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Asymptotic distribution method for structural reliability analysis in high dimensions

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In the reliability analysis of safety critical complex engineering structures, a very large number of the system parameters can be considered as random variables. The difficulty in computing the failure probability using the classical first- and second-order reliability methods (FORM and SORM) increases rapidly with the number of variables or 'dimension'. There are mainly two reasons behind this. The first is the increase in computational time with the increase in the number of random variables. In principle, this problem can be handled with superior computational tools. The second reason, which is perhaps more fundamental, is that there are some conceptual difficulties typically associated with high dimensions. This means that even when one manages to carry out the necessary computations, the application of existing FORM and SORM may still lead to incorrect results in high dimensions. This paper is aimed at addressing this issue. Based on the asymptotic distribution of quadratic form in Gaussian random variables, two formulations for the case when the number of random variables $n \to \infty$ is provided. The first is called 'strict asymptotic formulation' and the second is called 'weak asymptotic formulation'. Both approximations result in simple closed-form expressions for the probability of failure of an engineering structure. The proposed asymptotic approximations are compared with existing approximations and Monte Carlo simulations using numerical examples.

Keywords: reliability analysis; asymptotic methods; second-order reliability methods

1. Introduction

Uncertainties in specifying material properties, geometric parameters, boundary conditions and applied loadings are unavoidable in describing real-life engineering structural systems. Traditionally, this has been catered for in an *ad hoc* way through the use of safety factors at the design stage. Such an approach is unlikely to be satisfactory in today's competitive design environment, for example, in minimum weight design of aircraft structures. The situation may also arise when system safety is being jeopardized owing to the lack of detailed treatment of uncertainty at the design stage. For example, the finite probability of obtaining a resonance is unlikely to be captured by a safety factor-based approach given the intricate nonlinear relationships between the system parameters and the natural frequencies. For these reasons, a scientific

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and systematic approach is required to predict the probability of failure of a structure at the design stage. Accurate reliability assessment is also critical for optimal design of structures. Suppose the random variables describing the uncertainties in the structural properties and loading are considered to form a vector $\boldsymbol{x} \in \mathbb{R}^n$, where n is the number of random variables. The statistical properties of the system are fully described by the joint probability density function $p(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R}$. For a given set of variables $\boldsymbol{x}$, the structure will either fail under the applied (random) loading or it will be safe. The condition of the structure for every $\boldsymbol{x}$ can be described by a safety margin $g(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ such that the structure has failed if $g(\boldsymbol{x}) \leq 0$ and is safe if $g(\boldsymbol{x}) > 0$. Thus, the probability of failure is given by

$$P_{\mathrm{f}} = \int_{g(\boldsymbol{x}) \leq 0} p(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{1.1}$$

The function $g(\boldsymbol{x})$ is also known as the 'limit-state function' and the $(n-1)$-dimensional surface $g(\boldsymbol{x}) = 0$ is known as the 'failure surface'. The central theme of a reliability analysis is to evaluate the multidimensional integral (1.1). The exact evaluation of this integral, either analytically or numerically, is not possible for most practical problems because n is usually large and $g(\boldsymbol{x})$ is a highly nonlinear function of $\boldsymbol{x}$, which may not be available explicitly. Over the past three decades, there has been extensive research (e.g. Thoft-Christensen & Baker 1982; Madsen *et al.* 1986; Ditlevsen & Madsen 1996; Melchers 1999) to develop approximate numerical methods for the efficient calculation of the reliability integral. The approximate reliability methods can be broadly grouped into (i) first-order reliability method (FORM) and (ii) second-order reliability method (SORM). In FORM and SORM, it is assumed that all the basic random variables are transformed and scaled so that they are uncorrelated Gaussian random variables, each with zero mean and unit standard deviation.

The difficulty in computing the failure probability using the classical FORM and SORM increases rapidly with the number of variables or 'dimension'. There are mainly two reasons behind this. The first is the increase in computational time with the increase in the number of random variables. In principle, this problem can be handled with superior computational tools and powerful computing machines. The second, which is perhaps more fundamental, is that there are some conceptual difficulties associated typically with high dimensions. In the context of FORM, using the Tchebysheff bound, Veneziano (1979) has shown that the probability of failure depends on the dimension n, although the reliability index does not explicitly depend on n. In the context of SORM, using the χ^2-distribution, Fiessler *et al.* (1979) have shown that there can be significant difference of probability of failure in higher dimension for a fixed value of the reliability index. This means that even when one manages to carry out the necessary computations, the application of existing FORM and SORM may still lead to incorrect results in high dimensions. This paper is aimed at investigating this fundamental issue.

A new approach based on the asymptotic distribution of quadratic form of random variables is proposed in this paper. Two closed-form, asymptotically equivalent, approximate expressions of the integral in equation (1.1) are derived for the case when the number of random variables $n \rightarrow \infty$. It is assumed that the basic random variables are Gaussian or can be transformed to Gaussian, for example, using Rosenblatt transformation (Rosenblatt 1952). The first approximation is called 'strict asymptotic formulation' as it requires some asymptotic conditions to be satisfied strictly. The second approximation, called 'weak asymptotic formulation', relaxes some of the strict asymptotic requirements of the first approach. The proposed asymptotic approximations are compared with existing approximations and Monte Carlo simulations using numerical examples.

2. Review of classical FORM and SORM

For standard Gaussian basic variables $\boldsymbol{x} \sim \mathbb{N}_n(\boldsymbol{0}, \boldsymbol{I}_n)$, the joint probability density function is given by

$$p(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R} = (2\pi)^{-n/2} \mathrm{e}^{-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}/2}. \tag{2.1}$$

Substituting this into equation (1.1), we have

$$P_{\mathrm{f}} = (2\pi)^{-n/2} \int_{g(\boldsymbol{x}) \leq 0} \mathrm{e}^{-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}/2} \mathrm{d}\boldsymbol{x}. \tag{2.2}$$

The maximum contribution to the above integral comes from the neighbourhood where $(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}/2)$ is minimum subject to $g(\boldsymbol{x}) \leq 0$. This is the central concept behind the approximate analysis FORM and SORM. This 'minimum point', say $\boldsymbol{x}^*$, also known as the design point in structural reliability literature, is obtained from the following constrained optimization problem:

$$\boldsymbol{x}^* : \min\{(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})/2\} \quad \text{subject to} \quad g(\boldsymbol{x}) = 0. \tag{2.3}$$

We construct the Lagrangian $\mathcal{L}(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}/2 + \lambda g(\boldsymbol{x})$ and to obtain $\boldsymbol{x}^*$ explicitly $\partial\mathcal{L}(\boldsymbol{x})/\partial x_k = 0$ for $k=1,2, \ldots, n$. Substituting $\mathcal{L}(\boldsymbol{x})$, one obtains $\boldsymbol{x}^* = -\lambda \nabla g(\boldsymbol{x}^*)$. Taking transpose and multiplying, we have $(\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{x}^*) = \lambda^2(\nabla g^{\mathrm{T}}(\boldsymbol{x}^*)\nabla g(\boldsymbol{x}^*))$. From this, the Lagrange multiplier λ can be obtained as $\lambda = |\boldsymbol{x}^*|/|\nabla g|$. Hasofer & Lind (1974) defined the reliability index $\beta = |\boldsymbol{x}^*|$, which is the minimum distance of the failure surface from the origin in $\mathbb{R}^n$ (see figure 1 for a graphical illustration for the two-dimensional case). Using the value of λ, it is clear that, at the design point, the gradient vector to the failure surface and the vector from the origin are parallel, that is

$$\frac{\boldsymbol{x}^*}{\beta} = -\frac{\nabla g}{|\nabla g|} = \boldsymbol{\alpha}^*. \tag{2.4}$$

