Free-Vibration Analysis of Sandwich Panels with Randomly Irregular Honeycomb Core

T. Mukhopadhyay[1] and S. Adhikari[2]

Abstract: An analytical framework has been proposed to analyze the effect of random structural irregularity in honeycomb core for natural frequencies of sandwich panels. Closed-form formulas have been developed for the out-of-plane shear moduli of spatially irregular honeycombs following minimum potential energy theorem and minimum complementary energy theorem. Subsequently an analytical approach has been presented for free-vibration analysis of honeycomb core sandwich panels to quantify the effect of such irregularity following a probabilistic paradigm. Representative results have been furnished for natural frequencies corresponding to low vibration modes of a sandwich panel with high length-to-width ratio. The results suggest that spatially random irregularities in honeycomb core have considerable effect on the natural frequencies of sandwich panels. **DOI: 10.1061/(ASCE)EM.1943-7889.0001153.** © *2016 American Society of Civil Engineers.*

Author keywords: Sandwich panel; Irregular honeycomb core; Natural frequency; Random cell angle.

Introduction

An energy-based approach has been proposed for free-vibration analysis of sandwich panels with random geometric irregularity in the honeycomb core. Honeycomb sandwich panels are commonly used in various applications of civil, aerospace, and mechanical structures because of their high strength-to-weight ratios, desirable acoustic properties, and many other tailorable application-specific advantages (Zenkert 1995). Application of such sandwich panels as different structural components of aircraft is becoming increasingly popular from the basic need of aerospace industry for lightweight design that is capable of meeting the requirement of high structural rigidity. Lightweight bridge decks with sandwich panels are an active area of research in the bridge engineering research community. Over the last two decades, literature concerning vibration analysis of sandwich panels has been published following both finite-element (Chakrabarti and Sheikh 2004; Chalak et al. 2012; Long et al. 2012) and analytical approaches (Yongqiang and Zhiqiang 2008; Yu and Cleghorn 2005; Rao et al. 2004; Lee et al. 2007). Many review articles on this topic provide in-depth knowledge for deterministic analysis of such structures (Noor et al. 1996; Sayyad and Ghugal 2015). Scarpa and Tomlinson (2000) have presented free-vibration analysis of sandwich plates with auxetic honeycombs. The closed-form formulas provided by Gibson and Ashby (1999) to calculate effective elastic moduli of the honeycomb core without irregularity are found to be widely used in literature for free-vibration analysis of sandwich panels. However, spatial irregularity in the honeycomb core may occur due to uncertainty associated with manufacturing, uncertain distribution of intrinsic material properties, structural defects, variation in temperature, prestressing, and microstructural variability. It is imperative to consider the influence of such irregularities on global responses of the sandwich panel for performing a robust analysis. Although expressions for in-plane elastic moduli of irregular honeycombs have been recently derived (Mukhopadhyay and Adhikari 2016a, b), Out-of-plane elastic moduli of irregular honeycombs are required for bending and frequency analysis of sandwich panels with irregular cores.

Most of the investigations carried out so far concerning the analysis of sandwich panels are deterministic in nature and do not provide the necessary insight on the behavior of different structural responses generated from inherent statistical variations of stochastic material and geometric parameters. The state of available literature for vibration studies of sandwich structures with stochastic input parameters is very scarce. Stochastic finite-element analysis for deflection and free vibration response of soft-core sandwich plates with random material properties has been reported (Pandit et al. 2009, 2010). Stochastic free-vibration analysis and probabilistic failure analysis of laminated sandwich panels using Monte Carlo simulation coupled with finite-element modeling have been carried out recently (Dey et al. 2016; Kumar et al. 2015). The predominant approaches in these studies are to follow the expensive finite-element simulation using a perturbation-based method or Monte Carlo simulation that requires thousands of finite-element model evaluations. Existing investigations are mostly performed assuming a random variable-based approach, which neglects spatial irregularity of the honeycomb core. Such uncertainty modeling is rather unrealistic and often leads to poor quantification of the statistical variabilities associated with a structural system. The prime reason behind the neglect of spatial variability in the honeycomb core in stochastic analysis of sandwich structures is the computational cost associated with finite-element modeling. The simulation of randomly irregular honeycomb core geometry gives rise to the additional difficulty of creating a new finite-element mesh for each Monte Carlo sample. An analytical approach to avoid this would be a major computational convenience. Derivation of analytical expressions for global responses of sandwich panels with irregular honeycomb core requires closed-form formulas for equivalent elastic moduli of the cores. However, there is no analytical formula available in the literature to evaluate equivalent out-of-plane elastic properties of irregular honeycombs.

