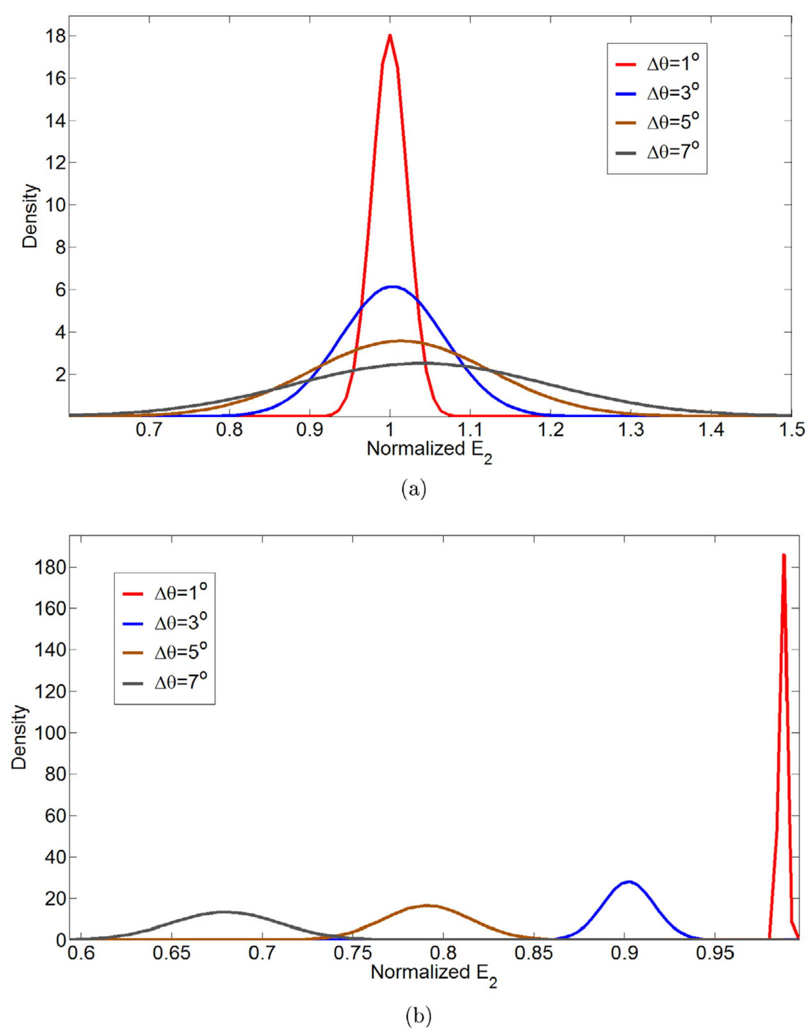
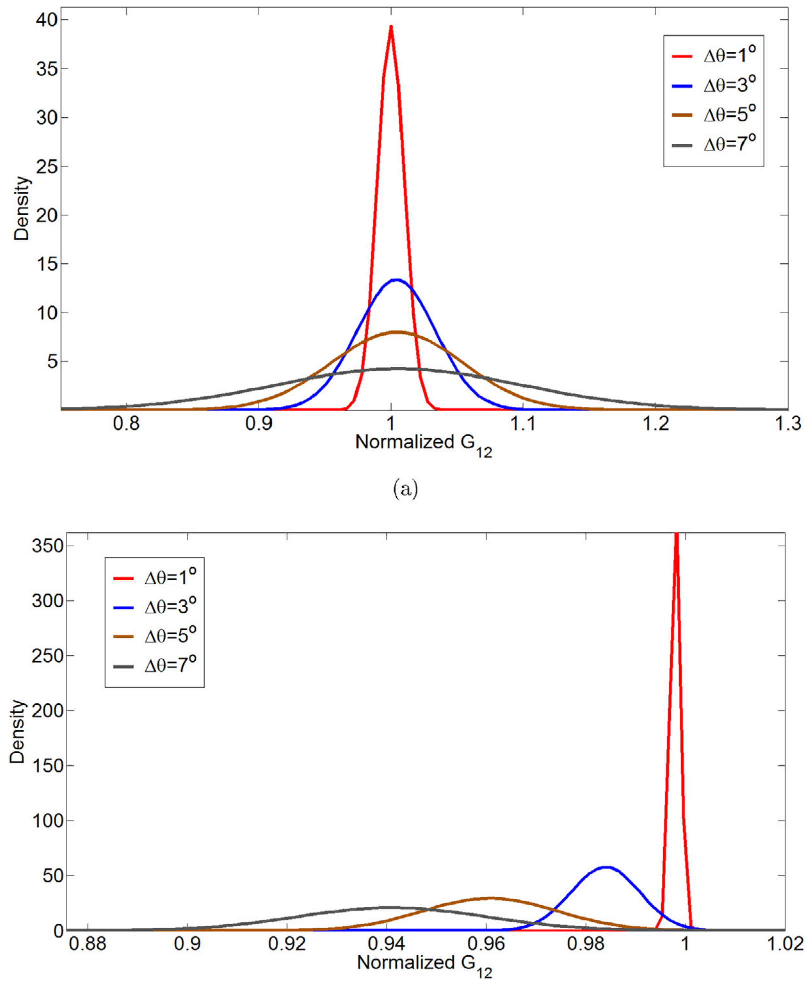


