INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2007; **69**:562–591
Published online 16 June 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.1781

Random matrix eigenvalue problems in structural dynamics

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SUMMARY

Natural frequencies and mode shapes play a fundamental role in the dynamic characteristics of linear structural systems. Considering that the system parameters are known only probabilistically, we obtain the moments and the probability density functions of the eigenvalues of discrete linear stochastic dynamic systems. Current methods to deal with such problems are dominated by mean-centred perturbation-based methods. Here two new approaches are proposed. The first approach is based on a perturbation expansion of the eigenvalues about an optimal point which is 'best' in some sense. The second approach is based on an asymptotic approximation of multidimensional integrals. A closed-form expression is derived for a general *r*th-order moment of the eigenvalues. Two approaches are presented to obtain the probability density functions of the eigenvalues. The first is based on the maximum entropy method and the second is based on a chi-square distribution. Both approaches result in simple closed-form expressions which can be easily calculated. The proposed methods are applied to two problems and the analytical results are compared with Monte Carlo simulations. It is expected that the 'small randomness' assumption usually employed in mean-centred-perturbation-based methods can be relaxed considerably using these methods.

Received 2 August 2005; Revised 7 April 2006; Accepted 19 April 2006

KEY WORDS: random eigenvalue problems; asymptotic analysis; statistical distributions; random matrices; linear stochastic systems

1. INTRODUCTION

Characterization of the natural frequencies and mode shapes play a key role in the analysis and design of engineering dynamic systems. The determination of natural frequency and mode shapes require the solution of an eigenvalue problem. Eigenvalue problems also arise in the context of the

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Contract/grant sponsor: Engineering and Physical Sciences Research Council (EPSRC); contract/grant number: GR/T03369/01
Contract/grant sponsor: Royal Society

stability analysis of structures. This problem could either be a differential eigenvalue problem or a matrix eigenvalue problem, depending on whether a continuous model or a discrete model is used to describe the given vibrating system. Description of real-life engineering structural systems is inevitably associated with some amount of uncertainty in specifying material properties, geometric parameters, boundary conditions and applied loads. When we take account of these uncertainties, it is necessary to consider *random eigenvalue problems*. The solution of the random eigenvalue problem is the first step to obtain the dynamic response statistics of linear stochastic systems. Random eigenvalue problems also arise in the stability analysis of linear systems with random imperfections. Several studies have been conducted on this topic since the mid-1960s. The study of probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics. The paper by Boyce [1], the book by Scheidt and Purkert [2] and the review papers by Ibrahim [3], Benaroya [4], Manohar and Ibrahim [5], and Manohar and Gupta [6] are useful sources of information on early work in this area of research and also provide a systematic account of different approaches to random eigenvalue problems.

In this paper, discrete linear systems or discretized continuous systems are considered. The random eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j \tag{1}$$

where λ_j and $\boldsymbol{\phi}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$ and $\mathbf{K}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$, the mass and stiffness matrices, are assumed to be smooth, continuous and at least twice differentiable functions of a random parameter vector $\mathbf{x} \in \mathbb{R}^m$. The vector $\mathbf{x}$ may consist of material properties, e.g. mass density, Poisson's ratio, Young's modulus; geometric properties, e.g. length, thickness, and boundary conditions. Statistical properties of the system are completely described by the joint probability density function $p_{\mathbf{x}}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$. For mathematical convenience we express

$$p_{\mathbf{x}}(\mathbf{x}) = \exp\{-L(\mathbf{x})\} \tag{2}$$

where $-L(\mathbf{x})$ is often known as the log-likelihood function. For example, if $\mathbf{x}$ is an m-dimensional multivariate Gaussian random vector with mean $\boldsymbol{\mu} \in \mathbb{R}^m$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times m}$ then

$$L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\|\boldsymbol{\Sigma}\| + \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \tag{3}$$

It is assumed that in general the random parameters are non-Gaussian and correlated, i.e. $L(\mathbf{x})$ can have any general form provided it is a smooth, continuous and at least twice differentiable function. It is further assumed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices so that all the eigenvalues are real and positive.

The central aim of studying random eigenvalue problems is to obtain the joint probability density function of the eigenvalues and the eigenvectors. The current literature on random eigenvalue problems arising in engineering systems is dominated by the mean-centred perturbation methods [7–16]. These methods work well when the uncertainties are small and the parameter distribution is Gaussian. In theory, any set of non-Gaussian random variables can be transformed into a set of standard Gaussian random variables by using numerical transformations such as the Rosenblatt transformation or the Nataf transformation. These transformations are however often complicated, numerically expensive and inconvenient to implement in practice. Some authors have proposed

DOI: 10.1002/nme

methods which are not based on the mean-centred perturbation method. Grigoriu [17] examined the roots of characteristic polynomials of real symmetric random matrices using the distribution of zeros of random polynomials. Lee and Singh [18] proposed a direct matrix product (Kronecker product) method to obtain the first two moments of the eigenvalues of discrete linear systems. More recently Nair and Keane [19] proposed a stochastic reduced basis approximation which can be applied to discrete or discretized continuous dynamic systems. Hála [20] and Mehlhose *et al*. [21] used a Ritz method to obtain closed-form expressions for moments and probability density functions of the eigenvalues (in terms of Chebyshev–Hermite polynomials). Szekely and Schueller [22], Pradlwarter *et al*. [23] and Du *et al*. [24] considered simulation-based methods to obtain eigensolution statistics of large systems. Ghosh *et al*. [25] used a polynomial chaos expansion for random eigenvalue problems. Adhikari [26] considered complex random eigenvalue problems associated with non-proportionally damped systems. Recently, Verhoosel *et al*. [27] proposed an iterative method that can be applied to non-symmetric random matrices also. Rahman [28] developed a dimensional decomposition method which does not require the calculation of eigensolution derivatives.

Under special circumstances when the matrix $\mathbf{H} = \mathbf{M}^{-1}\mathbf{K} \in \mathbb{R}^{N\times N}$ is Gaussian unitary ensemble (GUE) or Gaussian orthogonal ensemble (GOE) an exact closed-form expression can be obtained for the joint pdf of the eigenvalues using random matrix theory (RMT). See the book by Mehta [29] and references therein for discussions on RMT. RMT has been extended to other type of random matrices. If $\mathbf{H}$ has Wishart distribution then the exact joint pdf of the eigenvalues can be obtained from Muirhead [30, Theorem 3.2.18]. Edelman [31] obtained the pdf of the minimum eigenvalue (first natural frequency squared) of a Wishart matrix. A more general case when the matrix $\mathbf{H}$ has β-distribution has been obtained by Muirhead [30, Theorem 3.3.4] and more recently by Dumitriu and Edelman [32]. Unfortunately, the system matrices of real structures may not always follow such distributions and consequently some kind of approximate analysis is required.

In this paper, two new approaches are proposed to obtain the moments and probability density functions of the eigenvalues. The first approach is based on a perturbation expansion of the eigenvalues about a point in the $\mathbf{x}$-space which is 'optimal' is some sense (and in general different from the mean). The second approach is based on asymptotic approximation of multidimensional integrals. The proposed methods do not require the 'small randomness' assumption or Gaussian pdf assumption of the basic random variables often employed in literature. Moreover, they also eliminate the need to use intermediate numerical transformations of the basic random variables. In Section 2, perturbation-based methods are discussed. In Section 2.1, mean-centred perturbation methods are briefly reviewed. The perturbation method based on an optimal point is discussed in Section 2.2. In Section 3, a new method to obtain arbitrary order moments of the eigenvalues is proposed. Using these moments, some closed-form expressions of approximate pdf of the eigenvalues are derived in Section 4. In Section 5, the proposed analytical methods are applied to two problems and the results are compared with Monte Carlo simulations.

2. STATISTICS OF THE EIGENVALUES USING PERTURBATION METHODS

2.1. Mean-centred perturbation method

2.1.1. Perturbation expansion. The mass and the stiffness matrices are in general non-linear functions of the random vector $\mathbf{x}$. Denote the mean of $\mathbf{x}$ as $\boldsymbol{\mu} \in \mathbb{R}$, and consider that

$$\mathbf{M}(\boldsymbol{\mu}) = \overline{\mathbf{M}} \quad \text{and} \quad \mathbf{K}(\boldsymbol{\mu}) = \overline{\mathbf{K}} \tag{4}$$

DOI: 10.1002/nme

are the 'deterministic parts' of the mass and stiffness matrices, respectively. In general $\overline{\mathbf{M}}$ and $\overline{\mathbf{K}}$ are different from the mean matrices. The deterministic part of the eigenvalues:

$$\overline{\lambda}_j = \lambda_j(\boldsymbol{\mu}) \tag{5}$$

is obtained from the deterministic eigenvalue problem:

$$\overline{\mathbf{K}}\,\overline{\boldsymbol{\phi}}_j = \overline{\lambda}_j\,\overline{\mathbf{M}}\,\overline{\boldsymbol{\phi}}_j \tag{6}$$

The eigenvalues, $\lambda_j(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ are non-linear functions of the parameter vector $\mathbf{x}$. If the eigenvalues are not repeated, then each $\lambda_j(\mathbf{x})$ is expected to be a smooth and twice differentiable function since the mass and stiffness matrices are smooth and twice differentiable functions of the random parameter vector. In the mean-centred perturbation approach, the function $\lambda_j(\mathbf{x})$ is expanded by its Taylor series about the point $\mathbf{x} = \boldsymbol{\mu}$ as

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\mu}) + \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) + \tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) \tag{7}$$

Here $\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \in \mathbb{R}^m$ and $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu}) \in \mathbb{R}^{m\times m}$ are, respectively, the gradient vector and the Hessian matrix of $\lambda_j(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\mu}$, that is

$$\{\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\}_k = \left.\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}=\boldsymbol{\mu}} \tag{8}$$

and

$$\{\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\}_{kl} = \left.\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k \partial x_l}\right|_{\mathbf{x}=\boldsymbol{\mu}} \tag{9}$$

Expressions for the elements of the gradient vector and the Hessian matrix are given in Section A.1. Due to Equation (5), Equation (7) implies that the eigenvalues are effectively expanded about their corresponding deterministic value $\overline{\lambda}_j$.

Equation (7) represents a quadratic form in basic non-Gaussian random variables. The first-order perturbation, which is often used in practice, is obtained from Equation (7) by neglecting the Hessian matrix. In this case, the eigenvalues are simple linear functions of the basic random variables. This formulation is expected to produce acceptable results when the random variation in $\mathbf{x}$ is small. If the basic random variables are Gaussian, then first-order perturbation results in a Gaussian distribution of the eigenvalues. In this case, a closed-form expression for their joint probability density function can be obtained easily, see for example, Reference [7]. Recently, Adhikari [26] used the first-order perturbation method for complex eigenvalue problems arising in non-proportionally damped systems. However, if the elements of $\mathbf{x}$ are non-Gaussian then even the first-order perturbation method is not helpful because there is no general method to obtain the resulting pdf in a simple manner.