[1]Ph.D. Student, College of Engineering, Swansea Univ., Bay Campus, Swansea SA1 8EN, U.K. (corresponding author). E-mail: 800712@swansea.ac.uk

[2]Professor, College of Engineering, Swansea Univ., Bay Campus, Swansea SA1 8EN, U.K. E-mail: s.adhikari@swansea.ac.uk

Note. This manuscript was submitted on January 31, 2016; approved on June 16, 2016; published online on August 1, 2016. Discussion period open until January 1, 2017; separate discussions must be submitted for individual papers. This technical note is part of the ***Journal of Engineering Mechanics***, © ASCE, ISSN 0733-9399.

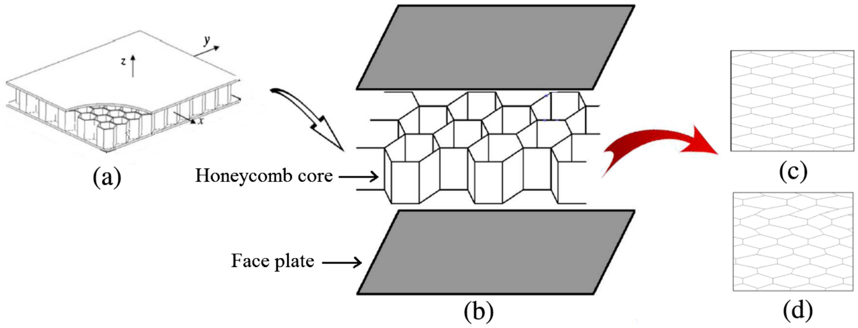

Fig. 1. (a) Sandwich panel; (b) components of honeycomb core sandwich panels; (c) regular honeycomb configuration; (d) irregular honeycomb with structural irregularity

This article proposes an efficient analytical framework leading to closed-form formulas for predicting equivalent elastic moduli of irregular honeycombs having spatially random variations in cell angles. Analytical formulations have subsequently been developed for natural frequencies of sandwich panels in conjunction with the proposed closed-form formulas for effective elastic moduli of irregular honeycomb cores (Fig. 1). To the best of authors' knowledge, this is the first such attempt for developing analytical formulas for free-vibration analysis of sandwich panels with randomly irregular honeycomb cores.

Analytical Formulation

Natural Frequencies

In this study a sandwich panel with a very high length-to-width ratio with uniform support along the two opposite long edges has been considered. For such ratios causing cylindrical bending, sandwich panel deformation may be considered independent of the length coordinate. Natural frequencies of such a panel with width a and core height h can be expressed as (Whitney 1987)

$$\omega_m = \frac{m^2\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}\sqrt{1-\frac{Sm^2\pi^2}{1+Sm^2\pi^2}}, \qquad m = 1, 2, 3, \ldots \tag{1}$$

where m = mode number of vibration; $S = D/G_{13}ha^2$ and D = contribution to sandwich bending stiffness from the face plates; and ρ = mass density.

From Eq. (1), it is evident that the natural frequency of the sandwich panel depends on the shear modulus G_{13} of the honeycomb core. Because this out-of-plane shear modulus is needed in the considered problem, closed-form formulas for G_{13} of irregular honeycombs are derived in the following paragraph. However, G_{23} can be derived following a similar analogy. In this work, Gibson and Ashby's (1999) approach for regular honeycombs has been extended to the case of irregular honeycombs.