Assuming that $g(\boldsymbol{x})$ is continuous, smooth and at least twice differentiable, in SORM, the actual function is replaced by its second-order Taylor series expansion about the design point

$$g(\boldsymbol{x}) \approx g(\boldsymbol{x}^*) + \nabla g(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^*) + \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^{\mathrm{T}} \boldsymbol{H}_g(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) = \tilde{g}(\boldsymbol{x}). \tag{2.5}$$

Dividing equation (2.5) by $|\nabla g|$ and noticing that $g(\boldsymbol{x}^*) = 0$ because $\boldsymbol{x}^*$ is on the failure surface, the second-order approximation of $g(\boldsymbol{x})$ can be expressed as

$$\frac{\tilde{g}(\boldsymbol{x})}{|\nabla g|} = \left(\beta + \frac{\beta^2}{2}\boldsymbol{\alpha}^{*\mathrm{T}}\frac{\boldsymbol{H}_g}{|\nabla g|}\boldsymbol{\alpha}^*\right) - \left(\boldsymbol{\alpha}^{*\mathrm{T}} + \beta\boldsymbol{\alpha}^{*\mathrm{T}}\frac{\boldsymbol{H}_g}{|\nabla g|}\right)\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\frac{\boldsymbol{H}_g}{|\nabla g|}\boldsymbol{x}. \tag{2.6}$$

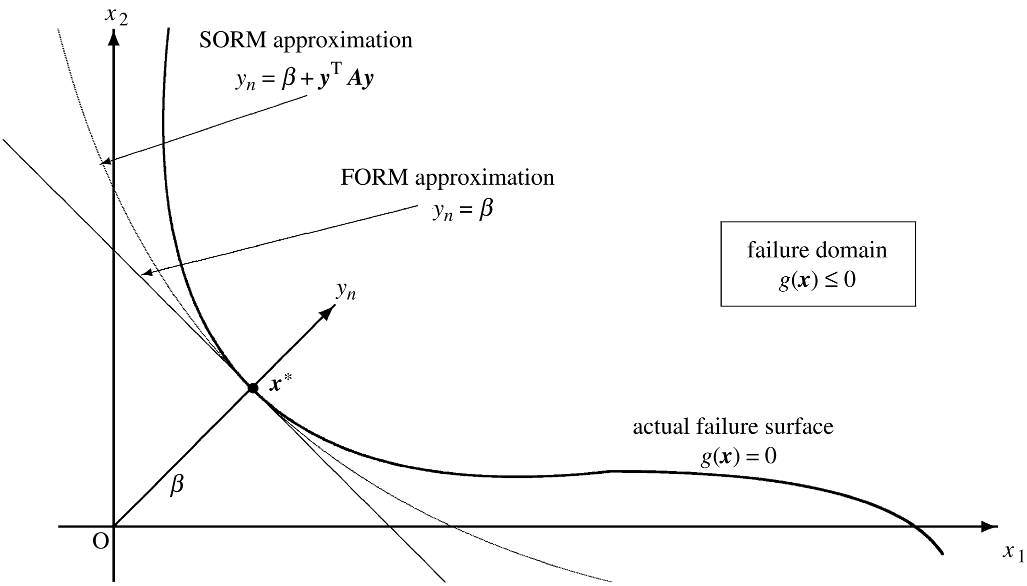

Figure 1. Geometric interpretation of approximate reliability analysis.

Madsen *et al.* (1986) proposed a parabolic approximation to equation (2.6), which keeps the intuitive notion of the reliability index. Construct an orthogonal matrix $\boldsymbol{R} \in \mathbb{R}^{n\times n}$ whose nth column is $\boldsymbol{\alpha}^*$, that is

$$\boldsymbol{R} = [\boldsymbol{R}_1 | \boldsymbol{\alpha}^*], \quad \text{where } \boldsymbol{R}_1 \in \mathbb{R}^{n\times n-1} \quad \text{and} \quad {\boldsymbol{\alpha}^*}^{\mathrm{T}} \boldsymbol{R}_1 = \boldsymbol{0}_{1\times(n-1)}. \tag{2.7}$$

The matrix $\boldsymbol{R}$ can be obtained, for example, by Gram-Schmidt orthogonalization. Using the orthogonal transformation $\boldsymbol{x} = \boldsymbol{R}\tilde{y}$ and partitioning $\tilde{\boldsymbol{y}}$ as

$$\tilde{\boldsymbol{y}} = \left\{ \frac{\boldsymbol{y}}{y_n} \right\}, \quad \text{where } \boldsymbol{y} \sim \mathbb{N}_{n-1}(\boldsymbol{0}_{n-1}, \boldsymbol{I}_{n-1}) \quad \text{and } y_n \sim \mathbb{N}_1(0,1), \tag{2.8}$$

from equation (2.6), one obtains

$$\frac{\tilde{g}}{|\nabla g|} = \left(\beta + \frac{\beta^2}{2} {\boldsymbol{\alpha}^*}^{\mathrm{T}} \frac{\boldsymbol{H}_g}{|\nabla g|} \boldsymbol{\alpha}^* \right) - \left(y_n + \beta {\boldsymbol{\alpha}^*}^{\mathrm{T}} \frac{\boldsymbol{H}_g}{|\nabla g|} \boldsymbol{R}\tilde{y} \right) + \tilde{\boldsymbol{y}}^{\mathrm{T}} \tilde{\boldsymbol{A}} \tilde{\boldsymbol{y}}. \tag{2.9}$$

Here,

$$\tilde{\boldsymbol{A}} = \frac{1}{2} \frac{\boldsymbol{R}^{\mathrm{T}} \boldsymbol{H}_g(\boldsymbol{x}^*) \boldsymbol{R}}{|\nabla g(\boldsymbol{x}^*)|} \in \mathbb{R}^{n\times n}. \tag{2.10}$$

For convenience, partition $\tilde{\boldsymbol{A}}$ as

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \tilde{\boldsymbol{A}}_{1n} \\ \tilde{\boldsymbol{A}}_{n1} & \tilde{A}_{nn} \end{bmatrix}, \quad \text{where} \quad \boldsymbol{A} \in \mathbb{R}^{(n-1)\times(n-1)} \quad \text{and} \quad \tilde{\boldsymbol{A}}_{n1} = \tilde{\boldsymbol{A}}_{1n}^{\mathrm{T}} \in \mathbb{R}^{1\times(n-1)}. \tag{2.11}$$

In view of this partition, equation (2.9) reads

$$\frac{\tilde{g}}{|\nabla g|} = -y_n + \beta + \boldsymbol{y}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{y} + \frac{\beta^2}{2} \boldsymbol{\alpha}^{*^{\mathrm{T}}} \frac{\boldsymbol{H}_g}{|\nabla g|} \boldsymbol{\alpha}^* - \beta \boldsymbol{\alpha}^{*^{\mathrm{T}}} \frac{\boldsymbol{H}_g}{|\nabla g|} \boldsymbol{R} \tilde{y} + 2 y_n \tilde{\boldsymbol{A}}_{n1} \boldsymbol{y} + \tilde{A}_{nn} y_n^2. \quad (2.12)$$

Keeping only second-order terms in $\boldsymbol{y}$, Madsen *et al.* (1986) have approximated equation (2.12) by a parabolic function

$$\frac{\tilde{g}}{|\nabla g|} \approx -y_n + \beta + \boldsymbol{y}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{y}. \quad (2.13)$$

The parabolic function in equation (2.13) is normally used in the classical SORM approximations. With this approximation the failure probability in equation (2.2) is given by

$$P_{\mathrm{f}} \approx \mathrm{Prob}\left[\frac{\tilde{g}}{|\nabla g|} \leq 0\right] \approx \mathrm{Prob}[y_n \geq \beta + \boldsymbol{y}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{y}] = \mathrm{Prob}[y_n \geq \beta + U], \quad (2.14)$$

where

$$U : \mathbb{R}^{n-1} \mapsto \mathbb{R} = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{y} \quad (2.15)$$

is a central quadratic form in standard Gaussian random variables.

