Complex Modes in Stochastic Systems

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Abstract

Linear dynamic systems must generally be expected to exhibit non-proportional damping. Non-proportionally damped linear systems do not possess classical normal modes but possess *complex modes*. In this paper we analyze the complex modes and complex natural frequencies arising in multiple degree-of-freedom non-proportionally damped discrete linear stochastic systems. Second-order statistics of the complex eigenvalues and eigenvectors are presented by assuming the randomness of the system to be so small that the first-order perturbation approach is valid. The proposed method is illustrated by considering numerical example based on a linear array of damped spring-mass oscillators. It is shown that the approach can predict the statistics of complex eigenvalues and eigenvectors with good accuracy when compared with independent Monte Carlo simulations.

Keywords: Complex modes, non-proportionately damped, discrete linear stochastic systems, spring-mass oscillators.

1 Introduction

Dynamics of linear systems with statistical parameter uncertainties is currently an active area of research. The primary concern here lies in the probabilistic modelling of the uncertainties in specifying elastic, mass and damping properties of the structure. Problems of undamped linear deterministic structural dynamics have been extensively treated in the existing literature using classical modal analysis. Now, classical modal analysis has also been generalized to deal with undamped stochastic systems. Over the years many efficient methods have emerged to solve the so called *random eigenvalue problems* that arise in the dynamics of undamped stochastic systems. Several review papers have appeared in this field which discuss the current as well as the earlier works[3,8,9].

It is well known that if the damping is 'proportional' then classical modal analysis also holds for damped systems. Conditions for existence of classical normal modes were derived by Caughey and O'Kelly[5]. It must be noted that proportional damping or 'classical damping' is purely a mathematical abstraction. There is no physical reason or mathematical basis for why a general system should behave like this. If the damping is not proportional then linear systems possess *complex modes* instead of real normal modes. Apart from the mathematical consistency, practical experience in modal testing also shows that most real life structures possess complex modes. As Sestieri and Ibrahim[13] have put it '... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality!' However, consideration of complex modes in stochastic structural dynamics has not been considered till now.

Central theme to this paper is the analysis of the complex modes in the dynamics of non-proportionally damped discrete linear stochastic systems. Due to the non-proportional nature of the damping, eigenvalues become complex random variables and eigenvectors become complex random processes. This fact makes the analysis quite different from the traditional random eigenvalue problems arising in the analysis of stochastic undamped systems where the random eigensolutions remain real. The complex eigen-analysis, however, may be performed by extending the currently available techniques of random eigenvalue problems in conjunction with the *first-order* formulation of the equations of motion. The first-order or state-space formalism, although straightforward, requires significant numerical effort to obtain statistics of the eigensolutions as the size of the problem doubles. Moreover, this approach also lacks some of the intuitive simplicity of the traditional 'N-space' based modal analysis method of random structural systems.

For these reasons, in this paper an approach is proposed to obtain the statistics of complex eigenvalues and eigenvectors in N-space. It is assumed that the randomness is small so that the first-order perturbation method can be applied. Attention is restricted to the second-order statistics of the complex eigensolutions. In Section 2, we briefly discuss the requisite mathematical background on linear multiple-degree-of-freedom discrete systems needed for further derivations. Derivatives of complex eigenvalues and eigenvectors are derived in Section 3. These expressions are employed in Section 4 to obtain the mean, autocorrelation and crosscorrelation of complex eigenvalues and eigenvectors. The results are expressed in terms of the covariance matrices of the system properties. In Section 5 the proposed method is applied to a non-proportionally damped random eight-degree-of-freedom system and the results are verified by an independent Monte Carlo simulation.

