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Polynomial chaos-based extended Padé expansion in structural dynamics

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SUMMARY

The response of a random dynamical system is totally characterized by its probability density function (pdf). However, determining a pdf by a direct approach requires a high numerical cost; similarly, surrogate models such as direct polynomial chaos expansions are not generally efficient, especially around the eigenfrequencies of the dynamical system. In the present study, a new approach based on Padé approximants to obtain moments and pdf of the dynamic response in the frequency domain is proposed. A key difference between the direct polynomial chaos representation and the Padé representation is that the Padé approach has polynomials in both numerator and denominator. For frequency response functions, the denominator plays a vital role as it contains the information related to resonance frequencies, which are uncertain. A Galerkin approach in conjunction with polynomial chaos is proposed for the Padé approximation. Another physics-based approach, utilizing polynomial chaos expansions of the random eigenmodes, is proposed and compared with the proposed Padé approach. It is shown that both methods give accurate results even if a very low degree of the polynomial expansion is used. The methods are demonstrated for two degree-of-freedom system with one and two uncertain parameters.

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1. INTRODUCTION

In order to determine the statistics of the random dynamical system response, several methods may be used such as Monte Carlo simulation (MCS) or polynomial chaos (PC) expansion (PCE) [1]. It is well known that the main drawback of MCS is its numerical cost. The PC method is an alternative that expands the dynamical response, X, on a set of orthogonal polynomials whose variables are mutually independent standard normal deviates. However, it turns out that the convergence of a PCE around the 'deterministic' resonances (i.e., related to the mean mass and stiffness matrices) is quite poor [2]: the polynomial expression of the solution is perhaps not suitable and can be improved.

An improvement may come from the numerical convergence acceleration of the probability density function (pdf): some researchers [3, 4] have already worked on the convergence acceleration [5] of the moments and the coefficients of the PCE. Even though they demonstrated that Aitken's transformation and its generalization were successfully applied to the sequences defined by the first two

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moments of the responses, it is still necessary to consider a quite high degree of the PCE in order to obtain an accurate estimation of the moments. Further improvement can be obtained by considering the Padé approximants (PAs) [6, 7]. Indeed, as the frequency response function of a random dynamical system is a rational function of the modal characteristics, which are random, it seems appropriate to estimate the solution in terms of a rational function that depends on the uncertain parameters [8, 9]. Thereby, the main contribution of the present study is to estimate the pdf of the responses with a generalization of the PAs [10], called here ‘extended’ Padé approximants (XPAs): they are rational functions where the numerator and the denominator are a linear combination of PC.

The modal analysis together with the principle of mode superposition is a powerful tool widely used for studying deterministic linear dynamical systems. An extension to uncertain dynamical linear systems has been developed. The first work on random mode determination in a structural dynamics framework is probably the paper published by Collins *et al.* [11]. This work was based on a perturbation approach and has been used by several authors [12, 13] and extended by Adhikari [14]. Lan *et al.* [15] used a stochastic collocation method to estimate the eigenpairs. Sall [16], Sarrouy [17], Ghanem [18], and Ghosh a[19] have estimated the random modes following a method proposed by Dessombz [20, 21], which relies on a PCE and will be employed in this paper. However, the random mode superposition has been used rarely to evaluate the random frequency response function. Hence, the second main contribution of this paper is to investigate the use of the random mode approach in order to obtain the pdf of the response of a linear dynamical system with uncertain parameters.

In summary, the main objective of this work is to derive the pdf of uncertain dynamical responses by investigating both the PA and the random mode approaches. The paper is organized as follows. The random dynamical system is described in the next section. Then the PA method is presented in Section 3, as well as the PCE. In Section 4, the random modes are described as a PCE. Finally, numerical simulations are performed on a two degree-of-freedom (dof) system and discussed in Sections 5 and 6. These examples are very simple, with a low number of dofs to make possible closed-form expressions of the exact solution, as well as the estimated solution with the PCE approaches. Further, they illustrate the methods very well and show how it is possible to extend them to systems with more dofs and with more uncertain parameters.

2. RANDOM DYNAMICAL SYSTEM

A linear random N-dof dynamical system excited with harmonic force vector, $\mathbf{F}$, is investigated.

The uncertain dynamical system is characterized by the mass, stiffness, and damping matrices ($\mathbf{M}$, $\mathbf{K}$, and $\mathbf{D}$), which depend on an r-element uncertain parameter vector, Ξ. The dynamical response, $\mathbf{X}(\omega, \Xi) \in \mathbb{R}^N$, is then the solution of the system

$$(-\omega^2\mathbf{M} + \imath\omega\mathbf{D} + \mathbf{K})\mathbf{X}(\omega, \Xi) = \mathbf{F}(\omega), \tag{1}$$

where ω is the circular frequency of the applied forces and $\imath^2 = -1$.

The uncertain matrices are written as

$$\mathbf{M}(\Xi) = \mathbf{M}_0 + \sum_{i=1}^{r} \xi_i \mathbf{M}_i, \tag{2}$$

$$\mathbf{K}(\Xi) = \mathbf{K}_0 + \sum_{i=1}^{r} \xi_i \mathbf{K}_i, \tag{3}$$

$$\mathbf{D}(\Xi) = \mathbf{D}_0 + \sum_{i=1}^{r} \xi_i \mathbf{D}_i, \tag{4}$$

where ξ_i represents the i-th uncertain parameter with zero mean and is the i-th element of the aforementioned defined random vector Ξ. The related so-called deterministic dynamical system is characterized by the mean matrices ($\mathbf{M}_0$, $\mathbf{K}_0$, and $\mathbf{D}_0$).

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3. POLYNOMIAL CHAOS AND PADÉ APPROXIMANTS

3.1. Polynomial chaos expansion

A brief presentation of the well-known PC method will be given in the following, mainly to define the notation. For the interested reader, an explicit solution with a PCE has been used for uncertain dynamical systems in references [2, 4]. The response of the dynamical system may be expanded in terms of PC Ψ_j [1] as

$$\mathbf{X}(\omega, \Xi) = \sum_{i=0}^{\infty} \mathbf{Y}_i(\omega)\, \Psi_i(\Xi) \quad \text{with } \forall i < j,\ \text{degree of } \Psi_i(\Xi) \leqslant \text{degree of } \Psi_j\Xi). \tag{5}$$

In the following, normalized Hermite or Legendre polynomials are used to build the PC set.

In practice, the PCE is truncated:

$$\mathbf{X}^P(\omega, \Xi) = \sum_{i=0}^{P} \mathbf{Y}_i^P(\omega)\, \Psi_j(\Xi), \tag{6}$$

where P depends on the number of random variables and the PC degree [1]. Coefficients $\mathbf{Y}_i^P$ are determined by replacing $\mathbf{X}^P$ by its expansion in Equation (1) and by using the orthogonality properties of the Hermite polynomials with respect to the Gaussian weight function. Then the coefficients are the solution of

$$\tilde{\mathbf{H}}^P(\omega)\, \mathbf{Y}^P = \tilde{\mathbf{F}}^P, \tag{7}$$

where [2]

$$\mathbf{C}_k \in \mathbb{R}^{(P+1)\times(P+1)}, \text{with}[C_k]_{IJ} =< k, I, J >, \tag{8}$$

$$\tilde{\mathbf{H}}^P = \sum_{k=0}^{r} \mathbf{C}_k \otimes (-\omega^2 \mathbf{M}_k + \imath\omega\, \mathbf{D}_k + \mathbf{K}_k) \in \mathbb{R}^{N(P+1)\times N(P+1),} \tag{9}$$

$$\mathbf{Y}^P = [\mathbf{Y}_0^T\ \mathbf{Y}_1^T\ \cdots\ \mathbf{Y}_P^T]^T \in \mathbb{R}^{N(P+1),} \tag{10}$$

$$\tilde{\mathbf{F}}^P = [\mu \mathbf{F}^T\ 0\ 0\ \cdots\ 0\ ]^T \in \mathbb{R}^{N(P+1).} \tag{11}$$

$\otimes$ denotes the Kronecker product, $(\bullet)^T$ denotes the transpose of $(\bullet)$, $\mu = \int_\Xi \Psi_0(\Xi)\mathcal{P}(\Xi)\, \mathrm{d}\Xi$, and $< i_1 \cdots i_n >$ is defined by

$$< i_1 \cdots i_n > =< \Psi_{i_1}(\Xi) \cdots \Psi_{i_n}(\Xi) > = \int_\Xi \left(\Psi_{i_1}(\Xi) \cdots \Psi_{i_n}(\Xi)\right) \mathcal{P}(\Xi)\, \mathrm{d}\Xi \tag{12}$$

with $\mathcal{P}(\Xi) = \prod_{\alpha=1}^{r} p_\alpha(\xi_\alpha)$ and $p_\alpha(\xi_\alpha)$ is the pdf of ξ_α, and $\mathrm{d}\Xi = \prod_{\alpha=1}^{r} \mathrm{d}\xi_\alpha$. When Hermite polynomials are used, a closed-form solution exists for $< ijk >$, which is given in Appendix A. Note also that the polynomials are normalized: $< ij >= \delta_{ij}$ (δ_{ij} is the Kronecker delta).

Once Equation (7) is solved, the pdf can then be estimated with an MCS directly applied to Equation (6). In the following, P is dropped for a sake of simplicity.

3.2. Rational function expansion: Padé approximants

A PA of a function $\mathcal{F}$ is a rational function derived from the Taylor series of $\mathcal{F}$. The PA converges much faster than the Taylor expansion [6, 7] when the function has poles. In this paper, $\mathcal{F} = \mathbf{X}(\Xi)$,

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the response of the uncertain system. First, the function is assumed to depend on one variable (i.e., $\Xi = \xi$). Indeed, the definition of the PA of a multivariate function is not obvious, for reasons that will be presented later.