When the second-order terms are retained in Equation (7), each $\lambda_j(\mathbf{x})$ results in a quadratic form in $\mathbf{x}$. If the elements of $\mathbf{x}$ are Gaussian, then it is possible to obtain descriptive statistics using the theory of quadratic forms as discussed next.

2.1.2. Eigenvalue statistics using theory of quadratic forms. Extensive discussions on quadratic forms in Gaussian random variables can be found in the books by Johnson and Kotz [33, Chapter 29]

DOI: 10.1002/nme

and Mathai and Provost [34]. Using the methods outlined in these references moments/cumulants of the eigenvalues are obtained in this section.

When $\mathbf{x}$ is a multivariate Gaussian random vector, the moment generating function of $\lambda_j(\mathbf{x})$, for any $s \in \mathbb{C}$, can be obtained from Equation (7) as

$$M_{\lambda_j}(s) = E[\exp\{s\lambda_j(\mathbf{x})\}]$$

$$= \int_{\mathbb{R}^m} \exp\{s\lambda_j(\boldsymbol{\mu}) + s\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) + \frac{s}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) - L(\mathbf{x})\}\,\mathrm{d}\mathbf{x} \quad (10)$$

where $L(\mathbf{x})$ is given by Equation (3). Using the transformation

$$\mathbf{y} = (\mathbf{x}-\boldsymbol{\mu}) \quad (11)$$

the integral in Equation (10) can be evaluated exactly as

$$M_{\lambda_j}(s) = (2\pi)^{-m/2}\|\boldsymbol{\Sigma}\|^{-1/2}\int_{\mathbb{R}^m} \exp\left\{s\overline{\lambda}_j + s\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})\mathbf{y} - \frac{1}{2}\mathbf{y}^{\mathrm{T}}[\boldsymbol{\Sigma}^{-1} - s\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})]\mathbf{y}\right\}\mathrm{d}\mathbf{y} \quad (12)$$

or

$$M_{\lambda_j}(s) = \frac{\exp\{s\overline{\lambda}_j + (s^2/2)\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})\boldsymbol{\Sigma}[\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})]^{-1}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\}}{\sqrt{\|\mathbf{I} - s\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\|}} \quad (13)$$

To obtain the pdf of $\lambda_j(\mathbf{x})$, the inverse Laplace transform of Equation (13) is required. The exact closed-form expression of the pdf can be obtained for few special cases only. Some approximate methods to obtain the pdf of $\lambda_j(\mathbf{x})$ will be discussed in Section 4.

If the mean-centred first-order perturbation is used then $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu}) = \mathbf{O}$ and from Equation (13) we obtain

$$M_{\lambda_j}(s) \approx \exp\left\{s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\right\} \quad (14)$$

This implies that $\lambda_j(\mathbf{x})$ is a Gaussian random variable with mean $\overline{\lambda}_j$ and variance $\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})$. However, for second-order perturbations in general the mean of the eigenvalues is not the deterministic value. The cumulants of $\lambda_j(\mathbf{x})$ can be obtained from

$$\kappa_j^{(r)} = \frac{\mathrm{d}^r}{\mathrm{d}s^r}\ln M_{\lambda_j}(s)|_{s=0} \quad (15)$$

Here $\kappa_j^{(r)}$ is the rth-order cumulant of jth eigenvalue and from Equation (13) we have

$$\ln M_{\lambda_j}(s) = s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})\boldsymbol{\Sigma}[\mathbf{I} - s\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})]^{-1}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) - \frac{1}{2}\ln\|\mathbf{I} - s\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\| \quad (16)$$

Using this expression and after some simplifications it can be shown that

$$\kappa_j^{(r)} = \overline{\lambda}_j + \tfrac{1}{2}\mathrm{Trace}(\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}) \quad \text{if } r = 1 \quad (17)$$

DOI: 10.1002/nme

$$\kappa_j^{(r)} = \frac{r!}{2}\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})[\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})]^{r-2}\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \frac{(r-1)!}{2}\mathrm{Trace}([\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}]^r) \quad \text{if } r \geqslant 2 \tag{18}$$

The mean and first few cumulants of the eigenvalues can be explicitly obtained as

$$\widehat{\lambda}_j = \kappa_j^{(1)} = \overline{\lambda}_j + \tfrac{1}{2}\mathrm{Trace}(\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}) \tag{19}$$

$$\mathrm{Var}[\lambda_j] = \kappa_j^{(2)} = \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \tfrac{1}{2}\,\mathrm{Trace}([\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}]^2) \tag{20}$$

$$\kappa_j^{(3)} = 3\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})[\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})]\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \mathrm{Trace}([\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}]^3) \tag{21}$$

and

$$\kappa_j^{(4)} = 12\mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\mu})[\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})]^2\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + 3\,\mathrm{Trace}([\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}]^4) \tag{22}$$

From the cumulants, the raw moments $\mu_j^{(r)} = E[\lambda_j^r]$ and the central moments $\mu_j^{\prime(r)} = E[(\lambda_j - \bar{\lambda}_j)^r]$ can be obtained using standard formulae [35, Chapter 26].

2.2. *Perturbation method based on an optimal point*

2.2.1. Perturbation expansion. In the mean-centred perturbation method, $\lambda_j(\mathbf{x})$ is expanded in a Taylor series about $\mathbf{x} = \boldsymbol{\mu}$. This approach may not be suitable for all problems, especially if $\mathbf{x}$ is non-Gaussian then $p_{\mathbf{x}}(\mathbf{x})$ may not be centred around the mean. Here we are looking for a point $\mathbf{x} = \boldsymbol{\alpha}$ in the $\mathbf{x}$-space such that the Taylor series expansion of $\lambda_j(\mathbf{x})$ about this point

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\alpha})(\mathbf{x}-\boldsymbol{\alpha}) + \tfrac{1}{2}(\mathbf{x}-\boldsymbol{\alpha})^{\mathrm{T}}\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})(\mathbf{x}-\boldsymbol{\alpha}) \tag{23}$$

is optimal in some sense. The idea of the optimal point is related to the saddle point approximation, see for example the book by MacKay [36, Chapter 29]. The optimal point $\boldsymbol{\alpha}$ can be selected in various ways. For practical applications the mean of the eigenvalues is often the most important. For this reason, the optimal point $\boldsymbol{\alpha}$ is selected such that the mean or the first moment of each eigenvalue is calculated most accurately. The mathematical formalism presented here is not restricted to this specific criteria and can be easily modified if any moment other than the first moment is required to be obtained more accurately. Using Equation (2), the mean of $\lambda_j(\mathbf{x})$ can be obtained as

$$\widehat{\lambda}_j = E[\lambda_j(\mathbf{x})] = \int_{\mathbb{R}^m} \lambda_j(\mathbf{x})\, p_{\mathbf{x}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^m} \lambda_j(\mathbf{x}) \mathrm{e}^{-L(\mathbf{x})}\, \mathrm{d}\mathbf{x} \tag{24}$$

or

$$\widehat{\lambda}_j = \int_{\mathbb{R}^m} \mathrm{e}^{-h_j(\mathbf{x})}\, \mathrm{d}\mathbf{x} \tag{25}$$

where

$$h_j(\mathbf{x}) = L(\mathbf{x}) - \ln \lambda_j(\mathbf{x}) \tag{26}$$

Evaluation of integral (25), either analytically or numerically, is in general difficult because (a) $\lambda_j(\mathbf{x})$ and $L(\mathbf{x})$ are complicated non-linear functions of $\mathbf{x}$, (b) an explicit functional form $\lambda_j(\mathbf{x})$

DOI: 10.1002/nme

is not easy to obtain except for very simple problems (usually an FE run is required to obtain λ_j for every $\mathbf{x}$), and (c) the dimension of the integral m is large. For these reasons some kind of approximation is required. From Equation (25), note that the maximum contribution to the integral comes from the neighbourhood where $h_j(\mathbf{x})$ is minimum. Therefore, the function $h_j(\mathbf{x})$ is expanded in a Taylor series about the point where $h_j(\mathbf{x})$ has its global minimum. By doing so, the error in evaluating integral (25) would be minimized. Thus, the optimal point can be obtained from

$$\frac{\partial h_j(\mathbf{x})}{\partial x_k} = 0 \quad \text{or} \quad \frac{\partial L(\mathbf{x})}{\partial x_k} = \frac{1}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} \quad \forall k \tag{27}$$

Combining the above equation for all k, at $\mathbf{x} = \boldsymbol{\alpha}$ we have

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha}) \mathbf{d}_L(\boldsymbol{\alpha}) \tag{28}$$

Equation (28) implies that at the optimal point the gradient vectors of the eigenvalues and log-likelihood function are parallel. The non-linear set of equations (28) have to be solved numerically. Due to the explicit analytical expression of $\mathbf{d}_{\lambda_j}$ in terms of the derivative of the mass and stiffness matrices, expensive numerical differentiation of $\lambda_j(\mathbf{x})$ at each step is not needed. Moreover, for most $p_{\mathbf{x}}(\mathbf{x})$, a closed-form expression of $\mathbf{d}_L(\mathbf{x})$ is available. For example, when $\mathbf{x}$ has multivariate Gaussian distribution, $L(\mathbf{x})$ is given by Equation (3). By differentiating this, we obtain

$$\mathbf{d}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \tag{29}$$

Substituting this in Equation (28), the optimal point $\boldsymbol{\alpha}$ can be obtained as

$$\boldsymbol{\alpha} = \boldsymbol{\mu} + \frac{1}{\lambda_j(\boldsymbol{\alpha})} \boldsymbol{\Sigma} \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) \tag{30}$$

This equation also gives a recipe for an iterative algorithm to obtain $\boldsymbol{\alpha}$. One starts with an initial $\boldsymbol{\alpha}$ in the right-hand side and obtains an updated $\boldsymbol{\alpha}$ in the left-hand side. This procedure can be continued until the difference between the values of $\boldsymbol{\alpha}$ obtained from both sides of Equation (30) is less than (l_2 vector norm can be used to measure the difference) a predefined small value. A good value to start the iteration process is $\boldsymbol{\alpha} = \boldsymbol{\mu}$, as in the case of mean-centred approach. Note that the solution of a deterministic eigenvalue problem is needed at each step of the iteration process. The form of Equation (23) is similar to that of Equation (7). As mentioned before, when the basic random variables are non-Gaussian, determination of moments and the pdf is not straightforward. However, when $\mathbf{x}$ is Gaussian, some useful statistics of the eigenvalues can be obtained in closed-form.

