Reliability Analysis Using Parabolic Failure Surface Approximation

S. Adhikari[1]

Abstract: In the second-order reliability method the failure surface is approximated by a general quadratic surface in the neighborhood of the design point. In this paper this general quadratic surface is further approximated by a parabolic surface. Several methods are proposed to obtain the probability content associated with this parabolic failure surface. It is assumed that the basic random variables are Gaussian. The proposed methods can be broadly grouped into: (1) nonasymptotic approximate methods, (2) exact methods, and (3) asymptotic distribution methods. Most of these methods result in a closed-form expression for the failure probability. For nonasymptotic approximations, a least-square approach and an optimal point expansion method using approximate probability density functions of a quadratic form in Gaussian random variables have been proposed. It is shown that such approximations give accurate results without significant numerical effort. Exact results, however, require greater numerical effort. The new asymptotic result is derived for the case when the number of random variables approaches infinity. Several numerical examples are provided to compare the proposed results with existing equivalent results and Monte Carlo simulations.

DOI: 10.1061/(ASCE)0733-9399(2004)130:12(1407)

CE Database subject headings: Structural reliability; Structural failures; Reliability analysis; Monte Carlo method.

Introduction

Reliability based structural analysis and design have two main objectives, namely: (1) to obtain the probability of failure of an existing structure under the action of applied loadings, and (2) to design a structure with prescribed reliability requirements. For both types of problem, it is required to calculate the probability of failure (or survival) of a structure, either to assess the risk associated with an existing structural facility, or to determine if a structural design has met the prescribed reliability criteria. Suppose the random variables describing the uncertainties in the structural properties and loading are considered to form a vector $\mathbf{x} \in \mathbb{R}^n$. The statistical properties of the system are fully described by the joint probability density function $p(\mathbf{x}):\mathbb{R}^n \mapsto \mathbb{R}$. For a given set of variables $\mathbf{x}$ the structure will either fail under the applied (random) loading or will be safe. The condition of the structure for every $\mathbf{x}$ can be described by a safety margin $g(\mathbf{x}):\mathbb{R}^n \mapsto \mathbb{R}$ such the structure has failed if $g(\mathbf{x}) \leqslant 0$ and is safe if $g(\mathbf{x}) > 0$. Thus, the probability of failure is given by

$$P_f = \int_{g(\mathbf{X}) \leqslant 0} p(\mathbf{x}) d\mathbf{x} \tag{1}$$

The function $g(\mathbf{x})$ is also known as the failure surface or the limit-state function. The central theme of a reliability analysis is to evaluate the multidimensional integral Eq. (1). The exact evaluation of this integral, either analytically or numerically, is not possible for most practical problems because: (1) n is large; (2) $p(\mathbf{x})$ is non-Gaussian; and (3) $g(\mathbf{x})$ is a highly nonlinear function of $\mathbf{x}$. Even direct Monte Carlo simulation is quite expensive because P_f is usually a small quantity.

Over the past three decades there has been extensive research (see for example, the books by Thoft-Christensen and Baker 1982; Madsen et al. 1986; Ditlevsen and Madsen 1996; Melchers 1999) to develop approximate numerical methods for the efficient calculation of the reliability integral. A review of earlier as well as current works may be found in Manohar and Gupta (2003). The approximate reliability methods can be broadly grouped into: (1) first-order reliability method (FORM), and (2) second-order reliability method (SORM). In FORM and SORM it is assumed that all the basic random variables are transformed and scaled so that they are uncorrelated Gaussian random variables, each with zero mean and unit standard deviation. In principle such a transformation can always be achieved, for example using the Rosenblatt (1952) transformation. In the transformed space Hasofer and Lind (1974) defined the reliability index

$$\beta = (\mathbf{x}^{*T}\mathbf{x}^{*})^{1/2} \tag{2}$$

Here $\mathbf{x}^*$, the "design point" or the "checking point," is the point on the failure surface with minimum distance from the origin. That is, x* is the solution of the following optimization problem:

$$\min\{(\mathbf{x}^T\mathbf{x})^{1/2}\}$$

$$\text{subject to } g(\mathbf{x}) = 0 \tag{3}$$

Once the reliability index and the design point is known, the probability of failure can be obtained using FORM or SORM. In FORM the failure surface is approximated by a hyperplane which is tangent to the failure surface at the design point. In SORM, the actual failure surface is approximated by a quadratic hypersurface in the neighborhood of the design point. It is possible for there to be multiple points with the same minimum distance from the origin. In which case one separately calculates the contributions

[1]Dept. of Aerospace Engineering, Univ. of Bristol, Bristol, U.K.; formerly, PhD, Dept. of Engineering, Univ. of Cambridge, Cambridge, U.K. E-mail: s.adhikari@bristol.ac.uk

Note. Associate Editor: Gerhart I. Schueller. Discussion open until May 1, 2005. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on October 1, 2002; approved on January 28, 2004. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 130, No. 12, December 1, 2004. ©ASCE, ISSN 0733-9399/2004/12-1407–1427/$18.00.

arising from all design points which are significant and adds them to obtain the total failure probability.

In general it is difficult to quantify the amount of error introduced in FORM or SORM. For standard Gaussian basic variables, Breitung (1984) proposed an asymptotic method (asymptotic SORM) which becomes exact in the limit when $\beta \to \infty$. In a later work, using the maximum likelihood approach in conjunction with the asymptotic analysis, Breitung (1991) obtained the failure probability in the space of non-Gaussian basic random variables. Here, like the failure surface, the log-likelihood function is also expressed by a second-order Taylor series in the neighborhood of the design point. This is equivalent to approximating the basic non-Gaussian probability density function (PDF) by a Gaussian PDF in the vicinity of the design point. In view of this, one can identify three different stages of approximations, and consequently errors, in SORM:

1. **Type 1:** error due to approximating a general nonlinear failure surface by a quadratic hypersurface in the neighborhood of the design point;
2. **Type 2:** error due to approximating a general PDF by a Gaussian PDF in the neighborhood of the design point; and
3. **Type 3:** any errors in evaluating the probability content after making the previous two assumptions.

In a recent paper Breitung (2002) pointed out the importance of distinguishing between these errors. Type 1 error is fundamental to SORM and cannot be improved unless higher order derivatives of the failure surface are used (Breitung and Richter 1996). Type 2 error can be avoided provided the basic variables are already Gaussian or transformed to Gaussian using numerical transformations. Breitung (1984, 1989, 1991) used asymptotic approximations at each stage and has shown that the three types of errors asymptotically vanish when $\beta \to \infty$. Extensive work has been done, for example by Hohenbichler and Rackwitz (1988), Naess (1987), Tvedt (1990), Köylüoğlu and Nielsen (1994), Cai and Elishakoff (1994), Zhao and Ono (1999a, b), Polidori et al. (1999), and Hong (1999), to improve the reliability assessment when the asymptotic condition does not hold. Majority of these works deal with Type 3 error. Another route to reducing all three types of errors is to use simulation methods, which, with the development of importance sampling techniques are becoming very efficient. Further discussions on developments in this area can be found in the recent review paper by Manohar and Gupta (2003) and the cited references therein.

This paper deals with Type 3 error. Three distinct methods, namely nonasymptotic approximate methods, exact methods, and asymptotic distribution methods, are proposed to reduce this error. It is assumed that the basic random variables are Gaussian so that Type 2 error is not present. Discussions are confined to parabolic failure surfaces and systems with a unique design point only.

Transformations of Failure Surface

General Quadratic Surface

For standard Gaussian basic variables $\mathbf{x} \sim \mathbb{N}_n(\mathbf{0}, \mathbf{I}_n)$ the joint PDF is given by

$$p(\mathbf{x}):\mathbb{R}^n \mapsto \mathbb{R} = (2\pi)^{-n/2} e^{-\mathbf{x}^T\mathbf{x}/2} \tag{4}$$

Assuming that $g(\mathbf{x})$ is continuous, smooth and at least twice differentiable, in SORM the actual failure surface is replaced by its second-order Taylor series expansion about the design point

$$g(\mathbf{x}) \approx g(\mathbf{x}^*) + \nabla g(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \tfrac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T\mathbf{H}_g(\mathbf{x}^*) \times (\mathbf{x} - \mathbf{x}^*) = \tilde{g}(\mathbf{x}) \tag{5}$$

Because $\mathbf{x}^*$ is on the failure surface, $g(\mathbf{x}^*) = 0$ and we obtain

$$\tilde{g}(\mathbf{x}) = \left(-\nabla g^T\mathbf{x}^* + \tfrac{1}{2}\mathbf{x}^{*T}\mathbf{H}_g\mathbf{x}^*\right) - (-\nabla g^T + \mathbf{x}^{*T}\mathbf{H}_g)\mathbf{x} + \tfrac{1}{2}\mathbf{x}^T\mathbf{H}_g\mathbf{x} \tag{6}$$

For notational convenience $(\mathbf{x}^*)$ is omitted. Fiessler et al. (1979), Madsen et al. (1986), and Tvedt (1990) have used orthogonal transformations to reduce Eq. (5) into suitable forms. At the design point the gradient vector to the failure surface and the vector from the origin are parallel, that is

$$\frac{\mathbf{x}^*}{\beta} = -\frac{\nabla g}{|\nabla g|} = \boldsymbol{\alpha}^* \tag{7}$$

where $\beta = |\mathbf{x}^*|$ and $\boldsymbol{\alpha}^* =$ units vector to $\mathbf{x}^*$. Dividing Equation (6) by $|\nabla_g|$ and using Eq. (7) the second-order approximation to the failure surface can be expressed as

$$\frac{\tilde{g}(\mathbf{x})}{|\nabla g|} = \left(\beta + \frac{\beta^2}{2}\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla_g|}\boldsymbol{\alpha}^*\right) - \left(\boldsymbol{\alpha}^{*T} + \beta\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla_g|}\right)\mathbf{x} + \frac{1}{2}\mathbf{x}^T\frac{\mathbf{H}_g}{|\nabla_g|}\mathbf{x} \tag{8}$$

Using the orthogonal transformation

$$\mathbf{x} = \mathbf{T}\tilde{\mathbf{y}} \tag{9}$$

where $\tilde{\mathbf{y}} \sim \mathbb{N}_n(\mathbf{0}, \mathbf{I}_n)$ and $\mathbf{T} \in \mathbb{R}^{n\times n}$ is the matrix of eigenvectors of $\mathbf{H}_g(\mathbf{x}^*)$, Eq. (8) can be transformed to

$$\frac{\tilde{g}}{|\nabla g|} = \tilde{a}_0 - \tilde{\mathbf{a}}_1^T\tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T\tilde{\mathbf{A}}_2\tilde{\mathbf{y}} \tag{10}$$

Here

$$\tilde{a}_0 = \beta + \frac{\beta^2}{2}\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla g|}\boldsymbol{\alpha}^* \in \mathbb{R} \tag{11}$$

$$\tilde{\mathbf{a}}_1^T = \left(\boldsymbol{\alpha}^{*T} + \beta\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla g|}\right)\mathbf{T} \in \mathbb{R}^{1\times n} \tag{12}$$

and

$$\tilde{\mathbf{A}}_2 = \frac{1}{|2\nabla g|}\mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n] \in \mathbb{R}^{n\times n} \tag{13}$$

The quadratic hypersurface in Eq. (10) can be elliptic, parabolic, or hyperbolic. The parabolic case appears when for some k, $\lambda_k = 0$ but $\mathbf{a}_{1_k} \neq 0$. Because $p(\mathbf{x})$ in Eq. (4) is rotationally symmetric, any orthogonal transformation (rotation) like Eq. (9) does not change the probability content of an infinitely small volume element. Therefore, the failure probability can be approximated as

$$P_f \approx \mathrm{Prob}[\tilde{g}/|\nabla g| \leq 0] = \mathrm{Prob}\left[\tilde{a}_0 + \sum_{k=1}^{n}(-\tilde{a}_{1_k}\tilde{y}_k + \tilde{A}_{2_{kk}}\tilde{y}_k^2) \leq 0\right] \tag{14}$$

By inverting the characteristic function of $\tilde{g}/|\nabla_g|$, Tvedt (1990) has proposed a complex-domain numerical integration method for the exact evaluation of Eq. (14).

Parabolic Approximation

Madsen et al. (1986) proposed a parabolic approximation to Eq. (5) which keeps the intuitive notion of the reliability index. Construct an orthogonal matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$ whose nth column is $\boldsymbol{\alpha}^*$, that is

$$\mathbf{R} = [\mathbf{R}_1 | \boldsymbol{\alpha}^*] \text{ where } \mathbf{R}_1 \in \mathbb{R}^{n\times n-1}$$

and

$$\boldsymbol{\alpha}^{*T}\mathbf{R}_1 = \mathbf{0}_{1\times(n-1)} \tag{15}$$

The matrix $\mathbf{R}$ can be obtained, for example, by Gram–Schmidt orthogonalization. Using the orthogonal transformation

$$\mathbf{x} = \mathbf{R}\tilde{\mathbf{y}} \tag{16}$$

and partitioning $\tilde{\mathbf{y}}$ as

$$\tilde{\mathbf{y}} = \begin{Bmatrix} \mathbf{y} \\ y_n \end{Bmatrix}$$

where

$$\mathbf{y} \sim \mathbb{N}_{n-1}(\mathbf{0}_{n-1}, \mathbf{I}_{n-1})$$

and

$$y_n \sim \mathbb{N}_1(0,1) \tag{17}$$

from Eq. (8) one obtains

$$\frac{\tilde{g}}{|\nabla g|} = \left(\beta + \frac{\beta^2}{2}\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla g|}\boldsymbol{\alpha}^*\right) - \left(y_n + \beta\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla g|}\mathbf{R}\tilde{\mathbf{y}}\right) + \tilde{\mathbf{y}}^T\tilde{\mathbf{A}}\tilde{\mathbf{y}} \tag{18}$$

Here

$$\tilde{\mathbf{A}} = \frac{1}{2}\frac{\mathbf{R}^T\mathbf{H}_g(\mathbf{x}^*)\mathbf{R}}{|\nabla g(\mathbf{x}^*)|} \in \mathbb{R}^{n\times n} \tag{19}$$

For convenience partition $\tilde{\mathbf{A}}$ as

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \tilde{\mathbf{A}}_{1n} \\ \tilde{\mathbf{A}}_{n1} & \tilde{\mathbf{A}}_{nn} \end{bmatrix} \tag{20}$$

where $\mathbf{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\tilde{\mathbf{A}}_{n1} = \tilde{\mathbf{A}}_{1n}^T \in \mathbb{R}^{1\times(n-1)}$

In view of this partition, Equation (18) reads

$$\frac{\tilde{g}}{|\nabla g|} = -y_n + \beta + \mathbf{y}^T\mathbf{A}\mathbf{y} + \frac{\beta^2}{2}\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla g|}\boldsymbol{\alpha}^* - \beta\boldsymbol{\alpha}^{*T}\frac{\mathbf{H}_g}{|\nabla g|}\mathbf{R}\tilde{\mathbf{y}} + 2y_n\tilde{\mathbf{A}}_{n1}\mathbf{y} + \tilde{A}_{nn}y_n^2 \tag{21}$$

Keeping only second-order terms in $\mathbf{y}$ and neglecting any cross terms Madsen et al. (1986) have approximated Eq. (21) by a parabolic surface

$$\frac{\tilde{g}}{|\nabla g|} \approx -y_n + \beta + \mathbf{y}^T\mathbf{A}\mathbf{y} \tag{22}$$

The matrix $\mathbf{A}$ can be diagonalized by a further orthogonal transformation using the eigenvectors of $\mathbf{A}$. The parabolic surface in Eq. (22) has been used by a number of authors, for example, Hohenbichler and Rackwitz (1988), Köylüoğlu and Nielsen (1994), Cai and Elishakoff (1994), Zhao and Ono (1999a, b), Polidori et al. (1999), and Hong (1999). Der-Kiureghian et al. (1987) and Der-Kiureghian and de Stefano (1991) have also used this parabolic surface as the basis for point fitted SORM. With this approximation the failure probability is given by

$$P_f \approx \text{Prob}\left[\frac{\tilde{g}}{|\nabla g|} \leqslant 0\right] \approx \text{Prob}[y_n \geqslant \beta + \mathbf{y}^T\mathbf{A}\mathbf{y}] \tag{23}$$

Denoting

$$U: \mathbb{R}^{n-1} \mapsto \mathbb{R} = \mathbf{y}^T\mathbf{A}\mathbf{y} \tag{24}$$

as a central quadratic form in standard Gaussian random variables, the failure probability can be rewritten as

$$P_f \approx \text{Prob}[y_n \geqslant \beta + U] = \int_{\mathbb{R}} \left\{\int_{\beta+u}^{\infty} \varphi(y_n)dy_n\right\} p_U(u)du = E[\Phi(-\beta - U)] \tag{25}$$

The exact probability density function of the quadratic form U is in general not available in closed form. For this reason it is difficult to calculate the expectation $E[\Phi(-\beta - U)]$ analytically. Several authors have used approximations of $E[\Phi(-\beta - U)]$ to obtain closed-form expressions of P_f. A selected collection of such expressions can be found in Zhao and Ono (1999a). In this paper closed-form expressions of P_f will be derived with an aim to improve on the accuracy of currently available formulae. Direct numerical integration methods, like the saddle-point integration method in Tvedt (1990), can be used to evaluate $E[\Phi(-\beta - U)]$ in Eq. (25) exactly and quite efficiently. Nevertheless, simple closed-form expressions are useful in many situations, for example; (1) to intuitively understand numerical results, and (2) in the initial design stage when sensitivity of the failure probability with respect to certain design variables is required.