Consider that the Taylor series expansion of the response, $\mathbf{X}^{Tay}$, is known, up to a given degree, m. A PA of $\mathbf{X}_k$ (k-th element of vector $\mathbf{X}$) is denoted $[M_k/N_k]_{\mathbf{X}_k^{Tay}}$, where M_k is the degree of the numerator and N_k is the degree of the denominator, and is given by

$$[M_k/N_k]_{\mathbf{X}_k^{Tay}}(\xi) = \frac{\sum_{i=0}^{M_k} N_{k,i}^{PA}(\omega)\, \xi^i}{\sum_{i=0}^{N_k} D_{k,i}^{PA}(\omega)\, \xi^i}. \tag{13}$$

The PA is such that

$$\mathbf{X}_k^{Tay}(\omega, \xi) - [M_k/N_k]_{X_k^{Tay}}(\xi) = O(\xi^{M_k+N_k+1}). \tag{14}$$

There are $M_k + N_k + 2$ unknowns, which are defined up to a multiplicative factor: so, usually, $D_{0,k}^{PA}$ is set equal to unity [22]. Hence, to calculate the $M_k + N_k + 1$ coefficients of the PA, m, the degree of the Taylor series expansion is equal to $M_k + N_k$, and then Equation (14) gives $M_k + N_k + 1$ equations.

This is more difficult for multivariate functions as several definitions may hold [22–26]. For the general case, a PA involves $\sharp M_k + \sharp N_k - 1$ unknowns (where $\sharp m$ denotes the number of coefficients of a multivariate polynomial of degree m), if we decide that the numerator (resp. denominator) must contain all terms up to degree M_k (resp. N_k). As a consequence, a Taylor series that has at least $\sharp M_k + \sharp N_k - 1$ coefficients is required to determine the PA unknowns. The problem comes from the relationship between a polynomial degree m and the number of coefficients involved in the definition of a multivariate polynomial with r variables, $\sharp m = (m + r)!/(m!r!)$. Indeed, in general, there does not exist m such that $\sharp m = \sharp M_k + \sharp N_k - 1$. If one considers that all the terms up to degree m must be kept, the problem leads to an overdetermined problem, and $\sharp m \geqslant \sharp M_k + \sharp N_k - 1$. However, one can keep the relation $\sharp m = \sharp M_k + \sharp N_k - 1$ and accept that some polynomials of degree m are not included in the PCE. Then, a decision must be made in the choice of the equations. This will be discussed further in the next section and in Section 6.2.1.

3.3. *Rational function expansion: extended Padé approximants*

In the stochastic finite element context, PCE is much more interesting than a Taylor series. Hence, it is suggested to replace monomial ξ^i, by PC $\Psi_i(\Xi)$. Such generalization had been defined and studied in many papers [6, 10, 27–30]. Chantrasmi *et al.* [31] have already used XPAs (Legendre–Padé approximants) for uncertainty propagation. They proposed multivariate approximants on the basis of a definition given by Guillaume *et al.* [25]. Their objective was to calculate the statistics (pdf) of the position and the strength of a shock in a fluid mechanics context, which involves strong discontinuities (shock waves).

In the present study, the interest of the XPAs for calculating the response pdf of a random dynamical system is twofold. First, they had been developed to accelerate the polynomial expansion convergence rate of a function. This property is important as it had been shown that the PCE has poor convergence properties around the deterministic eigenmodes [2]. Second, it is expected that the response of an uncertain dynamical system is a rational function of the uncertain parameters. Hence, the representation of the response with PAs seems to be more appropriate than a polynomial expansion.

The PAs are extended to a rational function such that the numerator and the denominator are developed in terms of PC as

$$[M_k/N_k]_{\mathbf{X}_k^{PC}}(\Xi) = \frac{\sum_{j=0}^{n_k} N_{k,j}^{XPA}(\omega)\, \Psi_j(\Xi)}{\sum_{j=0}^{d_k} D_{k,j}^{XPA}(\omega)\, \Psi_j(\Xi)}, \tag{15}$$

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where $n_k = \sharp M_k - 1$ and $d_k = \sharp N_k - 1$; k refers to the k-th dof. Similarly to the previous section, $D^{XPA}_{k,0}$ is equal to unity.

$N^{XPA}_{k,i}$ and $D^{XPA}_{k,i}$ are derived by comparing Equation (6) with Equation (15):

$$\sum_{i=0}^{P} \mathbf{Y}_{ik}(\omega)\, \Psi_j(\boldsymbol{\Xi}) = \frac{\sum_{j=0}^{n_k} N^{XPA}_{k,j}(\omega)\, \Psi_j(\boldsymbol{\Xi})}{1 + \sum_{j=1}^{d_k} D^{XPA}_{k,j}(\omega)\, \Psi_j(\boldsymbol{\Xi})}, \tag{16}$$

where $P = \sharp m - 1$ and m is the PCE degree of the response. This is transformed and reorganized as

$$\sum_{j=0}^{n_k} N^{XPA}_{k,j}(\omega)\Psi_j(\boldsymbol{\Xi}) - \sum_{j=1}^{d_k} D^{XPA}_{k,j}(\omega) \left(\sum_{i=0}^{P} \mathbf{Y}_{k,i}(\omega)\Psi_i(\boldsymbol{\Xi})\Psi_j(\boldsymbol{\Xi}) \right) = \sum_{i=0}^{P} \mathbf{Y}_{k,i}(\omega)\, \Psi_i(\boldsymbol{\Xi}). \tag{17}$$

The $n_k + d_k + 1$ coefficients $N^{XPA}_{k,j}$ and $D^{XPA}_{k,j}$ are then calculated by projecting Equation (17) on $\Psi_l(\boldsymbol{\Xi})$ for l from 0 to P': $P' + 1$ equations are obtained:

$$N^{XPA}_{k,l}(\omega)\, \mathrm{Ind}_{n_k}(l) - \sum_{j=1}^{d_k} D^{XPA}_{k,j}(\omega) \left(\sum_{i=0}^{P} \mathbf{Y}_{k,i}(\omega) < i\ j\ l > \right) = \mathbf{Y}_{k,l}(\omega)\, \mathrm{Ind}_P(l), \tag{18}$$

where $\mathrm{Ind}_n(l)$ is equal to unity if $0 \leqslant l \leqslant n$ and to zero otherwise. The factor $\mathrm{Ind}_P(l)$ in the right hand side of Equation (18) suggests that $P' \leqslant P$ otherwise it would mean that $\forall l > P, \mathbf{Y}_{k,l}(\omega) = 0$ in the 'exact' PCE (i.e., with all the terms from 0 to infinity) of the response. Such approximation cannot hold when the PCE does not converge quickly and P is low. As a consequence, in the following, P' is supposed to be lower or equal to P.

$\mathrm{Ind}_{n_k}(l)$ indicates that the coefficients of the denominator are determined first with the following equations:

$$\forall\, l\, /\, n_k + 1 \leqslant l \leqslant P' \quad \sum_{j=1}^{d_k} D^{XPA}_{k,j}(\omega) \left(\sum_{i=0}^{P} \mathbf{Y}_{k,i}(\omega) < i\ j\ l > \right) = -\mathbf{Y}_{k,l}(\omega). \tag{19}$$

To avoid getting an underdetermined system, $P' \geqslant n_k + d_k$. However, the last condition does not provide P and P'. The choice of P' may involve m', which is the degree of $\Psi_{P'}$ and then is an integer such that

$$m' \in \mathbb{N}, \quad \binom{m' - 1}{r} < P' \leqslant \binom{m'}{r}. \tag{20}$$

Equation (17) can be projected on all the polynomials whose degree is lower or equal to m': $P' + 1 = \sharp m'$. Hence, except if by chance $P' = \sharp m' - 1 = n_k + d_k$, the denominator coefficients are the solution of an overdetermined system. Further, as P' is assumed to be lower or equal to P, then $m' \leqslant m$. A further discussion on the choice of P, P', m, and m' is given in Section 6.2.1. The determination of a multivariate XPA has been discussed in several papers (e.g., [25, 26, 32]).

Once the denominator coefficients are determined, the numerator coefficients are obtained directly as

$$\forall\, l/0 \leqslant l \leqslant n_k \quad N^{XPA}_{k,l}(\omega) = \sum_{j=1}^{d_k} D^{XPA}_{k,j}(\omega) \left(\sum_{i=0}^{P} \mathbf{Y}_{k,i}(\omega) < i\ j\ l > \right) + \mathbf{Y}_{k,l}(\omega). \tag{21}$$

Finally, by performing an MCS on $[M_k/N_k]_{\mathbf{X}^{PC}_k}(\boldsymbol{\Xi})$, the pdf of the response may be estimated.

Note that in the single variate case, the XPA is determined easily: the PCE degree is $M_k + N_k$, and $P + 1 = P' + 1 = n_k + d_k + 1 = \sharp(M_k + N_k)$.

4. RANDOM MODES

A natural way to obtain the response of an N-dof dynamical system is to expand the solution on the eigenvectors

$$\mathbf{X}(t) = \sum_{k=1}^{N} q_k(t)\, \boldsymbol{\phi}_k, \tag{22}$$

where $\boldsymbol{\phi}_k$ is an eigenvector and q_k defines the deterministic modal coordinate for the k-th eigenvector.