2 Background

The equations of motion describing the free vibration of a linear, damped discrete system with N degrees-of-freedom are

$$\mathbf{M}\ddot{\mathrm{u}}(t) + \mathbf{C}\dot{\mathrm{u}}(t) + \mathbf{K}\mathrm{u}(t) = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass, damping and stiffness matrices respectively. $\mathrm{u}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $t \in \mathbb{R}^+$ denotes time. It is assumed that all the system matrices are symmetric. We consider randomness of the system matrices of the following form:

$$\mathbf{M} = \overline{\mathbf{M}} + \delta\mathbf{M}, \quad \mathbf{C} = \overline{\mathbf{C}} + \delta\mathbf{C}, \quad \text{and} \quad \mathbf{K} = \overline{\mathbf{K}} + \delta\mathbf{K}. \tag{2}$$

Here, $\overline{(\bullet)}$ and $\delta(\bullet)$ denotes the nominal (deterministic) and random parts of $(\bullet)$ respectively. Without any loss of generality it may be assumed that $\delta\mathbf{M}$, $\delta\mathbf{C}$ and $\delta\mathbf{K}$ are zero-mean random matrices. We further assume that the random parts of the system matrices are small and also they are such that

1. the symmetry of the system matrices is preserved,
2. the mass matrix $\mathbf{M}$ is positive definite, and
3. $\mathbf{C}$ and $\mathbf{K}$ are non-negative definite.

Note that no assumptions on the type of randomness, (for example Gaussian) is assumed at this stage. For the sake of generality we also consider that the elements of all the random matrices $\delta\mathbf{M}$, $\delta\mathbf{C}$ and $\delta\mathbf{K}$ are correlated to each other. The eigenvalue problem associated with (1) can be represented by

$$s_j^2\mathbf{M}\mathbf{u}_j + s_j\mathbf{C}\mathbf{u}_j + \mathbf{K}\mathbf{u}_j = 0, \tag{3}$$

where $s_j \in \mathbb{C}$ is the jth eigenvalue and $\mathbf{u}_j \in \mathbb{C}^N$ is the jth eigenvector. There are $2N$ eigenvalues and eigenvectors appearing in complex conjugate pairs. For convenience we arrange the eigenvalues as

$$s_1, s_2, \cdots, s_N, s_1^*, s_2^*, \cdots, s_N^*. \tag{4}$$

The aim of this paper is to obtain second-order statistics of the first N eigenvalues and eigenvectors.

If the system matrices are deterministic, several methods exist to obtain the eigensolutions. These methods mainly follow two routes, the state-space methods and methods in configuration space or 'N-space'. The state-space methods[10] although exact in nature, requires significant numerical effort to obtain the eigensolutions as the size of the problem doubles. Moreover, this approach also lacks some of the intuitive simplicity of the analysis based on 'N-space'. For these reasons the determination of eigenvalues and eigenvectors in N-space is very much desirable. Recently Adhikari[1] proposed a Neumann series based method in which the complex eigenvectors of non-conservative systems are expressed as a series in the corresponding undamped eigenvectors. This method can be used to obtain the complex eigensolutions up to any desired level of accuracy without using the state-space formalism. This motivates us towards developing procedures to obtain the statisctics of complex eigensolutions of stochastic linear systems in N-space. Our method is based on the derivatives of complex eigensolutions.

3 Derivatives of Complex Eigensolutions

In a recent paper, Adhikari[2] has derived first-order derivatives of complex natural frequencies and mode shapes. In the context of structural dynamics, the natural frequencies λ_j are defined as $s_j = j\lambda_j$. Here we slightly modify the approach outlines in Adhikari[2] to obtain first-order derivatives of the complex eigensolutions.

Suppose the system matrices are functions of some variable α so that $\mathbf{M}, \mathbf{C}, \mathbf{K} \equiv \mathbf{M}, \mathbf{C}, \mathbf{K}(\alpha)$. For notational convenience rewrite (3) as

$$\mathbf{F}_j\mathbf{u}_j = 0, \tag{5}$$

where the *regular matrix pencil*

$$\mathbf{F}_j \equiv \mathbf{F}(s_j, \alpha) = \left[s_j^2\mathbf{M} + s_j\mathbf{C} + \mathbf{K}\right] \quad \in \mathbb{C}^{N\times N}. \tag{6}$$

Differentiating (5) with respect to α one obtains

$$\frac{\partial \mathbf{F}_j}{\partial \alpha}\mathbf{u}_j + \mathbf{F}_j\frac{\partial \mathbf{u}_j}{\partial \alpha} = 0, \tag{7}$$

where $\frac{\partial \mathbf{F}_j}{\partial \alpha}$ may be obtained by differentiating (6) as

$$\frac{\partial \mathbf{F}_j}{\partial \alpha} = \frac{\partial \tilde{\mathbf{F}}_j}{\partial \alpha} + \frac{\partial s_j}{\partial \alpha}\mathbf{G}_j. \tag{8}$$