2.2.2. Eigenvalue statistics using theory of quadratic forms. For notational convenience we rewrite the optimal perturbation expansion (23) as

$$\lambda_j(\mathbf{x}) \approx c_j + \mathbf{a}_j^{\mathrm{T}} \mathbf{x} + \tfrac{1}{2} \mathbf{x}^{\mathrm{T}} \mathbf{A}_j \mathbf{x} \tag{31}$$

where the constants $c_j \in \mathbb{R}$, $\mathbf{a}_j \in \mathbb{R}^m$ and $\mathbf{A}_j \in \mathbb{R}^{m \times m}$ are given by

$$c_j = \lambda_j(\boldsymbol{\alpha}) - \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\alpha}) \boldsymbol{\alpha} + \tfrac{1}{2} \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha}) \boldsymbol{\alpha} \tag{32}$$

$$\mathbf{a}_j = \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) - \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha}) \boldsymbol{\alpha} \tag{33}$$

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$$\mathbf{A}_j = \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha}) \tag{34}$$

From Equation (31), the closed-form expression of the moment generating function of $\lambda_j(\mathbf{x})$ can be obtained exactly in a way similar to that for the case of the mean-centred perturbation method. Using Equation (3), it can be shown that

$$\begin{aligned} M_{\lambda_j}(s) &= E[\exp\{s\lambda_j(\mathbf{x})\}] \\ &= (2\pi)^{-m/2}\|\boldsymbol{\Sigma}\|^{-1/2}\int_{\mathbb{R}^m}\exp\{s(c_j + \mathbf{a}_j^{\mathrm{T}}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}_j\mathbf{x}) \\ &\quad -\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}\,\mathrm{d}\mathbf{x} \end{aligned} \tag{35}$$

This m-dimensional integral can be evaluated exactly to obtain

$$M_{\lambda_j}(s) = \frac{\exp\{sc_j - \frac{1}{2}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{2}(\boldsymbol{\mu}+s\boldsymbol{\Sigma}\mathbf{a}_j)^{\mathrm{T}}[\mathbf{I}-s\boldsymbol{\Sigma}\mathbf{A}_j]^{-1}(\boldsymbol{\mu}+s\boldsymbol{\Sigma}\mathbf{a}_j)\}}{\sqrt{\|\mathbf{I}-s\boldsymbol{\Sigma}\mathbf{A}_j\|}} \tag{36}$$

From the preceeding expression, the cumulants of the eigenvalues can be evaluated using Equation (15) as

$$\kappa_j^{(r)} = c_j + \tfrac{1}{2}\,\mathrm{Trace}(\mathbf{A}_j\boldsymbol{\Sigma}) + \tfrac{1}{2}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}_j\boldsymbol{\mu} + \mathbf{a}_j^{\mathrm{T}}\boldsymbol{\mu} \quad \text{if } r = 1 \tag{37}$$

and

$$\begin{aligned} \kappa_j^{(r)} &= \frac{(r-1)!}{2}\,\mathrm{Trace}([\mathbf{A}_j\boldsymbol{\Sigma}]^r) + \frac{r!}{2}\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]^{r-2}\boldsymbol{\Sigma}\mathbf{a}_j \\ &\quad + r!\boldsymbol{\mu}^{\mathrm{T}}[\mathbf{A}_j\boldsymbol{\Sigma}]^{r-1}\mathbf{A}_j\boldsymbol{\mu} + r!\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]^{r-1}\mathbf{A}_j\boldsymbol{\mu} \quad \text{if } r \geqslant 2 \end{aligned} \tag{38}$$

The mean and first few cumulants of the eigenvalues can be explicitly obtained as

$$\widehat{\lambda}_j = \kappa_j^{(1)} = c_j + \tfrac{1}{2}\mathrm{Trace}(\mathbf{A}_j\boldsymbol{\Sigma}) + \tfrac{1}{2}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}_j\boldsymbol{\mu} + \mathbf{a}_j^{\mathrm{T}}\boldsymbol{\mu} \tag{39}$$

$$\mathrm{Var}[\lambda_j] = \kappa_j^{(2)} = \tfrac{1}{2}\,\mathrm{Trace}([\mathbf{A}_j\boldsymbol{\Sigma}]^2) + \mathbf{a}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{a}_j + 2\boldsymbol{\mu}^{\mathrm{T}}[\mathbf{A}_j\boldsymbol{\Sigma}]\mathbf{A}_j\boldsymbol{\mu} + 2\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]\mathbf{A}_j\boldsymbol{\mu} \tag{40}$$

$$\kappa_j^{(3)} = \mathrm{Trace}([\mathbf{A}_j\boldsymbol{\Sigma}]^3) + 3\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]\mathbf{a}_j + 6\boldsymbol{\mu}^{\mathrm{T}}[\mathbf{A}_j\boldsymbol{\Sigma}]^2\mathbf{A}_j\boldsymbol{\mu} + 6\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]^2\mathbf{A}_j\boldsymbol{\mu} \tag{41}$$

and

$$\kappa_j^{(4)} = 3\,\mathrm{Trace}([\mathbf{A}_j\boldsymbol{\Sigma}]^4) + 12\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]^2\mathbf{a}_j + 24\boldsymbol{\mu}^{\mathrm{T}}[\mathbf{A}_j\boldsymbol{\Sigma}]^3\mathbf{A}_j\boldsymbol{\mu} + 24\mathbf{a}_j^{\mathrm{T}}[\boldsymbol{\Sigma}\mathbf{A}_j]^3\mathbf{A}_j\boldsymbol{\mu} \tag{42}$$

Since Equations (17), (18), (37) and (38) gives cumulants of arbitrary order, it is possible to reconstruct the pdf of the eigenvalues from them. However, when the elements of $\mathbf{x}$ are non-Gaussian, then neither the first-order perturbation nor the second-order perturbation methods are helpful because there is no general method to obtain the resulting statistics in a simple manner. In such cases, the method outlined in the next section might be more useful.

DOI: 10.1002/nme

3. METHOD BASED ON THE ASYMPTOTIC INTEGRAL

3.1. Multidimensional integrals in unbounded domains

In this section, the moments of the eigenvalues are obtained based on an asymptotic approximation of the multidimensional integral. Consider a function $f(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ which is smooth and at least twice differentiable. Suppose we want to evaluate an integral of the following form:

$$\mathscr{I} = \int_{\mathbb{R}^m} \exp\{-f(\mathbf{x})\}\, \mathrm{d}\mathbf{x} \tag{43}$$

This is an m-dimensional integral over the unbounded domain $\mathbb{R}^m$. The maximum contribution to this integral comes from the neighbourhood where $f(\mathbf{x})$ reaches its global minimum. Suppose that $f(\mathbf{x})$ reaches its global minimum at a *unique* point $\boldsymbol{\theta} \in \mathbb{R}^m$. Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0 \quad \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0} \tag{44}$$

Using this, $f(\mathbf{x})$ is expanded in a Taylor series about $\boldsymbol{\theta}$ and Equation (43) is rewritten as

$$\begin{aligned}\mathscr{I} &= \int_{\mathbb{R}^m} \exp\{-\{f(\boldsymbol{\theta}) + \tfrac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^{\mathrm{T}}\mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) + \varepsilon(\mathbf{x},\boldsymbol{\theta})\}\}\, \mathrm{d}\mathbf{x} \\ &= \exp\{-f(\boldsymbol{\theta})\} \int_{\mathbb{R}^m} \exp\{-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^{\mathrm{T}}\mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) - \varepsilon(\mathbf{x},\boldsymbol{\theta})\}\, \mathrm{d}\mathbf{x}\end{aligned} \tag{45}$$

where $\varepsilon(\mathbf{x}, \boldsymbol{\theta})$ is the error if only the terms up to second-order were retained in the Taylor series expansion. With suitable scaling of $\mathbf{x}$ the integral in Equation (43) can be transformed to the so-called 'Laplace integral'. Under special conditions, such integrals can be well approximated using asymptotic methods. The relevant mathematical methods and formal derivations are covered in detail in the books by Bleistein and Handelsman [37] and Wong [38]. Here we propose a somewhat different version of the asymptotic integrals. The error $\varepsilon(\mathbf{x}, \boldsymbol{\theta})$ depends on higher order derivatives of $f(\mathbf{x})$ at $\mathbf{x} = \boldsymbol{\theta}$. If they are small compared to $f(\boldsymbol{\theta})$ and the elements of $\mathbf{D}_f(\boldsymbol{\theta})$, their contribution will be negligible to the value of the integral. Therefore, we assume $f(\boldsymbol{\theta})$ and the elements of $\mathbf{D}_f(\boldsymbol{\theta})$ are large so that

$$\left|\frac{1}{f(\boldsymbol{\theta})}\mathscr{D}^{(j)}(f(\boldsymbol{\theta}))\right| \to 0 \quad \text{and} \quad \forall k, l, \left|\frac{1}{[\mathbf{D}_f(\boldsymbol{\theta})]_{kl}}\mathscr{D}^{(j)}(f(\boldsymbol{\theta}))\right| \to 0 \quad \text{for } j > 2 \tag{46}$$

where $\mathscr{D}^{(j)}(f(\boldsymbol{\theta}))$ is jth-order derivative of $f(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\theta}$. Under such assumptions, $\varepsilon(\mathbf{x}, \boldsymbol{\theta}) \to 0$. Therefore, the integral in (45) can be approximated as

$$\mathscr{I} \approx \exp\{-f(\boldsymbol{\theta})\} \int_{\mathbb{R}^m} \exp\{-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^{\mathrm{T}}\mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta})\}\, \mathrm{d}\mathbf{x} \tag{47}$$

If $\boldsymbol{\theta}$ is the global minimum of $f(\mathbf{x})$ in $\mathbb{R}^m$, the symmetric Hessian matrix $\mathbf{D}_f(\boldsymbol{\theta}) \in \mathbb{R}^{m \times m}$ is also expected to be positive definite. Now use the coordinate transformation

$$\boldsymbol{\xi} = (\mathbf{x} - \boldsymbol{\theta})\mathbf{D}_f^{-1/2}(\boldsymbol{\theta}) \tag{48}$$

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The Jacobian of this transformation is

$$\|\mathbf{J}\| = \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2} \tag{49}$$

Using Equation (48), the integral in Equation (47) can be evaluated as

$$\mathscr{I} \approx \exp\{-f(\boldsymbol{\theta})\} \int_{\mathbb{R}^m} \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2} \exp\{-\tfrac{1}{2}(\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi})\}\, \mathrm{d}\boldsymbol{\xi} \tag{50}$$

or

$$\mathscr{I} \approx (2\pi)^{m/2} \exp\{-f(\boldsymbol{\theta})\} \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2} \tag{51}$$

The analysis proposed here is somewhat different from the widely used Laplace's method of asymptotic approximation of integrals [38, Chapter IX, Theorem 3]. Here it is simply assumed that the higher order derivatives $\mathscr{D}^{(j)}(f(\boldsymbol{\theta}))$ are negligibly small compared to $f(\boldsymbol{\theta})$ and the elements of $\mathbf{D}_f(\boldsymbol{\theta})$. This approximation is expected to yield good results if the minimum of $f(\mathbf{x})$ around $\mathbf{x} = \boldsymbol{\theta}$ is sharp. If $\boldsymbol{\theta}$ is not unique then it is required to sum the contributions arising from all such optimal points separately. Equation (51) will now be used to obtain moments of the eigenvalues.