Outline of Paper

The various new methods to obtain expressions of P_f proposed in this paper are presented in the following sequence:

A. Methods using approximation of $\Phi(-\beta - u)$
   1. Taylor series of $\ln[\Phi(-\beta - u)]$ about $u = 0$
   2. Least-square method
   3. Optimal point expansion method.

B. Methods using approximation of the pdf of U
   1. Mean value approximation
   2. Patnaik's χ^2 type approximation
   3. Pearson's approximation

C. Methods for exact probability content for parabolic failure surfaces
   1. Series expansion methods
   2. Fourier series method
   3. Fourier sin transform method

D. Asymptotic distribution method for $n \to \infty$

In the next few sections these methods are explained in detail. Results obtained from each method are illustrated by numerical examples and compared with existing reliability approximations and Monte Carlo simulations. The last section summarizes the main outcomes of this paper.

Nonasymptotic Approximations to Failure Probability

Using quadratic approximation of the failure surface together with asymptotic analysis Breitung (1984) proved that

$$P_f \to \frac{\Phi(-\beta)}{\prod_{j=1}^{n-1}\sqrt{1+2\beta a_j}} \text{ when } \beta \to \infty \tag{26}$$

Here a_j, the eigenvalues of $\mathbf{A}$, can be related to the principal curvatures of the surface κ_j as $a_j = \kappa_j/2$. This result is important because it gives the asymptotic behavior P_f under general conditions. Some approximations to be proposed in this paper are nonasymptotic in the sense that we do not assume any parameter, like β in Eq. (26), is approaching infinity. The advantage of nonasymptotic approximations is that the approximations may work well when the asymptotic condition is not met. The disadvantage is that one cannot in general prove that an approximation approaches the exact value when some conditions (like an asymptotic condition) are fulfilled. However, for reliability approximations one would expect $P_f \to 0$ as $\beta \to \infty$. This limiting behavior is satisfied by all the approximate formulae to be derived in this paper.

As mentioned earlier, the main difficulty in obtaining the probability of failure from Eq. (25) is that the probability density function of the quadratic form U given in Eq. (24) is in general not available in closed form. However its moment generating function, characteristic function and moments of any order can be obtained. Extensive discussions on quadratic forms in Gaussian random variables can be found in the books by Johnson and Kotz (1970, Chapter 29) and Mathai and Provost (1992). For any $s \in \mathbb{C}$ the moment generating function of U can be obtained as

$$M_U(s) = E[e^{sU}] = \|\mathbf{I}_{n-1} - 2s\mathbf{A}\|^{-1/2} \tag{27}$$

Due to the simplicity of this expression, in the next section we propose three different approaches to approximate $\Phi(-\beta-u)$ in terms of exponentials of u. The matrix $\mathbf{A}$ is likely to be positive definite because the Hessian matrix $\mathbf{H}_g(\mathbf{x}^*)$ is expected to be positive definite in the projection on the tangential space. Some authors, for example Köylüoğlu and Nielsen (1994), Zhao and Ono (1999a), Polidori et al. (1999) have considered the cases when $\mathbf{A}$ is negative definite or more generally positive semidefinite. Here we have not considered such cases and throughout the paper it is assumed that $\mathbf{A}$ is a positive definite matrix.

Failure Probability Using Approximations of $\Phi(-\beta-U)$

Taylor Series of $\ln[\Phi(-\beta-u)]$ *about* $u=0$

The function $\Phi(-\beta-u)$ is smooth, continuous, and differentiable (of any order) for $u \in \mathbb{R}$. We write

$$\Phi(-\beta-u) = e^{\ln[\Phi(-\beta-u)]} \tag{28}$$

From the definition of U in Eq. (24) it is easy to see that $u \in \mathbb{R}^+$ as $\mathbf{A}$ is positive definite. The maximum of $\ln[\Phi(-\beta-u)]$ in $\mathbb{R}^+$ occurs at $u=0$. Therefore, the maximum contribution to the expectation of $\ln[\Phi(-\beta-U)]$ comes from the neighborhood of $u=0$. Expanding $\ln[\Phi(-\beta-u)]$ in a first-order Taylor series about $u=0$ we obtain

$$\Phi(-\beta-u) \approx \exp\ln[\Phi(-\beta)] - \frac{\varphi(\beta)}{\Phi(-\beta)}u = \Phi(-\beta)\exp\left(-\frac{\varphi(\beta)}{\Phi(-\beta)}u\right) \tag{29}$$

The reason for keeping only one term in the Taylor series is to exploit the expression of the moment generating function in Eq. (27). From Eq. (25) the probability of failure can be obtained by taking the expectation of Eq. (29) as

$$P_f = E[\Phi(-\beta-U)] \approx \Phi(-\beta)\left\|\mathbf{I}_{n-1} + 2\frac{\varphi(\beta)}{\Phi(-\beta)}\mathbf{A}\right\|^{-1/2} \tag{30}$$

The term $\varphi(\beta)/\Phi(-\beta)$ in Eq. (30) was first obtained by Hohenbichler and Rackwitz (1988) and later rederived by Köylüoğlu and Nielsen (1994) and Polidori et al. (1999) using different approaches. In this paper the credit for this formula is attributed to Hohenbichler and Rackwitz (1988). A minor difference between Eq. (30) and existing similar expressions in the cited references is that the eigenvalues of $\mathbf{A}$ are not required in Eq. (30). Taking an asymptotic expansion of $\varphi(\beta)/\Phi(-\beta)$ we obtain

$$\frac{\varphi(\beta)}{\Phi(-\beta)} \approx \beta + \beta^{-1} - 2\beta^{-3} + 10\beta^{-5} - 74\beta^{-7} + 706\beta^{-9} + \cdots \tag{31}$$

Breitung's approximation in Eq. (26) can be obtained from Eq. (30) by keeping only the first term in this series, which is asymptotically correct when $\beta \to \infty$. Numerical works in Köylüoğlu and Nielsen (1994) and Polidori et al. (1999) show that the formula Eq. (30) works reasonably well for smaller β. In terms of a simple formula, P_f expressed in Eq. (30) is possibly the best available in current literature.

Least-Square Method

The *only* source of error in formula (30) is from approximating $\Phi(-\beta-u)$ by an one-term exponential function in Eq. (29). After this approximation, the expectation operation is carried out *exactly* using the moment generating function of U. Several authors have proposed methods to improve this approximation. Hohenbichler and Rackwitz (1988) have proposed an importance sampling technique to reduce this error while Köylüoğlu and Nielsen (1994) have used higher-order terms in the McLaurin series expansion of the error term. The aim here is to see how far we can go with a simple single-exponential term approximation of $\Phi(-\beta-u)$. Therefore, we are looking for an approximation to $\Phi(-\beta-u)$ of the form

$$\Phi(-\beta-u) \approx c_1 e^{-c_2 u} \tag{32}$$

such that the error in this approximation is minimized in some sense. The error in representing $\Phi(-\beta-u)$ by Eq. (32) is given by

$$\varepsilon(u) = \Phi(-\beta-u) - c_1 e^{-c_2 u} \tag{33}$$

Recalling that $u \in \mathbb{R}^+$, the objective function can be defined by the l_2 norm of $\varepsilon(u)$ as

$$\psi = \int_0^\infty \varepsilon(u)^2 du = \int_0^\infty [\Phi(-\beta-u) - c_1 e^{-c_2 u}]^2 du \tag{34}$$

To minimize ψ with respect to c_1 and c_2 we must have

$$\frac{\partial \psi}{\partial c_i} = 0 \quad \text{for } i = 1,2 \tag{35}$$

After some simplifications of Eq. (35) one obtains

$$c_1 = 2c_2\mathcal{L}[\Phi(-\beta-u); c_2] \quad \text{for } i = 1 \tag{36}$$

and

Table 1. Values of c_1 and c_2 in Least-Square Method

β	c_1	c_2
0.00	5.5089583e-01	1.3075
0.25	4.3609324e-01	1.4497
0.50	3.3133310e-01	1.6024
0.75	2.4091628e-01	1.7650
1.00	1.6721652e-01	1.9366
1.25	1.1054756e-01	2.1165
1.50	6.9479790e-02	2.3038
1.75	4.1448781e-02	2.4977
2.00	2.3437948e-02	2.6975
2.25	1.2548254e-02	2.9024
2.50	6.3544929e-03	3.1120
2.75	3.0412682e-03	3.3255
3.00	1.3746774e-03	3.5426
3.25	5.8648912e-04	3.7627
3.50	2.3605347e-04	3.9857
3.75	8.9590399e-05	4.2110
4.00	3.2051520e-05	4.4385
4.25	1.0805111e-05	4.6679
4.50	3.4314611e-06	4.8990
4.75	1.0263359e-06	5.1316
5.00	2.8904481e-07	5.3656

$$c_1 = 4c_2^2 \mathcal{L}[u\Phi(-\beta - u); c_2] \quad \text{for } i = 2 \tag{37}$$

where $\mathcal{L}[f(u); s]$ is the Laplace transform of $f(u)$ in s. Substituting c_1 from Eq. (36) into Eq. (37) and using the Laplace transforms of $\Phi(-\beta - u)$ and $u\Phi(-\beta - u)$ from Eqs. (147) and (148) in the Appendix, one has

$$(2c_2^2 + 2\beta c_2 - 1)e^{c_2^2/2 + c_2\beta}\Phi(-\beta - c_2) - 2c_2\varphi(\beta) + \Phi(-\beta) = 0 \tag{38}$$

The constant c_2 can be obtained by solving this equation. Substituting the value of c_2 in Eq. (36) the constant c_1 can be calculated obtained as

$$c_1 = 2[\Phi(-\beta) - e^{c_2^2/2 + c_2\beta}\Phi(-\beta - c_2)] \tag{39}$$

An exact closed-form solution of Eq. (38) does not exist. However, numerical solution can be obtained easily using widely available nonlinear equation solvers (the function "fzero" in *MATLAB*® is used here). Because c_2 and consequently c_1 are functions of the reliability index β only, we can produce a unique table for the values of c_1 and c_2 for different values of β which can be directly used in practical applications. Table 1 shows the values of c_1 and c_2 for β ranging from 0 to 5 with steps of 0.25. This table is obtained by solving Eq. (38) with an initial guess of $c_2 = \varphi(\beta)/\Phi(-\beta)$.

The probability of failure corresponding to the approximation in Eq. (32) is given by

$$P_f = E[\Phi(-\beta - U)] \approx c_1 \|\mathbf{I}_{n-1} + 2c_2\mathbf{A}\|^{-1/2} \tag{40}$$

It is easy to see that Breitung's approximation in Eq. (26) and also the approximation in Eq. (30) are like special cases of Eq. (40). For Breitung's approximation $c_1 = \Phi(-\beta)$ and $c_2 = \beta$. For the approximation in Eq. (30) by Hohenbichler and Rackwitz (1988), $c_1 = \Phi(-\beta)$, which is the same as Breitung's, but $c_2 = \varphi(\beta)/\Phi(-\beta)$. A graphical comparison between the constants c_1 and c_2 obtained from the least-square method and those obtained in Breitung (1984) and Hohenbichler and Rackwitz (1988) is shown in Fig. 1. Although the values themselves are not very different, they make noticeable difference to the reliability estimate (see example 1 later) because P_f is usually a very small quantity.

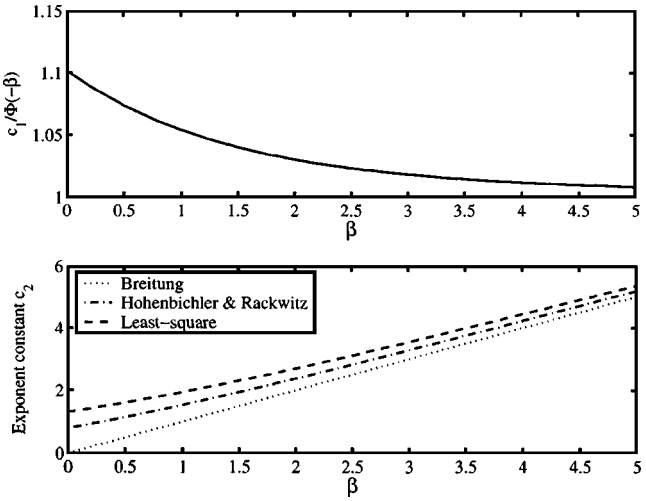

Fig. 1. Comparison of values of c_1 and c_2 between existing results and least square analysis

In Fig. 2, a comparison is drawn between $\Phi(-\beta - u)$ and various single-exponential term approximations. The plots are normalized by $\Phi(-\beta)$. Four representative values of β are selected and the range of u is considered as $0 \leq u \leq 4$. The curve obtained by the least-square approach is the closest to $\Phi(-\beta - u)$ for all four values of β. The difference between the other methods and the least-square method is greater for lower values of β and less for the higher values. Thus, when β is small the least-square approach is expected to give a better result (see example 1 later).

Optimal Point Expansion Method

The probability density function of U has not been considered explicitly in deriving the previous two approximations. From Eq. (25) we can write

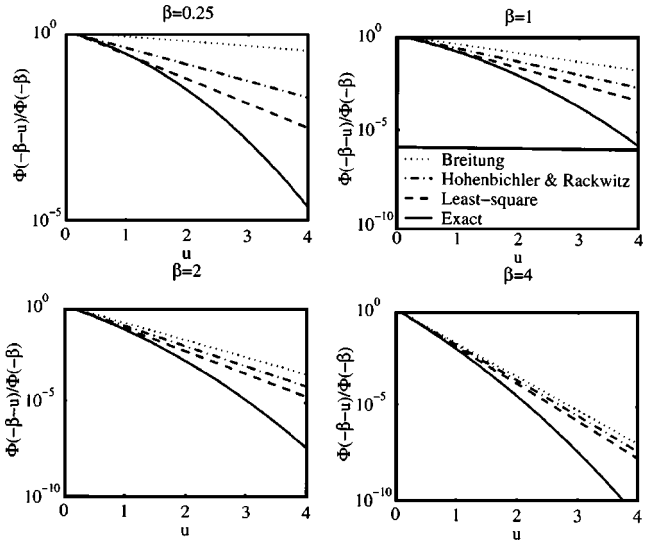

Fig. 2. Approximation of $\Phi(-\beta - u)$ using single exponential function

$$P_f = \int_{\mathbb{R}^+} \Phi(-\beta - u) p_U(u) du = \int_{\mathbb{R}^+} e^{\ln[\Phi(-\beta-u)]+\ln[p_U(u)]} du \tag{41}$$

The aim here is to expand the integrand in a first-order Taylor series about the most probable point or optimal point, say $u=u^*$. The optimal point is the point where the integrand reaches its maxima in $u \in \mathbb{R}^+$. Clearly, to obtain the maxima of the integrand, knowledge of $p_U(u)$ is also required. Because in general the exact closed-form expression of $p_U(u)$ is not available, an approximation to $p_U(u)$ is needed. Such an approximate $p_U(u)$ will only be used to find the maxima of the integrand and will *not* be used subsequently to calculate the expectation. The expectation operation will be carried out exactly by utilizing the expression of the moment generating function in Eq. (27). For the maxima of the integrand we must have

$$\frac{\partial}{\partial u}\{\ln[\Phi(-\beta - u)] + \ln[p_U(u)]\} = 0 \tag{42}$$

or

$$\frac{\varphi(\beta + u)}{\Phi(-(\beta + u))} = \frac{\dfrac{\partial p_U(u)}{\partial u}}{p_U(u)} \tag{43}$$

Because this relationship holds at the optimal point u^* we define a constant η as

$$\eta = \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} = \frac{1}{p_U(u^*)} \frac{\partial p_U(u^*)}{\partial u} \tag{44}$$

The choice of u^* will depend on $p_U(u)$ and will be discussed shortly. Taking a first-order Taylor series expansion of $\ln[\Phi(-\beta - u)]$ about $u=u^*$ we have

$$\ln[\Phi(-\beta - u)] \approx \ln[\Phi(-\beta - u^*)] - \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))}(u - u^*) \tag{45}$$

or

$$\Phi(-\beta - u) \approx \exp \ln[\Phi(-(\beta + u^*))] - \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))}(u - u^*) \tag{46}$$

Using Eq. (44) this reduces to

$$\Phi(-\beta - u) \approx \Phi(-\beta') e^{\eta u^*} e^{-\eta u} \tag{47}$$

where

$$\beta' = \beta + u^* \tag{48}$$

Taking the expectation of Eq. (47), the probability of failure using the optimal point expansion method can be expressed as

$$P_f \approx \Phi(-\beta') e^{\eta u^*} \|\mathbf{I}_{n-1} + 2\eta \mathbf{A}\|^{-1/2} \tag{49}$$

To obtain u^* and η knowledge of the PDF of U is required. Several approximations to $p_U(u)$ are available in the references, for example, see the books by Johnson and Kotz (1970, Chapter 29) and Mathai and Provost (1992). Fiessler et al. (1979) have suggested an approximation to $p_U(u)$ by a central χ^2 density of $(n-1)$ degrees-of-freedom by using the mean of the eigenvalues of **A**. This "mean curvature" approach is later used by Zhao and Ono (1999a, b). Here, we use two simple χ^2 approximations, namely, Patnaik's χ^2 type approximation (Patnaik 1949) and an approximation analogous to Pearson's three moment central χ^2 approximation to the distribution of a noncentral χ^2 (Pearson 1959). In the next sections it will be shown that within the scope of χ^2 approximations these two are better than the mean curvature approximation.