Using quadratic approximation of the failure surface, together with asymptotic analysis, Breitung (1984) proved that

$$P_{\mathrm{f}} \to \Phi(-\beta) \| \boldsymbol{I}_{n-1} + 2\beta \boldsymbol{A} \|^{-1/2} \quad \text{when } \beta \to \infty. \quad (2.16)$$

Here, $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function. The eigenvalues of $\boldsymbol{A}$, say, a_j, can be related to the principal curvatures of the surface κ_j as $a_{j=}\kappa_j/2$. This result is important because it gives the asymptotic behaviour P_{f} under general conditions. It should be noted that equation (2.16) is the exact asymptotic expression and it cannot be improved as long as the asymptotic behaviour of P_{f} in β is considered. Later, Hohenbichler & Rackwitz (1988) proposed the following formula

$$P_{\mathrm{f}} \approx \Phi(-\beta) \left\| \boldsymbol{I}_{n-1} + 2 \frac{\varphi(\beta)}{\Phi(-\beta)} \boldsymbol{A} \right\|^{-1/2}, \quad (2.17)$$

where $\varphi(\cdot)$ is the standard Gaussian probability density function. This expression is more accurate than equation (2.16) for lower values of β (although both are asymptotically equivalent) and it was also rederived by Köylüoğlu & Nielsen (1994) and Polidori *et al.* (1999) using different approaches. In FORM, the failure surface is approximated by a hyperplane at the design point. This implies that the Hessian matrix at the design point is assumed to be a null matrix. Substituting $\boldsymbol{H}_g(\boldsymbol{x}^*) = \boldsymbol{O}$ in the Taylor series expansion (2.5), it is easy to see that $\boldsymbol{A} = \boldsymbol{O}$ and consequently $U = 0$ in equation (2.14). Thus, from equation (2.14) or figure 1, one obtains the probability of failure

$$P_{\mathrm{f}} \approx \Phi(-\beta). \quad (2.18)$$

This is the simplest approximation to the integral (2.2). Breitung's formula (2.16) and the formula by Hohenbichler & Rackwitz (2.17) can be viewed as corrections to the FORM formula (2.18) to take account of the curvature of the failure surface at the design point.

If n is very large, then the computation of P_{f} using any available methods will be difficult. Nevertheless, it is useful to ask the following questions of fundamental interest.

(i) Suppose we have followed the 'usual route' and did all the calculations (i.e. obtained x^*, β and $\boldsymbol{A}$). Can we still expect the same level of accuracy from the classical FORM/SORM formulae in high dimensions as we do in low dimensions? If not, what are the exact reasons behind it?

(ii) From the point of view of classical FORM/SORM, what do we mean by 'high dimension'? Is it a problem-dependent quantity, or is it simply our perception based on available computational tools so that what we regard as a high dimension today may not be considered as a high dimension in the future when more powerful computational tools will be available?

We tried to answer these questions using the asymptotic distribution of the quadratic form (2.15) as $n\to\infty$. It will be shown that minor modifications to the classical FORM and SORM formulae can improve their accuracy in high dimensions. Based on an error analysis, we also attempt to provide a value of n above which the number of random variables can be considered as high from the point of view of classical FORM/SORM.

3. Asymptotic distribution of quadratic forms

Discussions on asymptotic distribution of quadratic forms may be found in Mathai & Provost (1992, section 4.6*b*). Here, one of the simplest forms of asymptotic distribution of U will be used. We start with the moment generating function of U

$$M_U(s) = E[\mathrm{e}^{sU}] = E[\mathrm{e}^{s\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y}}] = \|\boldsymbol{I}_{n-1} - 2s\boldsymbol{A}\|^{-1/2} = \prod_{k=1}^{n-1}(1-2sa_k)^{-1/2}. \quad (3.1)$$

Now construct a sequence of new random variables $q = U/\sqrt{n}$. The moment generating function of q,

$$M_q(s) = M_U(s/\sqrt{n}) = \prod_{k=1}^{n-1}(1-2sa_k/\sqrt{n})^{-1/2}. \quad (3.2)$$

From this,

$$\begin{aligned} \ln(M_q(s)) &= -\frac{1}{2}\sum_{k=1}^{n-1}\ln(1-2sa_k/\sqrt{n}) \\ &= \frac{1}{2}\sum_{k=1}^{n-1}2sa_k/\sqrt{n} + s^2(2a_k/\sqrt{n})^2/2 + s^3(2a_k/\sqrt{n})^3/3 + \ldots, \end{aligned} \quad (3.3)$$

provided that

$$|2sa_k| < 1, \quad \text{for} \quad k = 1, 2, \ldots, n-1. \tag{3.4}$$

Consider a case when a_k and n are such that the higher-order terms of s vanish as $n \to \infty$, that is, we assume n is large such that the following conditions hold:

$$\sum_{k=1}^{n-1} (2a_k/\sqrt{n})^2/2 < \infty \quad \text{or} \quad \frac{2}{n}\text{Trace}(\boldsymbol{A}^2) < \infty \tag{3.5}$$

and

$$\sum_{k=1}^{n-1} (2a_k/\sqrt{n})^r/r \to 0 \quad \text{or} \quad \frac{2^r}{n^{r/2} r}\text{Trace}(\boldsymbol{A}^r) \to 0, \quad \forall\, r \geq 3. \tag{3.6}$$

Under these assumptions, the series in equation (3.3) can be truncated after the quadratic term

$$\left.\begin{aligned} \ln(M_q(s)) &\approx \frac{1}{2}\sum_{k=1}^{n-1} s(2a_k/\sqrt{n}) + s^2(2a_k/\sqrt{n})^2/2, \\ &= \text{Trace}(\boldsymbol{A})s/\sqrt{n} + (2\,\text{Trace}(\boldsymbol{A}^2))s^2/2n. \end{aligned}\right\} \tag{3.7}$$

Therefore, the moment generating function of $U = q\sqrt{n}$ can be approximated by

$$M_U(s) \approx \mathrm{e}^{\text{Trace}(\boldsymbol{A})s + 2\ \text{Trace}\ (\boldsymbol{A}^2)\ s^2/2}. \tag{3.8}$$

From the uniqueness of the Laplace Transform pair, it follows that when the conditions (3.4)–(3.6) are satisfied, U asymptotically approaches a Gaussian random variable with mean Trace($\boldsymbol{A}$) and variance 2 Trace($\boldsymbol{A}^2$), that is

$$U \simeq \mathbb{N}_1(\text{Trace}(\boldsymbol{A}), 2\,\text{Trace}(\boldsymbol{A}^2)) \quad \text{when } n \to \infty. \tag{3.9}$$