3.2. Calculation of an arbitrary moment of the eigenvalues

An arbitrary rth-order moment of the eigenvalues can be obtained from

$$\begin{aligned} \mu_j^{(r)} = E[\lambda_j^r(\mathbf{x})] &= \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x})\, p_{\mathbf{x}}(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^m} \exp\{-(L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}))\}\, \mathrm{d}\mathbf{x}, \quad r = 1, 2, 3, \ldots \end{aligned} \tag{52}$$

The equation can be expressed in the form of Equation (43) by choosing

$$f(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}) \tag{53}$$

Differentiating the above equation with respect to x_k, we obtain

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = \frac{\partial L(\mathbf{x})}{\partial x_k} - \frac{r}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} \tag{54}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from Equation (44) by equating the above expression to zero. Therefore at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0 \quad \forall k \tag{55}$$

or

$$\frac{r}{\lambda_j(\boldsymbol{\theta})} \frac{\partial \lambda_j(\boldsymbol{\theta})}{\partial x_k} = \frac{\partial L(\boldsymbol{\theta})}{\partial x_k} \quad \forall k \tag{56}$$

or

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) r = \lambda_j(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta}) \tag{57}$$

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Equation (57) is similar to Equation (28) and needs to solved numerically to obtain $\boldsymbol{\theta}$. The steps to obtain $\boldsymbol{\theta}$ are similar to those described in Section 2.2.

The elements of the Hessian matrix $\mathbf{D}_f(\boldsymbol{\theta})$ can be obtained by differentiating Equation (54) with respect to x_l:

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_k \partial x_l} = \frac{\partial^2 L(\mathbf{x})}{\partial x_k \partial x_l} - r\left(-\frac{1}{\lambda_j^2(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} + \frac{1}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k \partial x_l}\right)$$

$$= \frac{\partial^2 L(\mathbf{x})}{\partial x_k \partial x_l} + \frac{1}{r}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}\right\}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\right\} - \frac{r}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k \partial x_l} \tag{58}$$

At $\mathbf{x} = \boldsymbol{\theta}$ we can use Equation (56) so that Equation (58) reads

$$\left.\frac{\partial^2 f(\mathbf{x})}{\partial x_k \partial x_l}\right|_{\mathbf{x}=\boldsymbol{\theta}} = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial x_k \partial x_l} + \frac{1}{r}\frac{\partial L(\boldsymbol{\theta})}{\partial x_k}\frac{\partial L(\boldsymbol{\theta})}{\partial x_l} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\frac{\partial^2 \lambda_j(\boldsymbol{\theta})}{\partial x_k \partial x_l} \tag{59}$$

Combining this equation for all k and l we have

$$\mathbf{D}_f(\boldsymbol{\theta}) = \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^{\mathrm{T}} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \tag{60}$$

where $\mathbf{D}_{\lambda_j}(\bullet)$ is defined in Equation (9). Using the asymptotic approximation (51), the rth moment of the eigenvalues can be obtained as

$$\mu_j^{(r)} \approx (2\pi)^{m/2}\lambda_j^r(\boldsymbol{\theta})\exp\{-L(\boldsymbol{\theta})\}\left\|\mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^{\mathrm{T}} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta})\right\|^{-1/2} \tag{61}$$

This is perhaps the most general formula to obtain the moments of the eigenvalues of linear stochastic dynamic systems. The optimal point $\boldsymbol{\theta}$ needs to be calculated by solving non-linear set of equations (57) for each λ_j and r. Several special cases arising from Equation (61) are of practical interest:

- *Mean of the eigenvalues*: The mean of the eigenvalues can be obtained by substituting $r = 1$ in Equation (61), that is

$$\widehat{\lambda}_j = \mu_j^{(1)} = (2\pi)^{m/2}\lambda_j(\boldsymbol{\theta})\exp\{-L(\boldsymbol{\theta})\}\|\mathbf{D}_L(\boldsymbol{\theta}) + \mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^{\mathrm{T}} - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta})\|^{-1/2} \tag{62}$$

- *Central moments of the eigenvalues*: Once the mean is known, the central moments can be expressed in terms of the raw moments $\mu_j^{(r)}$ using the binomial transform

$$\mu_j'^{(r)} = E[(\lambda_j - \widehat{\lambda}_j)^r] = \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\mu_j^{(k)}\widehat{\lambda}_j^{r-k} \tag{63}$$

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- *Random vector* $\mathbf{x}$ *has multivariate Gaussian distribution*: In this case $L(\mathbf{x})$ is given by Equation (3) and by differentiating Equation (29) we obtain

$$\mathbf{D}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1} \tag{64}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from Equation (57) as

$$\boldsymbol{\theta} = \boldsymbol{\mu} + \frac{r}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) \tag{65}$$

Using Equations (29) and (64), the Hessian matrix can be derived from Equation (60) as

$$\begin{aligned}\mathbf{D}_f(\boldsymbol{\theta}) &= \boldsymbol{\Sigma}^{-1} + \frac{1}{r}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \\ &= \boldsymbol{\Sigma}^{-1}\left(\mathbf{I} + \frac{1}{r}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\right) - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta})\end{aligned} \tag{66}$$

Therefore, the rth moment of the eigenvalues can be obtained from Equation (61) as

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\{-\tfrac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})\}\|\boldsymbol{\Sigma}\|^{-1/2}\|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2} \tag{67}$$

Using Equation (66) and recalling that for any two matrices $\mathbf{A}$ and $\mathbf{B}$, $\|\mathbf{A}\|\|\mathbf{B}\| = \|\mathbf{AB}\|$ we have

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\{-\tfrac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})\}\|\mathbf{I} + \widetilde{\mathbf{D}}_f(\boldsymbol{\theta})\|^{-1/2} \tag{68}$$

where

$$\widetilde{\mathbf{D}}_f(\boldsymbol{\theta}) = \frac{1}{r}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \tag{69}$$

The probability density function of the eigenvalues is considered in the next section.

4. PROBABILITY DENSITY FUNCTION OF THE EIGENVALUES

4.1. Maximum entropy probability density function

Once the cumulants/moments of the eigenvalues are known, the pdf of the eigenvalues can be obtained using the maximum entropy method (MEM). Because Equations (17), (18), (37), (38) and (61) can be used to calculate any arbitrary order cumulant and moment, the pdf can be obtained accurately by taking higher order terms. Here, following Kapur and Kesavan [39], a general approach is presented.

Since $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices, all the eigenvalues are real and positive. Suppose the pdf of λ_j is given by $p_{\lambda_j}(u)$ where $u \in \mathbb{R}$ is positive, that is $u \in [0, \infty]$. Considering that only first n moments are used, the pdf of each eigenvalue must satisfy the following constraints:

$$\int_0^\infty p_{\lambda_j}(u)\,\mathrm{d}u = 1 \tag{70}$$

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and

$$\int_0^\infty u^r p_{\lambda_j}(u)\,\mathrm{d}u = \mu_j^{(r)}, \quad r = 1, 2, 3, \ldots, n \tag{71}$$

Using Shannon's measure of entropy

$$\mathscr{S} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u)\,\mathrm{d}u \tag{72}$$

we construct the Lagrangian

$$\mathscr{L} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u)\,\mathrm{d}u - (\rho_0 - 1)\left[\int_0^\infty p_{\lambda_j}(u)\,\mathrm{d}u - 1\right] - \sum_{r=1}^{n} \rho_r \left[\int_0^\infty u^r p_{\lambda_j}(u)\,\mathrm{d}u - \mu_j^{(r)}\right] \tag{73}$$

where $\rho_r, r = 0, 1, 2, \ldots, n$ are Lagrange multipliers. The function $p_{\lambda_j}(u)$ which maximizes $\mathscr{L}$ can be obtained using the calculus of variations. Using the Euler–Lagrange equation the solution is given by

$$p_{\lambda_j}(u) = \exp\left\{-\rho_0 - \sum_{i=1}^{n} \rho_i u^i\right\} = \exp\{-\rho_0\} \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\}, \quad u \geqslant 0 \tag{74}$$

The Lagrange multipliers can be obtained from the constraint equations ((70) and (71)) as

$$\exp\{\rho_0\} = \int_0^\infty \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\} \mathrm{d}u \tag{75}$$

and

$$\exp\{\rho_0\}\mu_j^{(r)} = \int_0^\infty u^r \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\} \mathrm{d}u \quad \text{for } r = 0, 1, 2, \ldots, n \tag{76}$$

Closed-form expressions for ρ_r are in general not possible for all n. If we take $n = 2$, then the resulting pdf can be expressed as the truncated Gaussian density function

$$p_{\lambda_j}(u) = \frac{1}{\sqrt{2\pi}\sigma_j \Phi(\widehat{\lambda}_j/\sigma_j)} \exp\left\{-\frac{(u - \widehat{\lambda}_j)^2}{2\sigma_j^2}\right\}, \quad u \geqslant 0 \tag{77}$$

where σ_j is given by

$$\sigma_j^2 = \mu_j^{(2)} - \widehat{\lambda}_j^2 \tag{78}$$

The approach presented above can also be used in conjunction with the perturbation methods by transforming the cumulants obtained from Equations (17), (18), (37) and (38) to moments. The truncated Gaussian density function derived here ensures that the probability of any eigenvalues becoming negative is zero.

DOI: 10.1002/nme

4.2. Approximation by χ^2 probability density function

We use an approximation analogous to Pearson's [40] three moment central χ^2 approximation to the distribution of a noncentral χ^2. The eigenvalues are approximated as

$$\lambda_j \approx \eta_j + \gamma_j \chi^2_{\nu_j}(u) \tag{79}$$

where $\chi^2_{\nu_j}(u)$ is a central χ^2 density function with ν_j degrees of freedom (see Reference [35, Chapter 26]). The constants η_j, γ_j, and ν_j are obtained such that the first three moments of λ_j are equal to that of the approximated χ^2 pdf. The moment generating function of the approximated χ^2 pdf is given by

$$E[\exp\{-s(\eta_j + \gamma_j \chi^2_{\nu_j})\}] = \exp\{-s\eta_j\}(1 + 2s\gamma_j)^{-\nu_j/2} \tag{80}$$

Equating the first three moments we have

$$\eta_j + \nu_j \gamma_j = \mu_j^{(1)} \tag{81}$$

$$\eta_j^2 + 2\eta_j \nu_j \gamma_j + \nu_j^2 \gamma_j^2 + 2\nu_j \gamma_j^2 = \mu_j^{(2)} \tag{82}$$

and

$$\eta_j^3 + 3\eta_j^2 \nu_j \gamma_j + 3\eta_j \nu_j^2 \gamma_j^2 + 6\eta_j \nu_j \gamma_j^2 + \nu_j^3 \gamma_j^3 + 6\nu_j^2 \gamma_j^3 + 8\nu_j \gamma_j^3 = \mu_j^{(3)} \tag{83}$$

This set of coupled non-linear equations can be solved exactly in closed-form to obtain η_j, γ_j, and ν_j:

$$\eta_j = \frac{\mu_j^{(1)^2} \mu_j^{(2)} - 2\mu_j^{(2)^2} + \mu_j^{(1)} \mu_j^{(3)}}{2\mu_j^{(1)^3} - 3\mu_j^{(1)} \mu_j^{(2)} + \mu_j^{(3)}} \tag{84}$$

$$\gamma_j = \frac{2\mu_j^{(1)^3} - 3\mu_j^{(1)} \mu_j^{(2)} + \mu_j^{(3)}}{4(\mu_j^{(2)} - \mu_j^{(1)^2})} \tag{85}$$

and

$$\nu_j = 8 \frac{(\mu_j^{(2)} - \mu_j^{(1)^2})^3}{(2\mu_j^{(1)^3} - 3\mu_j^{(1)} \mu_j^{(2)} + \mu_3')^2} \tag{86}$$

Moments of $\lambda_j(\mathbf{x})$ obtained in Equation (61), can be used directly in the right-hand side of these equations. Alternatively, this approach can also be used in conjunction with the perturbation methods by transforming the cumulants obtained from Equations (17), (18), (37) and (38) to moments. Using the transformation in Equation (79), the approximate probability density function of $\lambda_j(\mathbf{x})$ is given by

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{\chi^2_{\nu_j}} \left(\frac{u - \eta_j}{\gamma_j} \right) = \frac{(u - \eta_j)^{\nu_j/2 - 1} \exp\{-(u - \eta_j)/2\gamma_j\}}{(2\gamma_j)^{\nu_j/2} \Gamma(\nu_j/2)} \tag{87}$$

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The two approximated pdf proposed here have simple forms but it should be noted that they are not exhaustive. Given the moments/cumulants, different probability density functions can be fitted using different methods. The application of the approximate pdfs derived here is illustrated in the next section.