Energy Theorems for Random Systems

For deriving the expression for G_{13} of irregular honeycombs, minimum potential energy theorem and minimum complementary energy theorem are used. These give the upper bound and the lower bound of G_{13}, respectively. The term *energy* refers to strain energy in this article. It is assumed that the shear stresses are uniform within the cell walls. A uniform shear strain γ_{13} caused by shear stress τ_{13} acting in the 1–2 plane is considered (Fig. 2). Direction 3 is perpendicular to the 1–2 plane. For irregular honeycombs, unlike regular honeycombs, all cell walls contribute in the analysis of out-of-plane shear moduli. According to the minimum potential energy theorem, the strain energy calculated from any postulated set of displacements which are compatible with the external boundary conditions and with themselves will be a minimum for the exact displacement distribution. Thus, for the irregular honeycomb as shown in Fig. 2

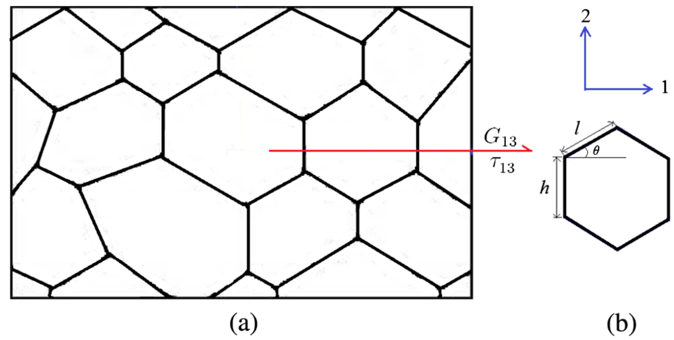

Fig. 2. (a) Spatially irregular honeycomb configuration; (b) unit cell of a regular honeycomb with cell angle θ

$$\frac{1}{2}G_{13}\nu_{13}^2 V \leq \frac{1}{2}\sum_i G_s \nu_i^2 V_i \tag{2}$$

where G_s = shear modulus of the cell wall material; $V(= LBd)$ and $V_i(= l_i t d)$ = total volume and volume of the ith cell wall, respectively; l_i, t, and d = length of ith cell wall, thickness of cell wall, and depth of honeycomb core, respectively; ν_i and ν_{13} = strain in the ith cell wall and global strain, respectively; and L and B = overall length and width of the entire irregular honeycomb, respectively.

From the concept of basic mechanics, it is evident that $\nu_i = \nu_{13}\cos\theta_i$, where θ_i = inclination angle of the ith cell wall with Direction 1. Thus, from Eq. (2), the upper bound of G_{13} can be obtained as

$$\frac{G_{13}}{G_s} \leq \frac{t}{LB}\sum_i l_i \cos^2\theta_i \tag{3}$$

The lower bound of G_{13} is obtained using minimum complementary energy theorem, which states that among the stress distributions that satisfy equilibrium at each point and that are in equilibrium with the external loads, the strain energy is a minimum for the exact stress distribution. By expressing this statement as an inequality, for the shear in Direction 1

$$\frac{1}{2}\frac{\tau_{13}^2}{G_{13}}V \le \frac{1}{2}\sum_i \frac{\tau_i^2}{G_s}V_i \tag{4}$$

From the condition of force equilibrium

$$\tau_{13}LB = \sum_i \tau_i t l_i \cos\theta_i \tag{5}$$

From Eqs. (4) and (5) the following inequality can be obtained

$$\frac{1}{G_{13}}\left(\frac{1}{LB}\sum_i \tau_i t l_i \cos\theta_i\right)^2 LBd \le \sum_i \frac{\tau_i^2}{G_s} l_i t d \tag{6}$$

Replacing $\tau_i = G_s \nu_i = G_s \nu_{13}\cos\theta_i$, the Eq. (6) becomes

$$\frac{t}{G_{13}LB}\left(\sum\nolimits_i G_s \nu_{13}\cos^2\theta_i l_i\right)^2 \le \sum_i \frac{1}{G_s}(G_s\nu_{13}\cos\theta_i)^2 l_i \tag{7}$$

Eq. (7) can be reduced to

$$\frac{t}{G_{13}LB}\left(\sum\nolimits_i \cos^2\theta_i l_i\right)^2 \le \frac{1}{G_s}\sum_i \cos^2\theta_i l_i \tag{8}$$

Dividing both side of Eq. (8) by $\sum_i \cos^2\theta_i l_i$, the lower bound of G_{13} can be obtained as

$$\frac{G_{13}}{G_s} \ge \frac{t}{LB}\sum_i l_i \cos^2\theta_i \tag{9}$$

The expressions of the lower and the upper bound of G_{13} for irregular honeycomb are noticed to be identical from Eqs. (3) and (9). Thus, the shear modulus G_{13} of an irregular honeycomb can be expressed as

$$\frac{G_{13}}{G_s} = \frac{t}{LB}\sum_i l_i \cos^2\theta_i \tag{10}$$

The expression for G_{13} of an irregular honeycomb reduces to the formula given by Gibson and Ashby (1999) in the special case of regular hexagonal honeycomb.