For practical problems, the minimum number of random variables required for the accuracy of this asymptotic distribution will be helpful. The error in neglecting higher order terms in series (3.3) is of the form

$$\sum_{k=1}^{n-1} (2sa_k/\sqrt{n})^r/r = \frac{1}{r}\left(\frac{2s}{\sqrt{n}}\right)^r \text{Trace}(\boldsymbol{A}^r), \quad \text{for } r \geq 3. \tag{3.10}$$

Values of s define the domain over which the moment generating function is used. For large β, it turns out that appropriate choice of s is $s = -\beta$ (see appendix A for explanation). Using this, we aim to derive a simple expression for the minimum value of n, which is sufficient for the application of the asymptotic distribution method. From the expression of error (3.10), assume there exist a small real number ϵ (allowable error) such that

$$\left|\frac{1}{r}\frac{(-2\beta)^r}{n^{r/2}}\text{Trace}(\boldsymbol{A}^r)\right| < \epsilon \quad \text{or} \quad n^{r/2} > \frac{(2\beta)^r}{r\epsilon}\text{Trace}(\boldsymbol{A}^r), \tag{3.11}$$

or

$$n > \frac{4\beta^2}{\sqrt[r]{r^2\varepsilon^2}}\left(\sqrt[r]{\mathrm{Trace}(\boldsymbol{A}^r)}\right)^2. \tag{3.12}$$

Given that $\boldsymbol{A}$ is a positive definite matrix, the critical value of n is obtained for $r=3$

$$n_{\min} = \frac{4\beta^2}{\sqrt[3]{9\epsilon^2}}\left(\sqrt[3]{\mathrm{Trace}(\boldsymbol{A}^3)}\right)^2. \tag{3.13}$$

From equation (3.13), the following points may be observed: (i) the minimum number of random variables required would be more if ϵ (error) is considered to be small (as expected) and $n_{\min} \propto 1/\epsilon^{2/3}$; (ii) if β is large, more random variables are needed to achieve a desired accuracy and $n_{\min} \propto \beta^2$; and (iii) if $\boldsymbol{A}$ has some large eigenvalues (principal curvatures), they would control the term in the bracket and consequently $n_{\min}$. In the next two sections, the asymptotic distribution (3.9) is used to obtain the probability of failure.

4. Failure probability using strict asymptotic formulation

In the strict asymptotic formulation, we start with equation (2.14). The probability of failure can be rewritten as

$$P_{\mathrm{f}} \approx \mathrm{Prob}[y_n \geq \beta + U] = \mathrm{Prob}[y_n - U \geq \beta]. \tag{4.1}$$

From (3.9), the asymptotic pdf of U is Gaussian; therefore, the variable $z = y_n - U$ is also a Gaussian random variable with mean $(-\mathrm{Trace}(\boldsymbol{A}))$ and variance $(1+2\,\mathrm{Trace}(\boldsymbol{A}^2))$. The probability of failure can be obtained from equation (4.1) as

$$P_{\mathrm{f}_{\mathrm{Strict}}} \to \Phi(-\beta_1), \quad \beta_1 = \frac{\beta + \mathrm{Trace}(\boldsymbol{A})}{\sqrt{1 + 2\mathrm{Trace}(\boldsymbol{A}^2)}} \quad \text{when } n \to \infty. \tag{4.2}$$

This is the exact asymptotic expression of P_{f} *after* making the parabolic failure surface assumption and it cannot be improved or changed asymptotically. If the failure surface is close to linear and the number of random variables is not very large, then it is expected that $\mathrm{Trace}(\boldsymbol{A}) = \mathrm{Trace}(\boldsymbol{A}^2) \to 0$, and it is easy to see that equation (4.2) reduces to the classical FORM formula (2.18). Therefore, the expressions derived here can be viewed as the 'correction' that needs to be applied to the classical FORM formula when a large number of random variables are considered. A simple geometric interpretation of this asymptotic expression can be given.

From equation (4.1), the failure domain is given by

$$y_n - U \geq \beta. \tag{4.3}$$

We have already shown that when $n \to \infty$

$$U \simeq \mathbb{N}_1(m, \sigma^2), \quad \text{with } m = \mathrm{Trace}(\boldsymbol{A}) \quad \text{and} \quad \sigma = \sqrt{2\,\mathrm{Trace}(\boldsymbol{A}^2)}. \tag{4.4}$$

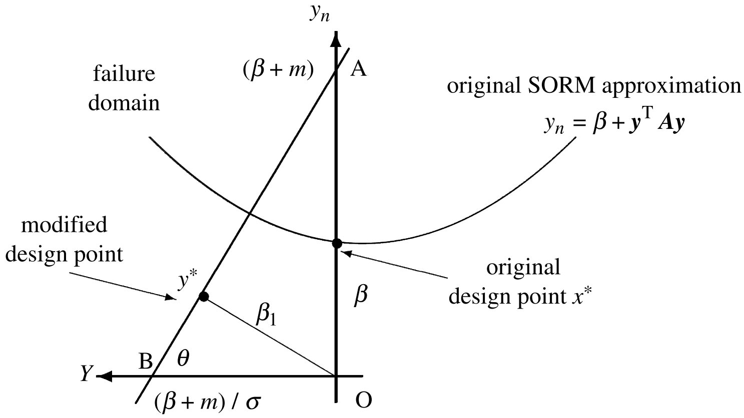

Figure 2. Geometric interpretation of the strict asymptotic formulation.

Using the standardizing transformation $Y=(U-m)/\sigma$, equation (4.3) can be rewritten as

$$\frac{y_n}{\beta+m}+\frac{Y}{-(\beta+m)/\sigma}\geq 1. \tag{4.5}$$

This implies that the original $(n-1)$-dimensional parabolic hypersurface asymptotically becomes a straight line in the two-dimensional $(y_n,\ Y)$-space, as shown in figure 2. Considering the triangle AOB, $\tan\theta=\mathrm{OA}/\mathrm{OB}=(\beta+m)/(\beta+m)/\sigma=\sigma$. Therefore, $\sin\theta=\tan\theta/\sqrt{1+\tan^2\theta}=\sigma/\sqrt{1+\sigma^2}$. Now, considering the triangle $\mathrm{OB}y^*$ and noticing that $\mathrm{O}y^*\perp\mathrm{AB}$, $\sin\theta=\mathrm{O}y^*/\mathrm{OB}=\beta_1/(\beta+m)/\sigma$. From this, the modified reliability index

$$\beta_1=\frac{\beta+m}{\sigma}\sin\theta=\frac{\beta+m}{\sigma}\frac{\sigma}{\sqrt{1+\sigma^2}}=\frac{\beta+m}{\sqrt{1+\sigma^2}}=\frac{\beta+\mathrm{Trace}(\boldsymbol{A})}{\sqrt{1+2\,\mathrm{Trace}(\boldsymbol{A}^2)}}. \tag{4.6}$$

Therefore, from figure 2, the failure probability $P_{\mathrm{f}_{\mathrm{strict}}}=\Phi(-\beta_1)$, which has been derived in equation (4.2). If n is small, m and σ are also expected to be small. This would shift the point B towards $-\infty$ in the Y-axis and point A towards the β level in the positive y_n-axis. That is, when m and σ approach to 0, line AB will rotate clockwise and, eventually, it will be parallel to the Y-axis with a shift of $+\beta$. In this situation, y^* will approach to the original design point in the y_n-axis and $\beta_1\to\beta$ as expected. This geometric analysis explains why classical SORM approximations based on the original design point $\boldsymbol{x}^*$ do not work well when a large number of random variables are considered.