Fig. 7. (a) Probabilistic characterization of E_2 considering randomly homogeneous system (b) probabilistic characterization of E_2 considering randomly inhomogeneous system (ratio of the elastic modulus for irregular lattices and that of regular lattices are plotted).

are obtained using five percent random variation (following a random uniform distribution) of the stochastic input parameters with respect to their corresponding mean values. It is interesting to notice that even though similar distribution of the input parameters are adopted, the final probability distributions of the in-plane elastic moduli are quite different from each other. Considering the response bounds of different in-plane elastic moduli due to combined effect, it is observed that E_2 is most affected by randomly homogeneous stochasticity, followed by E_1, G_{12}, ν_{21} and ν_{12}. However, the variation in response bounds for all the in-plane elastic moduli are quite significant from design point of view. Fig. 4(b) shows the results of a variance based sensitivity analysis



(a)

(b)

Fig. 8. (a) Probabilistic characterization of G_{12} considering randomly homogeneous system (b) probabilistic characterization of G_{12} considering randomly inhomogeneous system (ratio of the elastic modulus for irregular lattices and that of regular lattices are plotted).

[28,29], wherein the relative importance of the stochastic input parameters are quantified for all the in-plane elastic moduli. Such analysis could be of utmost importance for analysing uncertain systems and selective control on the stochastic input parameters.

Previous investigations [17] have reported that under-expansion in honeycomb cells increases the elastic moduli, while over-expansion has the opposite effect. The present study deals with the effects of spatially random distribution of under and over expanded cells with random distribution of intrinsic material property following a probabilistic framework. Fig. 5 shows the stochastic mean values for five in-plane elastic moduli considering a randomly inhomogeneous system. Less deviation between the results obtained using the proposed analytical approach and finite element method (FEM) simulation corroborates validity of the derived closed-form formulae. From the figure it is observed that the mean values of E_2, G_{12} and ν_{21} are significantly reduced by randomly inhomogeneous form of stochasticity corresponding to different degree of irregularity, while E_1 and ν_{12} are found to be least affected. Fig. 6 shows typical cellular lattice configuration corresponding to three different degree of irregularities. As E_2, G_{12} and ν_{21} are significantly affected by structural irregularity in the lattice configuration, probabilistic description for these elastic moduli are presented in Figs. 7–9 considering both randomly homogeneous and randomly inhomogeneous form of stochasticity. A clear trend is observed from the figures that the mean values reduce considerably with increasing degree of irregularity for randomly inhomogeneous system, while the mean values for randomly homogeneous systems remain practically unaltered. The standard deviation is found to increase with increasing degree of irregularity for both the cases. However, the response bound for a particular degree of irregularity is more in case of randomly homogeneous systems compared to randomly inhomogeneous system. The coefficient of variations corresponding to increasing degree of irregularity are plotted in Fig. 10 for E_2, G_{12} and ν_{21}, wherein it is observed that ν_{21} is most sensitive to randomly inhomogeneous stochasticity, followed by E_2 and G_{12} (considering slope of the curves). The influence of spatially random variation of intrinsic material property (randomly inhomogeneous stochasticity) on the in-plane elastic moduli are found to be negligible compared to the spatially random variation of structural geometry. As a typical instance, effect of spatially random variation of intrinsic material property on E_2 is presented in Fig. 11 for different degree of stochasticity. Comparing the results presented in Fig. 4 for E_s (intrinsic material property) with the results of Fig. 11, it can be observed that the elastic moduli are affected more in case of randomly homogeneous stochasticity compared to randomly inhomogeneous stochasticity. However, it is worthy to mention here that the two Poisson ratios remain unaffected under the influence of both the forms of