Patnaik's Approximation

In this approximation the quadratic form U is approximated by a central χ^2 distribution such that their first two central moments are the same. Thus

$$U \simeq \theta \chi^2_\nu \tag{50}$$

such that

$$E[U] = E[\theta \chi^2_\nu] \tag{51a}$$

and

$$\mathrm{Var}[U] = \mathrm{Var}[\theta \chi^2_\nu] \tag{51b}$$

From the equality of the moments in Eqs. (51*a*) and (51*b*) we have

$$\theta \nu = \mathrm{Trace}(\mathbf{A}) \tag{52a}$$

and

$$2\theta^2 \nu = 2\mathrm{Trace}(\mathbf{A}^2) \tag{52b}$$

By solving these two equations the parameters θ and ν can be obtained as

$$\theta = \frac{\mathrm{Trace}(\mathbf{A}^2)}{\mathrm{Trace}(\mathbf{A})} \tag{53a}$$

and

$$\nu = \frac{[\mathrm{Trace}(\mathbf{A})]^2}{\mathrm{Trace}(\mathbf{A}^2)} \tag{53b}$$

Using the transformation in Eq. (50) the approximate probability density function of U can be expressed as

$$p_U(u) \approx \frac{1}{\theta} p\chi^2_\nu\left(\frac{u}{\theta}\right) = \frac{u^{\nu/2-1} e^{-u/2\theta}}{(2\theta)^{\nu/2}\Gamma(\nu/2)} \tag{54}$$

Substituting $p_U(u)$ into Eq. (43) the equation for the optimal point can be simplified to

$$\frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} = \frac{1}{2}\left(\frac{\nu - 2}{u^*} - \frac{1}{\theta}\right) \tag{55}$$

An exact closed-form solution of this equation does not exist. However, it can be easily solved numerically (for example the function "fzero" in *MATLAB*® can be used) to obtain u^*. An approximate solution of Eq. (55) can be obtained by considering the asymptotic expansion of the ratio $\varphi(\beta+u^*)/\Phi(-(\beta+u^*))$, similar to that given in Eq. (31). Keeping only the first term, the left-hand side becomes $(\beta+u^*)$ and consequently we obtain

$$(\beta + u^*) \approx \frac{1}{2}\left(\frac{\nu - 2}{u^*} - \frac{1}{\theta}\right) \tag{56}$$

or

$$u^{*2} + \left(\frac{1}{2\theta} + \beta\right) u^* - (\nu - 2)/2 \approx 0 \tag{57}$$

Taking the positive solution of this quadratic equation the optimal point is obtained as

$$u^* \approx -\left(\frac{1}{4\theta}+\frac{\beta}{2}\right)+\frac{1}{4\theta}\sqrt{(2\theta\beta+1)^2+8\theta^2(\nu-2)} \tag{58}$$

Numerical calculations show that this approximate expression of u^* is accurate enough for all practical purposes. The constant η can be directly evaluated from Eq. (55) as

$$\eta=\frac{1}{2}\left(\frac{\nu-2}{u^*}-\frac{1}{\theta}\right) \tag{59}$$

In view of Eqs. (48) and (56) we also have

$$\beta' \approx \eta \tag{60}$$

Substituting η in Eq. (49) the probability of failure using Patnaik's approximation is given by

$$P_f \approx \Phi(-\beta')e^{1/2(\nu-2-u^*/\theta)}\left\|\mathbf{I}_{n-1}+\left(\frac{\nu-2}{u^*}-\frac{1}{\theta}\right)\mathbf{A}\right\|^{-1/2} \tag{61}$$

Next Pearson's approach is considered to approximate $p_U(u)$.

Pearson's Approximation

Pearson's approximation is similar to Patnaik's approximation except that an additional parameter γ is used such that

$$U \simeq \gamma+\theta\chi_\nu^2 \tag{62}$$

Mathai and Provost (1992) have mentioned that Pearson's approximation, which requires very little additional work compared to that of Patnaik, gives a much better fit to the true pdf of U. The parameters ν, θ and γ are selected such that the first three central moments of U and $\gamma+\theta\chi_\nu^2$ are the same, that is

$$\gamma+\theta\nu=\text{Trace}(\mathbf{A}) \tag{63a}$$

$$2\theta^2\nu=2\text{Trace}(\mathbf{A}^2) \tag{63b}$$

and

$$8\theta^3\nu=8\text{Trace}(\mathbf{A}^3) \tag{63c}$$

By solving these three equations simultaneously the parameters γ, θ, and ν can be obtained as

$$\gamma=\text{Trace}(\mathbf{A})-\frac{[\text{Trace}(\mathbf{A})]^2}{\text{Trace}(\mathbf{A}^3)} \tag{64a}$$

$$\theta=\frac{\text{Trace}(\mathbf{A}^3)}{\text{Trace}(\mathbf{A}^2)} \tag{64b}$$

and

$$\nu=\frac{[\text{Trace}(\mathbf{A}^2)]^3}{[\text{Trace}(\mathbf{A}^3)]^2} \tag{64c}$$

Using the transformation in Eq. (62) the approximate probability density function of U is given by

$$p_U(u)\approx\frac{1}{\theta}p\chi_\nu^2\left(\frac{u-\gamma}{\theta}\right)=\frac{(u-\gamma)^{\nu/2-1}e^{-(u-\gamma)/2\theta}}{(2\theta)^{\nu/2}\Gamma(\nu/2)} \tag{65}$$

Substituting $p_U(u)$ into Eq. (43) the equation for the optimal point can be simplified to

$$\frac{\varphi(\beta+u^*)}{\Phi(-(\beta+u^*))}=\frac{1}{2}\left(\frac{\nu-2}{u^*-\gamma}-\frac{1}{\theta}\right) \tag{66}$$

This equation can be solved exactly by using numerical methods to obtain u^*. With one-term asymptotic expansion of $\varphi(\beta+u^*)/\Phi(-(\beta+u^*))$, Eq. (66) can be approximated as

$$(\beta+u^*)\approx\frac{1}{2}\left(\frac{\nu-2}{u^*-\gamma}-\frac{1}{\theta}\right) \tag{67}$$

or

$$u^{*2}+\left(\frac{1}{2\theta}+\beta-\gamma\right)u^*-\left(\frac{1}{\theta}+2\beta\right)\frac{\gamma}{2}-(\nu-2)/2\approx 0 \tag{68}$$

Positive solution of this quadratic equation produces

$$u^* \approx -\left(\frac{1}{4\theta}+\frac{\beta-\gamma}{2}\right)+\frac{1}{4\theta}\sqrt{(2\theta(\beta-\gamma)+1)^2+8\gamma\theta(1+2\theta\beta)+8\theta^2(\nu-2)} \tag{69}$$

This approximate expression for u^* is accurate enough for all practical purposes. The constant η can be directly evaluated from Eq. (66) as

$$\eta=\frac{1}{2}\left(\frac{\nu-2}{u^*-\gamma}-\frac{1}{\theta}\right) \tag{70}$$

Substituting η in Eq. (49) the probability of failure using Pearson's approximation is given by

$$P_f \approx \Phi(-\beta')\exp\frac{u^*}{2}\left(\frac{\nu-2}{u^*-\gamma}-\frac{1}{\theta}\right)\times\left\|\mathbf{I}_{n-1}+\left(\frac{\nu-2}{u^*-\gamma}-\frac{1}{\theta}\right)\mathbf{A}\right\|^{-1/2} \tag{71}$$

Substituting $\gamma=0$ in Eqs. (69) and (70) recovers the Patnaik's approximation [Eqs. (58) and (59)]. This is expected as γ is the only additional term used in Pearson's approximation. Therefore, combining Patnaik's and Pearson's approximations, the failure probability using the optimal point expansion method can be expressed by

$$P_f \approx \Phi(-\eta)e^{\eta u^*}\|\mathbf{I}_{n-1}+2\eta\mathbf{A}\|^{-1/2} \tag{72}$$

In general u^* and η are given by Eqs. (69) and (70). The extra work required to use Eq. (72) is small compared to Breitung's formula (26) and Hohenbichler and Rackwitz's formula (30). In the numerical example it will be shown that this slight modification results in a very good approximation to the failure probability for systems with a large number of random variables, over a wide range of β values.

Example 1. Failure Probability Using Approximations of $\Phi(-\beta-u)$

We consider a problem for which the failure surface is *exactly* parabolic in the normalized space, as given by Eq. (22). The purpose of this hypothetical example is to understand how the proposed approximate formulae work after making a parabolic failure surface assumption. Therefore, the effect of Type 1 and Type 2 errors cannot and will not be investigated here. In numerical calculations we have fixed the number of random variables n and the trace of the coefficient matrix $\mathbf{A}$. Based on the values of n and Trace $(\mathbf{A})$ two cases are considered:

1. small number of random variables: $n-1=10$, and Trace $(\mathbf{A})=10$ and
2. large number of random variables: $n-1=50$, and Trace $(\mathbf{A})=5$.

When Trace $(\mathbf{A})=0$ the failure surface is effectively linear. Therefore, the more the value of Trace $(\mathbf{A})$ the more nonlinear the

failure surface becomes. The above two cases covers two different type of scenario that might arise in practical applications. It further considered that the eigenvalues of **A** are spaced in a cubic manner, that is $a_j = j^3\tilde{a}$, for $j = 1, 2, \cdots n-1$ and $\tilde{a} \geq 0$. Note that

$$\text{Trace}(\mathbf{A}) = \sum_{j}^{n-1} a_j = \tilde{a}\left(\frac{n(n-1)}{2}\right)^2 \tag{73}$$

therefore

$$\tilde{a} = \frac{\text{Trace}(\mathbf{A})}{(n(n-1)/2)^2} \tag{74}$$

Thus $\tilde{a}$ is defined uniquely for fixed values of n and Trace (**A**). To compare the relative accuracy of different methods some kind of indicator function is useful. Using the idea of relative error percentage proposed by Der Kiureghian et al. (1987) we define the relative error

$$e_i = \frac{|P_{f_{\text{approx}}} - P_{f_{\text{exact}}}|}{P_{f_{\text{exact}}}},\ i = 1, 2, \cdots \tag{75}$$

For better approximations e_i will be close to zero and vice versa. The exact probability of failure $P_{f_{\text{exact}}}$ is obtained by direct calculation of the expectation in Eq. (25) using numerical simulation. The samples of U are generated using Monte Carlo simulation. For each problem 3,000 samples have been generated and the result obtained from the Monte Carlo simulation is treated as a benchmark.

In total five approximate methods are considered, namely Breitung's formula in Eq. (26), Eq. (30) derived by Hohenbichler and Rackwitz (1988), the least-square formula given by Eq. (40) and two formulae derived using the optimal point expansion method in Eqs. (61) and (71). For Case 1, the probability of failure obtained using the five approximate methods and the exact probability of failure obtained using Monte Carlo simulation is shown in Fig. 3(a). For each method β is varied from 0 to 5. Because P_f is very small, the numerical values are normalized by dividing with $\Phi(-\beta)$, that is, the quantity $P_f/\Phi(-\beta)$ is plotted for all proposed methods. The relative error, defined in Eq. (75), corresponding to each method is shown in Fig. 3(b). From these figures it is clear that the results obtained from the three proposed methods are closest to the exact values within the range of $0 \leq \beta \leq 4$. The least-square method and the optimal point expansion method using Patnaik's approximation are closest over the full range of β. The optimal point expansion method using Pearson's approximation is accurate over the lower range of β values, but less so in the higher range.

Normalized probability of failure for Case 2 is shown in Fig. 4. The least-square method and the optimal point expansion methods perform better than the two existing methods. Both optimal point expansion methods, particularly using Pearson's pdf, yield very good approximation over the full range of β. The expressions of approximate u^* given by Eqs. (58) and (69) are used in this study. It was verified that the use of exact u^* obtained by solving the nonlinear Eqs. (57) and (68) does not make a significant difference to the result. From Fig. 4 it may be noted that the optimal point expansion methods give 100 times greater accuracy than the existing formulae without any significant increase of computation. This fact makes these methods very suitable for reliability calculations. Based on these results it is recommended that when the number of random variables is small the optimal point expansion method based on Patnaik's approximation should

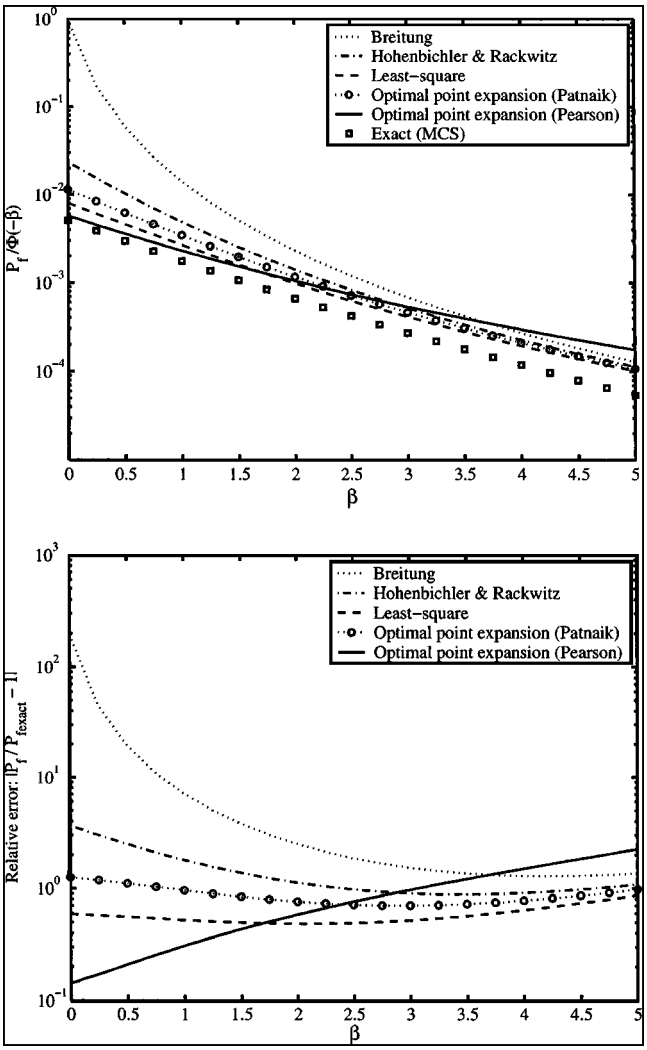

Fig. 3. Failure probability and relative error using one-term exponential approximation of $\Phi(-\beta - u)$. Case (1): $n-1=10$ and trace $(\mathbf{A})=10$.