Here the terms $\frac{\partial \tilde{\mathbf{F}}_j}{\partial \alpha}$ and $\mathbf{G}_j$ are defined by

$$\begin{aligned} \frac{\partial \tilde{\mathbf{F}}_j}{\partial \alpha} &= s_j^2 \frac{\partial \mathbf{M}}{\partial \alpha} + s_j \frac{\partial \mathbf{C}}{\partial \alpha} + \frac{\partial \mathbf{K}}{\partial \alpha} \\ \mathbf{G}_j &= 2s_j \mathbf{M} + \mathbf{C}. \end{aligned} \tag{9}$$

Premultiplying equation (7) by $\mathbf{u}_j^T$ one obtains the scalar equation

$$\mathbf{u}_j^T \frac{\partial \mathbf{F}_j}{\partial \alpha} \mathbf{u}_j + \mathbf{u}_j^T \mathbf{F}_j \frac{\partial \mathbf{u}_j}{\partial \alpha} = 0. \tag{10}$$

Taking the transpose of (5) and postmultiplying by $\frac{\partial \mathbf{u}_j}{\partial \alpha}$ it may be shown that the second term of the above equation vanishes. Now, substituting $\frac{\partial \mathbf{F}_j}{\partial \alpha}$ from (8) into (10) we obtain

$$\frac{\partial s_j}{\partial \alpha} = -\frac{\mathbf{u}_j^T \frac{\partial \tilde{\mathbf{F}}_j}{\partial \alpha} \mathbf{u}_j}{\mathbf{u}_j^T \mathbf{G}_j \mathbf{u}_j}. \tag{11}$$

Sestieri and Ibrahim[13] have shown that the expression of the denominator for the above equation, that is

$$\mathbf{u}_j^T \left[2s_j \mathbf{M} + \mathbf{C} \right] \mathbf{u}_j = \frac{1}{\gamma_j} \quad \text{(say)} \tag{12}$$

is the normalization constant for the jth complex mode. There are several ways in which the normalization constants can be selected. The one that is most consistent with traditional modal analysis practice is to choose $\gamma_j = 1/2s_j$. Observe that this degenerates to the familiar mass normalization relationship $\mathbf{u}_j^T \mathbf{M} \mathbf{u}_j = 1$ when the damping is zero. It should be noted that γ_j will be assumed constant and should not vary with the parameters. Now, combining (9), (11) and (12), the expression for the derivative of the jth complex eigenvalue can be expressed as

$$\frac{\partial s_j}{\partial \alpha} = -\gamma_j \mathbf{u}_j^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial \alpha} + s_j \frac{\partial \mathbf{C}}{\partial \alpha} + \frac{\mathbf{K}}{\partial \alpha} \right] \mathbf{u}_j. \tag{13}$$

To obtain the derivative of the eigenvectors it is required to convert the equations of motion into the state-space form and then relate the results to the eigenvectors of the second-order system. Following Adhikari[2] we have

$$\frac{\partial \mathbf{u}_j}{\partial \alpha} = \sum_{k=1}^{2N} a_{jk}^{(\alpha)} \mathbf{u}_k \tag{14}$$

where

$$\begin{aligned} a_{jk}^{(\alpha)} &= -\frac{\gamma_j}{s_j - s_k} \mathbf{u}_k^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial \alpha} + s_j \frac{\partial \mathbf{C}}{\partial \alpha} + \frac{\partial \mathbf{K}}{\partial \alpha} \right] \mathbf{u}_j \quad \forall k = 1, 2, \ldots, 2N, \neq j \\ \text{and} \quad a_{jj}^{(\alpha)} &= -\frac{\gamma_j}{2} \mathbf{u}_j^T \left[2s_j \frac{\partial \mathbf{M}}{\partial \alpha} + \frac{\partial \mathbf{C}}{\partial \alpha} \right] \mathbf{u}_j. \end{aligned} \tag{15}$$

Note that because the eigensolutions appear in complex conjugate pairs, in the above equations $s_k = s^*_{(k-N)}$, $\mathbf{u}_k = \mathbf{u}^*_{(k-N)}$ and $\gamma_k = \gamma^*_{(k-N)}$ for $k > N$.