The mass and stiffness matrices are random, so the eigenmodes, which will be denoted $\{\tilde{\omega}_k,\ \tilde{\boldsymbol{\phi}}_k\}$, are random as well. Then the random mode superposition reads

$$\mathbf{X}(t) = \sum_{n=1}^{N} \tilde{q}_n(t)\, \tilde{\boldsymbol{\phi}}_n, \tag{23}$$

where modal coordinate $\tilde{q}_n$ is random and depends on the random eigenmodes. Equation (23) holds not only to describe a steady-state response of a dynamical system but also for the transient response even if it has not been used in this latter context so far.

When force vector $\mathbf{F}$ is harmonic with frequency ω, the steady-state response is

$$\mathbf{X}(\omega) = \sum_{n=1}^{N} \tilde{q}_n(\omega)\, \tilde{\boldsymbol{\phi}}_n. \tag{24}$$

Modal coordinate $q_n(\omega)$ is derived by substituting Equation (24) in Equation (1) and by projecting this latter equation on each $\tilde{\boldsymbol{\phi}}_n$. Then the n-th modal equation is

$$(-\omega^2 + 2\tilde{\eta}_n\, \tilde{\omega}_n\omega + \tilde{\omega}_n^2)\, \tilde{q}_n(t) = \frac{\tilde{\boldsymbol{\phi}}_n^T\, \mathbf{F}}{\tilde{m}_n}, \tag{25}$$

where $\tilde{\eta}_n$ (resp. $\tilde{m}_n$) is the damping ratio (resp. the generalized modal mass) of mode n. In the following, the random damping may be calculated from the damping matrix:

$$\tilde{\eta}_n = \frac{\tilde{\boldsymbol{\phi}}_n^T\, \mathbf{D}\, \tilde{\boldsymbol{\phi}}_n}{2\, \tilde{\omega}_n\, \tilde{m}_n}. \tag{26}$$

Then the modal coordinate reads

$$\tilde{q}_n(t) = \frac{\tilde{\boldsymbol{\phi}}_n^T\, \mathbf{F}}{\tilde{m}_n\, (\tilde{\omega}_n^2 - \omega^2 + 2\tilde{\eta}_n\, \tilde{\omega}_n\omega)}. \tag{27}$$

Equation (27) shows that the response of the random dynamical system is a rational function of the random parameters, $\tilde{\boldsymbol{\phi}}_n$, $\tilde{\omega}_n$, $\tilde{\eta}_n$, and $\tilde{m}_n$. This is why the PA approach is appropriate as it consists in finding a rational function of the uncertain parameters.

The random eigenmodes can be determined with a MCS or a PCE. Considering the use of a PCE, they are expanded as follows [20, 21]:

$$\tilde{\omega}_k^2 = \omega_k^2 \left(\sum_{p=0}^{P} a_p^k\, \Psi_p(\boldsymbol{\Xi}) \right), \tag{28}$$

$$\tilde{\boldsymbol{\phi}}_k = \sum_{n=1}^{N} \tilde{\lambda}_n^k\, \boldsymbol{\phi}_n = \sum_{n=1}^{N} \left(\sum_{p=0}^{P} \lambda_{np}^k\, \Psi_p(\boldsymbol{\Xi}) \right) \boldsymbol{\phi}_n, \tag{29}$$

where $(\omega_k, \boldsymbol{\phi}_k)$ denotes the k-eigenmode of the deterministic system, defined in Section 2.

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$\{a_p^k, \{\lambda_{np}^k\}_{n=1\dots N}\}_{p=0\dots P}$ are the PC coefficients related to the PCE of random mode k.

Further the following mass normalization is applied

$$\boldsymbol{\phi}_k^T \mathbf{M}_0 \, \tilde{\boldsymbol{\phi}}_k = 1, \tag{30}$$

where $\mathbf{M}_0$ is the mean mass matrix. As a consequence,

$$\tilde{\lambda}_k^k = 1. \tag{31}$$

Then Equation (29) becomes

$$\tilde{\boldsymbol{\phi}}_k = \boldsymbol{\phi}_k + \sum_{\substack{n=1 \\ n\neq k}}^{N} \sum_{p=0}^{P} \lambda_{np}^k \, \Psi_p(\boldsymbol{\Xi}) \, \boldsymbol{\phi}_n. \tag{32}$$

Equations (28) and (32) show that the PCE of random mode k requires $N \times (P+1)$ unknowns. Projecting the eigenproblem

$$\left(\tilde{\mathbf{K}} - \tilde{\omega}_k^2 \, \tilde{\mathbf{M}}\right) \tilde{\boldsymbol{\phi}}_k = 0 \tag{33}$$

on each deterministic eigenmode $\{\boldsymbol{\phi}_n\}_{n=1\dots N}$ and each PC $\{\Psi_p(\boldsymbol{\Xi})\}_{p=0\dots P}$ gives the $N \times (P+1)$ related equations.

5. EXAMPLE 1

5.1. Two degree-of-freedom system with one uncertain parameter

The MCS, PCE, and random modes will be used to evaluate the pdf of $\mathbf{X}$ for the example shown in Figure 1. MCSs will serve as a reference for validating the results obtained with the XPA and random modes approaches. Stiffnesses k_1 and k_2 are assumed to be equal and uncertain:

$$k_1 = k_2 = \overline{k} \, (1 + \delta_K \, \xi), \tag{34}$$

where ξ is random variable. Thus, the uncertain stiffness matrix is

$$\mathbf{K} = \mathbf{K}_0 + \delta_K \xi \mathbf{K}_1 = \mathbf{K}_0 (1 + \delta_K \xi), \tag{35}$$

where

$$\mathbf{K}_0 = \mathbf{K}_1 = \overline{k} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{36}$$

In the following, ξ is either a truncated normal variable ($\xi \sim \mathcal{N}_{[-5;\,5]}(0;\,1)$) or a uniform random variable ($\xi \sim \mathcal{U}_{[-1;\,1]}$).

The characteristics of the system are listed in Tables I and II.

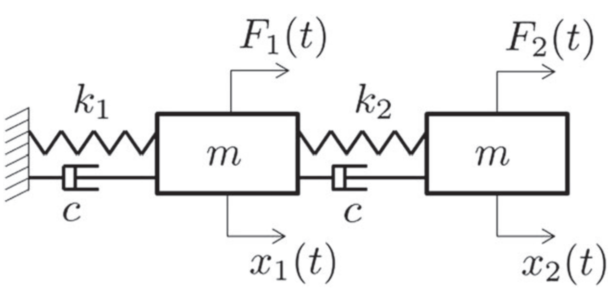

Figure 1. A two degree-of-freedom system with stochastic stiffness coefficients.

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Table I. System characteristics.

$\bar{k}$ (Nm^{-1})	m (kg)	c ($Nm^{-1}s^{-1}$)	δ_K (%)	F_{01} (N)	F_{02} (N)
15,000	1	1	5	1	0

Table II. Modal characteristics of the deterministic system.

Eigenfrequencies f (Hz)	12.05	31.54
Damping ratio (%)	0.25	0.66

5.2. ξ: truncated normal deviate

The mean and the standard deviation of the random stiffness can then be deduced from Table I. Note that if ξ had a normal law, the positiveness of the stiffness would be questionable. However, the ratio of standard deviation to the mean indicates that the probability to draw a negative stiffness is so low that the numerical estimation of this probability by a software like MATLAB is 0 and the probability to draw a stiffness lower than $0.75 \times \bar{k}$ is about $2.8\ 10^{-7}$. In the following, the number of samples is lower than one million. Hence, in practice, such statistical law could be used. However, to avoid such issue, the normal law is truncated so that $k \in [0.75\bar{k};\ 1.25\bar{k}]$: this corresponds to the mean plus/minus five standard deviations.

5.2.1. Probability density function: exact solution. The steady-state response $\mathbf{X} = [X_1 X_2]^T$ is the solution of the following equation:

$$(-\omega^2 \mathbf{M} + \mathbf{K})\ \mathbf{X}(\xi, \omega) = \mathbf{F}. \quad (37)$$

Thus, the exact solution, for each dof k, is the following rational function:

$$X_k(\xi, \omega) = \frac{N_{0,k} + N_{1,k}\ \xi}{1 + D_1\ \xi + D_2\ \xi^2} \quad (38)$$

with

$$\begin{aligned}
D_0 &= \bar{k}^2 + 2\ ic\omega\ \bar{k} - \omega^2(3\ \bar{k}\ m + c^2) - 3\ i\omega^3 m\ c + \omega^4 m^2 \\
D_1 &= \frac{1}{D_0}\left(2\ \bar{k}^2\ \delta_k - 3\ \bar{k}\ \delta_k\ \omega^2 m + 2\ ic\omega\ \bar{k}\ \delta_k\right) \\
D_2 &= \frac{1}{D_0}\left(\bar{k}^2\ \delta_k^2\right) \\
N_{0,1} &= \frac{1}{D_0}\left(\bar{k} - \omega^2 m + ic\omega\right) \\
N_{1,1} &= \frac{1}{D_0}\left(\bar{k}\ \delta_k\right) \\
N_{0,2} &= \frac{1}{D_0}\left(-\bar{k} - ic\omega\right) \\
N_{1,2} &= \frac{1}{D_0}\left(-\bar{k}\ \delta_k\right).
\end{aligned}$$