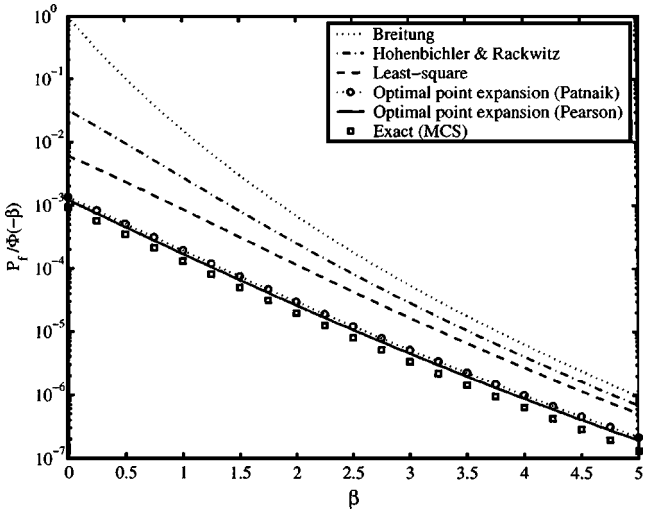

Fig. 4. Normalized failure probability using one-term exponential approximation of $\Phi(-\beta - u)$. Case (2): $n-1=50$, trace $(\mathbf{A})=5$.

be used, and when the number of random variables is large, the optimal point expansion method based on Pearson's approximation should be used.

From the results obtained in this section the following conclusions may be drawn:

1. Two new approximate methods to obtain the failure probability using parabolic hypersurface approximation, namely, the least-square method and the optimal point expansion method improve upon the equivalent existing approximations. These approximations are non-asymptomatic, therefore any asymptotic requirements, such as $\beta \to \infty$, are not needed.
2. Explicit calculation of the eigenvalues of the matrix **A** is not required to use any of the derived expressions for P_f.
3. The least square method can be implemented very easily using Table 1 provided for the numerical values of the two required constants.
4. When the number of random variables is small (say $n-1 \leqslant 20$) the optimal point expansion method based on Patnaik's approximation should be used to calculate P_f.
5. When the number of random variables is large (say $n-1 \geqslant 20$) the optimal point expansion method based on Pearson's approximation should be used to calculate P_f.

It should be noted that for the non-asymptotic case (i.e., when β is not very large) there will be errors associated with the quadratic approximation of the failure surface in Eq. (5). Such errors cannot be improved using the formulations derived here.

Failure Probability Using Approximations of Probability Density Functional of *U*

In the previous section only $\Phi(-\beta-u)$ is approximated. In this section we will approximate $p_U(u)$ to evaluate the failure probability using Eq. (25). Derivation of probability density and distribution functions of quadratic forms in random variables has received extensive attention in literature. Johnson and Kotz (1970, Chapter 29) and Mathai and Provost (1992, Chapter 4) have summarized significant contributions in this area. If the matrix **A** is an idempotent matrix then U becomes a χ^2 random variable with $(n-1)$ degrees-of-freedom (a nonsingular matrix **A** is said to be idempotent if $\mathbf{A}=\mathbf{A}^2$). In general **A** is not an idempotent matrix. In such a case it can be shown that the exact density of U can be represented by an infinite sum of central χ^2 random variables. Several other representations of the exact density of U are also available in the literature (see Mathai and Provost 1992, Chapter 4), for example, power series expansion, Laguerre series expansion, series expansion in terms of confluent hypergeometric function, and series expansion in zonal polynomials. Application of these results in the context of structural reliability analysis has not been adequately investigated. However, it is expected that the resulting expressions for P_f using these will be complicated. We consider the simplest case when $p_U(u)$ is approximated in terms of a single χ^2 random variable. Applicability of the two simple χ^2 approximations introduced in the previous section is investigated here. Comparisons are drawn with the mean value approximation suggested in the reliability literature.

Mean Value Approximation

This is the earliest (first suggested by Fiessler et al. 1979) and possibly the simplest approximation used in the context of structural reliability analysis. Here $p_U(u)$ is approximated by a central χ^2 density of $(n-1)$ degrees-of-freedom by using the mean of the eigenvalues of **A**. This is equivalent to replacing **A** by an idempotent matrix. Thus, $p_U(u)$ has the same form as Patnaik's approximation given by Eq. (54), but the constants θ and ν are defined by

$$\theta = \frac{1}{n-1}\text{Trace}(\mathbf{A}) \tag{76a}$$

and

$$\nu = n-1 \tag{76b}$$

The probability of failure using this approximation is given by

$$P_f \approx \frac{1}{(2\theta)^{(n-1)/2}\Gamma(\nu/2)}\int_0^\infty \Phi(-\beta-u)u^{(n-3)/2}e^{-u/2\theta}du \tag{77}$$

From Eq. (154) in the Appendix this integral can be evaluated exactly in closed form provided n is odd and $n \geqslant 3$. Otherwise numerical evaluation is needed over a finite interval. Recently Zhao and Ono (1999a, b) have reinvestigated this approximation and have provided empirical formulae to evaluate this integral. This approximation is expected to be accurate when the differences between the eigenvalues of **A** are very small.

Patnaik's χ^2 Type Approximation

Evaluation Using Exact Integral

The PDF of U using Patnaik's χ^2 approximation is given by Eq. (54). Using this, the probability of failure can be approximated from Eq. (25) as

$$P_f \approx \frac{1}{(2\theta)^{\nu/2}\Gamma(\nu/2)}\int_0^\infty \Phi(-\beta-u)u^{\nu/2-1}e^{-u/2\theta}du \tag{78}$$

From Eq. (154) in the Appendix a closed-form expression for this integral is possible provided $\nu \neq 0$ is an even integer. However, in view of Eq. (52*b*) ν is expected to be fractional and in general the integral in Eq. (78) has to be evaluated numerically over a finite interval. To obtain a closed-form expression further approximation of $\Phi(-\beta-u)$ is necessary. We consider the following two approximations:

First-Order Taylor Series Expansion of $\ln[\Phi(-\beta-u)]$

An approximation of $\Phi(-\beta-u)$ using a first-order Taylor series expansion of $\ln[\Phi(-\beta-u)]$ about the most the most probable point u^* is given in Eq. (47). Using this, together with η in Eq. (59), from Eq. (78) one has

$$P_f \approx \frac{\Phi(-\beta')}{(2\theta)^{\nu/2}\Gamma(\nu/2)}e^{1/2(\nu-2-u^*/\theta)}\int_0^\infty e^{-(u/2)(\nu-2-u^*/\theta)/u^*}u^{\nu/2-1}e^{-u/2\theta}du$$

$$= \Phi(-\beta')e^{1/2(\nu-2-u^*/\theta)}\left(\frac{\theta(\nu-2)}{u^*}\right)^{-\nu/2} \tag{79}$$

Second-Order Taylor Series Expansion of $\ln[\Phi(-\beta-u)]$

Expanding $\ln[\Phi(-\beta-u)]$ in a Taylor series about the most probable point u^* and retaining only the first two terms one has

$$\ln[\Phi(-\beta-u)] \approx \ln[\Phi(-\beta')] + \eta u^*(1-(\eta-\beta')u^*/2) - \eta(1-(\eta-\beta')u^*)u - \eta(\eta-\beta')u^2/2 \tag{80}$$

where β' and η are given in Eqs. (48) and (59). If the approximation in Eq. (60) is used then it is easy to see that Eq. (80) reduces to Eq. (45). This essentially implies that when the approximate u^* in Eq. (57) is used, the second-order approximation of $\ln[\Phi(-\beta-u)]$ given by Eq. (80) does not offer any improvement over the first-order approximation. Therefore, going to the second-order approximation for $\ln[\Phi(-\beta-u)]$ makes a difference only if u^* is obtained from the exact solution of Eq. (55).

Using $p_U(u)$ from Eq. (54) and substituting $\ln[\Phi(-\beta-u)]$ from Eq. (80) we have

$$P_f \approx \int_0^\infty e^{\ln[\Phi(-\beta-u)]} p_U(u)du = \frac{\Phi(-\beta')}{(2\theta)^{\nu/2}\Gamma(\nu/2)} e^{\mu_0} \int_0^\infty u^{\nu/2-1} e^{-\mu_1 u} e^{-\mu_2 u^2/2} du \tag{81}$$

where

$$\mu_0 = \eta u^*(1-(\eta-\beta')u^*/2) \tag{82}$$

$$\mu_1 = \eta(1-(\eta-\beta')u^*) + \frac{1}{2\theta} \tag{83}$$

$$\mu_2 = \eta(\eta-\beta') \tag{84}$$

The integral appearing in Eq. (81) can be evaluated exactly (Gradshteyn and Ryzhik 1994, Section 3.462) as

$$\int_0^\infty u^{\nu/2-1} e^{-\mu_1 u} e^{-\mu_2 u^2/2} du = (\mu_2)^{-\nu/4}\Gamma(\nu/2) e^{\mu_1^2/4\mu_2} D_{-\nu/2}\left(\frac{\mu_1}{\sqrt{\mu_2}}\right) \tag{85}$$

where $D_{-\nu/2}(\cdot)$ = parabolic cylinder function. Discussions on the parabolic cylinder function may be found in Gradshteyn and Ryzhik (1994, Sections 9.24–9.25) and Abramowitz and Stegun (1965, Chapter 19). Using this integral the probability of failure can be obtained from Eq. (81) as

$$P_f \approx \Phi(-\beta') e^{\mu_0 + \mu_1^2/4\mu_2} (2\theta\sqrt{\mu_2})^{-\nu/2} D_{-\nu/2}\left(\frac{\mu_1}{\sqrt{\mu_2}}\right) \tag{86}$$

Pearson's Approach

Evaluation Using Exact Integral

The PDF of U using Pearson's approximation is given in Eq. (65). Using this, from Eq. (25) the probability of failure can be approximated as

$$P_f \approx \frac{1}{(2\theta)^{\nu/2}\Gamma(\nu/2)} \int_0^\infty \Phi(-\beta-u)(u-\gamma)^{\nu/2-1} e^{-(u-\gamma)/2\theta} du \tag{87}$$

A closed-form expression of this integral is in general not possible and it has to be evaluated numerically by considering a finite interval. To obtain a closed-form expression, a first-order Taylor series expansion of $\ln[\Phi(-\beta-u)]$ is considered.

First-Order Taylor Series Expansion of $\ln[\Phi(-\beta-u)]$

An approximation to the random variable U using Pearson's approach is given in Eq. (62). The moment generating function of U can be obtained as

$$E[e^{-s(\gamma+\theta\chi_\nu^2)}] = e^{-s\gamma}(1+2s\theta)^{-\nu/2} \tag{88}$$

An approximation of $\Phi(-\beta-u)$ using a first-order Taylor series expansion of $\ln[\Phi(-\beta-u)]$ about the most the most probable point u^* is given in Eq. (47). Using this we have

$$P_f \approx \Phi(-\beta')E[e^{\eta u^*} e^{-\eta U}] \tag{89}$$

where η is given by Eq. (70). This expectation can be evaluated using the moment generating function in Eq. (88) and the final result can be simplified to

$$P_f \approx \Phi(-\beta') e^{(1/2)(\nu-2-(u^*-\gamma)/\theta)} \left(\frac{\theta(\nu-2)}{u^*-\gamma}\right)^{-\nu/2} \tag{90}$$

Example 2. Comparison of Approximate Probability Density Functions

The accuracy of P_f obtained using an approximate PDF of U depends on how well the true PDF of the quadratic form U is approximated. For this reason it is useful to compare the three χ^2 type approximations of U, namely, the mean value approximation, Patnaik's approximation and Pearson's approximation, with Monte Carlo simulation results. Fig. 5 shows the probability density function and the moment generating function of U for Case 1 described in Example 1. The histogram of U is obtained from 3,000 sample Monte Carlo simulation and has been scaled to have unit area. The moment generating function has been obtained by directly calculating the mean of e^{-sU}. Patnaik's approximation is the best for this case. Both the mean value and Pearson's approximations do not yield very accurate results. Pearson's approximation becomes worse in the lower tail of the PDF and in the higher range ($s>3$) of $M_U(-s)$.

The probability density function and the moment generating function of U for Case 2 described in Example 1 is shown Fig. 6. Unlike the previous case, Pearson's approximation provides the best fit, followed by Patnaik's approximation. The mean value approximation shows the greatest divergence from the simulation results. From the results obtained in this example we conclude that when the number of random variables is small Patnaik's approximation should be used, and when the number of random variables is large, Pearson's approximation should be used. In either case, both newly suggested approximations improve upon the mean value approximation.

Example 3. Failure Probability Using Exact Integration

In this example P_f obtained using the three χ^2 type approximations of U, namely, the mean value approximation, Patnaik's approximation, and Pearson's approximation are compared with Monte Carlo simulation, Breitung's formula in Eqs. (26) and (30) derived by Hohenbichler and Rackwitz (1988). A closed-form exact expression of P_f using χ^2 type approximations can be obtained only in special cases. In general Eqs. (77), (78), and (87) have to be evaluated numerically over a finite interval. Efficient algorithms for one dimensional numerical integration are widely available (for example the function "quadl" in *MATLAB*®). For the numerical examples considered here it was observed that the integral converges to a fixed value when $u \geqslant 15$.

Fig. 7 shows probability of failure (normalized by dividing with $\Phi(-\beta)$) for Case 1 with values of β ranging from 0 to 5. P_f

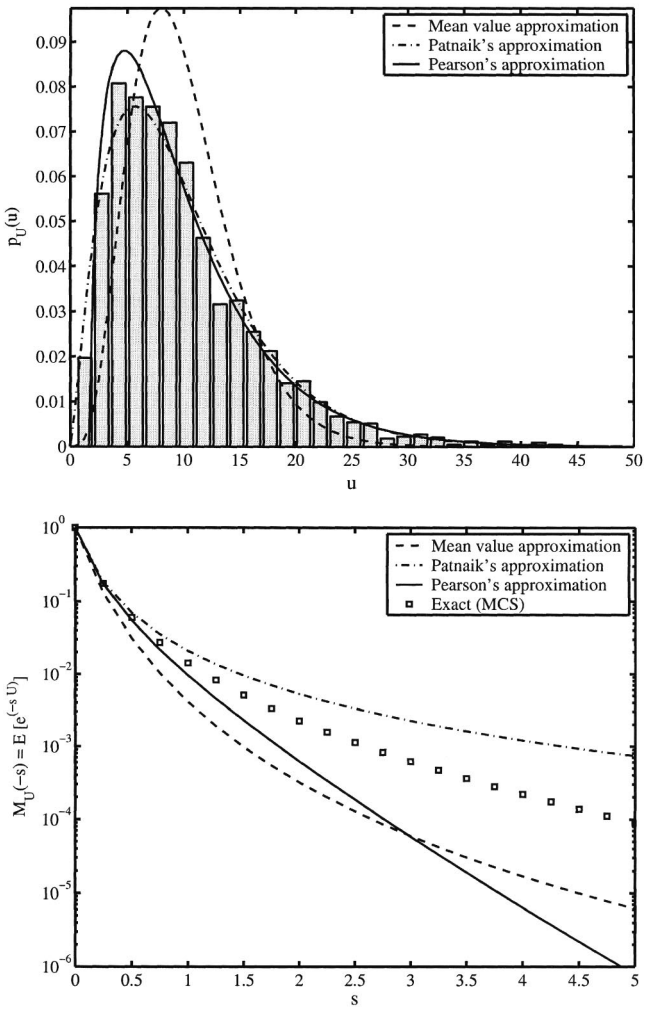

Fig. 5. Approximations of central quadratic form in Gaussian random variables, $U=\mathbf{y}^T\mathbf{A}\mathbf{y}$, $\mathbf{y}\sim\mathbb{N}_{n-1}(\mathbf{0}_{n-1},\mathbf{I}_{n-1})$. Case (1): $n-1=10$, trace $(\mathbf{A})=10$.