Equations (13)–(15) completely define the derivatives of the complex eigenvalues and eigenvectors with respect to an arbitrary parameter α. Observe that in the limit when $\mathbf{C} = 0$ these expressions reduce to the corresponding expressions for undamped systems derived by Fox and Kapoor[7]. Derivatives with respect to elements of the system matrices can be obtained from these equations by substituting $\alpha = K_{rs}$, $\alpha = C_{rs}$ and $\alpha = M_{rs}$. Using these substitutions, from (13) we have

$$\frac{\partial s_j}{\partial K_{rs}} = -\gamma_j \left(U_{rj} U_{sj}\right), \tag{16}$$

$$\frac{\partial s_j}{\partial C_{rs}} = s_j \frac{\partial s_j}{\partial K_{rs}}, \tag{17}$$

and

$$\frac{\partial s_j}{\partial M_{rs}} = s_j^2 \frac{\partial s_j}{\partial K_{rs}}. \tag{18}$$

Consider the derivative of the lth element of the jth eigenvector, denoted by U_{lj}. In (14), utilizing the above substitutions and using (15) one obtains

$$\frac{\partial U_{lj}}{\partial K_{rs}} = -\gamma_j \sum_{\substack{k=1 \\ k \neq j}}^{2N} \frac{(U_{rk} U_{sj})}{s_j - s_k} U_{lk}, \tag{19}$$

$$\frac{\partial U_{lj}}{\partial C_{rs}} = -\gamma_j 2 \left(U_{rj} U_{sj}\right) U_{lj} + s_j \frac{\partial U_{lj}}{\partial K_{rs}}, \tag{20}$$

and

$$\frac{\partial U_{lj}}{\partial M_{rs}} = -\gamma_j s_j \left(U_{rj} U_{sj}\right) U_{lj} + s_j^2 \frac{\partial U_{lj}}{\partial K_{rs}}. \tag{21}$$

Next, these relationships are utilized to obtain the statistics of the eigensolutions.

4 Statistics of Complex Eigensolutions

Collins and Thomson[6] were possibly the first to derive the expressions for statistics of undamped eigenvalues and eigenvectors of multiple-degrees-of-freedom systems using the first-order perturbation method. Later, several authors, for example Bucher and Brenner[4], Ramu an d Ganesan[11,12] have used similar approaches. Applicability of the method relies on the fact that the randomness of the system matrices is small. In this section we generalize these results to the case of complex eigensolutions arising in non-classically damped systems.

Consider $(\bar{s}_j, \bar{\mathbf{u}}_j)$ to be the jth eigensolution pair for the nominal system corresponding to (1), that is, $\bar{s}_j$ and $\bar{\mathbf{u}}_j$ satisfy the deterministic eigenvalue problem

$$\bar{s}_j^2 \overline{\mathbf{M}} \bar{\mathbf{u}}_j + \bar{s}_j \overline{\mathbf{C}} \bar{\mathbf{u}}_j + \overline{\mathbf{K}} \bar{\mathbf{u}}_j = 0. \tag{22}$$

The method proposed by Adhikari[1] may be used to obtain $\bar{s}_j$ and $\bar{\mathbf{u}}_j$ in N-space. If the random perturbations of the system matrices are small, s_j can be approximated by a first-order Taylor expansion as

$$s_j = \bar{s}_j + \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{\partial s_j}{\partial K_{rs}}\delta K_{rs} + \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{\partial s_j}{\partial C_{rs}}\delta C_{rs} + \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{\partial s_j}{\partial M_{rs}}\delta M_{rs}. \tag{23}$$