Note that normalized Hermite polynomials are related to the monomials

$$1 = \Psi_0(\xi), \quad (39)$$

$$\xi = \Psi_1(\xi), \quad (40)$$

$$\xi^2 = \sqrt{2}\ \Psi_2(\xi) + \Psi_0(\xi). \tag{41}$$

Then, expression (38) can easily be transformed into a rational function whose numerator and denominator are expanded in terms of the Hermite polynomials as

$$X_k(\xi,\omega) = \frac{N_{0,k}^{HP} + N_{1,k}^{HP}\ \Psi_1(\xi)}{1 + D_1^{HP}\ \Psi_1(\xi) + D_2^{HP}\ \Psi_2(\xi)} \tag{42}$$

with

$$\begin{aligned}
D_0^{HP} &= D_0(1+D_2) = (\overline{k}^2 + 2\ ic\omega\ \overline{k} - \omega^2(3\ \overline{k}\ m + c^2) - 3\ i\omega^3 m\ c + \omega^4 m^2)\left(1 + \overline{k}^2 \delta_k^2\right)\\
D_1^{HP} &= D_0 \frac{D_1}{D_0^{HP}} = \frac{1}{D_0^{HP}}\left(2\ \overline{k}^2 \delta_k - 3\ \overline{k}\ \delta_k\ \omega^2 m + 2\ ic\omega\ \overline{k}\ \delta_k\right)\\
D_2^{HP} &= D_0 \frac{D_2}{D_0^{HP}} = \frac{1}{D_0^{HP}}\left(\overline{k}^2 \delta_k^2\right)\\
N_{0,1}^{HP} &= D_0 \frac{N_{0,1}}{D_0^{HP}} = \frac{1}{D_0^{HP}}\left(\overline{k} - \omega^2 m + ic\omega\right)\\
N_{1,1}^{HP} &= D_0 \frac{N_{1,1}}{D_0^{HP}} = \frac{1}{D_0^{HP}}\left(\overline{k}\ \delta_k\right)\\
N_{0,2}^{HP} &= D_0 \frac{N_{0,2}}{D_0^{HP}} = \frac{1}{D_0^{HP}}\left(-\overline{k} - ic\omega\right)\\
N_{1,2}^{HP} &= D_0 \frac{N_{1,2}}{D_0^{HP}} = \frac{1}{D_0^{HP}}\left(-\overline{k}\ \delta_k\right).
\end{aligned}$$

Equation (42) shows that the exact solution is a rational function of the random parameter: deriving an estimation of the solution in terms of PAs, which are rational functions, is then appropriate.

The reference pdf is obtained with a direct MCS method together with a Latin Hypercube Sampling (LHS) with 10,000 samples of the random variable. It has been verified that the number of samples is sufficient for the convergence of the solution. The pdf is estimated at the first deterministic eigenfrequency, which seems to be the worst case [2]. The results are given in Figure 2(a).

5.3. *Probability density function: polynomial chaos expansion and extended Padé approximant*

The pdfs were also calculated directly from the PCE and with the Padé approach: they were compared with the reference pdf with the Kullback–Leibler divergence [33–35], D_{KL}, defined as

$$D_{KL}(p_{ref}(x)||p(x)) = \int_{D_x} p_{ref}(x)\ \ln\left(\frac{p_{ref}(x)}{p(x)}\right)\ \mathrm{d}x, \tag{43}$$

where D_x is the domain of a random variable x. D_{KL} is always nonnegative and is equal to zero when $p_{ref}(x) = p(x)$ almost everywhere.

An LHS with 10,000 samples was also performed directly on the PCE with $P = 500$ and $P = 501$: the pdfs are given in Figure 2(c) and (d). With a degree $P = 500$, a quite good estimation of the pdf is reached. However, the results are poor with $P = 501$. In fact, the parity influence on the first statistical moments was already noticed in [2].

A [0/1] PA pdf (i.e., $n_k = 0$ and $d_k = 1$) was derived with MCS (10,000 samples were used): it required a PCE with $P = 2$. The pdf is given in Figure 2(b). The quality of the results with such a low PCE degree is striking. In fact, increasing the numerator and denominator degree does not really improve the results. However, surprisingly, the only configuration that is not excellent is XPA [1/2] (Figure 3), even though this configuration should be the best, because the closed-form expression of the pdf is a rational function whose numerator (resp. denominator) degree is equal to 1 (resp. to 2). However, even this configuration accurately predicts the peak of the pdf, even though the tail is poorly predicted.

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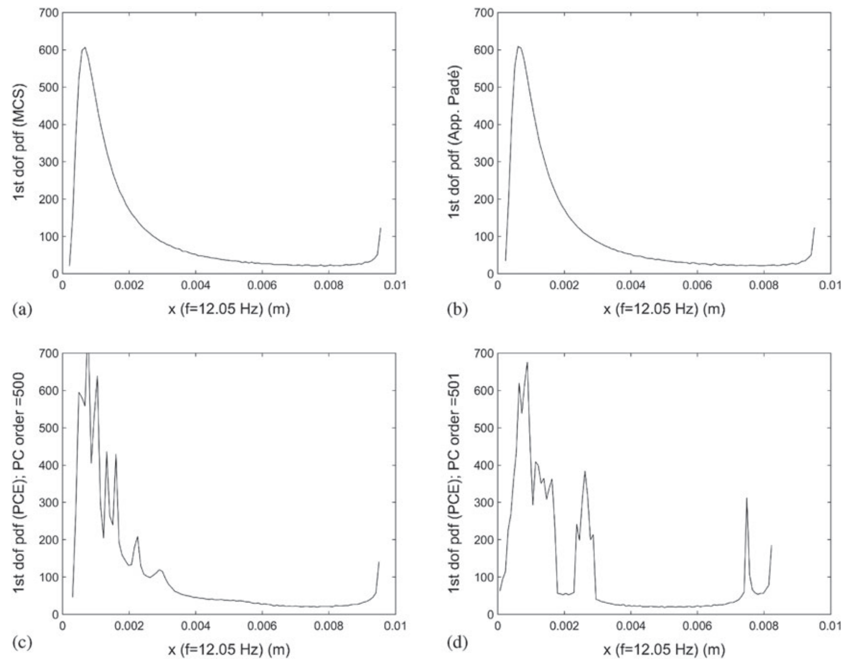

Figure 2. Probability density function of the response at the first deterministic eigenfrequency; (a) MCS (10,000 samples); (b) XPA ([0/1], $P = 1$); (c) PCE ($P = 500$); (d) PCE ($P = 501$). MCS, Monte Carlo simulation; PCE, polynomial chaos expansion.

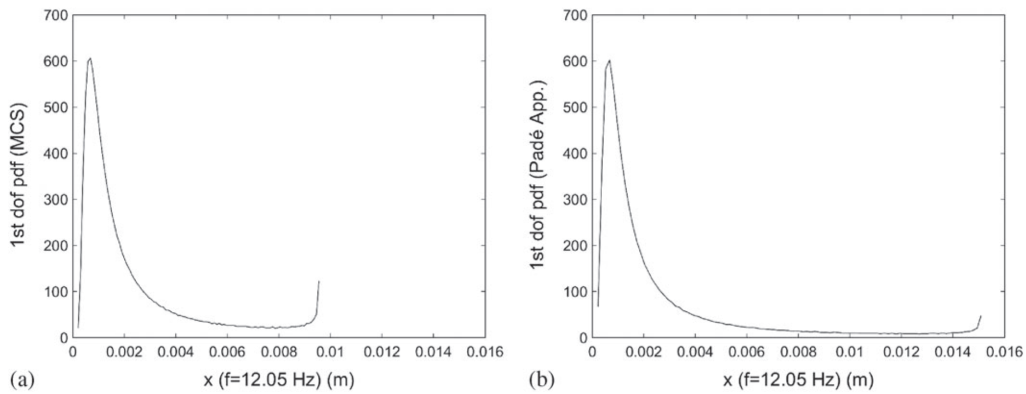

Figure 3. Probability density function of the response at the first deterministic eigenfrequency; (a) MCS (10,000 samples); (b) XPA ([1/2], $P = 3$). MCS, Monte Carlo simulation; XPA, extended Padé approximant.

The Kullback–Leibler divergences of the pdf calculated with the PCE approach and the Padé technique are listed in Table III: the results confirm the qualitative conclusions given from Figures 2 and 3. In particular, the divergences show that estimating the pdf with the Padé technique is much more efficient than with the PCE approach. Further, the Padé [1/2] divergence is quite low despite some dissimilarities: this is because only the tails of the distribution are not similar.

5.3.1. Mean and standard deviation: Monte Carlo simulation and extended Padé approximant. In [2], it was shown that the mean and the standard deviations are two slowly convergent sequences. A solution to improve the convergence rate was proposed in [4]. Knowing the pdf, any moments of the statistical distribution may be derived. If the pdf is well estimated with a low degree XPA, the moments must be very well estimated as well.

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Table III. Kullback–Leibler divergence – Example 1 – truncated normal deviate.

pdf	PCE 500	PCE 501	Padé [0/1]	Padé [1/2]	Padé [2/2]
D_{KL}	0.38	2.00	510^{-3}	0.09	1.510^{-4}

PCE, polynomial chaos expansion; pdf, probability density function.