5. APPLICATION EXAMPLES

5.1. A two-DOF system

5.1.1. System model and computational methodology. A simple two-degree-of-freedom undamped system has been considered to illustrate a possible application of the expressions developed so far. The main purpose of this example is to understand how the proposed methods compare with the existing methods. Figure 1 shows the example, together with the numerical values of the masses and spring stiffnesses. The system matrices for the example are given by

$$\mathbf{M}=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K}=\begin{bmatrix} k_1+k_3 & -k_3 \\ -k_3 & k_2+k_3 \end{bmatrix} \tag{88}$$

It is assumed that only the stiffness parameters k_1 and k_2 are uncertain so that $k_i = \bar{k}_i(1+\varepsilon_i x_i)$, $i=1, 2$ and $\bar{k}_i$ denote the deterministic values of the spring constants. Here $\mathbf{x}=\{x_1, x_2\}^{\mathrm{T}} \in \mathbb{R}^2$ is a vector of standard Gaussian random variables, that is $\boldsymbol{\mu}=\mathbf{0}$ and $\boldsymbol{\Sigma}=\mathbf{I}$. The numerical values of the 'strength parameters' are considered as $\varepsilon_1=\varepsilon_2=0.25$. The strength parameters are selected so that the system matrices are almost surely positive definite. Noting that $\mathbf{M}$ is independent of $\mathbf{x}$ and $\mathbf{K}$ is a linear function of $\mathbf{x}$, the derivative of the system matrices with respect to the random vector $\mathbf{x}$ can be obtained as

$$\frac{\partial \mathbf{K}}{\partial x_1}=\varepsilon_1\begin{bmatrix} \bar{k}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_2}=\varepsilon_2\begin{bmatrix} 0 & 0 \\ 0 & \bar{k}_2 \end{bmatrix} \tag{89}$$

$$\frac{\partial \mathbf{M}}{\partial x_i}=\mathbf{O} \quad \text{and} \quad \frac{\partial^2 \mathbf{K}}{\partial x_i \partial x_j}=\mathbf{O} \tag{90}$$

We calculate the raw moments and the probability density functions of the two eigenvalues of the system. Recall that the eigenvalues obtained from Equation (1) are the square of the natural

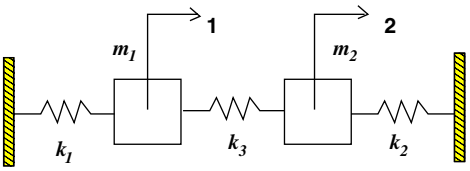

Figure 1. The undamped two-degree-of-system, $m_1=1\,\mathrm{kg}$, $m_2=1.5\,\mathrm{kg}$, $\bar{k}_1=1000\,\mathrm{N/m}$, $\bar{k}_2=1100\,\mathrm{N/m}$ and $k_3=100\,\mathrm{N/m}$.

DOI: 10.1002/nme

frequencies ($\lambda_j = \omega_j^2$). The following six methods are used to obtain the moments and the pdfs:

1. *Mean-centred first-order perturbation*: This case arises when $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})$ in the Taylor series expansion (7) is assumed to be a null matrix so that only the first-order terms are retained. This is the simplest approximation, and as mentioned earlier, results in a Gaussian distribution of the eigenvalues. Recalling that for this problem, $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$, the resulting statistics for this special case can be obtained from Equations (19) and (20) as

$$\widehat{\lambda}_j = \overline{\lambda}_j \tag{91}$$

and

$$\mathrm{Var}[\lambda_j] = \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\mathbf{0})\mathbf{d}_{\lambda_j}(\mathbf{0}) \tag{92}$$

The gradient vector $\mathbf{d}_{\lambda_j}(\mathbf{0})$ can be obtained from Equation (A2) using the system derivative matrices (89) and (90).

2. *Mean-centred second-order perturbation*: In this case all the terms in Equation (7) are retained. This approximation results in a quadratic form in the Gaussian random variables. The resulting statistics can be obtained from Equations (17) and (18) by substituting $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$. The elements of the Hessian matrix $\mathbf{D}_{\lambda_j}(\mathbf{0})$ can be obtained from Equation (A4) and using the system derivative matrices (89) and (90).
3. *Optimal point first-order perturbation*: This case arises when $\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})$ in the Taylor series expansion (23) is assumed to be a null matrix so that only the first-order terms are retained. Like its mean-centred counterpart, this approach also results in a Gaussian distribution of the eigenvalues. From Equation (31) we have

$$c_j = \lambda_j(\boldsymbol{\alpha}) - \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\alpha})\boldsymbol{\alpha}, \quad \mathbf{a}_j = \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) \quad \text{and} \quad \mathbf{A}_j = \mathbf{O} \tag{93}$$

The equation for the optimal point $\boldsymbol{\alpha}$ is obtained from Equation (30) as

$$\boldsymbol{\alpha} = \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})/\lambda_j(\boldsymbol{\alpha}) \quad \text{or} \quad \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\boldsymbol{\alpha} \tag{94}$$

Using these equations, the mean and the variance can be obtained as special cases of Equations (39) and (40) as

$$\widehat{\lambda}_j = \lambda_j(\boldsymbol{\alpha}) - \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\alpha})\boldsymbol{\alpha} = \lambda_j(\boldsymbol{\alpha}) - \lambda_j(\boldsymbol{\alpha})\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} \tag{95}$$

or

$$\widehat{\lambda}_j = \lambda_j(\boldsymbol{\alpha})(1 - |\boldsymbol{\alpha}|^2) \tag{96}$$

and

$$\mathrm{Var}[\lambda_j] = \mathbf{d}_{\lambda_j}^{\mathrm{T}}(\boldsymbol{\alpha})\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j^2(\boldsymbol{\alpha})|\boldsymbol{\alpha}|^2 \tag{97}$$

4. *Optimal point second-order perturbation*: In this case all the terms in Equation (23) are retained. Like the mean-centred approach, this approximation also results in a quadratic form in the Gaussian random variables, but with different coefficients. The resulting statistics can be obtained from Equations (37) and (38).

DOI: 10.1002/nme

5. *Method based on the asymptotic integral*: In this case, the moments can be obtained using Equation (61). For the standardized Gaussian random vector, substituting $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$ in Equation (67) the moment formula can be simplified to

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-\frac{1}{2}|\boldsymbol{\theta}|^2\right\} \left\| \mathbf{I} + \frac{1}{r}\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}/r - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2} \tag{98}$$

and

$$\boldsymbol{\theta} = r\mathbf{d}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}), \quad r = 1, 2, 3, \ldots \tag{99}$$

The vector $\boldsymbol{\theta}$ needs to be calculated for each r and j from Equation (99) using the iterative approach discussed before. The moments obtained from Equation (98) can be used to obtain the pdf using the approach given in Section 4.

6. *Monte Carlo simulation*: The samples of two independent Gaussian random variables x_1 and x_2 are generated and the eigenvalues are computed directly from Equation (1). A total of 15 000 samples are used to obtain the statistical moments and pdf of both the eigenvalues. The results obtained from the Monte Carlo simulation are assumed to be the benchmark for the purpose of comparing the five analytical methods described above.

5.1.2. Numerical results. Figure 2 shows the percentage error for the first four raw moments of the first eigenvalue. The percentage error for an arbitrary kth moment of an eigenvalue obtained using any one of the five analytical methods is given by

$$\text{Error}_{i\text{th method}} = \frac{|\{\mu_j^{(r)}\}_{i\text{th method}} - \{\mu_j^{(r)}\}_{\text{MCS}}|}{\{\mu_j^{(r)}\}_{\text{MCS}}} \times 100, \quad i = 1, \ldots, 5 \tag{100}$$

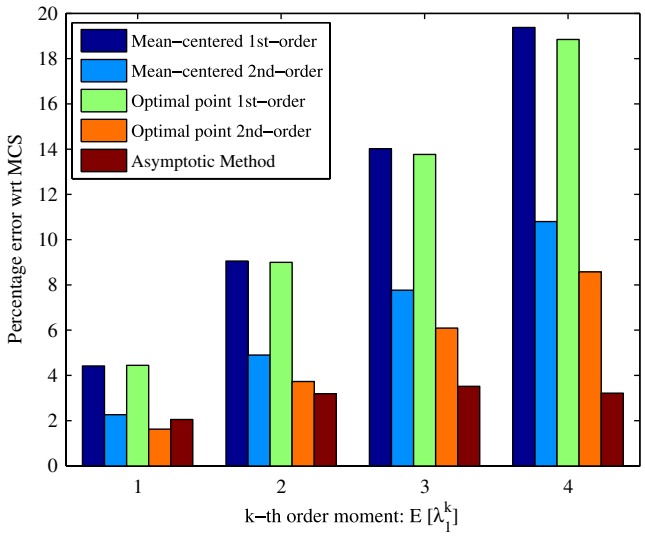

Figure 2. Percentage error for the first eigenvalue.