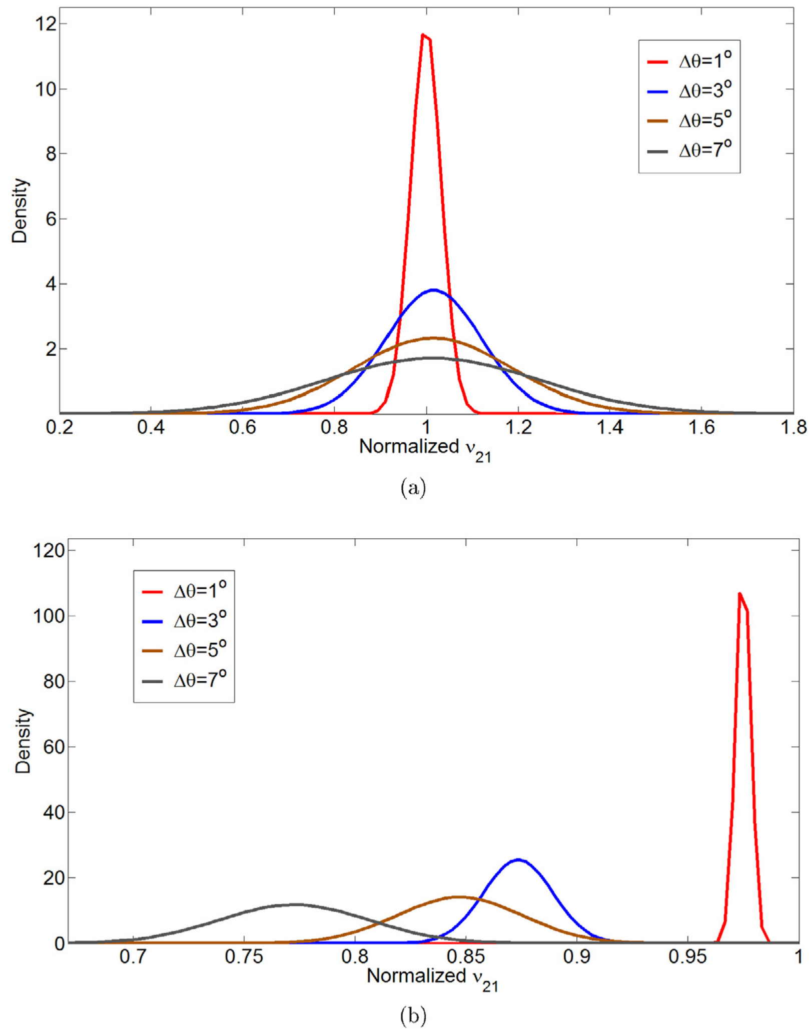


Fig. 9. (a) Probabilistic characterization of ν_{21} considering randomly homogeneous system (b) probabilistic characterization of ν_{21} considering randomly inhomogeneous system (ratio of the elastic modulus for irregular lattices and that of regular lattices are plotted).