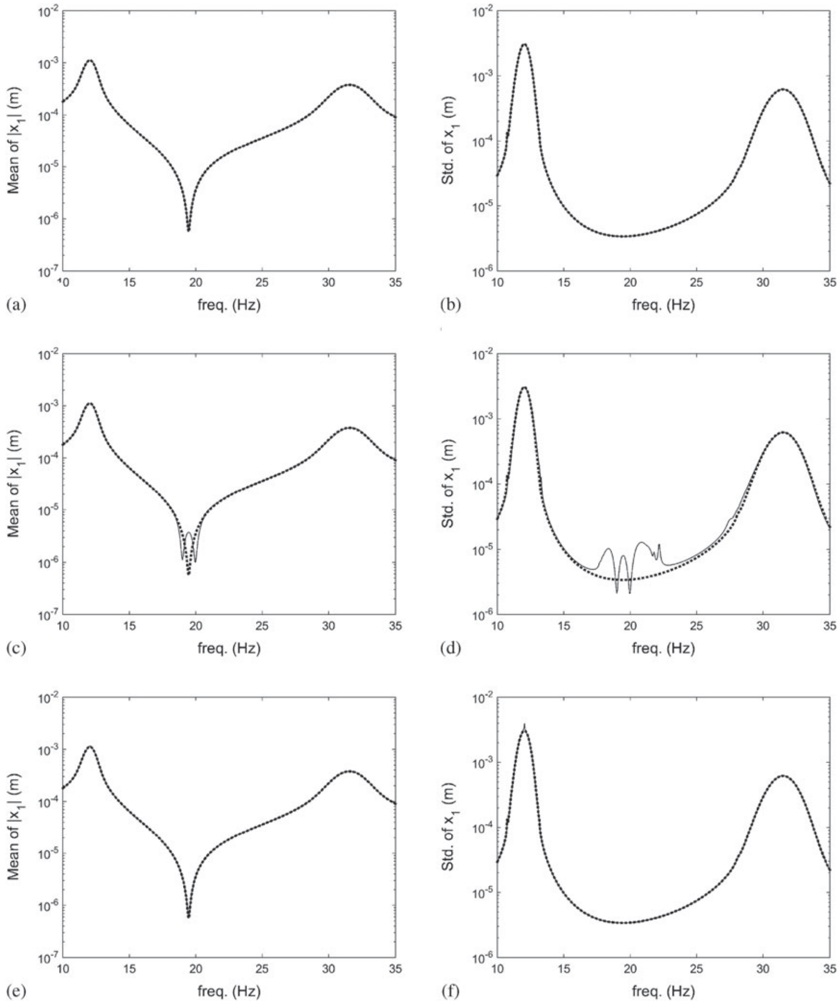

Figure 4. First moments (XPA: solid lines; MCS: dotted line) for several XPAs; (a) [2/2] XPA mean; (b) [2/2] XPA standard deviation; (c) [0/1] XPA mean; (d) [0/1] XPA standard deviation; (e) [1/2] XPA mean; (f) [1/2] XPA standard deviation. MCS, Monte Carlo simulation; XPA, extended Padé approximant.

The first two moments are given in Figure 4 for several XPAs. Figure 4(a) and (b) shows that with $P = 5$, it is possible to obtain excellent estimates of the first two moments. The XPA approach is then much more efficient than the Aitken method proposed in [4], as shown in Figure 5 where $P = 20$. It has been observed that a [0/1] XPA gives an excellent pdf at the first eigenfrequencies. However, Figure 4(c) and (d) shows that the moment estimation is poor about the deterministic antiresonant frequency. On the contrary, the moment is very well estimated with a [1/2] XPA, even around the deterministic eigenfrequencies, that is, where the pdf was not well estimated (Figure 4(e) and (f)).

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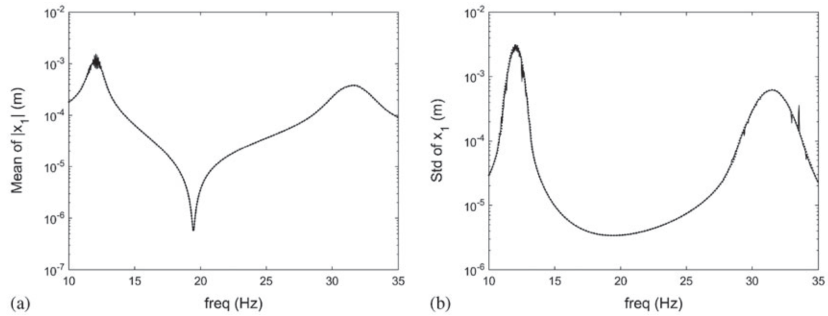

Figure 5. First moments (Aitken method [4]: solid lines; MCS: dotted line); (a) Aitken mean; (b) Aitken std. MCS, Monte Carlo simulation; std, standard deviation.

5.3.2. Random modes: exact solution. The deterministic modes are solutions to the following equation:

$$\left(\mathbf{K}_0 - \omega_k^2\, \mathbf{M}\right) \boldsymbol{\phi}_k = 0 \tag{44}$$

whereas the random modes are solution to

$$\left(\mathbf{K}_0(1 + \delta_K \xi) - \tilde{\omega}_k^2\, \mathbf{M}\right) \tilde{\boldsymbol{\phi}}_k = 0. \tag{45}$$

Then it is easy to derive the expression of the random modes as functions of the deterministic modes as

$$\tilde{\omega}_k^2 = \omega_k^2\,(1 + \delta_K \xi), \tag{46}$$

$$\tilde{\boldsymbol{\phi}}_k = \boldsymbol{\phi}_k. \tag{47}$$

In this particular case, the random eigenvectors are equal to the deterministic ones: this occurs because the random stiffness matrix is proportional to the deterministic stiffness matrix.

5.3.3. Random modes: polynomial chaos expansion. In the following, if index k is equal to 1, then index k' is equal to 2 and vice versa. Random mode k is determined according to the method indicated previously and then is expanded according to Equations (28) and (29). Thus, the following equation has to be solved:

$$\left(\mathbf{K}_0(1 + \delta_K \xi) - \omega_k^2 \left(\sum_{p=0}^{P} a_p^k \Psi_p(\xi)\right) \mathbf{M}\right) \left(\boldsymbol{\phi}_k + \sum_{p=0}^{P} \lambda_{k'p}^k \Psi_p(\xi) \boldsymbol{\phi}_{k'}\right) = 0. \tag{48}$$

$\{\omega_k, \boldsymbol{\phi}_k\}$ are the deterministic eigenmodes of the dynamical system defined in Equation (44). Multiplying Equation (48) by each eigenvector and using the orthogonality properties give

$$(1 + \delta_K \xi) - \sum_{p=0}^{P} a_p^k \Psi_p(\xi) = 0, \tag{49}$$

$$\omega_{k'}^2 \sum_{p=0}^{P} \lambda_{k'p}^k (\Psi_p(\xi) + \delta_K \xi \Psi_p(\xi)) - \omega_k^2 \sum_{p=0}^{P} \sum_{q=0}^{P} a_p^k \lambda_{k'q}^k \Psi_p(\xi) \Psi_q(\xi) = 0. \tag{50}$$

Note that $\Psi_0(\xi) = 1$ and $\Psi_1(\xi) = \xi$. Multiplying the last two equations by $\Psi_m(\xi)$ in the random space gives

$$a_m^k = (< m > + \delta_K < 1m >), \tag{51}$$

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$$\omega_{k'}^2 \sum_{p=0}^{P} \lambda_{k'p}^k (< mp > + \delta_K < 1mp >) - \omega_k^2 \sum_{p=0}^{P} \sum_{q=0}^{P} a_p^k \lambda_{k'q}^k < mpq >= 0. \tag{52}$$

Solving Equations (51) and (52) gives

$$a_0^k = 1, \tag{53}$$

$$a_1^k = \delta_K, \tag{54}$$

$$\forall p > 1 \; a_p^k = 0, \tag{55}$$

$$\forall p \in \mathbb{N} \lambda_p^k = 0. \tag{56}$$

Then the random mode k estimate is

$$\tilde{\omega}_k^2 = \omega_k^2(\Psi_0(\xi) + \delta_K \Psi_1(\xi)) = \omega_k^2(1 + \delta_K \xi), \tag{57}$$

$$\tilde{\boldsymbol{\phi}}_k = \boldsymbol{\phi}_k. \tag{58}$$

Comparing Equations (46) and (47) with the last two equations proves that a PCE of degree 1 gives the exact random modes and therefore the exact solution of the uncertain problem.

This result may be extended to all the dynamical systems with an uncertain stiffness matrix that verifies Equation (35), but the result does not hold in general, in particular, when the mass matrix is uncertain or when the number of uncertain parameters is greater than one.

5.4. ξ: uniform deviate

The interval of the random stiffness can then be deduced from Table I.

The reference pdf is obtained with a direct MCS method together with an LHS with 10,000 samples of the random variable. It has been verified that the number of samples is sufficient for the convergence of the solution. The pdf is estimated at the first deterministic eigenfrequency. The results are given in Figure 6(a).

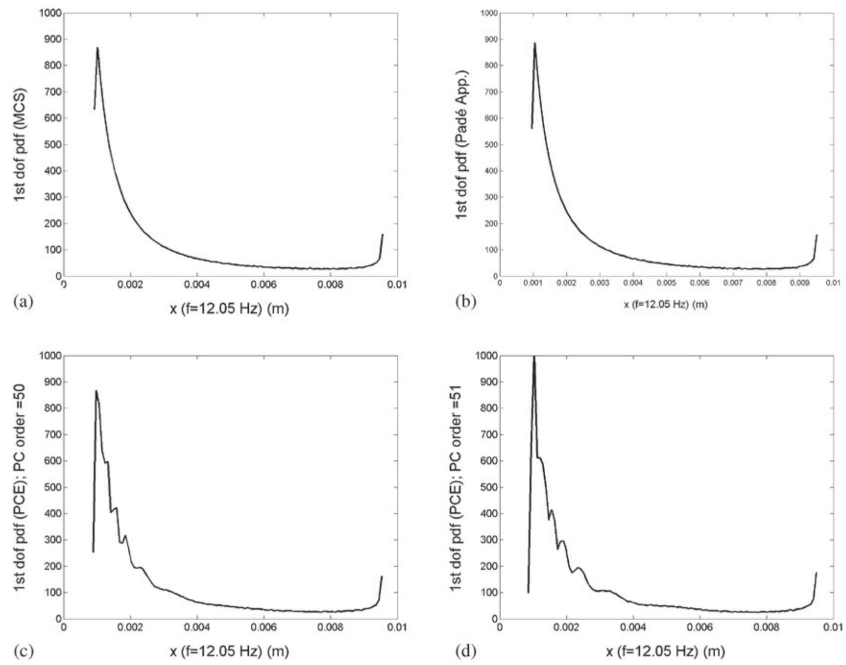

Figure 6. Probability density function of the response at the first deterministic eigenfrequency; (a) MCS (10,000 samples); (b) XPA ([0/2], $P = 2$); (c) PCE ($P = 50$); (d) PCE ($P = 51$). dof, degree of freedom; MCS, Monte Carlo simulation; PC, polynomial chaos; PCE, PC expansion; pdf, probability density function; XPA, extended Padé approximant.