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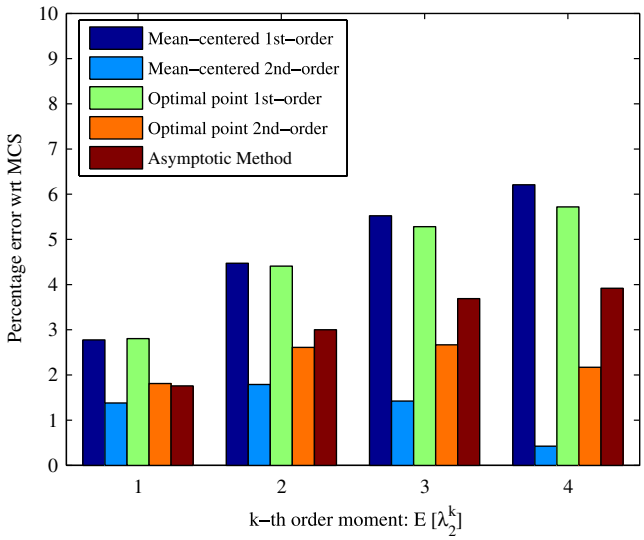

Figure 3. Percentage error for the second eigenvalue.

Figure 3 shows the percentage error for the first four raw moments of the second eigenvalue. For both eigenvalues, the error corresponding to the mean-centred first-order perturbation method is more than the other four methods. The error corresponding to the optimal point first-order perturbation method follows next. Moments obtained from mean-centred and optimal point second-order perturbation methods are more accurate compared to their corresponding first-order counterparts. In general, the moments obtained from the asymptotic formula equation (61) turn out to be quite accurate. The absolute errors for the second eigenvalue are less compared to the first eigenvalue. For the first eigenvalue, the moments obtained from the asymptotic formula turn out to be the most accurate, while for the second eigenvalue, the mean-centred second-order perturbation method yields the most accurate results.

Now consider the probability density function of the eigenvalues. Figures 4 and 5, respectively, show the pdf of the first and the second eigenvalue obtained from the five methods described earlier. The pdf corresponding to first five methods are obtained using the χ^2 distribution in Equation (87). The constants appearing in this equation are calculated from the moments using Equations (84)–(86). In the same plots, the normalized histograms of the eigenvalues obtained from the Monte Carlo simulation are also plotted. For the first eigenvalue, the pdf from the second-order perturbation methods are accurate in the lower and in the upper tail. For the second eigenvalue, the pdf from the asymptotic moment method is accurate over the whole curve.

5.2. A three-DOF system with closely spaced eigenvalues

5.2.1. System model and computational methodology. A three-degree-of-freedom undamped spring–mass system, taken from Reference [41], is shown in Figure 6. The main purpose of this example is to understand how the proposed methods work when some of the system

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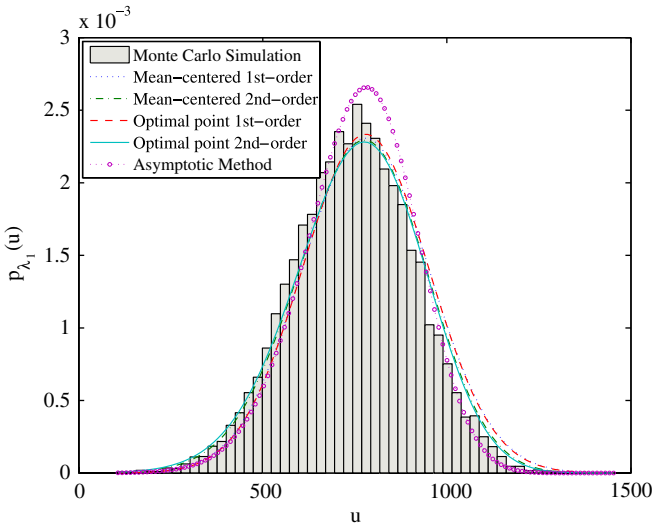

Figure 4. Probability density function of the first eigenvalue.

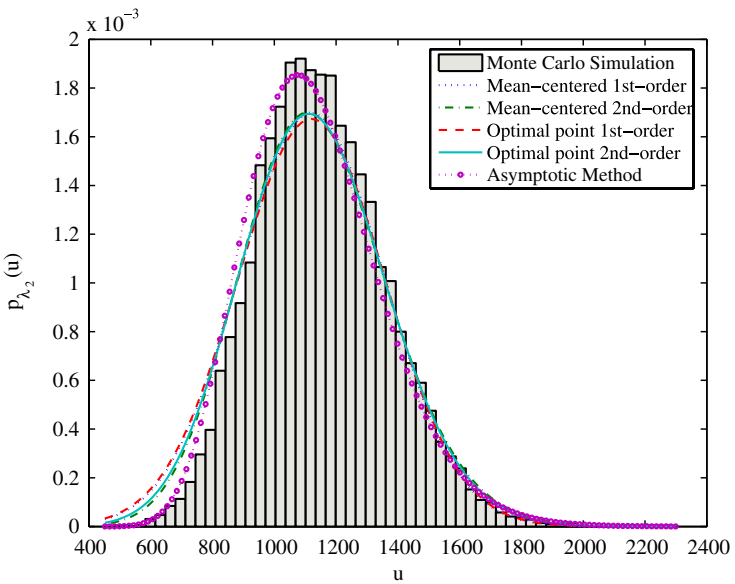

Figure 5. Probability density function of the second eigenvalue.

eigenvalues are closely spaced. This is an interesting case because it is well known that closely spaced eigenvalues are parameter sensitive. We will investigate how the parameter uncertainty affects the eigenvalue distribution in such cases. This study has particular relevance to the dynamics of nominally symmetric rotating machines, for example, turbine blades with random imperfections.

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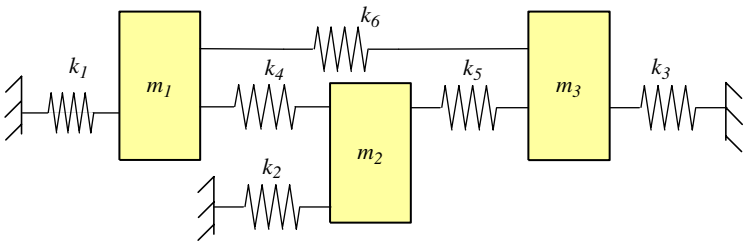

Figure 6. The three-degree-of-freedom random system.

The mass and stiffness matrices of the example system are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_4 + k_5 + k_2 & -k_5 \\ -k_6 & -k_5 & k_5 + k_3 + k_6 \end{bmatrix} \tag{101}$$

It is assumed that all mass and stiffness constants are random. The randomness in these parameters are assumed to be of the following form:

$$m_i = \overline{m}_i(1 + \varepsilon_m x_i), \quad i = 1, 2, 3 \tag{102}$$

$$k_i = \overline{k}_i(1 + \varepsilon_k x_{i+3}), \quad i = 1, \ldots, 6 \tag{103}$$

Here $\mathbf{x} = \{x_1, \ldots, x_9\}^{\mathrm{T}} \in \mathbb{R}^9$ is the vector of random variables. It is assumed that all random variables are Gaussian and uncorrelated with zero mean and unit standard deviation, that is $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$. Therefore, the mean values of m_i and k_i are given by $\overline{m}_i$ and $\overline{k}_i$. The numerical values of both the 'strength parameters' ε_m and ε_k are fixed at 0.15. In order to obtain statistics of the eigenvalues using the methods developed in this paper, the gradient vector and the Hessian matrix of the eigenvalues are required. As shown in Section A.1, this in turn requires the derivative of the system matrices with respect to the entries of $\mathbf{x}$. For most practical problems, which usually involve finite element modelling, these derivatives need to be determined numerically. However, for this simple example the derivatives can be obtained in closed-form and they are given in Section A.2.

We calculate the moments and the probability density functions of the three eigenvalues of the system. The following two sets of physically meaningful parameter values are considered:

- *Case 1*: *All eigenvalues are well separated.*
  For this case $\overline{m}_i = 1.0\,\mathrm{kg}$ for $i = 1, 2, 3$; $\overline{k}_i = 1.0\,\mathrm{N/m}$ for $i = 1, \ldots, 5$ and $k_6 = 3.0\,\mathrm{N/m}$.
- *Case 2*: *Two eigenvalues are close.*
  All parameter values are the same except $k_6 = 1.275\,\mathrm{N/m}$.

The moments of the eigenvalues for the above two cases are calculated from Equations (98) and (99). The moments are then used to obtain σ_j from Equation (78) and the constants in Equations (84)–(86). Using these constants, the truncated Gaussian pdf and the χ^2 pdf of the eigenvalues are obtained from Equations (77) and (87), respectively. These results are compared with Monte Carlo simulation. The samples of the nine independent Gaussian random variables

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$x_i, i = 1, \ldots, 9$ are generated and the eigenvalues are computed directly from Equation (1). A total of 15 000 samples are used to obtain the statistical moments and histograms of the pdf of the eigenvalues. The results obtained from Monte Carlo simulation are assumed to be the benchmark for the purpose of comparing the analytical methods. For the purpose of determining the accuracy, we again calculate the percentage error associated with an arbitrary rth moment using Equation (100). The results for the two cases are presented and discussed in the next subsection.

5.2.2. Numerical results. Case 1: All eigenvalues are well separated. When all of the eigenvalues are well separated, their derivatives with respect to the system parameters generally behave well. For the given parameter values, the eigenvalues of the corresponding deterministic system is given by

$$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4 \quad \text{and} \quad \overline{\lambda}_3 = 8 \tag{104}$$

The random 'scatter' of the eigenvalues is shown in Figure 7 for 1000 samples (out of 15 000) from the Monte Carlo simulation. It can be seen that the highest eigenvalue has the maximum scatter and because the eigenvalues are well separated, there is very little statistical overlap between them. Figure 8 shows the percentage error for the first four moments of the eigenvalues. These errors are reasonably small considering that the strength of randomness for all nine random variables is 15%. The error associated with higher eigenvalues are higher and also for a fixed eigenvalue, the higher order moments have more errors. Now consider the probability density function of the eigenvalues. The pdfs of the first eigenvalue obtained from the two methods are shown

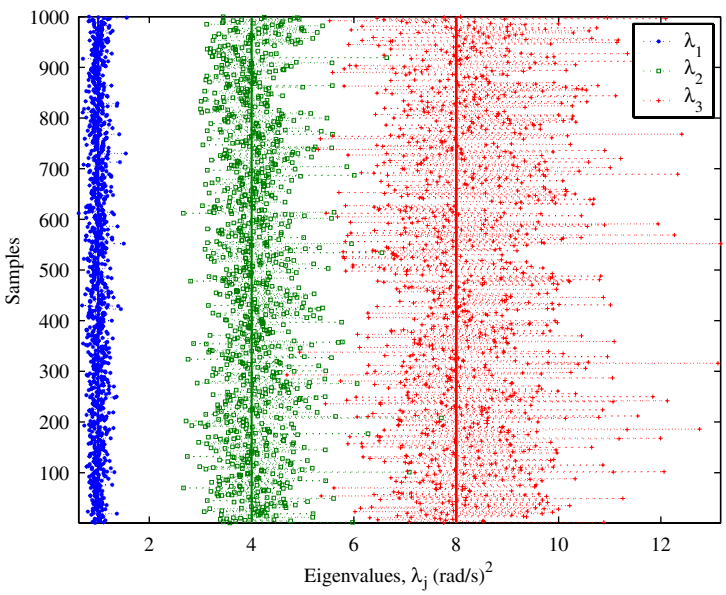

Figure 7. Statistical scatter in the eigenvalues: Case 1.