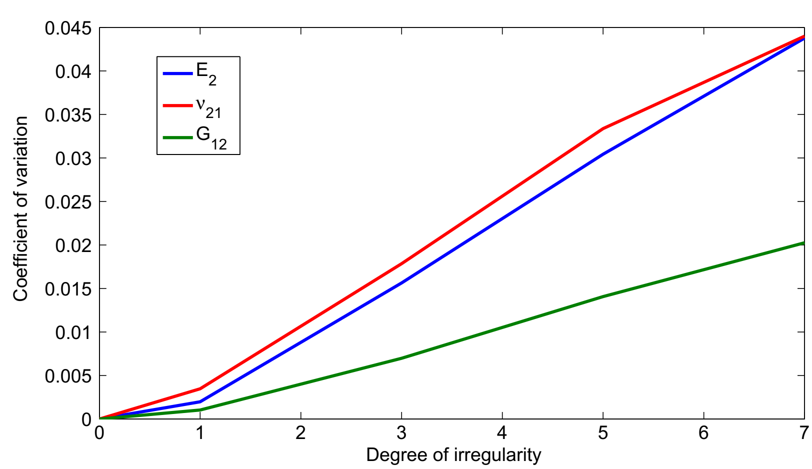


Fig. 10. Coefficient of variation for different degree of irregularity (considering randomly inhomogeneous system).

stochasticity in intrinsic material properties. This is because of the fact that the Poisson's ratios are not dependent on the intrinsic material properties of the lattice materials as evident from Eqs. (8) and (9).

Objective of this article is to present the effect of stochasticity and quantify the response bounds for the effective elastic moduli of the irregular lattices. To ensure that the spatially random input parameters for all the realizations of Monte Carlo simulation are

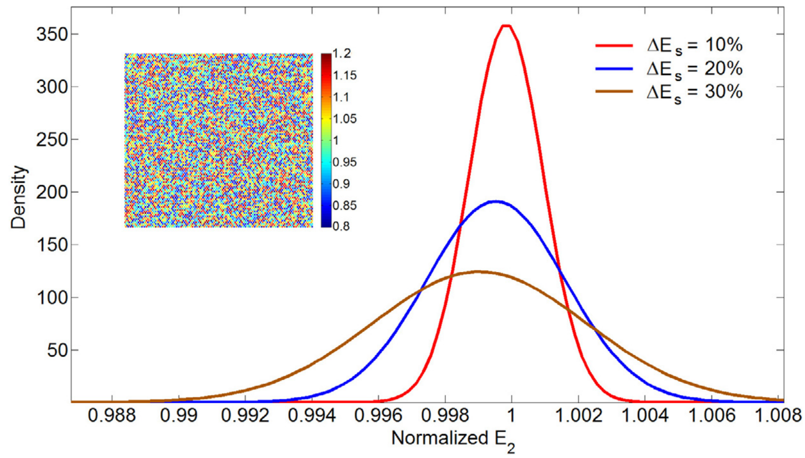


Fig. 11. Effect of spatial variation (randomly inhomogeneous system) of intrinsic material property (E_s) for E_2 (ratio of the elastic modulus for irregular lattices and that of regular lattices are plotted). Spatially random distribution of intrinsic elastic modulus (corresponding to $\Delta E_s = 20\%$) is shown for a typical random realization in the inset, wherein the ratio of random intrinsic elastic modulus and the deterministic value of intrinsic elastic modulus is plotted.

chosen with equal probability within their respective bounds, the random uniform distribution is adopted in this article. It can be noted in this context that the response bounds of the stochastic effective elastic moduli are not expected to vary significantly if other input distributions are considered. The overall probabilistic descriptions of the effective elastic moduli will depend on the adopted input parameter distribution. However, the proposed bottom up framework for analysing stochastic mechanics of metamaterials is applicable for other input parameter distributions, as the micro-mechanics of random lattices is not dependent on the input parameter distributions. It is interesting to notice that even though all the input distributions of the spatially random stochastic material and geometric parameters are considered to be randomly uniform, the distributions of the effective elastic moduli follow a Gaussian nature. This observation can be explained by central limit theorem [30–32] which states that: when the effect of independent random variables are combined, their sum tends toward a Gaussian distribution (commonly known as a bell curve) even if the original variables themselves are not normally distributed. The observations in case of randomly inhomogeneous system of the present article are in quite good agreement with this theorem of probability.