The value of n given by equation (3.13) can be viewed as the borderline between the low and high dimension. Beyond this value of n, a significant 'Trace effect' can be observed and, consequently, the modified reliability index β_1 instead of β should be used.

5. Failure probability using weak asymptotic formulation

The expression of P_{f} given by (4.2) cannot be improved asymptotically. However, there is scope for 'improvements' if one does not strictly apply the asymptotic condition $n \to \infty$. The advantage of such non-asymptotic approximations is that the approximations may work well even when the asymptotic condition is not met. The disadvantage is that a non-asymptotic approximation will have unquantified errors and one generally cannot prove that such errors will vanish when the asymptotic condition is fulfilled. Nevertheless, it is worth perusing a non-asymptotic approximation because real-life structural systems have finite number of random variables.

Rewriting equation (2.14), the failure probability can be expressed as

$$P_\mathrm{f} \approx \mathrm{Prob}[y_n \geq \beta + U] = \int_{\mathbb{R}} \left\{ \int_{\beta+\mu}^{\infty} \varphi(y_n) \mathrm{d}y_n \right\} p_U(u) \mathrm{d}u = E[\Phi(-\beta - U)]. \quad (5.1)$$

Here, $p_U(u)$ is the probability density function of U and $E[\cdot]$ is the expectation operator. Extensive discussions on quadratic forms in Gaussian random variables can be found in the books by Johnson & Kotz (1970, ch. 29) and Mathai & Provost (1992). In general, a simple closed-form expression of $p_U(u)$ is not available. For this reason, it is difficult to calculate the expectation $E[\Phi(-\beta - U)]$ analytically. Several authors have used approximations of $E[\Phi(-\beta - U)]$ to obtain closed-form expressions of P_f. A selected collection of such expressions can be found in Zhao & Ono (1999). Here, asymptotic distribution of U in (3.9) is used to obtain P_f from equation (5.1).

From the definition of U in equation (2.15), note that $u \in \mathbb{R}^+$ because $\boldsymbol{A}$ is a positive definite matrix, we rewrite equation (5.1) as

$$P_\mathrm{f} \approx \int_{\mathbb{R}^+} \Phi(-\beta - u) p_U(u) \mathrm{d}u = \int_{\mathbb{R}^+} \mathrm{e}^{\ln[\Phi(-\beta-u)] + \ln[p_U(u)]} \mathrm{d}u. \quad (5.2)$$

The aim here is to expand the integrand in a first-order Taylor series about the most probable point or optimal point, say, $u = u^*$. The optimal point is the point where the integrand in equation (5.2) reaches its maxima in $u \in \mathbb{R}^+$. The asymptotic approximation of $p_U(u)$ in equation (3.9) will only be used to find the maxima of the integrand and will *not* be used subsequently to calculate the expectation. The expectation operation will be carried out exactly by utilising the expression of the moment generating function in equation (3.1). For this reason, this approach is called weak asymptotic formulation.

For the maxima of the integrand in equation (5.2), we must have

$$\frac{\partial}{\partial u} \{ \ln[\Phi(-\beta - \mu)] + \ln[p_U(u)] \} = 0. \quad (5.3)$$

Recalling that

$$p_U(u) = (2\pi)^{-1/2} \sigma^{-1} \mathrm{e}^{-(u-m)^2/(2\sigma^2)}, \quad (5.4)$$

where m and σ are given in equation (4.4), equation (5.3) results in

$$\frac{\varphi(\beta + u)}{\Phi(-(\beta + u))} = \frac{m - u}{\sigma^2}. \quad (5.5)$$

Because this relationship holds at the optimal point u^*, we define a constant η as

$$\eta = \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} = \frac{m - u^*}{\sigma^2}. \quad (5.6)$$

Taking a first-order Taylor series expansion of $\ln[\Phi(-\beta-u)]$ about $u=u^*$, we have

$$\ln[\Phi(-\beta - u)] \approx \ln[\Phi(-\beta - u^*)] - \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))}(u - u^*) \tag{5.7}$$

or

$$\Phi(-\beta - u) \approx \exp\left(\ln[\Phi(-(\beta + u^*))] - \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))}(u - u^*)\right). \tag{5.8}$$

Using equation (5.6), this reduces to

$$\Phi(-\beta - u) \approx \Phi(-\beta_2)\mathrm{e}^{\eta u*}\mathrm{e}^{-\eta u}, \tag{5.9}$$

where

$$\beta_2 = \beta + u^*. \tag{5.10}$$

Taking the expectation of equation (5.9) and utilising the expression of the moment generating function in equation (3.1), the probability of failure can be expressed as

$$P_{\mathrm{f}} \approx \Phi(-\beta_2)\mathrm{e}^{\eta u*}\|\boldsymbol{I}_{n-1} + 2\eta\boldsymbol{A}\|^{-1/2}. \tag{5.11}$$

The optimal point u^* should be obtained by solving the nonlinear equation (5.6). An exact closed-form solution of this equation does not exist. However, it can be easily solved numerically (e.g. the function 'fzero' in MATLAB can be used) to obtain u^*. An approximate solution of equation (5.6) can be obtained by considering the asymptotic expansion of the ratio $\phi(\beta+u^*)/\Phi(-(\beta+u^*))$

$$\frac{\varphi(\beta+u^*)}{\Phi(-(\beta+u^*))} \approx (\beta+u^*) + (\beta+u^*)^{-1} - 2(\beta+u^*)^{-3} + 10(\beta+u^*)^{-5} - \ldots. \tag{5.12}$$

Keeping only the first term, the left-hand side of equation (5.6) becomes $(\beta+u^*)$ and, consequently, we obtain

$$\eta \approx (\beta+u^*) \approx \frac{m-u^*}{\sigma^2} \text{ or } u^* \approx \frac{m-\beta\sigma^2}{1+\sigma^2}, \tag{5.13}$$

so that

$$\beta_2 = \beta + u^* \approx \frac{\beta+m}{1+\sigma^2} = \frac{\beta + \mathrm{Trace}(\boldsymbol{A})}{1+2\,\mathrm{Trace}(\boldsymbol{A}^2)}. \tag{5.14}$$

In view of equations (5.10) and (5.13), it is also clear that

$$\eta \approx \beta_2. \tag{5.15}$$

Using this, from equation (5.13), u^* can be expressed in terms of β_2 as

$$u^* \approx -(\beta_2\sigma^2 - m) = -(2\beta_2\mathrm{Trace}(\boldsymbol{A}^2) - \mathrm{Trace}(\boldsymbol{A})). \tag{5.16}$$

Now, substituting η from equation (5.15) and u^* from equation (5.16) in equation (5.11), the failure probability using weak asymptotic formulation can be finally obtained as

$$P_{\mathrm{f}_{\mathrm{weak}}} \to \frac{\Phi(-\beta_2)\mathrm{e}^{-(2\beta_2^2 \mathrm{Trace}\,(\boldsymbol{A}^2) - \beta_2 \mathrm{Trace}(\boldsymbol{A}))}}{\sqrt{\|\boldsymbol{I}_{n-1} + 2\beta_2 \boldsymbol{A}\|}}, \text{ when } n \to \infty, \tag{5.17}$$

where β_2 is defined in equation (5.14). If the number of random variables is not very large, then it is expected that $\mathrm{Trace}(\boldsymbol{A}) = \mathrm{Trace}(\boldsymbol{A}^2) \to 0$. In that case, it is easy to see that $\beta_2 \to \beta$ and equation (5.17) reduces to the classical Breitung's SORM formula (2.16). Therefore, equation (5.17) can be viewed as the 'correction' that needs to be applied to the classical SORM formula when a large number of random variables are considered. Unlike the strict formulation, a simple geometric explanation of this expression cannot be given. In addition, note that the modified reliability indices β_1 and β_2 for the two formulations are not identical.