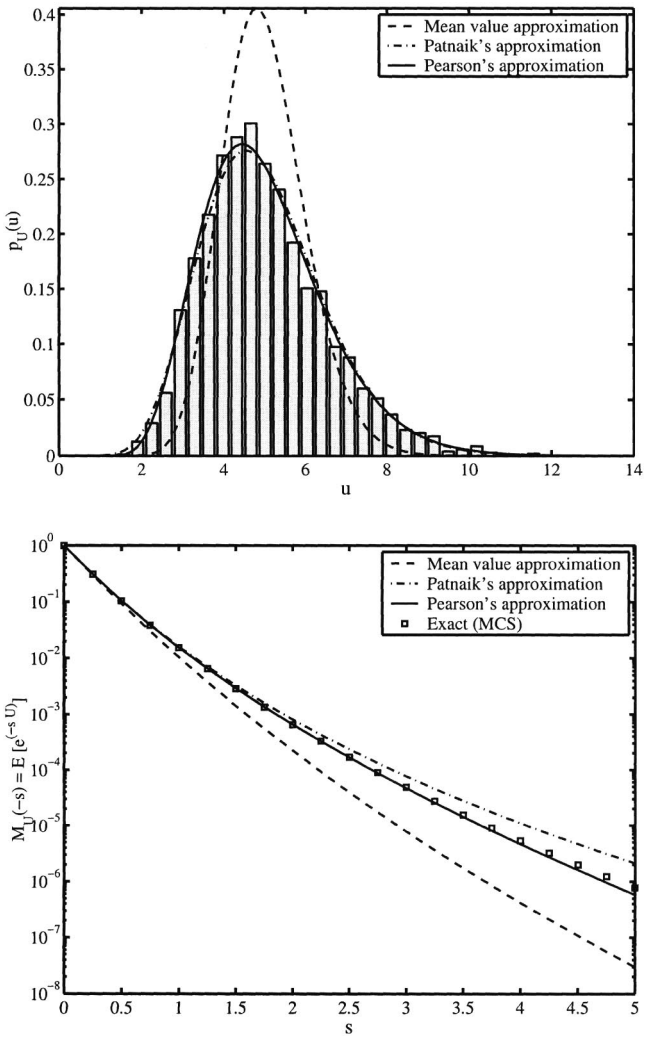

Fig. 6. Approximations of central quadratic form in Gaussian random variables, $U=\mathbf{y}^T\mathbf{A}\mathbf{y}$, $\mathbf{y}\sim\mathbb{N}_{n-1}(\mathbf{0}_{n-1},\mathbf{I}_{n-1})$. Case (2): $n-1=50$, trace $(\mathbf{A})=5$.

using Patnaik's approximation produces the best result for this case. Both the mean value approximation and Pearson's approximations underestimate P_f and are quite far from the exact value. Pearson's approximation is worse, especially for higher values of β.

Probability of failure for case 2 with values of β ranging from 0 to 5 is shown in Fig. 8. Unlike the previous case, P_f using Pearson's approximation becomes the best, followed by Patnaik's approximation. Especially, P_f using Pearson's approximation is very accurate over the complete range of β. The mean value approximation consistently underestimates P_f and turns out to be the worse in this case.

It is useful to understand as to why Pearson's approximation works so well for high values of n and less so otherwise. The key to understanding this feature lies in the nature of the approximate PDF. For the calculation of failure probability the lower tail of the PDF of U is very important because P_f is usually very small. On closer inspection of the plots of the PDF in Figs. 5(a) and 6(a) it can be noticed that in the lower tail Pearson's approximation is not very good when n is small and very good when n is high. This analysis suggests that in order to get a good approximation to the

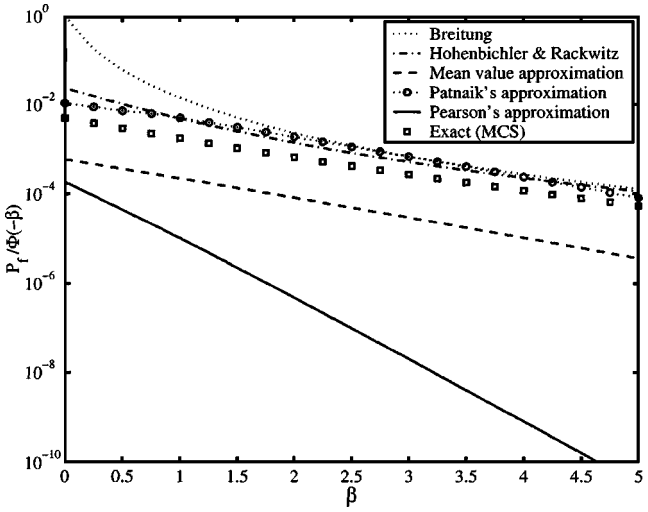

Fig. 7. Normalized failure probability using exact integration with approximate $p_U(u)$. Case (1): $n-1=10$, trace $(\mathbf{A})=10$.

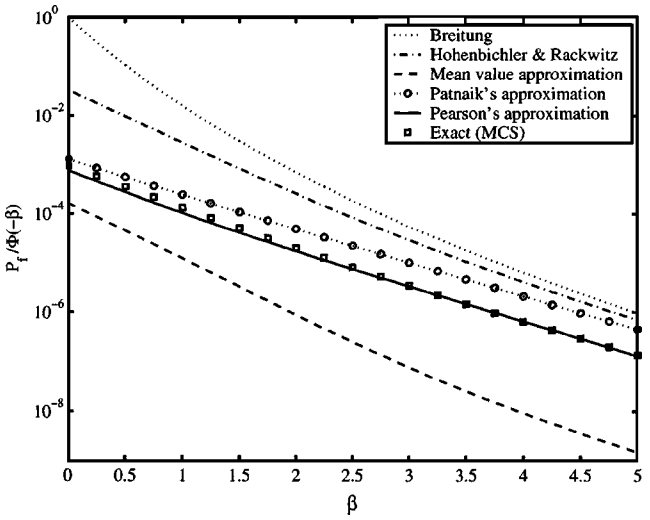

Fig. 8. Normalized failure probability using exact integration with approximate $p_U(u)$. Case (2): $n-1=50$, trace $(\mathbf{A})=5$.

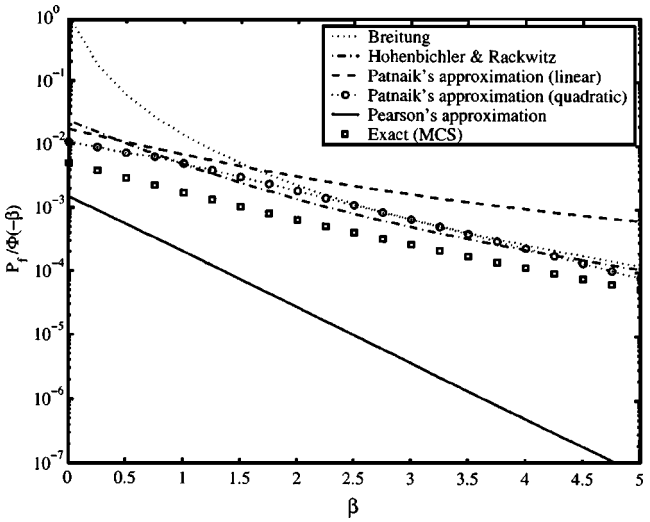

Fig. 9. Normalized failure probability using approximate integration and approximate $p_U(u)$. Case (1): $n-1=10$, trace $(\mathbf{A})=10$.

failure probability using an approximate PDF, it is necessary that the approximate PDF should fit the lower tail of the PDF accurately.

Recently Zhao and Ono (1999a, b) have considered P_f obtained from Eq. (77) as the benchmark. The results obtained here suggest that this method of benchmarking may yield misleading results because in many cases the mean value approximated PDF can be grossly inaccurate. As a consequence, even an exact numerical integration using an inaccurate PDF produces an incorrect P_f, which cannot be used as a basis for comparing the accuracy of other existing methods.

Example 4. Failure Probability Using Approximate Integration

Three methods, namely Patnaik's pdf with linear and quadratic approximations of $\ln[\Phi(-\beta-u)]$ and Pearson's PDF with linear approximation of $\ln[\Phi(-\beta-u)]$, that is, Eqs. (79), (86), and (90) are considered. Results obtained using these three formulae are compared with existing formulae proposed by Breitung (1984) and Hohenbichler and Rackwitz (1988) and with Monte Carlo simulation. Eqs. (79), (86), and (90) are expected to provide most inaccurate results as in deriving them approximations for both $p_U(u)$ and $\Phi(-\beta-u)$ are utilized. However, the advantage of using these formulae are that they are simple to use and require little computational effort to evaluate P_f. The purpose of this example is to verify if these formulae can at all be used for practical purposes.

Fig. 9 shows the probability of failure for Case 1 with values of β ranging from 0 to 5. The approximations using Patnaik's PDF produce acceptable results. The linear approximation becomes inaccurate for large β. The quadratic approximation is quite good for small β and gives similar accuracy as Breitung's and Hohenbichler and Rackwitz (1988)'s formulae for large β. As expected, the approximation using Pearson's PDF is not good because n is small.

Fig. 10 shows the probability of failure for Case 2. All three approximate formulae produce acceptable results. In particular P_f calculated using Pearson's PDF with linear approximation of $\ln[\Phi(-\beta-u)]$ [Eq. (90)] yields a very good result. These results suggest that even though formulae (79), (86), and (90) are derived using several approximations, they still provide acceptable results. The results are especially good when the number of random variables is large. This fact is promising because in real-life structural reliability problems the number of random variables is expected to be quite large.

From the results obtained in these representative examples, and also others not reported here, the following conclusions may be drawn:

1. When the number of random variables is small $(n-1\leqslant 20)$, Patnaik's approximation should be used to approximate the PDF of U and to calculate P_f.
2. When the number of random variables is large $(n-1\geqslant 20)$, Pearson's approximation should be used to approximate the PDF of U and to calculate P_f. Pearson's approximation becomes very good for $n>50$.
3. The mean value approximation is in general inaccurate when compared to the previous two approaches without providing any computational advantage.

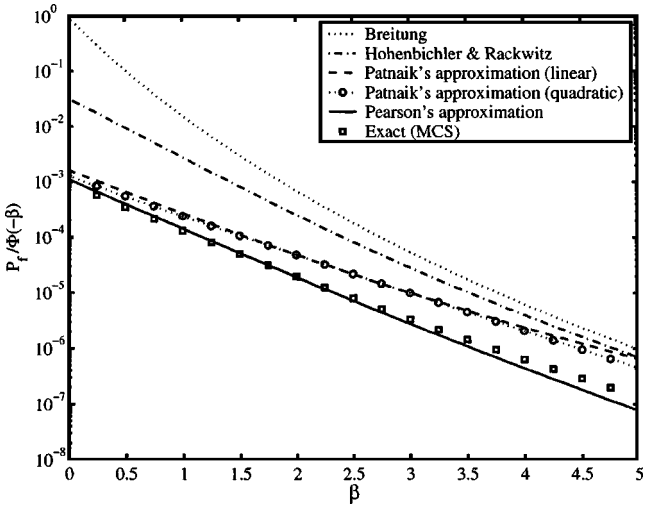

Fig. 10. Normalized failure probability using approximate integration and approximate $p_U(u)$. Case (2): $n-1=50$, trace $(\mathbf{A})=5$.

Exact Probability Content for Parabolic Failure Surfaces

The exact probability of failure with parabolic failure surface assumption may be obtained by an exact evaluation of the expectation operation in Eq. (25). It can be done in two ways: (1) by using the exact probability density function function of U, or (2) by exact representation of $\Phi(-\beta-u)$ in terms of other functions such that the expression for the exact moment generating function in Eq. (27) can be utilized. The method proposed by Tvedt (1990) belongs to the first approach. Here the second approach will be used to obtain the exact probability of failure. It should be noted that in this context "exact" means "exact after making the parabolic failure surface approximation." Therefore, errors that occur due to the parabolic failure surface approximation itself cannot be improved.

Series Expansion Methods

Power Series Expansion

The function $\Phi(-\beta-u)$ can be expanded as an infinite power series in u. Expanding $\Phi(-\beta-u)$ in a McLaurin series provides

$$\Phi(-\beta-u)=\Phi(-\beta)+\sum_{k=0}^{\infty}\frac{d^{k+1}}{du^{k+1}}\Phi(-\beta-u)\big|_{u=0}\frac{u^{k+1}}{(k+1)!} \tag{91}$$

Calculating the general kth order derivative of $\Phi(-\beta-u)$, the above equation can be expressed as

$$\Phi(-\beta-u)=\Phi(-\beta)-\varphi(\beta)\sum_{k=0}^{\infty}b_{k+1}u^{k+1} \tag{92}$$

where

$$b_{k+1}=\frac{(-1)^k 2^{-k/2}H_k(\beta/\sqrt{2})}{(k+1)!} \tag{93}$$

and $H_k(\cdot)$=Hermite polynomial of order k (see Abramowitz and Stegun 1965, Chapter 22 for discussions on the Hermite polynomials). Using Eq. (92), the probability of failure can be obtained from Eq. (25) as

$$P_f=E[\Phi(-\beta-U)]=\Phi(-\beta)-\varphi(\beta)\sum_{k=0}^{\infty}b_{k+1}\mathcal{M}_U^{(k+1)} \tag{94}$$

where

$$\mathcal{M}_U^{(k+1)}=E[U^{k+1}] \tag{95}$$

A method to obtain moments of arbitrary order of the quadratic form U is given in Mathai and Provost (1992, Section 3.2b). The first three moments are given by

$$\mathcal{M}_U^{(1)}=\text{Trace}(\mathbf{A}) \tag{96}$$

$$\mathcal{M}_U^{(2)}=2\text{Trace}(\mathbf{A}^2)+[\text{Trace}(\mathbf{A})]^2 \tag{97}$$

$$\mathcal{M}_U^{(3)}=8\text{Trace}(\mathbf{A}^3)+6\text{Trace}(\mathbf{A}^2)\text{Trace}(\mathbf{A})+[\text{Trace}(\mathbf{A})]^3 \tag{98}$$

The expression in Eq. (94) is exact if an infinite number of terms are retained in the series. The series is convergent if $\text{Trace}(\mathbf{A})<1$ and it is observed that the rate of convergence is poor, especially if $\text{Trace}(\mathbf{A})$ is not small. For large β using a finite number of terms may result in a negative P_f which is obviously incorrect. For these reasons Eq. (94) should be used only if β and $\text{Trace}(\mathbf{A})$ are small.

Series Expansion in Terms of e^{-u}

There are two main reasons behind the choice of e^{-u} for a series expansion: (1) any "powers series" in e^{-u} is always convergent for $u\in\mathbb{R}^+$, and (2) calculation of the expectation operation involving the powers of e^{-u} is straightforward as the moment generating function of U is known exactly.

For any $u\in\mathbb{R}^+$ we can define a variable $y\in\mathbb{R}^+$ such that

$$u=-\ln(y) \tag{99}$$

or

$$v=e^{-u} \tag{100}$$

Since $u=0$ implies $v=1$, we are interested in the Taylor series of $\Phi(-\beta+\ln(v))$ around $v=1$, which can be expressed as

$$\Phi(-\beta+\ln(v))=\Phi(-\beta)+\varphi(\beta)\sum_{k=1}^{\infty}\frac{\delta_k}{k!}(v-1)^k \tag{101}$$

where

$$\delta_k\varphi(\beta)=\frac{d^k}{dv^k}\Phi(-\beta+\ln(v))\big|_{v=1} \tag{102}$$

The expression of δ_k valid for all values of k is defined by Eq. (158) in the Appendix. Substituting v from Eq. (100), Eq. (101) can be rewritten as

$$\Phi(-\beta-u)=\Phi(-\beta)+\varphi(\beta)\sum_{k=1}^{\infty}\frac{\delta_k}{k!}(e^{-u}-1)^k \tag{103}$$

Now construct a function

$$\epsilon(u)=\frac{\Phi(-\beta-u)-\Phi(-\beta)}{\varphi(\beta)} \tag{104}$$

which has the following limiting properties:

$$\lim_{u\to 0}\epsilon(u)=0 \tag{105}$$

and

$$\lim_{u\to\infty}\epsilon(u)=-\frac{\Phi(-\beta)}{\varphi(\beta)} \tag{106}$$

From Eqs. (103) and (104) it is obvious that

$$\epsilon(u)=\sum_{k=1}^{\infty}\frac{\partial_k}{k!}(e^{-u}-1)^k \tag{107}$$

Conditions (105) and (106) must be satisfied by this series expansion. Condition (105) is naturally satisfied because $\lim_{u\to 0}e^{-u}=1$. To satisfy condition (106) we propose an approach by considering that for all practical purposes the series Eq. (107) must be truncated. Retaining a finite number of terms, say m, Eq. (107) can be approximated as

$$\epsilon(u)\approx\sum_{k=1}^{m-1}\frac{\delta_k}{k!}(e^{-u}-1)^k+\frac{\delta_m}{m!}(e^{-u}-1)^m \tag{108}$$

This expression will become exact when $m\to\infty$. The aim here is to modify the last term so that condition (106) is satisfied. Taking

the limit $u \to \infty$ in Eq. (108), from Eq. (106) the required condition can be expressed as

$$\sum_{k=1}^{m-1} \frac{\delta_k}{k!}(-1)^k + \frac{\delta_m}{m!}(-1)^m = -\frac{\Phi(-\beta)}{\varphi(\beta)} \tag{109}$$

or

$$\delta_m = m\,!\,(-1)^{(m-1)}\left(\frac{\Phi(-\beta)}{\varphi(\beta)} + \sum_{k=1}^{m-1}\frac{\delta_k}{k!}(-1)^k\right) \tag{110}$$