Fig. 12 shows the contour of nodes for deformed lattices with regular and irregular configurations. Typical location of nodes for a regular lattice subjected to stress in direction-2 with different levels is shown in Fig. 12(a), wherein it is evident that the movement of nodes for a particular stress level become higher as the distance from support increases due to a cumulative effect. The deformation of the nodes conforms a non-auxetic effect as the cell angle is considered to have a positive value in this study. The movement of different nodes are found to increase in direction-2 for higher level of stresses, as expected. Fig. 12(b) presents simulation bound of movements for different nodes in randomly irregular lattice (randomly inhomogeneous stochasticity) considering 2500 random realizations, while the movement of nodes for two individual random realizations with spatially varying structural configurations are shown in Fig. 12(c–d). It is interesting to notice that movement of the nodes in direction-2 increases significantly for spatially random structural geometries compared to the regular configuration implying a reduction in E_2 due to the effect of structural irregularity. The results presented in the preceding portion of this section (refer to Figs. 5 and 7) are in good agreement with this observation.

An ergodic behaviour [33] of spatially irregular hexagonal lattices (randomly inhomogeneous stochasticity) is identified on the basis of the computationally efficient analytical formulae for in-plane elastic moduli. For large number of RUCEs with random structural configuration, the ensemble statistics become close to the statistics of a single realization. As the number of RUCE increases for a particular degree of irregularity, the spread of the values for in-plane elastic moduli are found to reduce significantly, while the mean does not change considerably. A typical result is presented in Fig. 13(a) for $\Delta\theta = 5^\circ$ showing the spread of values for E_2 considering different number of RUCEs. The coefficient of variation for E_2 with increasing number of spatially random RUCEs is presented in Fig. 13(b), wherein it is evident that the coefficient of variation converges to a value $\sim 10^{-3}$ for large number of RUCEs. It can be noted here that the statistics presented in Fig. 13(b) corresponding to each of the number of RUCEs is based on 2500 realizations; thus the mean of all the realizations converges to a constant value for large number of RUCEs (as the standard deviation becomes very less). However, the converged mean value for large number of RUCEs are significantly lesser in case of irregular lattices compared to that of the regular lattice. The coefficient of variation for ν_{21} and G_{12} show similar trend for large number of RUCEs conforming the ergodicity of random lattices as shown in Fig. 14. Emergence of such ergodic behaviour in stochastic metamaterials can have considerable impact in designing such lattices accounting the effect of uncertainty in global properties.

4. Conclusion

Probabilistic descriptions of the five in-plane elastic moduli for stochastic hexagonal two-dimensional metamaterials are characterized following a computationally efficient analytical framework. Randomly homogeneous and randomly inhomogeneous form of stochasticity are considered in the intrinsic material property distribution and structural irregularity, which are often encountered due to manufacturing and operational uncertainties.The proposed analytical expressions are capable of obtaining the equivalent elastic moduli of irregular lattices accounting the spatial variability of structural geometry (including cell wall thickness (t)) and intrinsic material property (E_s) for randomly over and under expanded cells. From the closed-form analytical formulae, it can be noticed that longitudinal Young's modulus, transverse Young's modulus and