An important conclusion emerging from the above expression is that, when the system has sufficient degrees of freedom so that there are enough terms in the above summation, the *central limit theorem* assures that both real and imaginary parts of the random component of s_j would be Gaussian even if δK_{rs}, δC_{rs} and δM_{rs} are non-Gaussian. Thus, randomness of the system property matrices may be modelled by Gaussian random variables provided that the conditions for physically realistic systems outlined in Section 2 are satisfied. Assuming the randomness to be Gaussian and taking the expectation of (23), it is clear that

$$\langle s_j \rangle = \bar{s}_j \tag{24}$$

because $\langle \delta K_{rs} \rangle = \langle \delta C_{rs} \rangle = \langle \delta M_{rs} \rangle = 0$. Equation (24) indicates that the mean of the complex eigenvalues is the same as that of the undamped eigenvalues. Since the system matrices are real, from (23), the covariance between two complex eigenvalues may be obtained as

$$\begin{aligned}\mathrm{cov}(s_j, s_k) = \langle (s_j - \bar{s}_j)\rangle \langle (s_k - \bar{s}_k)^* \rangle = \sum_{r=1}^{N}\sum_{s=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N} \Bigg[& \frac{\partial s_j}{\partial K_{rs}} \frac{\partial s_k^*}{\partial K_{pq}} \mathrm{cov}(K_{rs}, K_{pq}) \\ & + \frac{\partial s_j}{\partial C_{rs}} \frac{\partial s_k^*}{\partial C_{pq}} \mathrm{cov}(C_{rs}, C_{pq}) + \frac{\partial s_j}{\partial M_{rs}} \frac{\partial s_k^*}{\partial M_{pq}} \mathrm{cov}(M_{rs}, M_{pq}) \\ & + 2\frac{\partial s_j}{\partial K_{rs}} \frac{\partial s_k^*}{\partial C_{pq}} \mathrm{cov}(K_{rs}, C_{pq}) + 2\frac{\partial s_j}{\partial C_{rs}} \frac{\partial s_k^*}{\partial M_{pq}} \mathrm{cov}(C_{rs}, M_{pq}) \\ & + 2\,\frac{\partial s_j}{\partial M_{rs}} \frac{\partial s_k^*}{\partial K_{pq}} \mathrm{cov}(M_{rs}, K_{pq}) \Bigg].\end{aligned} \tag{25}$$

At this stage it is useful to express the results in matrix form. Construct the vectors

$$\mathbf{s} = \{s_1, s_2, \cdots, s_N\}^T \in \mathbb{C}^N \quad \text{and} \quad \delta\kappa = \mathrm{vec}\,(\delta\mathbf{K}) \in \mathbb{R}^{N^2}$$

where the operation vec(•) denotes a new (large) vector by staking consecutive columns of (•). Similarly, $\delta\mathbf{C}$ and $\delta\mathbf{M}$ can also be defined. Now, rewrite (23) for $j = 1, 2, \cdots, N$ as

$$\mathbf{s} - \bar{\mathbf{s}} = \boldsymbol{\mathcal{D}}_{\mathbf{s}} \begin{Bmatrix} \delta\mathcal{K} \\ \delta\mathcal{C} \\ \delta\mathcal{M} \end{Bmatrix}. \tag{26}$$

In the above equation, $\boldsymbol{\mathcal{D}}_{\mathbf{s}}$, the matrix containing derivatives of s_j with respect to elements of the system matrices is given by

$$\boldsymbol{\mathcal{D}}_{\mathbf{s}}^T = \begin{bmatrix} \dfrac{\partial s_1}{\partial \mathcal{K}} & \dfrac{\partial s_2}{\partial \mathcal{K}} & \cdots & \dfrac{\partial s_N}{\partial \mathcal{K}} \\ \dfrac{\partial s_1}{\partial \mathcal{C}} & \dfrac{\partial s_2}{\partial \mathcal{C}} & \cdots & \dfrac{\partial s_N}{\partial \mathcal{C}} \\ \dfrac{\partial s_1}{\partial \mathcal{M}} & \dfrac{\partial s_2}{\partial \mathcal{M}} & \cdots & \dfrac{\partial s_N}{\partial \mathcal{M}} \end{bmatrix} \in \mathbb{R}^{3N^2 \times N}, \tag{27}$$

where $\frac{\partial s_j}{\partial \boldsymbol{\mathcal{K}}} \equiv \text{vec}\left(\frac{\partial s_j}{\partial K_{rs}}\right) \in \mathbb{R}^{N^2}$ for some fixed j. Similarly $\frac{\partial s_j}{\partial \boldsymbol{\mathcal{C}}}$ and $\frac{\partial s_j}{\partial \boldsymbol{\mathcal{M}}}$ can also be defined. Thus, the matrix $\boldsymbol{\mathcal{D}}_{\mathbf{s}}$ can be calculated by applying (16), (17) and (18) for the nominal system. Now, from (26), the covariance matrix of the eigenvalues, $\boldsymbol{\Sigma}_{\mathbf{s}}$, is obtained as