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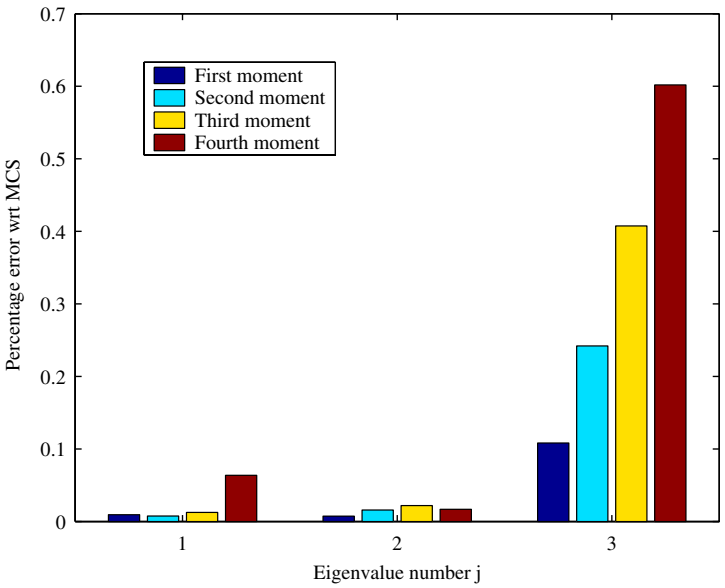

Figure 8. Percentage error for first four moments of the eigenvalues: Case 1.

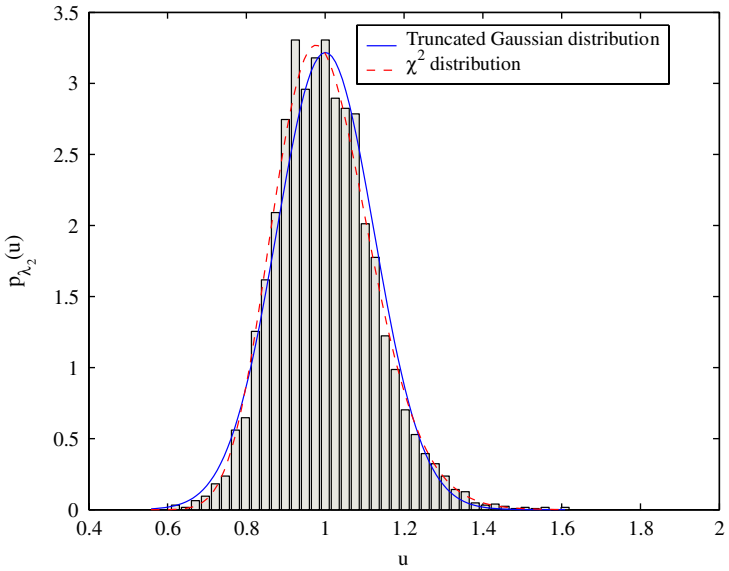

Figure 9. Probability density function of the first eigenvalue: Case 1.

in Figure 9. On the same plot, normalized histograms of the eigenvalue obtained from the Monte Carlo simulation are also shown. Both approximate methods match well with the Monte Carlo simulation result. This is expected since the first three moments are obtained accurately (less

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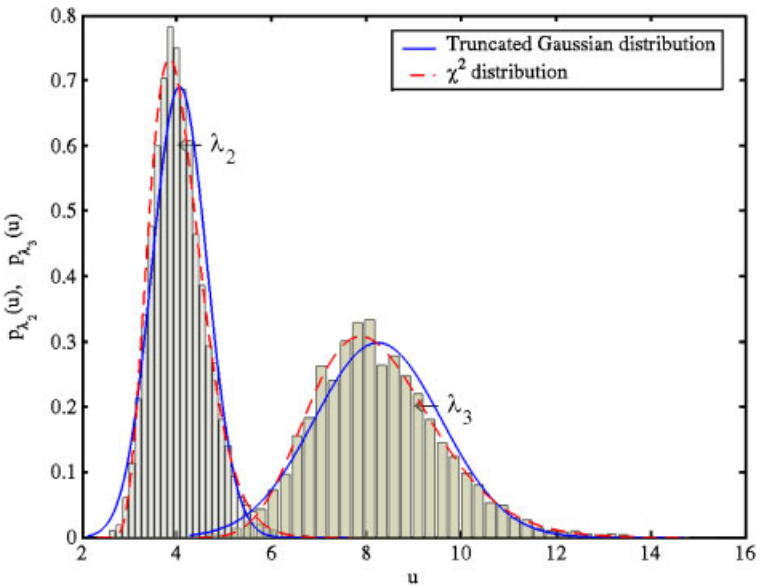

Figure 10. Probability density functions of the second and third eigenvalues: Case 1.

than 0.2% error as seen in Figure 8). The probability density functions of the second and third eigenvalues are shown in Figure 10. The region of statistical overlap is indeed small and can be verified from the plot of the actual samples in Figure 7. Again, both approximate methods match well with the Monte Carlo simulation result.

Case 2: *Two eigenvalues are close.* When some eigenvalues are closely spaced, their derivatives with respect to the system parameters may not behave well [41]. Indeed, if repeated eigenvalues exist, the formulation proposed here breaks down. The purpose of studying this case is to investigate how the proposed methods work when there are closely spaced eigenvalues so that there is a significant statistical overlap between them. For the given parameter values, the eigenvalues of the corresponding deterministic system are calculated as

$$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4 \quad \text{and} \quad \overline{\lambda}_3 = 4.55 \tag{105}$$

Clearly $\overline{\lambda}_2$ and $\overline{\lambda}_3$ are close to each other. The random scatter of the eigenvalues is shown in Figure 11 for 1000 samples from the Monte Carlo simulation. It can be seen that the third eigenvalue has the maximum scatter and because the second and the third eigenvalues are close, there is significant statistical overlap between them. Figure 12 shows the percentage error for the first four moments of the eigenvalues. The general trend of these errors are similar to the previous case except that the magnitudes of the errors corresponding to second and third eigenvalues are higher. This is expected because these two eigenvalues are close to each other.

The probability density function of the first eigenvalue obtained from the two methods are shown in Figure 13. On the same plot, normalized histograms of the eigenvalue obtained from Monte Carlo simulation are also shown. As in the previous case, both approximate methods match well with the Monte Carlo simulation result. This is expected since the first three moments are obtained accurately for this case also. The probability density functions of the second and third

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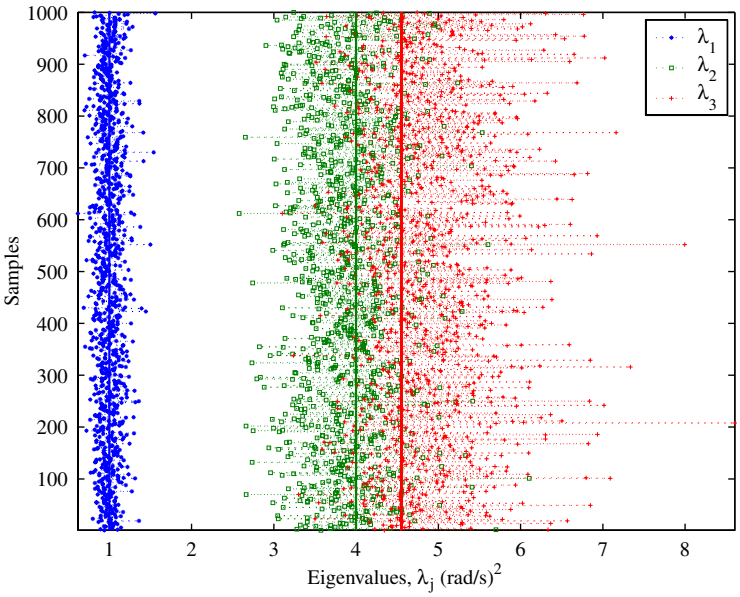

Figure 11. Statistical scatter in the eigenvalues: Case 2.

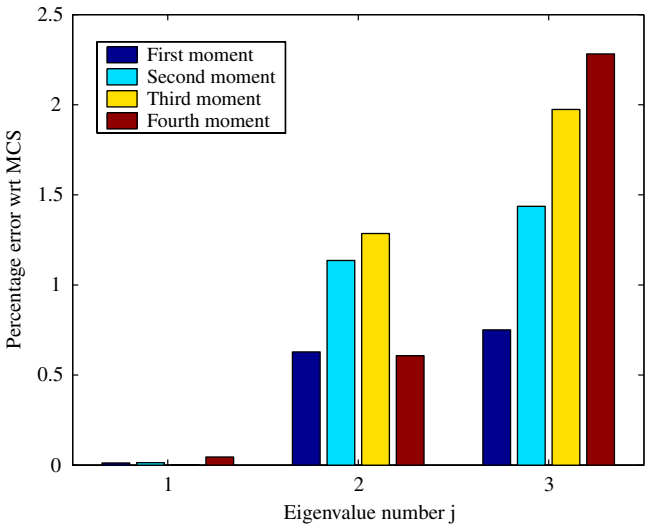

Figure 12. Percentage error for first four moments of the eigenvalues: Case 2.

eigenvalues are shown in Figure 14. There is a significant region of statistical overlap which can also be verified from the plot of the actual samples in Figure 11. In this case, the truncated Gaussian density function performs better than the χ^2 density function. However, none of the approximate methods match the Monte Carlo simulation result as well as in the previous case.

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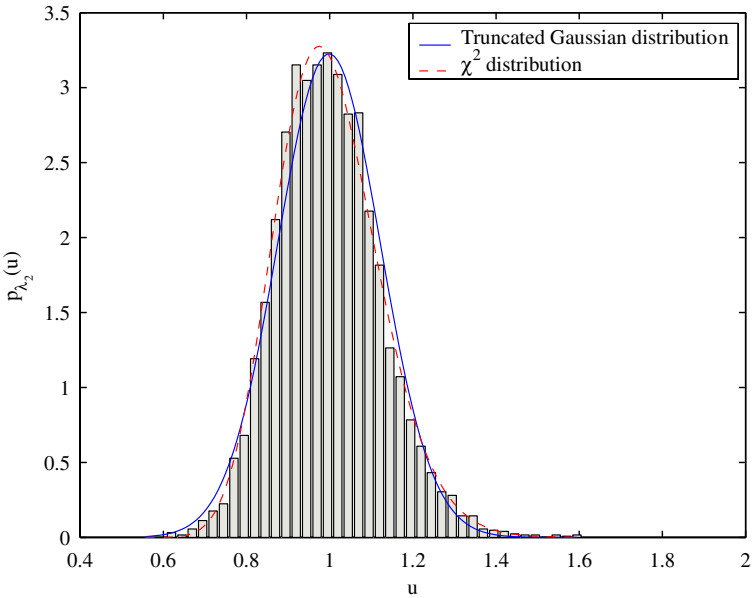

Figure 13. Probability density function of the first eigenvalue: Case 2.