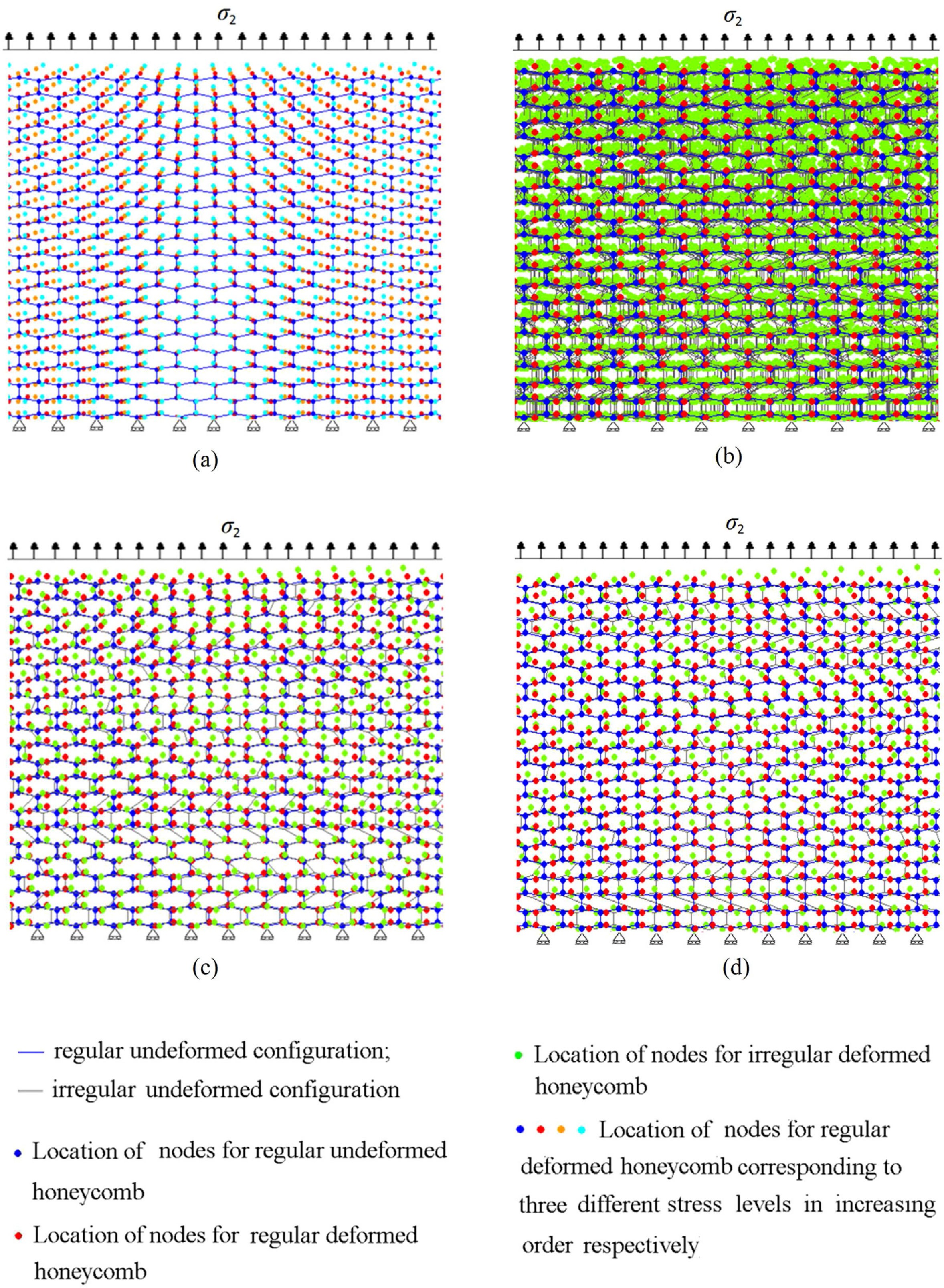


Fig. 12. (a) Typical contour of nodes for a regular hexagonal lattice subjected to stresses with different levels in direction-2 (b) simulation bound of movements for different nodes in randomly irregular lattice considering 2500 random realizations (c–d) movement of nodes for two random realizations with spatially varying structural configurations.

shear modulus are dependent on both structural geometry and material properties of the irregular lattice, while the Poisson's ratios depend only on structural geometry.

An important observation of this study is that, even though the effect of random variations in cell angle on E_1 and ν_{12} is negligible, E_2, ν_{21} and G_{12} reduce significantly with increasing degree of