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{s}} &= \langle (\mathbf{s}-\bar{\mathbf{s}})\,(\mathbf{s}-\bar{\mathbf{s}})^{*^T} \rangle \\ &= \boldsymbol{\mathcal{D}}_{\mathbf{s}} \left\langle \begin{Bmatrix} \delta\boldsymbol{\mathcal{K}} \\ \delta\boldsymbol{\mathcal{C}} \\ \delta\boldsymbol{\mathcal{M}} \end{Bmatrix} \begin{Bmatrix} \delta\boldsymbol{\mathcal{K}} \\ \delta\boldsymbol{\mathcal{C}} \\ \delta\boldsymbol{\mathcal{M}} \end{Bmatrix}^T \right\rangle \boldsymbol{\mathcal{D}}_{\mathbf{s}}^{*^T} = \boldsymbol{\mathcal{D}}_{\mathbf{s}} \boldsymbol{\Sigma}_{kcm} \boldsymbol{\mathcal{D}}_{\mathbf{s}}^{*^T}. \end{aligned} \tag{28}$$

Here, $\boldsymbol{\Sigma}_{kcm} \in \mathbb{R}^{3N^2 \times 3N^2}$, the joint covariance matrix of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are defined as

$$\boldsymbol{\Sigma}_{kcm} = \begin{bmatrix} \langle \delta\boldsymbol{\mathcal{K}} \delta\boldsymbol{\mathcal{K}}^T \rangle & \langle \delta\boldsymbol{\mathcal{K}} \delta\boldsymbol{\mathcal{C}}^T \rangle & \langle \delta\boldsymbol{\mathcal{K}} \delta\boldsymbol{\mathcal{M}}^T \rangle \\ \langle \delta\boldsymbol{\mathcal{C}} \delta\boldsymbol{\mathcal{K}}^T \rangle & \langle \delta\boldsymbol{\mathcal{C}} \delta\boldsymbol{\mathcal{C}}^T \rangle & \langle \delta\boldsymbol{\mathcal{C}} \delta\boldsymbol{\mathcal{M}}^T \rangle \\ \langle \delta\boldsymbol{\mathcal{M}} \delta\boldsymbol{\mathcal{K}}^T \rangle & \langle \delta\boldsymbol{\mathcal{M}} \delta\boldsymbol{\mathcal{C}}^T \rangle & \langle \delta\boldsymbol{\mathcal{M}} \delta\boldsymbol{\mathcal{M}}^T \rangle \end{bmatrix}. \tag{29}$$

Statistics of the eigenvectors can be obtained following a similar procedure as the eigenvalues. For small random perturbations of the system matrices, $\mathbf{u}_j$ can be approximated by a first-order Taylor expansion and consequently we have

$$\mathbf{u}_j - \bar{\mathbf{u}}_j = \boldsymbol{\mathcal{D}}_{\mathbf{u_j}} \begin{Bmatrix} \delta\boldsymbol{\mathcal{K}} \\ \delta\boldsymbol{\mathcal{C}} \\ \delta\boldsymbol{\mathcal{M}} \end{Bmatrix}. \tag{30}$$

In the above equation, $\boldsymbol{\mathcal{D}}_{\mathbf{u_j}}$, the matrix containing derivatives of $\mathbf{u}_j$ with respect to elements of the system matrices, is given by