Truncating the series in the left-hand side of Eq. (103) after the mth term and substituting this expression of δ_m, it is hoped that the left-hand side of Eq. (103) will approach to exact $\Phi(-\beta-u)$ for large values of m. Numerical results suggest that the convergence of this series is much better than the powers series approach described in the previous section. Note that

$$E[(e^{-u}-1)^k] = E\left[\sum_{r=0}^{k}\binom{k}{r}(e^{-u})^r(-1)^{k-r}\right]$$
$$= \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\|\mathbf{I}_{n-1} + 2r\mathbf{A}\|^{-1/2} \tag{111}$$

Using this, the probability of failure can be obtained by taking the expectation of Eq. (103) as

$$P_f = \Phi(-\beta) + \varphi(\beta)\sum_{k=1}^{\infty}\frac{\delta_k}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\|\mathbf{I}_{n-1} + 2r\mathbf{A}\|^{-1/2} \tag{112}$$

Fourier Series Method

We express $\Phi(-\beta-u)$ as

$$\Phi(-\beta-u) = \Phi(-\beta)(e^{-\mu u} - \rho(u)) \tag{113}$$

where

$$\rho(u) = e^{-\mu u} - \frac{\Phi(-\beta-u)}{\Phi(-\beta)} \tag{114}$$

and $\mu = \varphi(\beta)/\Phi(-\beta)$, which is similar to that derived by Hohenbichler and Rackwitz (1988). The aim here is to express $\rho(u)$ by a Fourier sin series

$$\rho(u) = \sum_{k=1}^{\infty}\alpha_k \sin\left(\frac{k\pi u}{L}\right) \tag{115}$$

where $(0,L)$ is the domain over which it is required to express $\Phi(-\beta-u)$ by a Fourier series. For numerical calculations it is sufficient to consider $L=5\pi$ as beyond this limit $\Phi(-\beta-u)$ is practically zero for all values of β. For large β even smaller values of L may be sufficient. The Fourier coefficients α_k are given by

$$\alpha_k = \frac{2}{L}\int_0^L \rho(u)\sin\left(\frac{k\pi u}{L}\right)du \tag{116}$$

The above coefficients can be obtained in a closed-form by taking the imaginary part of the integral $\int_0^L \rho(u)e^{i\omega u}du$ (with $\omega = k\pi/L$) considered in Eq. (146) in the Appendix. After some algebra, it can be shown that

$$\alpha_k = \frac{2}{L}\mathcal{I}\left(\frac{k\pi}{L}\right) \tag{117}$$

where the function $\mathcal{I}(\omega):\mathbb{R}\to\mathbb{R}$ is given by

$$\mathcal{I}(\omega) = \frac{e^{-\mu L}(\omega\cos(\omega L) + \mu\sin(\omega L)) - \omega}{\mu^2+\omega^2} + \frac{\Phi(-\beta-L)-\Phi(-\beta)}{\omega\Phi(-\beta)}$$
$$-\frac{e^{-\omega^2/2}}{\omega\Phi(-\beta)}\Re\{e^{-i\beta\omega}\Phi(-\beta-L+i\omega) - \Phi(-\beta+i\omega)\} \tag{118}$$

Taking the expectation of Eq. (113) and substituting $\rho(u)$ from Eq. (115) one has

$$P_f = \Phi(-\beta)\left(\|\mathbf{I}_{n-1} + \mu\mathbf{A}\|^{-1/2} - \sum_{k=1}^{\infty}\alpha_k E\left[\sin\left(\frac{k\pi}{L}U\right)\right]\right) \tag{119}$$

For any $\omega \in \mathbb{R}$, from the moment generating function in Eq. (27) we can have

$$E[\sin(\omega U)] = \Im\{E[e^{i\omega U}]\} = \Im\{\|\mathbf{I}_{n-1} - 2i\omega\mathbf{A}\|^{-1/2}\} \tag{120}$$

Using this for $\omega = k\pi/L$, Eq. (119) provides

$$P_f = \Phi(-\beta)\left\|\mathbf{I}_{n-1} - 2\frac{\varphi(\beta)}{\Phi(-\beta)}\mathbf{A}\right\|^{-1/2} - \Phi(-\beta)\sum_{k=1}^{\infty}\alpha_k\Im$$
$$\times\left\{\left\|\mathbf{I}_{n-1} - 2i\frac{k\pi}{L}\mathbf{A}\right\|^{-1/2}\right\} \tag{121}$$

The second part of the right-hand side of this equation can be viewed as a correction to the Hohenbichler and Rackwitz's formula in Eq. (30).

Fourier Sin Transform Method

This method is similar to the Fourier series method except that the function $\rho(u)$ is expressed in terms of a Fourier sin transform rather than a Fourier sin series. Thus, in place of the Fourier series in Eq. (115) we now have the Fourier integral

$$\rho(u) = \frac{2}{\pi}\int_0^{\infty}\hat{\rho}(\omega)\sin(\omega u)d\omega \tag{122}$$

where $\hat{\rho}(\omega)$ is defined as

$$\hat{\rho}(\omega) = \int_0^{\infty}\rho(u)\sin(\omega u)du \tag{123}$$

Using Eq. (147) in the Appendix and after some simplifications the preceding integral can be obtained in a closed form as

$$\hat{\rho}(\omega) = -\frac{\mu^2}{\omega(\mu^2+\omega^2)} + \frac{e^{-\omega^2/2}}{\omega\Phi(-\beta)}\Re\{e^{-i\beta\omega}\Phi(-\beta-i\omega)\} \tag{124}$$

Taking the expectation of Eq. (113) and substituting $\rho(u)$ from Eq. (122) one has

$$P_f = \Phi(-\beta)\left(\|\mathbf{I}_{n-1} + \mu\mathbf{A}\|^{-1/2} - \frac{2}{\pi}\int_0^{\infty}\hat{\rho}(\omega)E[\sin(\omega U)]d\omega\right) \tag{125}$$

Substituting $\hat{\rho}(\omega)$ from Eq. (124) and the expression of $E[\sin(\omega U)]$ from Eq. (120) we have

$$P_f = \Phi(-\beta)\left\|\mathbf{I}_{n-1} + 2\frac{\varphi(\beta)}{\Phi(-\beta)}\mathbf{A}\right\|^{-1/2} + \frac{2}{\pi}\int_0^\infty \left(\frac{\mu^2\Phi(-\beta)}{\omega(\mu^2+\omega^2)} - \frac{e^{-\omega^2/2}}{\omega}\Re\{e^{-i\beta\omega}\Phi(-\beta - i\omega)\}\right)\Im\{\|\mathbf{I}_{n-1} - 2i\omega\mathbf{A}\|^{-1/2}\}d\omega \tag{126}$$

The second part of the right-hand side of this equation can be viewed as a correction to the Hohenbichler and Rackwitz's formula in Eq. (30). Based on an earlier work of Rice (1980), Tvedt (1990) obtained the probability of failure exactly using an integral in the complex domain. Like Tvedt's result, Eq. (126) is also an exact expression for P_f. Because Eq. (126) is an integral over the real domain, all well known techniques for the numerical evaluation of one dimensional real integrals can be directly used to obtain P_f.

Example 5. Exact Failure Probability Using Parabolic Failure Surface Approximation

In this example the probability of failure obtained using the three exact methods, namely, series expansion in terms of e^{-u}, Fourier series method, and Fourier sin transform method, are compared with Monte Carlo simulation, Breitung's formula in Eqs. (26) and (30) derived by Hohenbichler and Rackwitz (1988). Formulae for P_f corresponding to these three methods are given in Eqs. (112), (121), and (126). Fig. 11 shows the probability of failure [normalized by dividing with $\Phi(-\beta)$] and corresponding relative error for Case 1 with values of β ranging from 0 to 5. The series expansion formula in Eq. (112) with 15 terms and the Fourier transform integral in Eq. (126) with an upper limit of 5 yield very accurate results. The Fourier series method works well for higher values of β, but less so for the lower values.

It should be noted that these results are exact only in a mathematical sense and numerical methods are required for practical implementation of these formulae. Accuracy of the numerical results obtained using these formulae depend on several factors, for example: (1) the number of terms retained in the series expansion Eq. (112); (2) the domain of the integration used to evaluate the Fourier coefficients given by Eq. (116); (3) the number of terms retained in the Fourier series Eq. (121); (4) the method of numerical integration, the selection of a finite upper limit in place of an infinite limit in the integral Eq. (126); and (5) the computation of cumulative normal distribution with a complex argument. In contrast to the approximate formulae derived in the previous sections, here the final results will depend on the details of the numerical methods selected. From the nature of these formulae, it is also obvious that a greater numerical effort provides more accurate results. Computational cost can be prohibitively expensive especially when the dimension of **A** is large and high accuracy is required. Nevertheless, as an alternative to Monte Carlo simulation, these formulae can be used as a benchmark.

Failure Probability Using Asymptotic Distribution

We consider the case when the number of random variables is very large, that is, when asymptotically $n \to \infty$. From Eqs. (22) and (23) the probability of failure can be rewritten as

$$P_f \approx \text{Prob}\left[\frac{\tilde{g}}{|\nabla g|} \leqslant 0\right] = \text{Prob}[z \geqslant \beta] \tag{127}$$

where

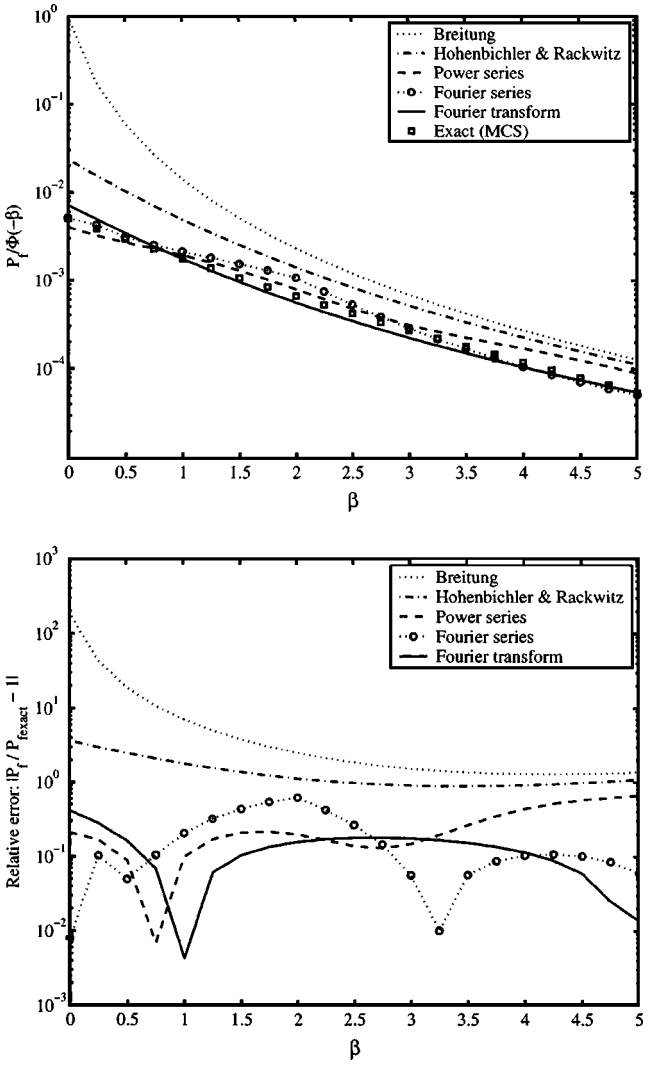

Fig. 11. Failure probability and relative error using exact methods, $n-1=10$, trace $(\mathbf{A})=10$

$$z = y_n - \mathbf{y}^T\mathbf{A}\mathbf{y} \in \mathbb{R} \tag{128}$$

From the central limit theorem it is expected that when $n \to \infty$ the random variable z will asymptotically approach a Gaussian random variable. Discussions on asymptotic distribution of quadratic forms may be found in Mathai and Provost (1992, Section 4.6b). Here one of the simplest forms of asymptotic distribution of z will be used to obtain P_f.

We start with the moment generating function of z

$$M_z(s) = E[e^{sz}] = E[e^{sy_n - s\mathbf{y}^T\mathbf{A}\mathbf{y}}] \tag{129}$$

Because y_n and $\mathbf{y}^T\mathbf{A}\mathbf{y}$ are independent random variables this equation reduces to

$$M_z(s) = E[e^{sy_n}]E[e^{-s\mathbf{y}^T\mathbf{A}\mathbf{y}}] \tag{130}$$

Considering the eigenvalues of **A** we have

$$M_z(s) = e^{s^2/2}\prod_{k=1}^{n-1}(1 + 2sa_k)^{-1/2} \tag{131}$$

Now construct a new random variable $q = z/\sqrt{n}$. The moment generating function of q is given by

$$M_q(s) = M_z(s/\sqrt{n}) = e^{s^2/2n}\prod_{k=1}^{n-1}(1+2sa_k/\sqrt{n})^{-1/2} \tag{132}$$

From this

$$\ln(M_q(s)) = s^2/2n - \frac{1}{2}\sum_{k=1}^{n-1}\ln(1+2sa_k/\sqrt{n})$$

$$= s^2/2n - \frac{1}{2}\sum_{k=1}^{n-1}2sa_k/\sqrt{n} - s^2(2a_k/\sqrt{n})^2/2$$

$$+ s^3(2a_k/\sqrt{n})^3/3 - \cdots \tag{133}$$

provided

$$|2sa_k| < 1, \quad \text{for } k = 1,2,\cdots,n-1 \tag{134}$$

Consider a case when a_k and n are such that the higher-order terms of s vanish as $n\to\infty$, i.e., we assume n is large such that the following conditions hold:

$$\sum_{k=1}^{n-1}(2a_k/\sqrt{n})^2/2 < \infty$$

or

$$\frac{2}{n}\text{Trace}(\mathbf{A}^2) < \infty \tag{135}$$

and

$$\sum_{k=1}^{n-1}(2a_k/\sqrt{n})^r/r \to 0$$

or

$$\frac{2^r}{n^{r/2}r}\text{Trace}(\mathbf{A}^r) \to 0, \quad \forall \quad r \geqslant 3 \tag{136}$$

Under these assumptions, the series in Eq. (133) can be truncated after the quadratic term so that

$$\ln(M_q(s)) \approx s^2/2n - \frac{1}{2}\sum_{k=1}^{n-1}s(2a_k/\sqrt{n}) - s^2(2a_k/\sqrt{n})^2/2$$

$$= -\text{Trace}(\mathbf{A})s/\sqrt{n} + (1+2\text{Trace}(\mathbf{A}^2))s^2/2n \tag{137}$$

Therefore, the moment generating function of $z=q\sqrt{n}$ can be approximated by

$$M_z(s) \approx e^{-\text{Trace}(\mathbf{A})s+(1+2\text{Trace}(\mathbf{A}^2))s^2/2} \tag{138}$$

From the uniqueness of the Laplace transform pair it follows that when conditions (134)–(136) are satisfied, z asymptotically approaches a Gaussian random variable with mean $(-\text{Trace}(\mathbf{A}))$ and variance $(1+2\ \text{Trace}(\mathbf{A}^2))$, that is

$$z \simeq \mathbb{N}_1(-\text{Trace}(\mathbf{A}), \sqrt{1+2\text{Trace}(\mathbf{A}^2)}) \text{ when } n\to\infty \tag{139}$$

Using the PDF of z, the asymptotic probability of failure can be obtained from Eq. (127) as

$$P_f \to \Phi\left(-\frac{\beta+\text{Trace}(\mathbf{A})}{\sqrt{1+2\text{Trace}(\mathbf{A}^2)}}\right) \text{ when } n\to\infty \tag{140}$$

Like Breitung's result in Eq. (26), this is another asymptotic result. Here the asymptotic behavior when $n\to\infty$, instead of $\beta\to\infty$, is considered. If the failure surface is linear or close to linear then $\text{Trace}(\mathbf{A})=\text{Trace}(\mathbf{A}^2)\to 0$, and it is easy to see that Eq. (140) reduces to the classical FORM formula $P_f \approx \Phi(-\beta)$. In many real-life problems the number of random variables is expected to be large. In such situations this asymptotic result may turn out to be useful.