6. Numerical results and discussions

We consider a problem for which the failure surface is *exactly* parabolic in the normalized space, as given by equation (2.13). The purpose of this numerical study is to understand how the proposed approximation works after making the parabolic failure surface assumption. Therefore, the effect of errors resulting from the parabolic failure surface assumption itself cannot, and will not, be investigated here.

In numerical calculations, we have fixed the number of random variables n and the Trace of the coefficient matrix $\boldsymbol{A}$. It is assumed that the eigenvalues of $\boldsymbol{A}$ are uniform positive random numbers. Based on the values of n and $\mathrm{Trace}(\boldsymbol{A})$, three cases are considered:

Case (i) a small number of random variables: $n-1=35$ and $\mathrm{Trace}(\boldsymbol{A})=1$;
Case (ii) a large number of random variables: $n-1=200$ and $\mathrm{Trace}(\boldsymbol{A})=1$;
Case (iii) a large number of random variables: $n-1=500$ and $\mathrm{Trace}(\boldsymbol{A})=2$.

When $\mathrm{Trace}(\boldsymbol{A})=0$ the failure surface is effectively linear. Therefore, the higher the value of $\mathrm{Trace}(\boldsymbol{A})$, the more nonlinear the failure surface becomes. Probability of failure obtained using the two asymptotic expressions is compared with Breitung's asymptotic result, the formula (2.17) derived by Hohenbichler & Rackwitz (1988) and Monte Carlo simulation. Monte Carlo simulation is carried out by generating 10000 samples of the quadratic form (2.15) and by numerically calculating the expectation operation (5.1) for each value of β.

Figure 3 shows probability of failure (normalized by dividing with $\Phi(-\beta)$) for values of β ranging from 0 to 6 for case (i). For this problem, the minimum number of random variables required for the applicability of the asymptotic distribution can be obtained from (3.13). Considering $\epsilon=0.01$, it can be shown from equation (3.13) that $n_{\min}=176$. Although this condition is not satisfied here, the results obtained from the weak asymptotic formulation are accurate. The results obtained from the strict asymptotic formulation are not accurate, especially when β is high. However, this is expected as the asymptotic condition has not been met for this case.

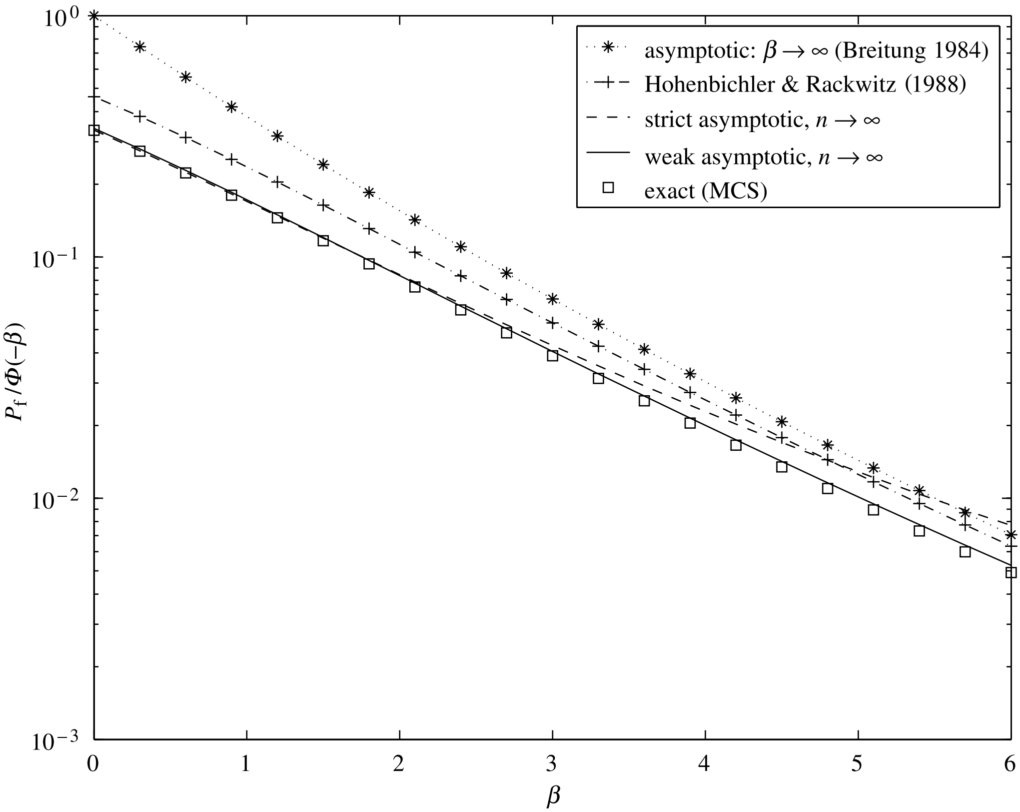

Figure 3. Normalized failure probability, case (i): $n-1=35$, Trace($\boldsymbol{A}$)=1.

Results obtained from the asymptotic analysis improve when the number of random variables becomes large. Figure 4 shows the probability of failure for case (ii). As expected, with more random variables, results obtained from both asymptotic formulations match well with the Monte Carlo simulation result. For this case, the maximum value of the curvature (a_j) is 0.0097, which implies that the failure surface is almost linear. Even in such cases, it is interesting to note the difference between the results obtained from existing approximations and the proposed methods. This difference becomes more prominent for higher n and higher values of Trace($\boldsymbol{A}$) as can be seen in figure 5 for case (iii). For this case, the maximum value of the curvature (a_j) is 0.008 implying that the failure surface is more linear compared with the previous two cases. For this case also, the failure probability obtained from both asymptotic formulations match well with the Monte Carlo simulation result.

From these selective numerical examples, the following points may be noted.