Results and Discussion

Representative results are presented for natural frequencies of a sandwich panel corresponding to low frequency bending modes with different degree of structural irregularity in a honeycomb core following the proposed analytical approach (refer Fig. 3). Random variation in the cell angles ($\Delta\theta$) have been considered in this study following uniform distribution. A typical irregular honeycomb configuration having spatially random variation in cell angles is shown in Fig. 4. Table 1 presents the results obtained for regular honeycomb using the formulas developed in the analytical formulation section for G_{13}, wherein good agreement with Gibson and Ashby's (1999) results corroborates validation of the proposed closed-form formulas.

Fig. 5 presents the effect of spatial irregularity on equivalent shear modulus of an irregular honeycomb with random distribution of cell angles with different magnitudes considering a mean cell angle $\theta = 30°$. The quantity $\Delta\theta$ represents the degree of irregularity that provides respective bounds as $(\theta + \Delta\theta, \theta - \Delta\theta)$ for spatially random variation of the cell angles. The results have been obtained following a probabilistic framework with 10,000 samples for each degree of randomness.

First three natural frequencies corresponding to bending modes have been studied. In case of low frequency bending modes it has

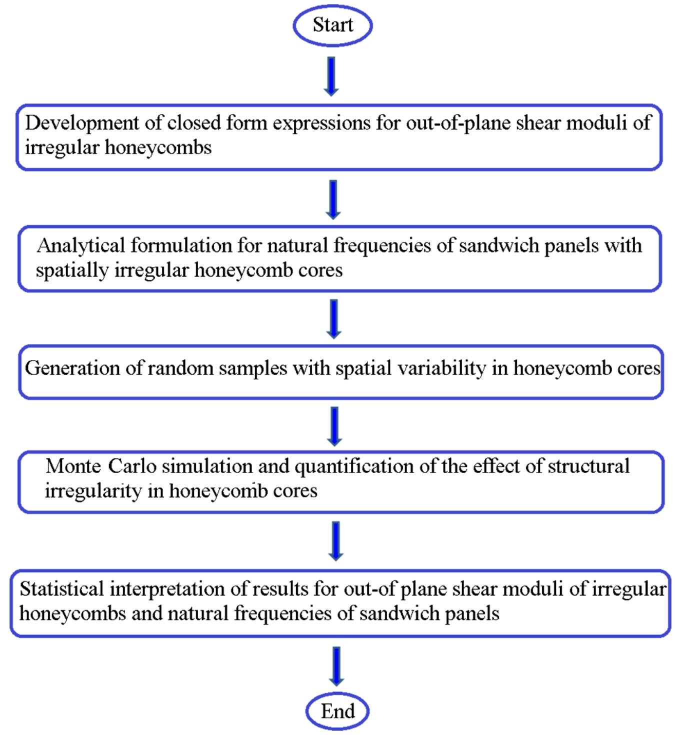

Fig. 3. Analytical framework for natural frequency of sandwich panels with irregular honeycomb cores

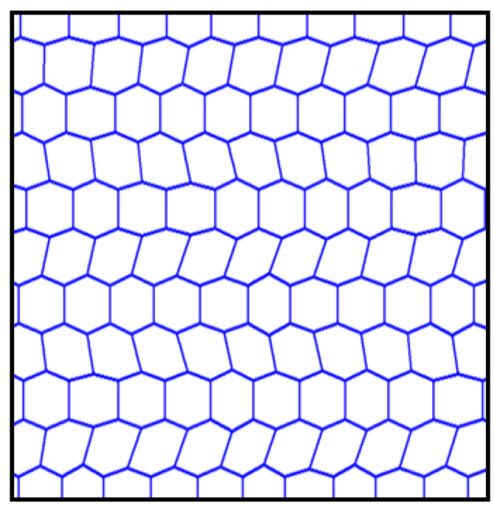

Fig. 4. Typical representation of irregularity in structural configuration of a honeycomb having spatially random variation in cell angles

Table 1. Nondimensional Shear Modulus (G_{13}/G_s) for a Regular Honeycomb with $t/l = 0.08$ and $h/l = 1$ [See Fig. 2(b)]