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Table IV. Kullback–Leibler divergence – Example 1 – uniform deviate.

pdf	PCE $P = 50$	PCE $P = 51$	PCE $P = 2$	Padé [0/2]
D_{KL}	510^{-3}	0.39	7.6	10^{-3}

PCE, polynomial chaos expansion; pdf, probability density function.

5.4.1. Probability density function: polynomial chaos expansion and extended Padé approximant. The pdfs are also calculated directly from the PCE and with the Padé approach: they are plotted in Figure 6(b)–(d), and they are compared with the reference pdf. The Kullback–Leibler divergences of the pdf calculated with the PCE approach and the Padé technique are listed in Table IV.

As indicated in [36], a PCE with Legendre polynomials (uniform distribution) converges much quicker than with the Hermite polynomials (normal distribution): the results are quite good with $P = 50$ whereas in the previous case, they were poor with $P = 500$.

The results are excellent with a [0/2] XPA (Table IV), which requires a PCE with $P = 2$: however, the pdf calculated with a PCE with $P = 2$ is far from the MCS pdf, as indicated with the Kullback–Leibler divergence given in Table IV.

5.4.2. Random modes: polynomial chaos expansion. Deriving the calculations made in Section 5.3.3 with the normalized Legendre polynomials leads to the same results: the random modes obtained with a PCE are the exact random modes. Note that the second normalized Legendre polynomial is $\Psi_1(\xi) = \sqrt{3}\,\xi$, that is, $\xi = \Psi_1(\xi)/\sqrt{3}$. As a consequence, Equation (51) is slightly modified: $a_m^k = (< m > +\delta_K < 1m >)/\sqrt{3}$.

6. EXAMPLE 2

6.1. Two degree-of-freedom system with two uncertain parameters

The example shown in Figure 1 is studied with uncertain stiffnesses k_1 and k_2:

$$k_1 = \bar{k}(1 + \delta_K \xi_1), \tag{59}$$

$$k_2 = \bar{k}(1 + \delta_K \xi_2), \tag{60}$$

where ξ_1 and ξ_2 are two independent normal random variables. In the following, ξ_i is either a truncated normal variable ($\xi_i \sim \mathcal{N}_{[-5;\,5]}(0;\,1)$) or a uniform random variable ($\xi_i \sim \mathcal{U}_{[-1;\,1]}$). The characteristics of the system are listed in Table I. Thus, the uncertain stiffness matrix is

$$\mathbf{K} = \mathbf{K}_0 + \xi_1\ \mathbf{K}_1 + \xi_2\ \mathbf{K}_2, \tag{61}$$

where

$$\mathbf{K}_0 = \bar{k} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \tag{62}$$

$$\mathbf{K}_1 = \bar{k}\ \delta_K \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{63}$$

$$\mathbf{K}_2 = \bar{k}\ \delta_K \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{64}$$

The response of the system is

$$X_1(\xi_1, \xi_2, \omega) = \frac{-\omega^2 m + \imath\omega c + a_2}{\omega^4 m^2 - \omega^3 3\imath cm - \omega^2(m(a_1 + 2a_2) + c^2) + \omega\imath c(a_1 + 3a_2) + a_1 a_2} \tag{65}$$

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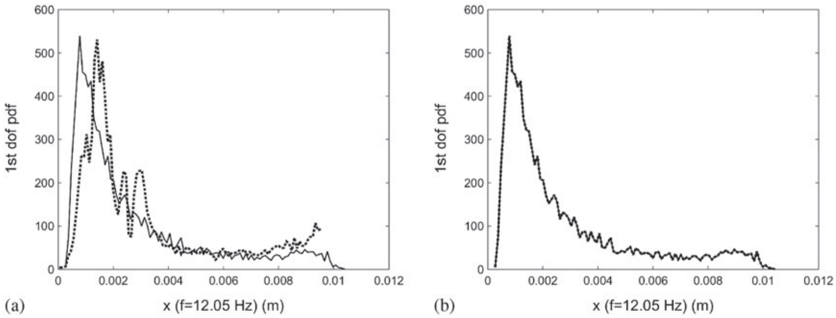

Figure 7. Probability density function of the response at the first deterministic eigenfrequency; normal deviates; MCS (solid line) versus (a) PCE (degree 50, $P = 1325$ – dotted line) and (b) XPA ([1/2], $P = 14$ – dotted line)). dof, degree of freedom; PCE, polynomial chaos expansion; pdf, probability density function; XPA, extended Padé approximant.

Table V. Kullback–Leibler divergence – Example 2 – truncated normal deviate.

pdf	PCE 50	Padé [1/2]	Mode + PCE
D_{KL}	0.32	0	0.008

PCE, polynomial chaos expansion; pdf, probability density function.

$$X_2(\xi_1, \xi_2, \omega) = \frac{1 + \delta_k \xi_2 + \imath\omega c}{\omega^4 m^2 - \omega^3 3\imath cm - \omega^2(m(a_1 + 2a_2) + c^2) + \omega\imath c(a_1 + 3a_2) + a_1 a_2} \tag{66}$$

with $a_1 = k_1/\overline{k} = 1 + \delta_k \xi_1$ and $a_2 = k_2/\overline{k} = 1 + \delta_k \xi_2$.

The reference pdf is still obtained with an LHS with 10,000 samples. The pdf was estimated at the first deterministic eigenfrequency, and the results are plotted in Figure 7 (normal deviates) and in Figure 9 (uniform deviates).

6.2. *Truncated normal deviates*

Both random variables ξ_1 and ξ_2 are drawn according to a truncate normal law to avoid any negative stiffness: $\xi_i \sim \mathcal{N}_{[-5;\,5]}(0;\ 1)$. Then, random stiffness k_i is in the intervalle given by the mean plus/minus five standard deviations.

6.2.1. Probability density function: polynomial chaos expansion and extended Padé approximant. The pdf was estimated with a PCE of degree 50, which required 1326 terms in the expansion. Figure 7(a) shows that the quality of the results is poor, even though the expansion requires more terms: the Kullback–Leibler divergences are listed in Table V.

The pdf was also calculated with the XPA approach. The notation of Section 3.3 is used. To have the smallest systems of equations as possible, m' is chosen minimal: it is the lowest integer such that $\sharp m' \geqslant n_k + d_k + 1$. Then P' is such that $n_k + d_k + 1 \leqslant P' + 1 \leqslant \sharp m'$. If $P' + 1$ is chosen equal to $\sharp m'$, Equation (17) is projected on all the PC of degree lower or equal to m'. If $P' + 1$ is chosen equal to $n_k + d_k + 1$, the system is minimal. Limiting the number of PCE coefficients suggests that $m = m'$ is a suitable choice. However, the numerical experiments show that the XPA approach is a little more efficient when $m \geqslant m' + 1$. In practice, the XPA was determined with $m = m' + 1$, $P + 1 = \sharp m$ (the response is expanded on all the PCs of degree lower or equal to m). Further, all the simulations have shown that the results are exactly the same if $P' + 1 = \sharp m'$ or if $P' + 1 = n_k + d_k + 1$.

Then, the XPA [1/2] results, which necessitate a PCE of degree $m = 4$ ($P + 1 = \sharp m$ =15 terms in the expansion), and a projection on $P' + 1 = n + d + 1 = 8$ Hermite polynomials of degree lower or equal to $m' = m - 1 = 3$, are equal to the MCS results (Figure 7(b)) as indicated by a divergence equal to zero. This is in a perfect agreement with Equation (65) as the numerator degree is equal to 1 and the denominator degree is equal to 2, if the response is considered as a function of the random variates (i.e., for a given frequency ω). Further, Equation (65) shows that the response has no term in ξ_1 in the numerator: it was found that the XPA has no term in ξ_1 in the numerator as well. The numerical results have shown that any XPA gives the rational function calculated with the XPA [1/2], if the requested degree for numerator (resp. denominator) is greater or equal to 1 (resp. 2). Hence, this approach is very efficient for this case study as it is possible to find the analytical results given by Equation (65).

On the contrary, the PCE of degree 50 had a divergence equal to 0.32, which indicates that the results are not in very good agreement with the reference pdf.