(i) For a fixed value of β and $\boldsymbol{A}$, the weak asymptotic formulation is more accurate for smaller values of n (say, n_1) compared with the strict asymptotic formulation. The convergence of the proposed formulations for increasing values of n, when β is ranging from 3 to 6 is shown in figure 6. The strict asymptotic formulation becomes accurate when n is more than that given by equation (3.13) (e.g. n_2). Although it was not possible to

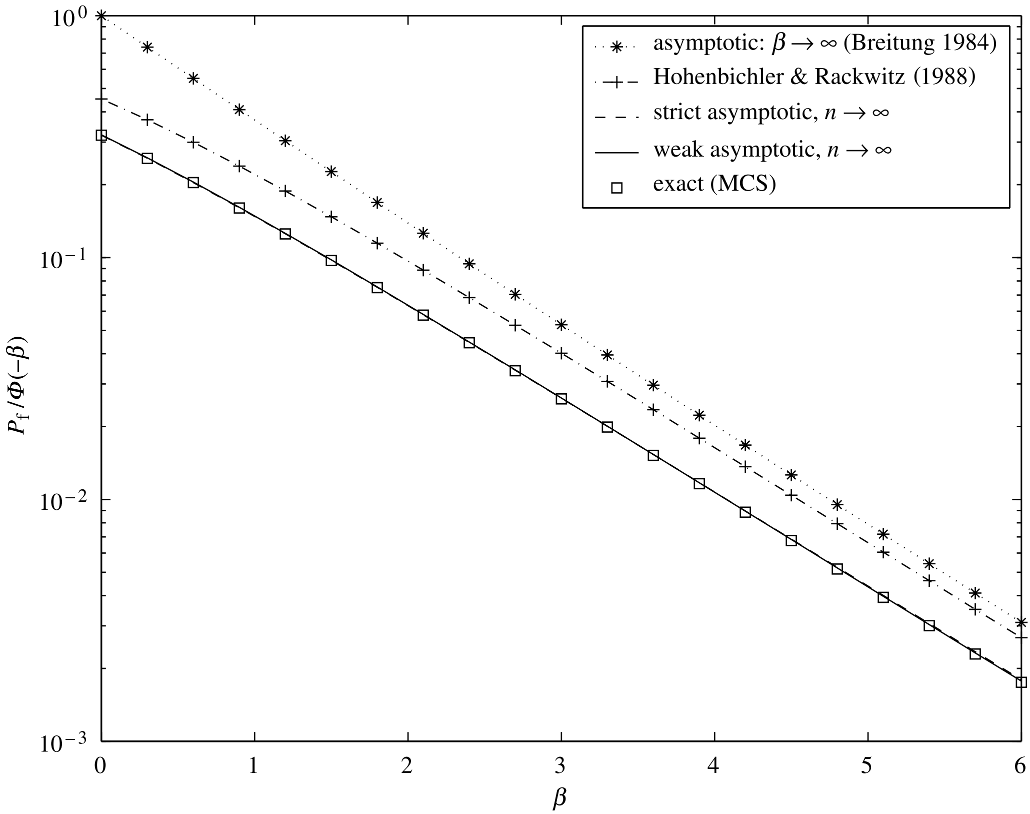

Figure 4. Normalized failure probability, case (ii): $n-1=200$, Trace($\boldsymbol{A}$)=1.

obtain an expression of n_1, it can be conceived conceptually (see figure 6). Overall, the applicability of the approximate analytical methods for structural reliability calculations as a function of number of random variables can be summarized in figure 7. When $n<n_2$, the existing FORM/SORM are applicable, but they may not be very accurate if $n>n_1$. When $n>n_1$, the weak asymptotic formulation can provide accurate result and when $n>n_2$, both of the proposed formulations yield similar results. Again, we recall that these conclusions are based on the validity of the parabolic failure surface approximation (2.13).

(ii) For a fixed value of $\boldsymbol{A}$, from equation (3.13), it can be seen that the results from both approaches will be more accurate if β is small. This fact can also be observed in the numerical results shown in figures 3–6. However, proposed asymptotic approximations are based on the parabolic failure of surface approximation (2.13), which is expected to be accurate when β is high. These two conflicting demands can only be met when n is significantly large.

(iii) When n is large, the computational cost to accurately obtain β and $\boldsymbol{A}$ can be prohibitive. In recent years, there have been significant developments in numerical simulation methods specifically tailored to deal with reliability problems in high dimensions (see Au & Beck 2003). Proposed formulae

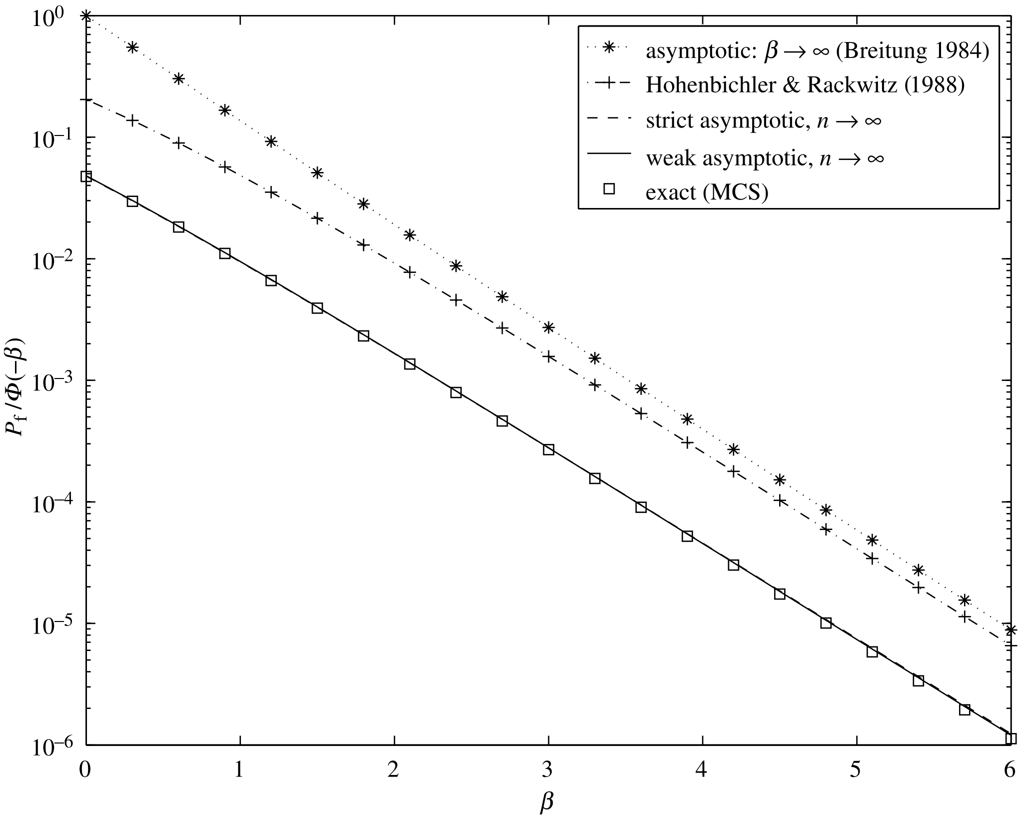

Figure 5. Normalized failure probability, case (iii): $n-1=500$, Trace$(\boldsymbol{A})=2$.

nevertheless provide an alternative that can give physical insight and can be used in the early stages of reliability based optimal design. Moreover, the modified design point and the asymptotic density function can be used for importance sampling in high dimension. However, Further research is needed in this area.