Cell angle (θ)	Gibson and Ashby (1999)	Eq. (10)
30°	0.046188	0.046190
45°	0.033137	0.033141

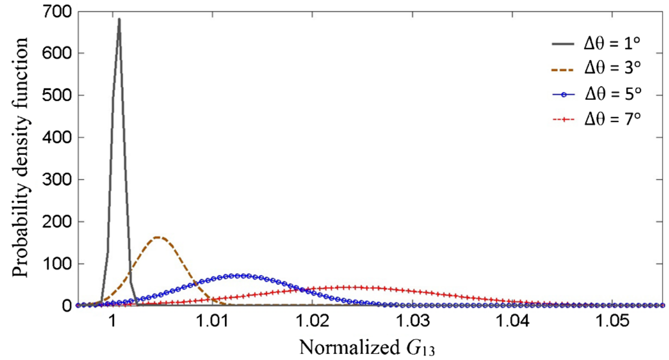

Fig. 5. Nondimensional shear modulus G_{13} ($G_{13\text{irregular}}/G_{13\text{regular}}$) for an irregular honeycomb; the subscripts "irregular" and "regular" indicate the shear moduli of irregular and regular honeycombs, respectively

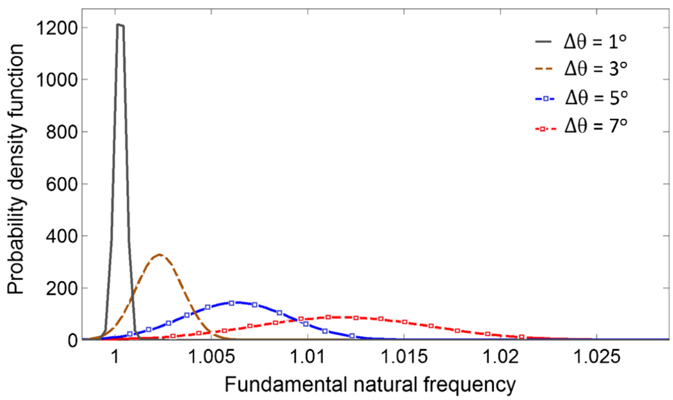

Fig. 6. Nondimensional natural frequencies ($\omega_{\text{irregular}}/\omega_{\text{regular}}$) for an irregular honeycomb; the subscripts "irregular" and "regular" indicate the natural frequencies of irregular and regular honeycombs, respectively

been observed that variation in the nature of the probability distribution of normalized natural frequencies obtained by the ratio ($\omega_{\text{irregular}}/\omega_{\text{regular}}$) among different modes is very small. In this article, the subscripts irregular and regular indicate the G_{13} values and natural frequencies corresponding to irregular and regular honeycomb respectively. The variation in normalized natural frequencies due to irregularity in the honeycomb core is shown in Fig. 6. Negligible variation in the ratio ($\omega_{\text{irregular}}/\omega_{\text{regular}}$) among different low frequency vibration modes can be explained easily by studying Eq. (1). Using this equation for the irregular and regular core and taking the ratio

$$\frac{\omega_{\text{irregular}}}{\omega_{\text{regular}}} = \sqrt{\frac{G_{13\text{irregular}}}{G_{13\text{regular}}}} \sqrt{\frac{G_{13\text{regular}} \times \frac{ha^2}{D\pi^2} + m^2}{G_{13\text{irregular}} \times \frac{ha^2}{D\pi^2} + m^2}} \tag{11}$$

In this expression of ($\omega_{\text{irregular}}/\omega_{\text{regular}}$), the sensitivity of m is found to be very low, as $[G_{13\text{regular}} \times (\text{ha}^2/D\pi^2)]$ and $[G_{13\text{irregular}} \times (\text{ha}^2/D\pi^2)]$ are on the order of 10^3. The probability distribution of low-frequency bending modes in actual values can be obtained using this nondimensional plot (Fig. 6) by multiplying the deterministic natural frequency (ω_{regular}) of the corresponding mode. Therefore, it can be understood that for low-frequency vibration modes, standard deviation and response bounds for a particular degree of irregularity increase with the increase in mode number