Minimum Value of n Required for the Accuracy of the Asymptotic Approach

Here we aim to derive a simple expression for the minimum value of n which is sufficient for a desired accuracy of the asymptotic distribution method. Since $a_k \in \mathbb{R}^+ \forall k$, condition (134) can be rewritten as

$$s < \frac{1}{2\max(a_k)} \tag{141}$$

Regarding the truncation of Eq. (133) after the second-order term, assume that there exist a small real number ϵ such that

$$\frac{1}{r}\left(\frac{2s}{\sqrt{n}}\right)^r \text{Trace}(\mathbf{A}^r) < \epsilon \quad \forall \quad r \geqslant 3 \tag{142}$$

This ensures that the error in truncating the series Eq. (133) is small. Assuming equality in Eq. (141) and substituting s in Eq. (142) one obtains

$$\frac{2^r}{r}\frac{1}{n^{r/2}}\left(\frac{1}{2\max(a_k)}\right)^r \text{Trace}(\mathbf{A}^r) < \epsilon \tag{143}$$

or

$$n > \frac{1}{\sqrt[r]{r^2\epsilon^2}}\left(\frac{\sqrt[3]{\text{Trace}(\mathbf{A}^r)}}{\max(a_k)}\right)^2 \quad \forall \quad r \geqslant 3 \tag{144}$$

Since $\mathbf{A}$=positive definite matrix, the critical value of n is obtained for $r=3$. Therefore, the minimum value of n can be obtained from

$$n > \frac{1}{\sqrt[3]{9\epsilon^2}}\left(\frac{\sqrt[3]{\text{Trace}(\mathbf{A}^3)}}{\max(a_k)}\right)^2 \tag{145}$$

The following points may be observed from this expression: (1) the minimum number of random variables required for a desired accuracy from the asymptotic distribution approach would be more if ϵ (the error) is considered to be small (as expected), (2) $n \propto 1/\epsilon^{2/3}$, and (3) the convergence would be better if the maximum eigenvalue of $\mathbf{A}$ is large compared to the other eigenvalues.

Example 6. Failure Probability Using Asymptotic Distribution

In this example probability of failure obtained using the asymptotic Gaussian distribution is compared with both Breitung's asymptotic result and the formula (30) derived by Hohenbichler and Rackwitz (1988). Fig. 12 shows asymptotic probability of failure (normalized by dividing with $\Phi(-\beta)$) for values of β ranging from 0 to 5. It is assumed that $n-1=100$ and $\text{Trace}(\mathbf{A})=1.5$. The eigenvalues of $\mathbf{A}$ are spaced in a cubic manner (as in Example 1). For this problem, the minimum number of random variables required for the applicability of the asymptotic distribution formula can be obtained from Eq. (145). Considering $\epsilon=0.001$, it can be shown from Eq. (145) that $n>231$ or $n-1>230$. Although this condition is not satisfied here, the results obtained from this approach is acceptable, especially for lower values of β. This numerical example also shows that the condition derived in Eq. (145) is only a sufficient condition and not a necessary condition.

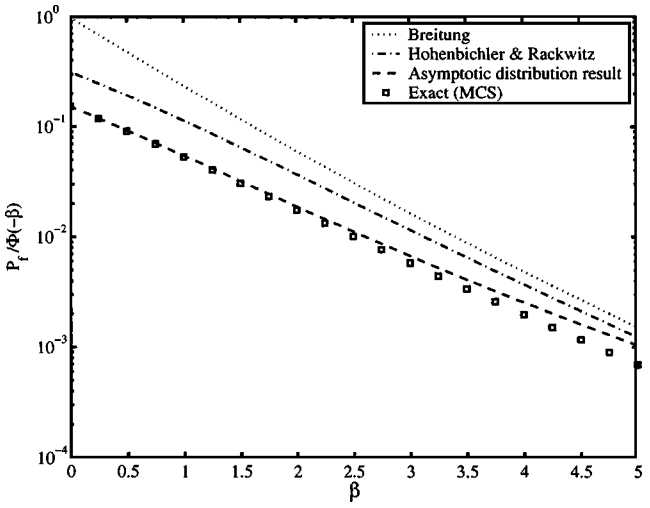

Fig. 12. Asymptotic failure probability, $n-1=100$, trace $(\mathbf{A})=1.5$

Results obtained from the asymptotic analysis improve when the number of random variables becomes large. Fig. 13 shows asymptotic probability of failure for $n-1=250$ and Trace$(\mathbf{A})=1.5$. For this case the condition in Eq. (145) is satisfied since $n-1>230$. As expected, asymptotic P_f match very well with the Monte Carlo simulation result for this case.

Summary and Conclusions

Probability of failure after approximating the failure surface by a parabolic hypersurface in the neighborhood of the design point has been considered. It is assumed that the basic random variables are Gaussian. Several methods to obtain the probability of failure based on these assumptions have been proposed. The proposed methods can be broadly grouped into: (1) nonasymptotic approximate methods, (2) exact methods, and (3) asymptotic distribution methods. Further divisions of each of these methods are summarized in Fig. 14. Table 2 summarizes the main results of the paper. From the numerical examples it is shown that the accuracy provided by the optimal point expansion methods is sufficient for practical purposes. These methods require very little numerical effort and produce excellent accuracy over a wide range of reliability indices and number of random variables. When the number of random variables becomes large further simplifications are possible. One can either use the approximate pdf using Pearson's method or the asymptotic formula. The new asymptotic formula is simple and numerical results suggest that it produces excellent accuracy when the asymptotic conditions are met. Some exact formulae for the failure probability have been derived. Although the computational cost is high for the exact methods, like Monte Carlo simulation, they can be used as a benchmark. The formulae derived here provide exact or accurate results *after* making the parabolic failure surface assumption. Fundamental errors associated with the parabolic failure surface assumption itself cannot be improved using the proposed methods. Further research is needed to extend these methods to obtain failure probability estimates associated with general quadratic failure surfaces with non-Gaussian basic random variables.

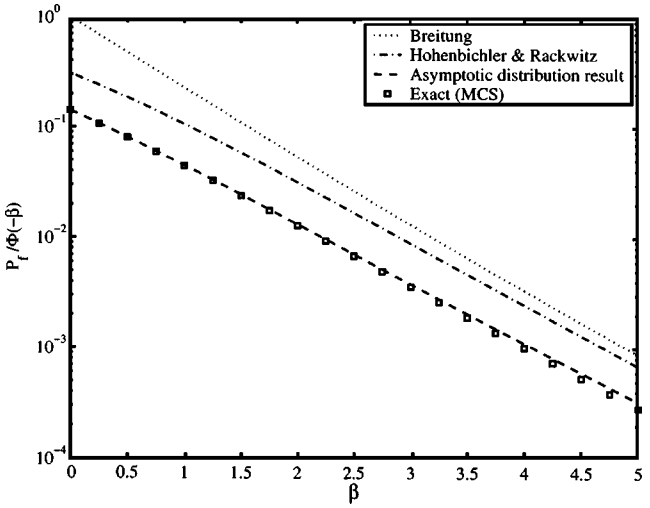

Fig. 13. Asymptotic failure probability, $n-1=250$, trace $(\mathbf{A})=1.5$

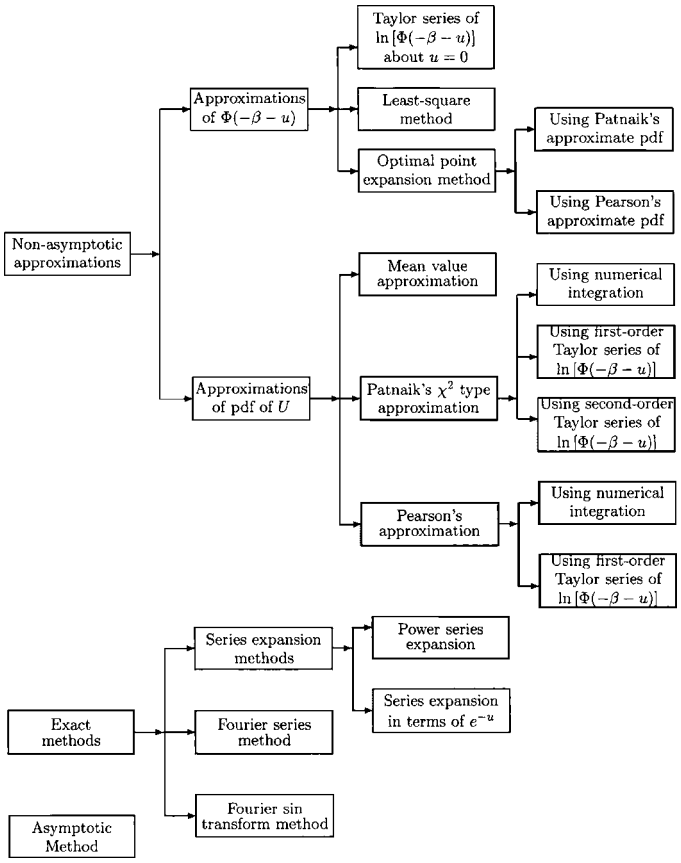

Fig. 14. Summary of methods proposed for determination of failure probability using parabolic failure surface approximation

Acknowledgments

The writer is indebted to Professor R. S. Langley from the University of Cambridge for his participation in numerous helpful discussions. The writer is also grateful to Andrew J Grime from the University of Cambridge for his careful reading of the manuscript and an anonymous referee for useful comments regarding the asymptotic distribution results. Financial support provided by the EPSRC is greatfully acknowledged.

Table 2. Summary of Formulae for Determination of Failure Probability Using Parabolic Failure Surface Approximation

Method		P_f	Remarks
Asymptotic, $\beta\to\infty$ (Breitung 1984).		$\Phi(-\beta)\lVert\mathbf{I}+2\beta\mathbf{A}\rVert^{-1/2}$	Works well when the asymptotic condition is met.
A	1. Taylor series of $\ln[\Phi(-\beta-u)]$ about $u=0$ (Hohenbichler and Rackwitz 1988).	$\Phi(-\beta)\lVert\mathbf{I}+2\mu\mathbf{A}\rVert^{-1/2}$	Works well when β is large and Trace ($\mathbf{A}$) is small.
	Least-square approximation of $\Phi(-\beta-u)$.	$c_1\lVert\mathbf{I}+2c_2\mathbf{A}\rVert^{-1/2}$	Works well especially for small β. Accuracy is independent of the probabilistic characteristics of U.
	3. Optimal point expansion based on Patnaik's and Pearson's approximation for the PDF of a quadratic form.	$\Phi(-\eta)e^{\eta u^*}\lVert\mathbf{I}+2\eta\mathbf{A}\rVert^{-1/2}$, $\eta=(\nu-2/u^*-\gamma-1/\theta)/2$. For Patnaik's approximation $\gamma=0$	In terms of cost effectiveness, these are the best methods. For small n use Patnaik's approximation, for large n use Pearson's approximation.
B	1. Numberical integration using Patnaik's approximate PDF.	$\int_0^\infty\Phi(-\beta-u)1/\theta p_{\chi_\nu^2}(u/\theta)du$	Works very well for large β small n.
	2. Patnaik's approximate PDF with first-order approximation of $\ln[\Phi(-\beta-u)]$.	$\Phi(-\beta')e^{(1/2)(\nu-2-u^*/\theta)}\left(\theta(\nu-2)/u^*\right)^{-\nu/2}$	Accuracy is poor for small n and large β. Works reasonably well otherwise.
	3. Patnaik's approximate PDF with second-order approximation of $\ln[\Phi(-\beta-u)]$.	$\Phi(-\beta')e^{\mu_0+\mu_1^2/4\mu_2}(2\theta\sqrt{\mu_2})^{-\nu/2}\times D_{-\nu/2}\left(\mu_1/\sqrt{\mu_2}\right)$	Achieves similar accuracy to exact integration method with Patnaik's approximate PDF.
	4. Numerical integration using Pearson's approximate PDF.	$\int_0^\infty\Phi(-\beta-u)1/\theta p_{\chi_\nu^2}(u-\gamma/\theta)du$	Accuracy is very poor for small n and large β. Works very well for large n for any β.
	5. Pearson's approximatee PDF with first-order approximation of $\ln[\Phi(-\beta-u)]$.	$\Phi(-\beta')\exp\left(\nu-2/2-(u^*-\gamma)/2\theta\right)\left(\theta(\nu-2)/u^*-\gamma\right)^{-\nu/2}$	Accuracy is poor for small n and large β. For large n works very well.
C	1. Power series expansion.	$\Phi(-\beta)-\varphi(\beta)\Sigma_{k=1}^\infty b_k\mathrm{E}[U^k]$	Accuracy is very poor, especially when Trace ($\mathbf{A}$) large.
	2. Series expansion in the powers of e^{-u}.	$\Phi(-\beta)+\varphi(\beta)\Sigma_{k=1}^\infty\delta_k/k!\,\Sigma_{r=0}^k(-1)^{k-r}\binom{k}{r}\lVert\mathbf{I}+2r\mathbf{A}\rVert^{-1/2}$	Accuracy is good for small β and Trace ($\mathbf{A}$). Otherwise requires many terms.
	3. Fourier series expansion.	$\Phi(-\beta)\lVert\mathbf{I}+2\mu\mathbf{A}\rVert^{-1/2}-\Phi(-\beta)\Sigma_{k=1}^\infty\alpha_k\Im\{\lVert\mathbf{I}-2ik\pi/L\mathbf{A}\rVert^{-1/2}\}$	Like the previous method, but requires more computation, especially if $\mathbf{A}$ is a large matrix.
	4. Fourier sin transform.	$\Phi(-\beta)\lVert\mathbf{I}+2\mu\mathbf{A}\rVert^{-1/2}+2/\pi\int_0^\infty\left(\mu^2\Phi(-\beta)/\omega(\mu^2+\omega^2)-\exp\left(-\omega^2/2/\omega\right)\Re\{e^{-i\beta\omega}\Phi(-\beta-i\omega)\}\right)\Im\lVert\mathbf{I}-2i\omega\mathbf{A}\rVert^{-1/2}d\omega$	Numerical integration over a finite interval is required. Numerically inefficient for large n and β.
D	Asymptotic, $n\to\infty$.	$\Phi\left(-\beta+\mathrm{Trace}(\mathbf{A})/\sqrt{1+2\mathrm{Trace}(\mathbf{A}^2)}\right)$	Works very well if the asymptotic conditions in Eqs. (135) and (136) are satisfied.

Appendix. Some Mathematical Details Used in the Paper

Some Integrals Involving $\Phi(\cdot)$

A few useful indefinite and definite integrals involving the cumulative normal distribution function $\Phi(\cdot)$ are derived here. These integrals specifically arise in reliability problems and may not be found readily in the standard tables of integrals. First consider the indefinite integral

$$\int\Phi(-\beta-u)e^{au}du=\frac{1}{a}\Phi(-\beta-u)-\frac{1}{a}e^{a^2/2-a\beta}\Phi(-\beta-u+a) \tag{146}$$

This result is derived using integration by parts. The Laplace transform of $\Phi(-\beta-u)$ can be obtained by substituting $a=-s$ in Eq. (146). Taking the appropriate limits and noting that $\Phi(-\infty)=0$ one obtains

$$\mathcal{L}[\Phi(-\beta-u);s]=\int_0^\infty \Phi(-\beta-u)e^{-su}du$$

$$=\frac{1}{s}\Phi(-\beta)-\frac{1}{s}e^{s^2/2+s\beta}\Phi(-\beta-s) \quad (147)$$

The Laplace transform of $u\,\Phi(-\beta-u)=(\partial/\partial s)\mathcal{L}[\Phi(-\beta-u);s]$. So using Eq. (147) we have

$$\int_0^\infty u\Phi(-\beta-u)e^{-su}du=-\frac{\partial}{\partial s}\left\{\frac{1}{s}\Phi(-\beta)-\frac{1}{s}e^{s^2/2+s\beta}\Phi(-\beta-s)\right\}$$

$$=\frac{1}{s^2}\Phi(-\beta)-\left(1+\frac{\beta}{s}-\frac{1}{s^2}\right)e^{s^2/2+s\beta}$$

$$\times\Phi(-\beta-s)-\frac{1}{s}\varphi(\beta+s) \quad (148)$$

This formula can be generalized for any integer ν as

$$\int_0^\infty u^\nu\Phi(-\beta-u)e^{-su}du$$

$$=(-1)^\nu\frac{d^\nu}{ds^\nu}\left\{\frac{1}{s}\Phi(-\beta)-\frac{1}{s}e^{s^2/2+s\beta}\Phi(-\beta-s)\right\} \quad (149a)$$

$$=\frac{\nu!}{s^{\nu+1}}\Phi(-\beta)-(-1)^\nu\frac{d^\nu}{ds^\nu}\left\{\frac{1}{s}e^{s^2/2+s\beta}\Phi(-\beta-s)\right\} \quad (149b)$$

Assuming $f_1=1/s$, $f_2=e^{s\beta+s^2/2}$, and $f_3=\Phi(-\beta-s)$ for any integer j we can derive the following generalized derivatives:

$$f_1^{(j)}=\frac{d^j}{ds^j}\frac{1}{s}=(-1)^j\frac{j!}{s^{j+1}} \quad (150)$$

$$f_2^{(j)}=\frac{d^j}{ds^j}e^{s\beta+s^2/2}=e^{s\beta+s^2/2}\sum_{k=0}^{[j/2]}\frac{j!}{k!(j-2k)!}2^{-k}(\beta+s)^{j-2k} \quad (151)$$

$$f_3^{(j)}=\frac{d^j}{ds^j}\Phi(-\beta-s)=(-1)^j2^{-(j-1)/2}\varphi(\beta+s)H_{j-1}\left(\frac{\beta+s}{\sqrt{2}}\right) \quad (152)$$

The second part of Eq. (149*b*) can be obtained by successive application of the Leibnitz' rule of general differentiation of a product as

$$\frac{d^\nu}{ds^\nu}\{f_1f_2f_3\}=\sum_{r_1=0}^{\nu}\sum_{r_2=0}^{\nu-r_1}\binom{\nu}{r_1}\binom{\nu-r_1}{r_2}f_1^{(r_1)}f_2^{(r_2)}f_3^{(\nu-r_1-r_2)} \quad (153)$$

Using this, from Eq. (149*b*) we have

$$\int_0^\infty u^\nu\Phi(-\beta-u)e^{-su}du=\frac{\nu!}{s^{\nu+1}}\Phi(-\beta)-(-1)^\nu\sum_{r_1=0}^{\nu}\sum_{r_2=0}^{\nu-r_1}\binom{\nu}{r_1}$$

$$\times\binom{\nu-r_1}{r_2}f_1^{(r_1)}f_2^{(r_2)}f_3^{(\nu-r_1-r_2)} \quad (154)$$

where $f_i^{(j)}$ for $i=1, 2, 3$ are given by Eq. (150)–(152). Although this is the exact expression of the integral, it becomes increasingly complicated for large ν. In such cases numerical integration over a finite domain may be preferable.