7. Conclusions

The demands of modern engineering design have led structural engineers to model a structure using random variables in order to handle uncertainties. Two approximations to calculate the probability of failure of an engineering structure when the number of random variables used for mathematical modelling $n \to \infty$ are provided. It is assumed that the basic random variables are Gaussian and the failure surface is approximated by a parabolic hypersurface in the neighbourhood of the design point. The new approximations are based on the asymptotic distribution of a central quadratic form in Gaussian random variables. The main outcome of the asymptotic analysis is that the conventional reliability index β needs to be modified when $n \to \infty$. A simple geometric explanation is given for this fact. Two formulations—namely, *strict asymptotic formulation* and *weak*

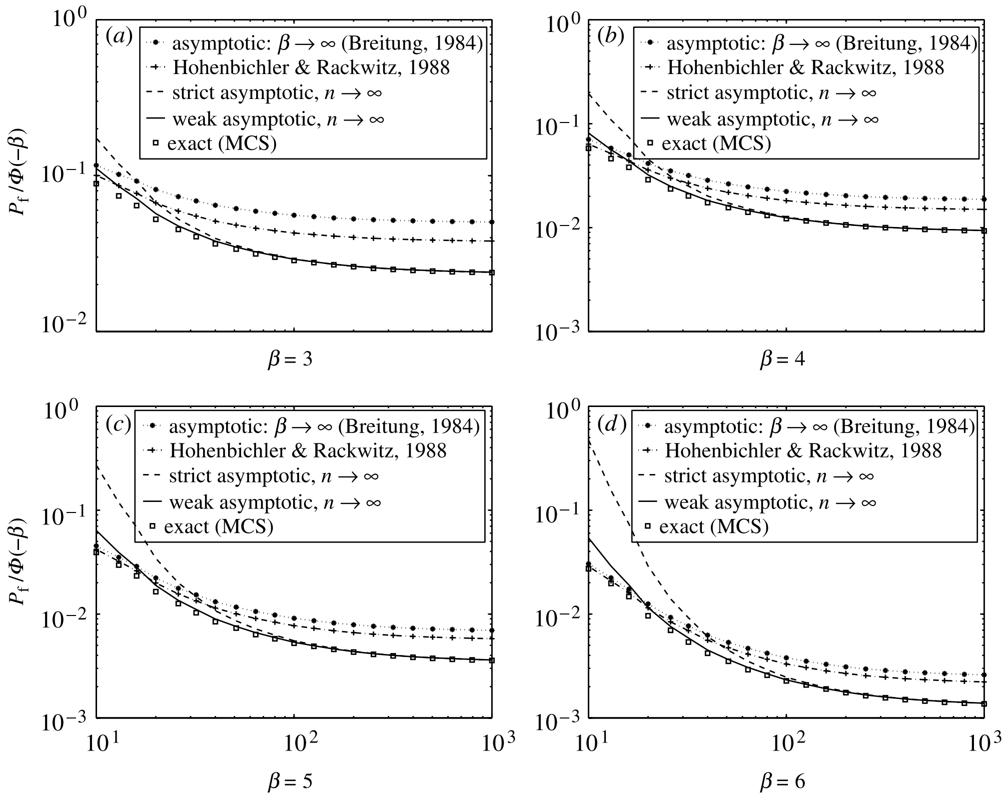

Figure 6. Normalized failure probability for different values of β when Trace($\boldsymbol{A}$)=1. (*a*) β=3; (*b*) β=4; (*c*) β=5; (*d*) β=6.

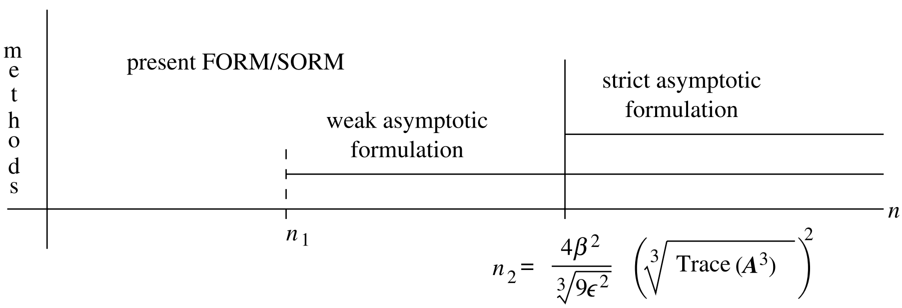

Figure 7. Approximate analytical methods for structural reliability calculations as a function of number of random variables n.

asymptotic formulation—are presented. Both approximations result in simple closed-form expressions:

$$P_{f_{\text{strict}}} \rightarrow \Phi(-\beta_1), \quad \beta_1 = \frac{\beta + \text{Trace}(\boldsymbol{A})}{\sqrt{1 + 2\,\text{Trace}(\boldsymbol{A}^2)}},$$

$$P_{\mathrm{f}_{\mathrm{weak}}} \rightarrow \frac{\Phi(-\beta_2)\mathrm{e}^{-(2\beta_2^2 \mathrm{Trace}(\boldsymbol{A}^2) - \beta_2 \mathrm{Trace}(\boldsymbol{A}))}}{\sqrt{\|\boldsymbol{I}_{n-1} + 2\beta_2 \boldsymbol{A}\|}}, \beta_2 = \frac{\beta + \mathrm{Trace}(\boldsymbol{A})}{1 + 2\,\mathrm{Trace}(\boldsymbol{A}^2)}.$$

For a small number of variables, the Trace effects may not be significant and, in such cases, it is easy to see that these two formulae reduce to the classical FORM and SORM formulae, respectively. A closed-form expression for the minimum number of random variables required to apply these asymptotic formulae is derived. Beyond this value of n, the reliability problem can be considered as high dimensional because the Trace effects become significant. The proposed approximations are compared with some existing approximations and Monte Carlo simulations using numerical examples. The results obtained from both asymptotic formulations match well with the Monte Carlo simulation results in high dimensions. Numerical studies show that the weak formulation generally applicable for a low number of variables compared with the strict formulation. In many real-life problems, the number of random variables is expected to be large. In such situations, the asymptotic results derived here will be useful.

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Appendix A. The choice of s in the moment generating function of $\boldsymbol{U}$

To select the value of s we begin with the approximation of P_{f} in equation (5.1) as

$$P_{\mathrm{f}} \approx \mathrm{E}[\Phi(-\beta - U)]. \tag{A 1}$$

Because $u \in \mathbb{R}^+$ as $\boldsymbol{A}$ is positive definite, the maxima of $\ln[\Phi(-\beta - u)]$ in $\mathbb{R}^+$ occurs at $u=0$. Therefore, the maximum contribution to the expectation of $\ln[\Phi(-\beta - U)]$ comes from the neighbourhood of $u=0$. Expanding $\ln[\Phi(-\beta - u)]$ in a first-order Taylor series about $u=0$ we obtain

$$\Phi(-\beta - u) \approx \exp\left(\ln[\Phi(-\beta)] - \frac{\varphi(\beta)}{\Phi(-\beta)} u\right) = \Phi(-\beta)\exp\left(-\frac{\varphi(\beta)}{\Phi(-\beta)} u\right). \tag{A 2}$$

The reason for keeping only one term in the Taylor series is to exploit the expression of the moment generating function in equation (3.1). Substituting $\Phi(-\beta - u)$ in equation (A 1), we have

$$P_{\mathrm{f}} \approx \Phi(-\beta)\mathrm{E}\left[\exp\left(-\frac{\varphi(\beta)}{\Phi(-\beta)} u\right)\right] = \Phi(-\beta) M_U\left(s = -\frac{\varphi(\beta)}{\Phi(-\beta)}\right). \tag{A 3}$$

The preceding equation indicates that in order to calculate P_{f}, the appropriate choice of s to be used in the moment generating function of U is given by

$$s = -\frac{\varphi(\beta)}{\Phi(-\beta)}. \tag{A 4}$$

If β is large, then using the asymptotic series (5.12), we have $s = (-\varphi(\beta)/\Phi(-\beta)) \approx -\beta$. This analysis explains the rationale behind choosing $s = -\beta$ is equation (3.10).

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