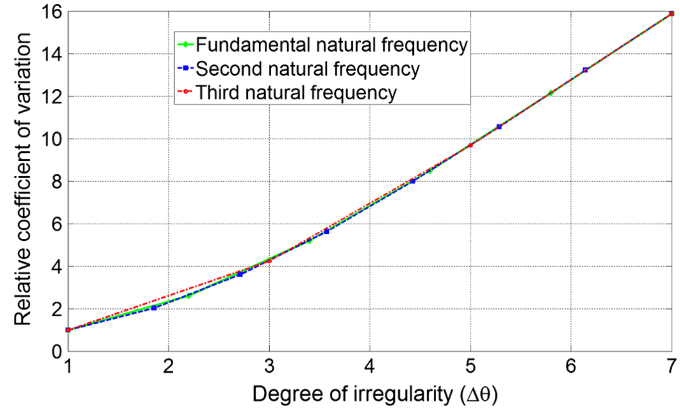

Fig. 7. Relative coefficient of variation for the first three natural frequencies corresponding to different degrees of irregularity in the honeycomb core; the relative coefficient of variation for a particular degree of irregularity ($\Delta\theta$) is obtained by dividing the corresponding coefficient of variation by the coefficient of variation for $\Delta\theta = 1$, where the coefficient of variation = ratio of mean and standard deviation for a distribution

of vibration depending on the corresponding deterministic natural frequency. From Figs. 5 and 6, it is evident that the mean and response bound for both G_{13} and natural frequencies increase with higher degrees of irregularity. Statistical analysis results for the first three natural frequencies showing relative coefficients of variation for different degree of irregularities are presented in Fig. 7, where the relative effect of increasing degrees of irregularity with respect to $\Delta\theta = 1$ is shown as a measure of sensitivity for spatial structural irregularity. Although the mean of the frequencies does not change very significantly with respect to the irregularity of the honeycomb core (from Fig. 6—an approximate 1.5% increase for $\Delta\theta = 7$)—the relative coefficient of variation shows drastically different behavior. Fig 7 shows that the value of the relative coefficient of variation corresponding to $\Delta\theta = 3$ is 4.2, whereas it increases to a value of 16 for $\Delta\theta = 7$. This significant increase in the coefficient of variation with respect to degree of irregularity in the honeycomb core demonstrates the importance of incorporating the effect of such variability in the subsequent design and analysis process.

Using the closed-form analytical formulas for out-of-plane elastic moduli developed in this article along with the in-plane elastic moduli (Mukhopadhyay and Adhikari 2016a, b) for irregular honeycombs, bending and deflection analysis of sandwich panels with spatially irregular honeycomb cores can also be carried out by applying standard expressions for bending and deflection of sandwich panels. Further, optimum design of sandwich panels considering stochastic static and dynamic response bounds as constraints can be easily performed using the proposed formulas. Noteworthy is the fact that the analytical expressions for global responses of sandwich panels will provide a major convenience in terms of computational cost in such analyses. Otherwise, an optimization process involving uncertainty and reliability analysis requires thousands of expensive finite-element simulations to be carried out (each with different random structural geometry and remeshing of the entire structure).

Conclusion

A closed-form expression for predicting the equivalent out-of-plane shear moduli of irregular honeycombs is developed based on simultaneous employment of the minimum potential energy theorem and the minimum complementary energy theorem and subsequent exploitation of two contradictory mathematical inequalities. Thereby an analytical approach is proposed for analyzing natural frequencies of a sandwich panel with an irregular honeycomb core. Results have been presented for natural frequencies of low bending vibration modes considering different degrees of irregularity through a probabilistic framework. The analyses provide thorough insight into the effect of irregularity on the shear modulus G_{13} and the natural frequencies of a honeycomb core sandwich panel. The spatial irregularity in honeycomb core has considerable effect on the global response of the sandwich panel, such as the natural frequencies. Thus it is very important to consider the effect of such irregularity in further analysis and design for a robust design in lightweight sandwich structures.

The novelty of this article includes both the development of a closed-form expression for out-of-plane elastic moduli of irregular honeycombs and the analytical framework for free-vibration analysis of sandwich panels with irregular honeycomb cores. The proposed closed-form expressions for out-of-plane shear moduli of irregular honeycombs can also be used to analyze responses for other structural applications of honeycombs. These formulas can be extended further to predict equivalent out-of-plane elastic moduli of irregular honeycombs having spatial variation in material properties and cell wall thickness. The analytical framework presented here can be used for efficient uncertainty quantification in the free-vibration analysis of sandwich panels considering spatial variability of material and geometric properties based on random field approaches.