Coefficients δ_k in Eq. (101)

Before deriving the coefficients explicitly first consider the formula for the kth derivative of a composite function. Suppose $f(x)=F(y)$ and $y=\varphi(x)$. Then, from Gradshteyn and Ryzhik (1994, Section 0.43) we have

$$\frac{d^k}{dx^k}f(x)=\sum_{j=1}^{k}\frac{U_j}{j!}F^{(j)}(y) \quad (155)$$

where

$$U_j=\sum_{r=1}^{j-1}\binom{j}{r}(-1)^ry^r\frac{d^k}{dx^k}y^{j-r} \quad (156)$$

From Eq. (102) we have

$$\delta_k=\frac{1}{\varphi(\beta)}\frac{d^k}{dv^k}\Phi\,(F(v))|_{v=1} \quad (157)$$

where $F(v)=-\beta+\ln(v)$. This derivative can be obtained using Eqs. (155) and (156) as

$$\delta_k=\frac{1}{\varphi(\beta)}\sum_{j=1}^{k}\delta_j'^{(k)}\Phi^{(j)}(-\beta) \quad (158)$$

where

$$\Phi^{(j)}(-\beta)=\frac{d^j}{dx^j}\Phi\,(-\beta+x)|_{x=0}=(-1)^{2j}2^{-(j-1)/2}\varphi(\beta)H_{j-1}\left(\frac{\beta}{\sqrt{2}}\right) \quad (159)$$

It may be shown that the constants $\delta_j'^{(k)}$ are the coefficients associated with x^j in the polynomial expression of

$$\frac{\Gamma(x+1)}{\Gamma(x-k+1)}$$

Using this relationships the first ten terms can be explicitly obtained as

$$\delta_1=1 \quad (160)$$

$$\delta_2=-1+\beta \quad (161)$$

$$\delta_3=1-3\beta+\beta^2 \quad (162)$$

$$\delta_4=8\beta-6\beta^2+\beta^3 \quad (163)$$

$$\delta_5=-8-20\beta+29\beta^2-10\beta^3+\beta^4 \quad (164)$$

$$\delta_6 = 60 + 34\beta - 135\beta^2 + 75\beta^3 - 15\beta^4 + \beta^5 \quad (165)$$

$$\delta_7 = -394 + 126\beta + 619\beta^2 - 525\beta^3 + 160\beta^4 - 21\beta^5 + \beta^6 \quad (166)$$

$$\delta_8 = 2{,}632 - 2{,}514\beta - 2.632\beta^2 + 3{,}654\beta^3 - 1{,}540\beta^4 + 301\beta^5 - 28\beta^6 + \beta^7 \quad (167)$$

$$\delta_9 = -18{,}542 + 28{,}008\beta + 7{,}580\beta^2 - 25{,}704\beta^3 + 14{,}469\beta^4 - 3{,}780\beta^5 + 518\beta^6 - 36\beta^7 + \beta^8 \quad (168)$$

$$\delta_{10} = 138{,}870 - 285{,}774\beta + 36{,}900\beta^2 + 181{,}040\beta^3 - 137{,}025\beta^4 + 45{,}381\beta^5 - 8{,}190\beta^6 + 834\beta^7 - 45\beta^8 + \beta^9 \quad (169)$$

Notation

The following symbols are used in this paper:

$\mathbf{A}$ = a symmetric matrix containing first $(n-1)\times(n-1)$ block of $\tilde{\mathbf{A}}$;
$\tilde{\mathbf{A}}$ = transformed and scaled Hessian matrix defined in Eq. (19);
a_j = jth eigenvalue of $\mathbf{A}$;
b_k = constants in power series expansion of $\Phi(-\beta-u)$;
$\mathbb{C}$ = space of complex numbers;
c_1, c_2 = real scalar constants in least-square approximation of $\Phi(-\beta-u)$;
D_{-n} = parabolic cylinder function;
$\mathrm{diag}[\mathbf{v}]$ = diagonal matrix containing vector $\mathbf{v}$ in diagonal;
$\mathrm{E}[\cdot]$ = expectation operator;
$g(\mathbf{x})$ = failure surface in space of basic variables;
$\tilde{g}(\mathbf{x})$ = quadratic approximation of failure surface;
$\mathbf{H}_g(\mathbf{x}^*)$ = Hessian matrix of $g(\mathbf{x})$ evaluated at $\mathbf{x}^*$, $[\mathbf{H}_g(\mathbf{x}^*)]_{ij} = (\partial^2 g(\mathbf{x})/\partial x_i \partial x_j)|_{\mathbf{x}=\mathbf{x}^*}$;
$H_k(\cdot)$ = Hermite polynomial of order k;
$\mathbf{I}_k$ = unit matrix of order k;
$\mathcal{I}(\omega)$ = function of ω defined in Eq. (118);
$\Im(\cdot)$ = imaginary part of $(\cdot)$;
i = unit imaginary number, $i=\sqrt{-1}$;
$[j/2]$ = integer part of $j/2$, for example $[3/2]=1$, $[5/2]=2$, $[6/2]=3$ etc.;
$\binom{k}{r}$ = binomial coefficients;
L = upper limit of integral in Fourier coefficients of $\Phi(-\beta-u)$;
$\mathcal{L}[f(u);s]$ = Laplace transform of function $f(u)$ in s, $\mathcal{L}[f(u);s]=\int_0^\infty f(u)e^{-su}du$;
$M_U(s)$ = moment generating function of U;
$\mathcal{M}_U^j$ = jth raw moment of quadratic form U;
m = number of terms in series expansion of $\Phi(-\beta-u)$ in e^{-u};
$\mathbb{N}_n(\mathbf{m},\mathbf{\Sigma})$ = n dimensional Gaussian random vector with mean $\mathbf{m}\in\mathbb{R}^n$ and covariance matrix $\mathbf{\Sigma}\in\mathbb{R}^{n\times n}$;
n = number of basic random variables;
P_f = probability of failure;
$p(\mathbf{x})$ = probability density function of $\mathbf{x}$;
$\mathrm{Prob}[\cdot]$ = probability of the event $(\cdot)$;
$p_U(u)$ = probability density function of u;
$p_{\chi^2_\nu}$ = probability density function of χ^2_ν;
PDF = probability density function;
q = scaled random variable used for asymptotic distribution;
$\mathbf{R}$ = orthogonal transformation matrix defined in Eq. (15);
$\mathbb{R}$ = space of real numbers;
$\mathbb{R}^+$ = space of positive real numbers;
$\Re(\cdot)$ = real part of $(\cdot)$;
s = a scalar complex variable;
$\mathbf{T}$ = matrix of eigenvectors of $\mathbf{H}_g(\mathbf{x}^*)$;
$\mathrm{Trace}(\cdot)$ = trace of matrix $(\cdot)$;
U = quadratic form $\mathbf{y}^T\mathbf{A}\mathbf{y}$ in standard Gaussian random variables;
u = real scalar variable for values of randomvariable U;
u^* = optimal expansion point;
$\mathrm{Var}[\cdot]$ = variance of $(\cdot)$;
$\mathbf{x}$ = basic random variables;
$\mathbf{x}^*$ = design point;
$\mathbf{y}$ = $n-1$ dimensional uncorrelated Gaussian random vector;
$\tilde{\mathbf{y}}$ = n dimensional uncorrelated Gaussian random vector;
y_n = standard Gaussian random variable for nth coordinate;
z = random variable used for asymptotic distribution;
α^* = unit vector to the design point;
α_k = Fourier coefficients;
β = Hasofer and Lind reliability index;
β' = modified reliability index, $\beta'=\beta+u^*$;
$\Gamma(\cdot)$ = Gamma function;
γ = translation constant in Pearson's approximate pdf;
δ_k = constants in series expansion of $\Phi(-\beta-u)$ in e^{-u};
$\delta_j'^{(k)}$ = constants in series expansion of δ_k;
ϵ = small real number;
$\epsilon(u)$ = error term in series expansion of $\Phi(-\beta-u)$ in e^{-u};
$\varepsilon(u)$ = error in least-square approximation;
η = ratio $\varphi(\beta+u^*)/\Phi(-(\beta+u^*))$;
θ = multiplication constant in Patnaik's and Pearson's approximate probability density functional;
κ_j = jth principal curvature of failure surface at design point;
λ_k = kth eigenvalue of Hessian matrix $\mathbf{H}_g(\mathbf{x}^*)$;
μ = ratio $\varphi(\beta)/\Phi(-\beta)$;
ν = degrees of freedom for χ^2 probability density functional;
$\rho(u)$ = error term in Fourier series expansion of $\Phi(-\beta-u)$;
$\hat{\rho}(\omega)$ = Fourier sin transform of $\rho(\omega)$;
$\Phi(\cdot)$ = standard Gaussian cumulative distribution function;

$\varphi(\cdot)$ = standard Gaussian probability density function;
χ^2_ν = χ^2 random variable with ν degrees of freedom;
ψ = objective function for least-square approximation;
ω = real scalar variable used in Fourier series expansion;
$\nabla_g(\mathbf{x})$ = gradient vector of $g(\mathbf{x})$; $\{\nabla_g(\mathbf{x})\}_i = \partial g(\mathbf{x})/\partial x_i$;
$(\cdot)^T$ = matrix transpose;
$(\cdot)^{(j)}$ = jth order derivative of $(\cdot)$;
$|\cdot|$ = l_2 norm of a vector;
$\|\cdot\|$ = determinant of a matrix;
$\approx$ = approximately equal to;
$\forall$ = for all;
$\in$ = belongs to;
$\mapsto$ = maps into;
$\propto$ = proportional to;
$\rightarrow$ = approaches to;
$\sim$ = distributed as; and
$\simeq$ = approximately distributed as.

References

Abramowitz, M., and Stegun, I. A. (1965). *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, Dover, New York.

Breitung, K. (1984). "Asymptotic approximations for multinormal integrals." *J. Eng. Mech.*, 110(3), 357–366.

Breitung, K. (1989). "Asymptotic approximations for probability integrals." *Probab. Eng. Mech.*, 4(4), 187–190.

Breitung, K. (1991). "Probability approximations by log likelihood maximization." *J. Eng. Mech.*, 117(3), 457–477.

Breitung, K. (2002). "Asymptotic methods in reliability." *Proc., 3rd Int. Conf. on Mathematical Methods in Reliability Methodology and Practice*, H. Langseth and B. Lindqvist, eds., Trondheim, Norway, 3–6.

Breitung, K., and Richter, W. D. (1996). "A geometric approach to an asymptotic expansion for large deviation probabilities of Gaussian random vectors." *J. Multivariate Anal.*, 58(1), 1–20.

Cai, G. Q., and Elishakoff, I. (1994). "Refined second-order reliability analysis." *Struct. Safety*, 14(4), 267–276.

Der Kiureghian, A., and de Stefano, M. (1991). "Efficient algorithm for second order reliability analysis." *J. Eng. Mech.*, 117(12), 2904–2923.

Der Kiureghian, A., Lin, H.-Z., and Hwang, S.-J. (1987). "Second-order reliability approximations." *J. Eng. Mech.*, 113(8), 1208–1225.

Ditlevsen, O., and Madsen, H. O. (1996). *Structural reliability methods*, Wiley, Chichester, West Sussex, U.K.

Fiessler, B., Rackwitz, R., and Neumann, H. J. (1979). "Quadratic limit states in structural reliability." *J. Eng. Mech. Div.*, 105(4), 661–676.

Gradshteyn, I. S., and Ryzhik, I. M. (1994). *Table of integrals, series and products*, 5th Ed., Academic, Boston.

Hasofer, A. M., and Lind, N. C. (1974). "Exact and invariant second moment code format." *J. Eng. Mech. Div.*, 100(1), 111–121.

Hohenbichler, M., and Rackwitz, R. (1988). "Improvement of second-order reliability estimates by importance sampling." *J. Eng. Mech.*, 14(12), 2195–2199.

Hong, H. P. (1999). "Simple approximations for improving second-order reliability estimates." *J. Eng. Mech.*, 125(5), 592–595.

Johnson, N. L., and Kotz, S. (1970). *Distributions in statistics: Continuous univariate distributions-2, The Houghton Mifflin Series in Statistics*, Houghton Mifflin, Boston.

Köylüoğlu, H. U., and Nielsen, S. R. K. (1994). "New approximations for SORM integrals." *Struct. Safety*, 13(4), 235–246.

Madsen, H. O., Krenk, S., and Lind, N. C. (1986). *Methods of structural safety*, Prentice-Hall, Englewood Cliffs, N.J.

Manohar, C. S., and Gupta, S. (2003). "Modeling and evaluation of structural reliability: Current status and future directions." *Research reviews in structural engineering*, K. S. Jagadish and R. N. Iyengar, eds., Golden Jubilee Publications, Indian Institute of Science, Bangalore, India.

Mathai, A. M., and Provost, S. B. (1992). *Quadratic forms in random variables: Theory and applications*, Marcel Dekker, New York.

Melchers, R. E. (1999). *Structural reliability analysis and prediction*, 2nd Ed., Wiley, Chichester, West Sussex, U.K.

Naess, A. (1987). "Bounding approximations to some quadratic limit states." *J. Eng. Mech.*, 113(10), 1474–1492.

Patnaik, P. B. (1949). "The non-central χ^2 and f-distributions and their applications." *Biometrika*, 36, 202–232.

Pearson, E. S. (1959). "Note on an approximation to the distribution of non-central χ^2." *Biometrika*, 46, 364–364.

Polidori, D. C., Beck, J. L., and Papadimitriou, C. (1999). "New approximations for reliability integrals." *J. Eng. Mech.*, 125(4), 466–475.

Rice, S. O. (1980). "Distribution of quadratic forms in normal variables-Evaluation by numerical integration." *SIAM (Soc. Ind. Appl. Math.) J. Sci. Stat. Comput.*, 1, 438—448.

Rosenblatt, M. (1952). "Remarks on a multivariate transformation." *Ann. Math. Stat.*, 23, 470–472.

Thoft-Christensen, P., and Baker, M. J. (1982). *Structural reliability theory and its applications*, Springer, Berlin.

Tvedt, L. (1990). "Distribution of quadratic forms in normal space: application to structural reliability." *J. Eng. Mech.*, 116(6), 1183–1197.

Zhao, Y.-G., and Ono, T. (1999a). "New approximations for SORM: Part 1." *J. Eng. Mech.*, 125(1), 79–85.

Zhao, Y.-G., and Ono, T. (1999b). "New approximations for SORM: Part 2." *J. Eng. Mech.*, 125(1), 86–93.