6.2.2. Random modes: Monte Carlo simulation solution. The random modes are solutions of

$$\left(\mathbf{K}_0 + \xi_1 \mathbf{K}_1 + \xi_2 \mathbf{K}_2 - \tilde{\omega}_k^2 \mathbf{M}\right) \tilde{\boldsymbol{\phi}}_k = 0. \tag{67}$$

Then the random modes are

$$\overline{\omega}_1^2 = \frac{\overline{k}}{2m}(a_1 + 2\,a_2 - \sqrt{a_1^2 + 4\,a_2^2}), \tag{68}$$

$$\overline{\boldsymbol{\phi}}_1 = \begin{bmatrix} 2a_2 \\ a_1 + \sqrt{a_1^2 + 4\,a_2^2} \end{bmatrix}, \tag{69}$$

$$\overline{\omega}_2^2 = \frac{\overline{k}}{m}(a_1 + 2\,a_2 + \sqrt{a_1^2 + 4\,a_2^2}), \tag{70}$$

$$\overline{\boldsymbol{\phi}}_2 = \begin{bmatrix} 2a_2 \\ a_1 - \sqrt{a_1^2 + 4\,a_2^2} \end{bmatrix}. \tag{71}$$

The MCS gives the mean of the random modes as

$$\overline{\omega}_1^2 = 5723 \text{ (rad/s)}^2, \tag{72}$$

$$\overline{\boldsymbol{\phi}}_1 == \begin{bmatrix} 0.5256 \\ 0.8507 \end{bmatrix}, \tag{73}$$

$$\overline{\omega}_2^2 = 39277 \text{ (rad/s)}^2, \tag{74}$$

$$\overline{\boldsymbol{\phi}}_2 = \begin{bmatrix} -0.8505 \\ 0.5260 \end{bmatrix}. \tag{75}$$

6.2.3. Random modes: polynomial chaos expansion. Random mode k is determined according to the method indicated previously and then is expanded according to Equations (28) and (29). Then the following equation has to be solved:

$$\left(\mathbf{K}_0 + \xi_1 \mathbf{K}_1 + \xi_2 \mathbf{K}_2 - \omega_k^2 \left(\sum_{p=0}^{P} a_p^k \Psi_p(\boldsymbol{\Xi})\right) \mathbf{M}\right) \left(\boldsymbol{\phi}_k + \sum_{p=0}^{P} \lambda_{k'p}^k \Psi_p(\boldsymbol{\Xi}) \boldsymbol{\phi}_{k'}\right) = 0. \tag{76}$$

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Note that the PCs are numbered so that $\Psi_0(\Xi) = 1$, $\Psi_1(\Xi) = \xi_1$, and $\Psi_2(\Xi) = \xi_2$. Then Equation (76) may be written as

$$\left(\Psi_0(\Xi)\mathbf{K}_0 + \Psi_1(\Xi)\mathbf{K}_1 + \Psi_2(\Xi)\mathbf{K}_2 - \omega_k^2\left(\sum_{p=0}^{P} a_p^k\Psi_p(\Xi)\right)\mathbf{M}\right)\left(\boldsymbol{\phi}_k + \sum_{p=0}^{P}\lambda_{k'p}^k\Psi_p(\Xi)\boldsymbol{\phi}_{k'}\right) = 0. \tag{77}$$

Equation (77) is projected on each $\Psi_m(\Xi)$ in the random space:

$$\begin{aligned}\forall m = 0\cdots P, &\left(<0m>\mathbf{K}_0 + <1m>\mathbf{K}_1 + <2m>\mathbf{K}_2 - \omega_k^2 a_m^k\mathbf{M}\right)\boldsymbol{\phi}_k \\ &- \omega_k^2\left(\sum_{p=0}^{P}\sum_{q=0}^{P} a_p^k\lambda_{k'q}^k <pqm>\right)\mathbf{M}\boldsymbol{\phi}_{k'} \\ &+ \sum_{p=0}^{P}\lambda_{k'p}^k\left(<0pm>\mathbf{K}_0 + <1pm>\mathbf{K}_1 + <2pm>\mathbf{K}_2\right)\boldsymbol{\phi}_{k'} = 0.\end{aligned} \tag{78}$$

Projecting Equation (78) onto the deterministic eigenvectors gives the set equations required to solve for the unknowns. Hence, pre-multiplying Equation (78) by $\boldsymbol{\phi}_k$ gives

$$\begin{aligned}&\sum_{p=0}^{P}\left(<0pm>\boldsymbol{\phi}_k^T\mathbf{K}_0\boldsymbol{\phi}_{k'} + <1pm>\boldsymbol{\phi}_k^T\mathbf{K}_1\boldsymbol{\phi}_{k'} + <2pm>\boldsymbol{\phi}_k^T\mathbf{K}_2\boldsymbol{\phi}_{k'}\right)\lambda_{k'p}^k - \omega_k^2 a_m^k \\ &= -\left(<0m>\boldsymbol{\phi}_k^T\mathbf{K}_0\boldsymbol{\phi}_k + <1m>\boldsymbol{\phi}_k^T\mathbf{K}_1\boldsymbol{\phi}_k + <2m>\boldsymbol{\phi}_k^T\mathbf{K}_2\boldsymbol{\phi}_k\right),\end{aligned} \tag{79}$$

and pre-multiplying Equation (78) by $\boldsymbol{\phi}_{k'}$ gives

$$\begin{aligned}&\sum_{p=0}^{P}\left(<0pm>\boldsymbol{\phi}_{k'}^T\mathbf{K}_0\boldsymbol{\phi}_{k'} + <1pm>\boldsymbol{\phi}_{k'}^T\mathbf{K}_1\boldsymbol{\phi}_{k'} + <2pm>\boldsymbol{\phi}_{k'}^T\mathbf{K}_2\boldsymbol{\phi}_{k'}\right)\lambda_{k'p}^k \\ &- \omega_k^2\left(\sum_{p=0}^{P}\sum_{q=0}^{P} a_p^k\lambda_{k'q}^k <pqm>\right) \\ &= -\left(<0m>\boldsymbol{\phi}_{k'}^T\mathbf{K}_0\boldsymbol{\phi}_k + <1m>\boldsymbol{\phi}_{k'}^T\mathbf{K}_1\boldsymbol{\phi}_k + <2m>\boldsymbol{\phi}_{k'}^T\mathbf{K}_2\boldsymbol{\phi}_k\right).\end{aligned} \tag{80}$$

Equations (79) and (80) hold for $m = 0\cdots P+1$, and a matrix equation is derived

$$\begin{bmatrix}\boldsymbol{\phi}_k^T\mathbf{K}_1\boldsymbol{\phi}_{k'}\ \mathbf{S}_1 + \boldsymbol{\phi}_k^T\mathbf{K}_2\boldsymbol{\phi}_{k'}\ \mathbf{S}_2 & -\omega_k^2\ \mathbf{I}_{P+1} \\ \boldsymbol{\phi}_{k'}^T\mathbf{K}_2\boldsymbol{\phi}_{k'}\ \mathbf{S}_2 & \mathbf{0}_{P+1}\end{bmatrix}\mathbf{Y}^k - \omega_k^2\begin{bmatrix}\mathbf{0}_{P+1} \\ \mathbf{f}_{NL}(\mathbf{Y}^k)\end{bmatrix} = -\begin{bmatrix}\mathbf{b} \\ \mathbf{b'}\end{bmatrix}, \tag{81}$$

where

$$\mathbf{Y}^k \in \mathbb{R}^{2(P+1)},\ \mathbf{Y}^k = \begin{bmatrix}\boldsymbol{\lambda}_{k'}^k \\ \mathbf{a}^k\end{bmatrix}$$

$$\mathbf{S}_k \in \mathbb{R}^{(P+1)\times(P+1)} \qquad \mathbf{S}_{k,ij} = <ijk>$$

$$\mathbf{f}_{NL} \in \mathbb{R}^{(P+1)\times 1},\ \mathbf{f}_{NL,i}\left(\mathbf{Y}^k\right) = \frac{1}{2}\left(\mathbf{Y}^k\right)^T\begin{bmatrix}\mathbf{0}_{P+1} & \mathbf{S}_i \\ \mathbf{S}_i & \mathbf{0}_{P+1}\end{bmatrix}\mathbf{Y}^k$$

$$\mathbf{b} \in \mathbb{R}^{(P+1)},\ \mathbf{b} = \begin{bmatrix}\boldsymbol{\phi}_k^T\mathbf{K}_0\boldsymbol{\phi}_k \\ \boldsymbol{\phi}_k^T\mathbf{K}_1\boldsymbol{\phi}_k \\ \boldsymbol{\phi}_k^T\mathbf{K}_2\boldsymbol{\phi}_k \\ 0 \\ \vdots \\ 0\end{bmatrix} \text{and } \mathbf{b'} \in \mathbb{R}^{(P+1)},\ \mathbf{b'} = \begin{bmatrix}0 \\ \boldsymbol{\phi}_{k'}^T\mathbf{K}_1\boldsymbol{\phi}_k \\ \boldsymbol{\phi}_{k'}^T\mathbf{K}_2\boldsymbol{\phi}_k \\ 0 \\ \vdots \\ 0\end{bmatrix}.$$

$\mathbf{0}_{P+1} \in \mathbb{R}^{(P+1)\times(P+1)}$ is the null matrix, and $\mathbf{I}_{P+1} \in \mathbb{R}^{(P+1)\times(P+1)}$ is the identity matrix.

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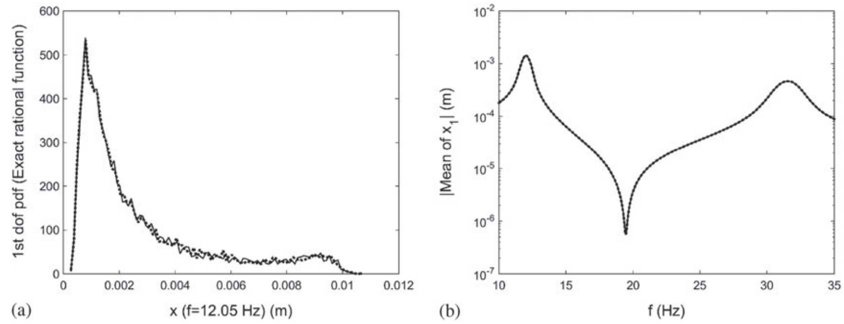

Figure 8. Normal deviates; random modes solution: MCS (solid lines) versus PCE (dotted line) – (a) pdf of x_1 and (b) mean frequency response. dof, degree of freedom; MCS, Monte Carlo simulation; PCE, polynomial chaos expansion; pdf, probability density function.