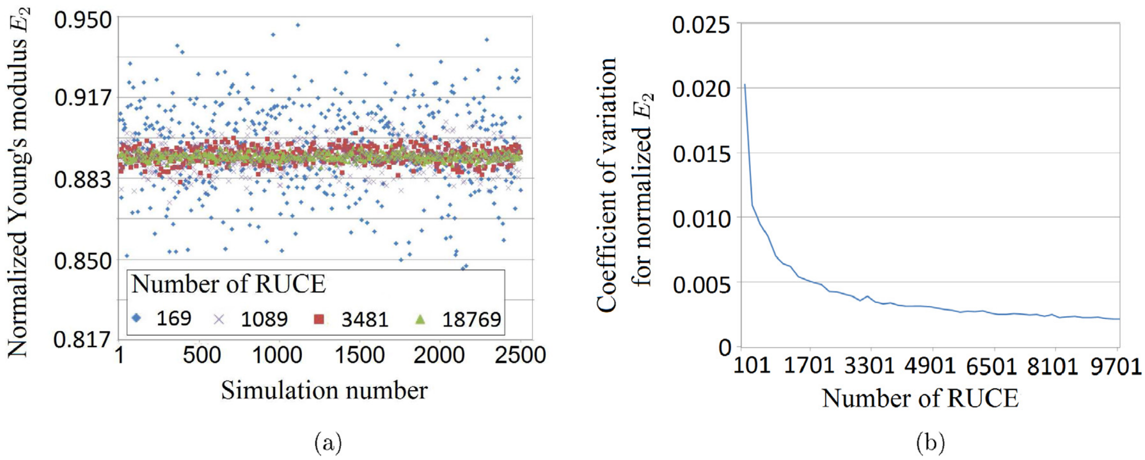


Fig. 13. (a) Spread of E_2 for different number of RUCEs (b) coefficient of variation for E_2 with increasing number of spatially random RUCEs.

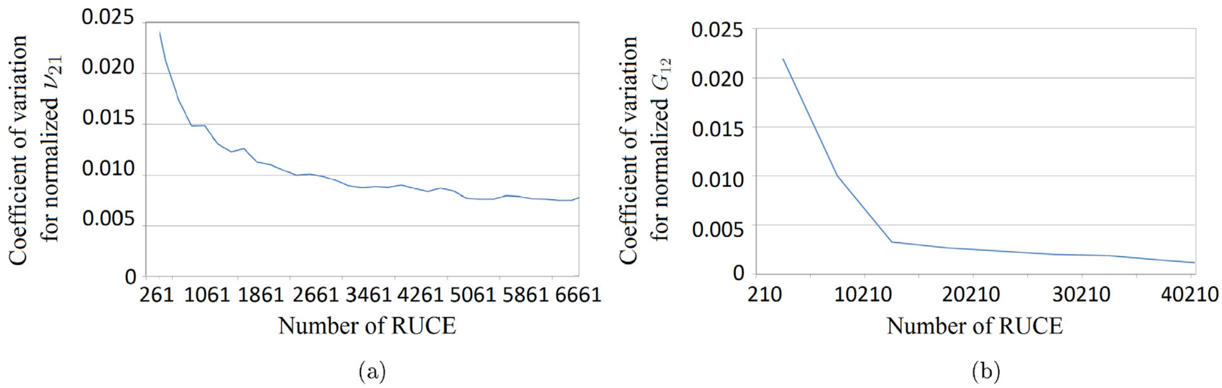


Fig. 14. (a) Coefficient of variation for ν_{21} with increasing number of spatially random RUCEs (b) coefficient of variation for G_{12} with increasing number of spatially random RUCEs.

irregularity. The ensemble mean values reduce considerably with increasing degree of irregularity for randomly inhomogeneous system, while the mean values for randomly homogeneous systems remain practically unaltered. The standard deviation is found to increase with increasing degree of irregularity for both the cases. However, the response bound for a particular degree of irregularity is more in case of randomly homogeneous systems compared to randomly inhomogeneous system. An ergodic behaviour is identified in spatially irregular lattices for randomly inhomogeneous stochasticity. The uncertainty in elastic moduli of hexagonal lattice metamaterials owing to random variations in cell angle and intrinsic material properties would have considerable effect on the subsequent analysis, design and control process. The proposed conceptual physics based analytical framework to efficiently characterize random variabilities in two dimensional metamaterials provides a comprehensive insight for prospective future investigations to quantify the effect of stochasticity on other metamaterials.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.compstruct.2016.11.080.

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