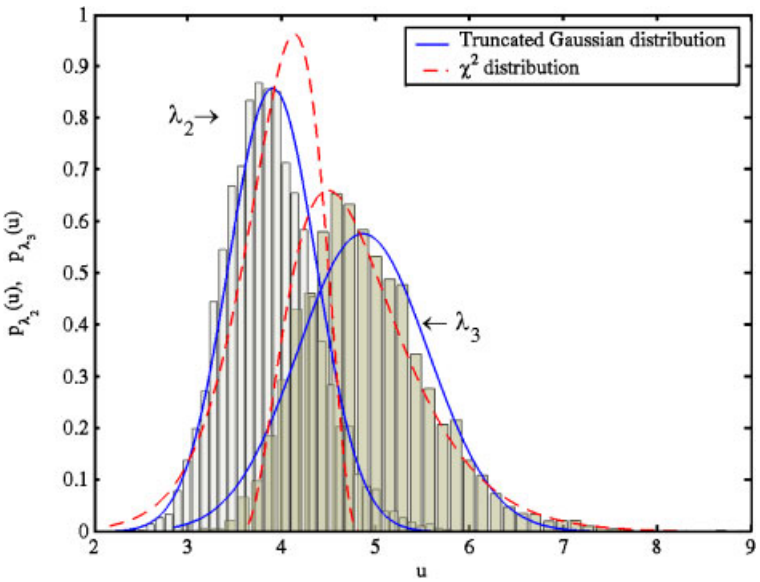

Figure 14. Probability density functions of the second and third eigenvalues: Case 2.

6. CONCLUSIONS

The statistics of the eigenvalues of discrete linear dynamic systems with parameter uncertainties have been considered. It is assumed that the mass and stiffness matrices are smooth and at least

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twice differentiable functions of a set of random variables. The random variables are in general considered to be non-Gaussian. The usual assumption of small randomness employed in most mean-centred-based perturbation analysis is not employed in this study. Two methods, namely (a) optimal point expansion method, and (b) asymptotic moment method, are proposed. The optimal point is obtained so that the mean of the eigenvalues are estimated most accurately. Both methods are based on an unconstrained optimization problem. Moments and cumulants of arbitrary orders are derived for both the approaches. Two simple approximations for the probability density function of the eigenvalues are derived. One is in terms of a truncated Gaussian random variable obtained using the maximum entropy principle. The other is a χ^2 random variable approximation based on matching the first three moments of the eigenvalues. Both formulations yield closed-form expressions of the pdf which can be computed easily.

The proposed formulae are applied to two problems. The moments and the pdf match encouragingly well with the corresponding Monte Carlo simulation results. However, when some eigenvalues are closely spaced, the proposed methods do not produce very accurate results. Further research is required to deal with systems with closely spaced or repeated eigenvalues. In order to obtain dynamic response statistics and system reliability, joint probability density functions of the eigenvalues and eigenvectors are required. Future studies will extend the proposed methods to obtain joint statistics of the eigenvalues.

APPENDIX A

A.1. Gradient vector and Hessian matrix of the eigenvalues

The eigenvectors of symmetric linear systems are orthogonal with respect to the mass and stiffness matrices. The eigenvectors are mass normalized, that is,

$$\boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_j = 1 \tag{A1}$$

Using this and differentiating Equation (1) with respect to x_k it can be shown that [42] for any $\mathbf{x}$

$$\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} = \boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}}\boldsymbol{\mathcal{G}}_{jk}(\mathbf{x})\boldsymbol{\phi}_j(\mathbf{x}) \tag{A2}$$

where

$$\boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) = \left[\frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_k} - \lambda_j(\mathbf{x})\frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k}\right] \tag{A3}$$

Differentiating Equation (1) with respect to x_k, and x_l, Plaut and Huseyin [43] have shown that, providing the eigenvalues are distinct,

$$\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k \partial x_l} = \boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}}\left[\frac{\partial^2 \mathbf{K}(\mathbf{x})}{\partial x_k \partial x_l} - \lambda_j(\mathbf{x})\frac{\partial^2 \mathbf{M}(\mathbf{x})}{\partial x_k \partial x_l}\right]\boldsymbol{\phi}_j(\mathbf{x})$$

$$-\left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}}\frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k}\boldsymbol{\phi}_j(\mathbf{x})\right)\left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}}\boldsymbol{\mathcal{G}}_{jl}(\mathbf{x})\boldsymbol{\phi}_j(\mathbf{x})\right)$$

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$$-\left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}}\frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_l}\boldsymbol{\phi}_j(\mathbf{x})\right)\left(\boldsymbol{\phi}_j(\mathbf{x})^{\mathrm{T}}\boldsymbol{\mathcal{G}}_{jk}(\mathbf{x})\boldsymbol{\phi}_j(\mathbf{x})\right)$$

$$+2\sum_{r=1}^{N}\frac{\left(\boldsymbol{\phi}_r(\mathbf{x})^{\mathrm{T}}\boldsymbol{\mathcal{G}}_{jk}(\mathbf{x})\boldsymbol{\phi}_j(\mathbf{x})\right)\left(\boldsymbol{\phi}_r(\mathbf{x})^{\mathrm{T}}\boldsymbol{\mathcal{G}}_{jl}(\mathbf{x})\boldsymbol{\phi}_j(\mathbf{x})\right)}{\lambda_j(\mathbf{x})-\lambda_r(\mathbf{x})} \tag{A4}$$

Equations (A2) and (A4) completely define the elements of the gradient vector and Hessian matrix of the eigenvalues.

A.2. Derivative of the system matrices with respect to the random variables

The derivatives of $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ with respect to elements of $\mathbf{x}$ can be obtained from Equation (101) together with Equations (102) and (103). For the mass matrix we have

$$\frac{\partial \mathbf{M}}{\partial x_1}=\begin{bmatrix}\overline{m}_1\varepsilon_m & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \frac{\partial \mathbf{M}}{\partial x_2}=\begin{bmatrix}0 & 0 & 0\\ 0 & \overline{m}_2\varepsilon_m & 0\\ 0 & 0 & 0\end{bmatrix},\quad \frac{\partial \mathbf{M}}{\partial x_3}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \overline{m}_3\varepsilon_m\end{bmatrix} \tag{A5}$$

All other $\partial\mathbf{M}/\partial x_i$ are null matrices. The derivatives of the stiffness matrix are

$$\frac{\partial \mathbf{K}}{\partial x_4}=\begin{bmatrix}\overline{k}_1\varepsilon_k & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \frac{\partial \mathbf{K}}{\partial x_5}=\begin{bmatrix}0 & 0 & 0\\ 0 & \overline{k}_2\varepsilon_k & 0\\ 0 & 0 & 0\end{bmatrix},\quad \frac{\partial \mathbf{M}}{\partial x_6}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \overline{k}_3\varepsilon_k\end{bmatrix}$$

$$\frac{\partial \mathbf{K}}{\partial x_7}=\begin{bmatrix}\overline{k}_4\varepsilon_k & -\overline{k}_4\varepsilon_k & 0\\ -\overline{k}_4\varepsilon_k & \overline{k}_4\varepsilon_k & 0\\ 0 & 0 & 0\end{bmatrix},\ \frac{\partial \mathbf{K}}{\partial x_8}=\begin{bmatrix}0 & 0 & 0\\ 0 & \overline{k}_5\varepsilon_k & -\overline{k}_5\varepsilon_k\\ 0 & -\overline{k}_5\varepsilon_k & \overline{k}_5\varepsilon_k\end{bmatrix},\ \frac{\partial \mathbf{M}}{\partial x_9}=\begin{bmatrix}\overline{k}_6\varepsilon_k & 0 & -\overline{k}_6\varepsilon_k\\ 0 & 0 & 0\\ -\overline{k}_6\varepsilon_k & 0 & \overline{k}_6\varepsilon_k\end{bmatrix} \tag{A6}$$

and all other $\partial\mathbf{K}/\partial x_i$ are null matrices. Also note that all of the first-order derivative matrices are independent of $\mathbf{x}$. For this reason, all the higher order derivatives of the $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ matrices are null matrices.

APPENDIX B: NOMENCLATURE

$\mathbf{a}_j$	constant in optimal perturbation expansion
$\mathbf{A}_j$	constant in optimal perturbation expansion
c_j	constant in optimal perturbation expansion
$\mathbf{d}_{(\bullet)}(\mathbf{x})$	gradient vector of $(\bullet)$ at $\mathbf{x}$
$\mathbf{D}_{(\bullet)}(\mathbf{x})$	Hessian matrix of $(\bullet)$ at $\mathbf{x}$

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$E[\bullet]$	mathematical expectation operator
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
$L(\mathbf{x})$	negative of the log-likelihood function
m	number of basic random variables
$M_{\lambda_j}(s)$	moment generating function of the eigenvalues
$\mathbf{M}$	mass matrix
n	number of moments used for pdf construction
N	degrees-of-freedom of the system
$\mathbf{O}$	null matrix
$p_{(\bullet)}$	probability density function of $(\bullet)$
$\mathbf{x}$	basic random variables

Greek letters

$\boldsymbol{\alpha}$	optimal point for perturbation method
$\varepsilon_m, \varepsilon_k$	strength parameters associated with mass and stiffness coefficients
η_j, γ_j, ν_j	parameters of χ^2-approximated pdf of λ_j
$\boldsymbol{\theta}$	optimal point for asymptotic method
$\kappa_j^{(r)}$	rth-order cumulant of the eigenvalues
λ_j	eigenvalues of the system
$\boldsymbol{\mu}$	mean of parameter vector $\mathbf{x}$
$\mu_j^{(r)}$	rth-order (raw) moment of the eigenvalues
$\mu_j'^{(r)}$	rth-order central moment of the eigenvalues
ρ_r	Lagrange multipliers, $r = 0, 1, 2, \ldots, n$
σ_j	standard deviation of λ_j
$\boldsymbol{\Sigma}$	covariance matrix
$\boldsymbol{\phi}_j$	eigenvectors of the system
Φ	cumulative Gaussian distribution function
$\chi^2_{\nu_j}(u)$	χ^2 density function with ν_j degrees-of-freedom
$\mathscr{L}$	Lagrangian
$\overline{(\bullet)}$	deterministic value of $(\bullet)$
$(\bullet)^{\mathrm{T}}$	matrix transpose
$\approx$	approximately equal to
$\mathbb{R}$	space of real numbers
$\|\bullet\|$	determinant of matrix $(\bullet)$
$\in$	belongs to
$\mapsto$	maps into
$\|\bullet\|$	l_2 norm of $(\bullet)$
$\widehat{(\bullet)}$	mean of $(\bullet)$

Abbreviations

dof	degrees-of-freedom
exp	exponential function
pdf	probability density function

DOI: 10.1002/nme

ACKNOWLEDGEMENTS

S. Adhikari gratefully acknowledges the support of the Engineering and Physical Sciences Research Council through the award of an Advanced Research Fellowship. M. I. Friswell gratefully acknowledges the support of the Royal Society through a Royal Society-Wolfson Research Merit Award.

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DOI: 10.1002/nme