The nonlinear Equation (81) is solved with a Newton–Raphson method and gives the following estimate of the random modes for a PC degree equal to 1:

$$\tilde{\omega}_1^2 = \omega_1^2\,(0.9988 + 0.0362\xi_1 + 0.0138\xi_2)\,, \tag{82}$$

$$\tilde{\boldsymbol{\phi}}_1 = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2\,(-0.0003 - 0.01\xi_1 + 0.01\xi_2)\,, \tag{83}$$

$$\tilde{\omega}_2^2 = \omega_2^2\,(1.0002 + 0.0138\xi_1 + 0.0362\xi_2)\,, \tag{84}$$

$$\tilde{\boldsymbol{\phi}}_2 = \boldsymbol{\phi}_2 + \boldsymbol{\phi}_1\,(0.0003 + 0.01\xi_1 - 0.01\xi_2)\,. \tag{85}$$

The mean modes are

$$\overline{\omega}_1^2 = 5723\ (\text{rad/s})^2, \tag{86}$$

$$\overline{\boldsymbol{\phi}}_1 == \begin{bmatrix} 0.5255 \\ 0.8508 \end{bmatrix}, \tag{87}$$

$$\overline{\omega}_2^2 = 39277\ (\text{rad/s})^2, \tag{88}$$

$$\overline{\boldsymbol{\phi}}_2 = \begin{bmatrix} -0.8508 \\ 0.5255 \end{bmatrix}. \tag{89}$$

Hence, comparing the last equations with Equations (72)–(75) shows that the results obtained with a PCE of low degree are very accurate.

Equations (23) and (27) give, for each frequency, the distribution of the uncertain response. Figure 8 compares the results with the random modes obtained from MCS and a PCE of degree 1: the pdf of the response evaluated at the first deterministic eigenfrequency is given in Figure 8(a) whereas the mean frequency response is plotted in Figure 8(b). The results are very good even with a very low PCE degree; this is confirmed by the low value of the Kullback–Leibler divergence given in Table V.

6.3. Uniform deviates

As already mentioned, the PCE converges much quicker with Legendre polynomials. Figure 9(a) shows that the pdf calculated with a PCE of degree 30 is not very different from the reference pdf. Similarly to the case with the normal deviate, the XPA is very efficient as it is equal to the reference pdf (Figure 9(b)), which is indicated by a Kullback–Leibler divergence equal to zero (Table VI).

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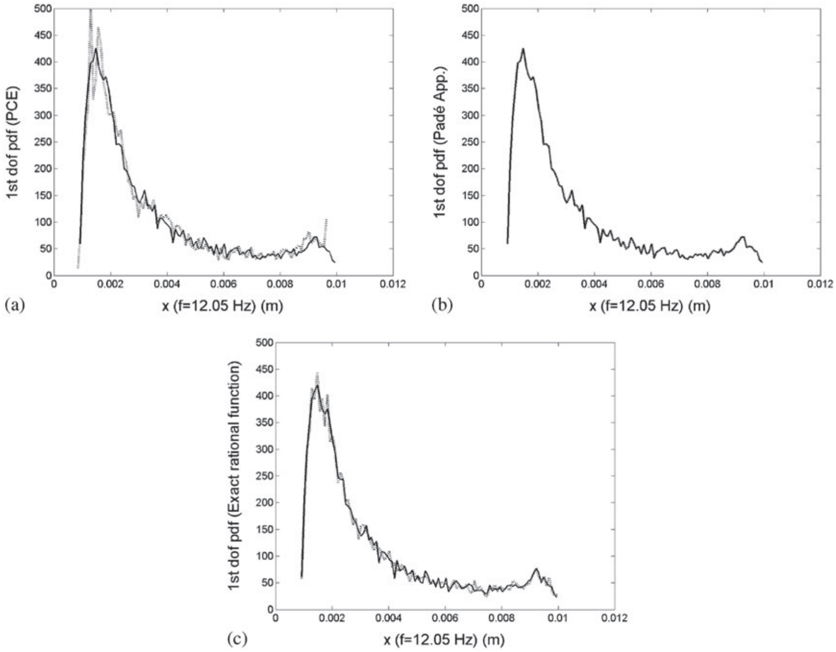

Figure 9. Probability density function of the response at the first deterministic eigenfrequency; uniform deviates; MCS (solid line) versus (a) PCE (degree 30, $P = 495$ – dotted line), (b) XPA ([1/2], $P = 14$ – dotted line), and (c) random modes solution. dof, degree of freedom; MCS, Monte Carlo simulation; PCE, polynomial chaos expansion; pdf, probability density function; XPA, extended Padé approximant.

Table VI. Kullback–Leibler divergence – Example 2 – uniform deviate.

pdf	PCE 30	Padé [1/2]	Mode + PCE
D_{KL}	0.28	0	0.005

PCE, polynomial chaos expansion; pdf, probability density function.

Figure 9(c) shows that the random mode approach is also very efficient: the Kullback–Leibler divergence is very low (Table VI) whereas the degree of the PCE to calculate the random modes is equal to one ($P = 2$).

The mean modes obtained from a PCE of degree 1 and an MCS are

$$\overline{\omega}_1^2 = 5727\ (\text{rad/s})^2, \tag{90}$$

$$\overline{\boldsymbol{\phi}}_1 == \begin{bmatrix} 1 \\ 1.6188 \end{bmatrix}, \tag{91}$$

$$\overline{\omega}_2^2 = 39273\ (\text{rad/s})^2, \tag{92}$$

$$\overline{\boldsymbol{\phi}}_2 = \begin{bmatrix} 1.6188 \\ -1 \end{bmatrix}. \tag{93}$$

The mean modes obtained from a PCE are

$$\overline{\omega}_1^2 = 5727\ (\text{rad/s})^2, \tag{94}$$

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$$\overline{\boldsymbol{\phi}}_1 == \begin{bmatrix} 1 \\ 1.6184 \end{bmatrix}, \tag{95}$$

$$\overline{\omega}_2^2 = 39273 \text{ (rad/s)}^2, \tag{96}$$

$$\overline{\boldsymbol{\phi}}_2 = \begin{bmatrix} 1.6184 \\ -1 \end{bmatrix}. \tag{97}$$

The previous equations show that the PCE approach to calculate the mean random modes is very efficient in this case.

7. CONCLUSION

The problem of obtaining pdf of the dynamic response in the frequency domain of a damped linear stochastic system is considered in this paper. A numerical approach and a physical approach were presented to estimate the response pdf of a random dynamical system and illustrated on two simple case studies. Both approaches exploit PCEs in different ways compared with PCE applied directly to the frequency domain response. The numerical approach relies on the (multivariate) PAs derived from a PCE of the response, whereas the mechanical approach requires the estimation of the random modes with a PCE. The examples show the efficiency of both approaches. In particular, it is possible to estimate the first two moments of the response with a very low degree PCE: these methods are even more efficient than the one proposed in [4]. This study also suggests that the random modes, which may be easily calculated with a PCE, might be as efficient as the deterministic modes in the study of a deterministic linear structure.

APPENDIX A: CLOSED-FORM SOLUTION OF <*IJL*> ([37, P. 390, EX. 87)]

A polynomial chaos is a function of multiple independent random variables, $\Xi = (\xi_1, \ldots, \xi_r)$, and may be written as

$$\Psi_i(\Xi) = \psi_{i_1}(\xi_1) \times \cdots \times \psi_{i_r}(\xi_r) = \prod_{\alpha=1}^{r} \psi_{i_\alpha}(\xi_\alpha). \tag{A1}$$

In the following, $\psi_{i_\alpha} = H_{i_\alpha}$ where $H_{i_\alpha}(\xi_\alpha)$ is a normalized Hermite polynomial; $\sum_{\alpha=1}^{r} i_\alpha$ is the degree of Ψ_i, i may be either a multi-index $(i_1, \ldots, i_r)$ or a single index defined from the multi-index $(i_1, \ldots, i_r)$ through a mapping.

The triple product $< ijl >=< \Psi_i, \Psi_j, \Psi_l >$ is

$$< i\, j\, l >= \prod_{\alpha=1}^{r} \int \cdots \int H_{i_\alpha}(\xi_\alpha)\, H_{j_\alpha}(\xi_\alpha)\, H_{l_\alpha}(\xi_\alpha)\, p(\xi_\alpha)\, \mathrm{d}\xi_\alpha \tag{A2}$$

$$= \prod_{\alpha=1}^{r} < H_{i_\alpha}, H_{j_\alpha}, H_{l_\alpha} >, \tag{A3}$$

where $< H_{i_\alpha}, H_{j_\alpha}, H_{l_\alpha} >$:

$$\text{if } s_\alpha \text{ is odd, } < H_{i_\alpha}, H_{j_\alpha}, H_{l_\alpha} >= 0, \tag{A4}$$

$$\text{if } s_\alpha \text{ is even, } < H_{i_\alpha}, H_{j_\alpha}, H_{l_\alpha} >= \frac{\sqrt{i_\alpha!\, j_\alpha!\, l_\alpha!}}{(s_\alpha - i_\alpha)!\, (s_\alpha - j_\alpha)!\, (s_\alpha - l_\alpha)!}\, \mathrm{Ind}_{\max(i_\alpha, j_\alpha, l_\alpha)}(s_\alpha) \tag{A5}$$

with $s_\alpha = (i_\alpha + j_\alpha + l_\alpha)/2$, and function $\mathrm{Ind}_m(l)$ is defined in Section 3.3.

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