Acknowledgments

TM acknowledges financial support from Swansea University through the award of a Zienkiewicz Scholarship. SA acknowledges financial support from the Royal Society of London through the Wolfson Research Merit Award.

References

Chakrabarti, A., and Sheikh, A. (2004). "Vibration of laminate faced sandwich plate by a new refined element." *J. Aerosp. Eng.*, 10.1061/(ASCE)0893-1321(2004)17:3(123), 123–134.

Chalak, H. D., Chakrabarti, A., Iqbal, M. A., and Sheikh, A. (2012). "Vibration of laminated sandwich beams with soft core." *J. Vib. Control*, 18(10), 1422–1435.

Dey, S., Mukhopadhyay, T., Naskar, S., Dey, T. K., Chalak, H. D., and Adhikari, S. (2016). "Probabilistic characterization for dynamics and stability of laminated soft core sandwich plates." *J. Sandwich Struct. Mater.*, in press.

Gibson, L., and Ashby, M. F. (1999). *Cellular solids structure and properties*, Cambridge University Press, Cambridge, U.K.

Kumar, A., Chakrabarti, A., Bhargava, P., and Chowdhury, R. (2015). "Probabilistic failure analysis of laminated sandwich shells based on higher order zigzag theory." *J. Sandwich Struct. Mater.*, 17(5), 546–561.

Lee, C. R., Kam, T. Y., and Sun, S. J. (2007). "Free-vibration analysis and material constants identification of laminated composite sandwich plates." *J. Eng. Mech.*, 10.1061/(ASCE)0733-9399(2007)133:8(874), 874–886.

Long, R., Barry, O., and Oquamanam, D. C. D. (2012). "Finite element free vibration analysis of soft-core sandwich beams." *AIAA J.*, 50(1), 235–238.

Mukhopadhyay, T., and Adhikari, S. (2016a). "Effective in-plane elastic properties of auxetic honeycombs with spatial irregularity." *Mech. Mater.*, 95, 204–222.

Mukhopadhyay, T., and Adhikari, S. (2016b). "Equivalent in-plane elastic properties of irregular honeycombs: An analytical approach." *Int. J. Solids Struct.*, 91, 169–184.

Noor, A. K., Burton, W. S., and Bert, C. W. (1996). "Computational models for sandwich panels and shells." *Appl. Mech. Rev.*, 49(3), 155–199.

Pandit, M., Singh, B., and Sheikh, A. (2009). "Stochastic perturbation-based finite element for deflection statistics of soft core sandwich plate with random material properties." *Int. J. Mech. Sci.*, 51(5), 363–371.

Pandit, M., Singh, B., and Sheikh, A. (2010). "Stochastic free vibration response of soft core sandwich plates using an improved higher-order zigzag theory." *J. Aerosp. Eng.*, 10.1061/(ASCE)0893-1321(2010)23:1(14), 14–23.

Rao, M. K., Scherbatiuk, K., Desai, Y. M., and Shah, A. H. (2004). "Natural vibrations of laminated and sandwich plates." *J. Eng. Mech.*, 10.1061/(ASCE)0733-9399(2004)130:11(1268), 1268–1278.

Sayyad, A. S., and Ghugal, Y. M. (2015). "On the free vibration analysis of laminated composite and sandwich plates: A review of recent literature with some numerical results." *Compos. Struct.*, 129, 177–201.

Scarpa, F., and Tomlinson, G. (2000). "Theoretical characteristics of the vibration of sandwich plates with in-plane negative poisson's ratio values." *J. Sound Vib.*, 230(1), 45–67.

Whitney, J. M. (1987). *Structural analysis of laminated anisotropic plates*, Technomic Publishing Company, Lancaster, PA.

Yongqiang, L., and Zhiqiang, J. (2008). "Free flexural vibration analysis of symmetric rectangular honeycomb panels with SCSC edge supports." *Compos. Struct.*, 83(2), 154–158.

Yu, S. D., and Cleghorn, W. L. (2005). "Free flexural vibration analysis of symmetric honeycomb panels." *J. Sound Vib.*, 284(1–2), 189–204.

Zenkert, D. (1995). *An introduction to sandwich construction*, Chameleon Press, London.