$$\boldsymbol{\mathcal{D}}_{\mathbf{u_j}}^T = \begin{bmatrix} \frac{\partial U_{1j}}{\partial \boldsymbol{\mathcal{K}}} & \frac{\partial U_{2j}}{\partial \boldsymbol{\mathcal{K}}} & \cdots & \frac{\partial U_{Nj}}{\partial \boldsymbol{\mathcal{K}}} \\ \frac{\partial U_{1j}}{\partial \boldsymbol{\mathcal{C}}} & \frac{\partial U_{2j}}{\partial \boldsymbol{\mathcal{C}}} & \cdots & \frac{\partial U_{Nj}}{\partial \boldsymbol{\mathcal{C}}} \\ \frac{\partial U_{1j}}{\partial \boldsymbol{\mathcal{M}}} & \frac{\partial U_{2j}}{\partial \boldsymbol{\mathcal{M}}} & \cdots & \frac{\partial U_{Nj}}{\partial \boldsymbol{\mathcal{M}}} \end{bmatrix} \in \mathbb{R}^{3N^2 \times N}, \tag{31}$$

where $\frac{\partial U_{lj}}{\partial \boldsymbol{\mathcal{K}}} \equiv \text{vec}\left(\frac{\partial U_{lj}}{\partial K_{rs}}\right) \in \mathbb{R}^{N^2}$ for some fixed l, j. Similarly $\frac{\partial U_{lj}}{\partial \boldsymbol{\mathcal{C}}}$ and $\frac{\partial U_{lj}}{\partial \boldsymbol{\mathcal{M}}}$ can also be defined. Thus, the matrix $\boldsymbol{\mathcal{D}}_{\mathbf{u_j}}$ can be calculated by applying equations (19), (20) and (21) for the nominal system. Applying equation (30) for the jth and kth set, the covariance matrix of jth and kth eigenvectors, $\boldsymbol{\Sigma}_{\mathbf{u_j u_k}}$ is obtained as

$$\boldsymbol{\Sigma}_{\mathbf{u_j u_k}} = \langle \left(\mathbf{u}_j - \bar{\mathbf{u}}_j\right) (\mathbf{u}_k - \bar{\mathbf{u}}_k)^{*^T} \rangle = \boldsymbol{\mathcal{D}}_{\mathbf{u_j}} \boldsymbol{\Sigma}_{kcm} \boldsymbol{\mathcal{D}}_{\mathbf{u_k}}^{*^T}. \tag{32}$$

5 Numerical Example

An eight-DOF system consisting of a linear array of spring–mass oscillators and dampers is considered to illustrate a possible use of the expressions developed so far. Figure 1 shows the model system.

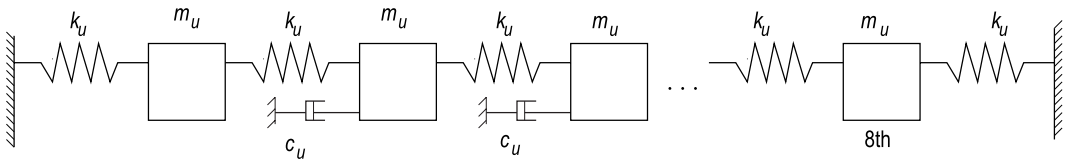

Fig. 1 Linear array of 8 spring–mass oscillators; nominal system: $m_u = 1\,\text{Kg}$, $k_u = 10\,\text{N/m}$ and $c_u = 0.1\,\text{Nm/s}$.

Eight masses, each of nominal mass $m_u = 1\,\text{kg}$, are connected by springs of nominal stiffness $k_u = 10\,\text{N/m}$. The masses corresponding to the second to sixth units have viscous dampers with nominal value $c_u = 0.1\,\text{Nm/s}$ connected to the ground. It is assumed that the mass, stiffness and damping associated with all the units are random. Randomness associated with each unit has the following form $m_{u_j} = m_u\left(1 + \epsilon_{m_j} g_j\right)$, $k_{u_j} = k_u\left(1 + \epsilon_{k_j} g_j\right)$ and $c_{u_j} = c_u\left(1 + \epsilon_{c_j} g_j\right)$. Here $g_j, \forall j$ are assumed to be uncorrelated, identically distributed, zero-mean, unit-standard-deviation Gaussian random variables ($N(0, 1)$). For this assumption, the joint covariance matrix $\mathbf{\Sigma}_{kcm}$, defined in (29), becomes a diagonal matrix. Numerical values of the 'strength parameters', ϵ_{m_j}, ϵ_{k_j} and ϵ_{c_j} are assumed to be 0.1, that is, we consider 10% randomness for all the parameter values.

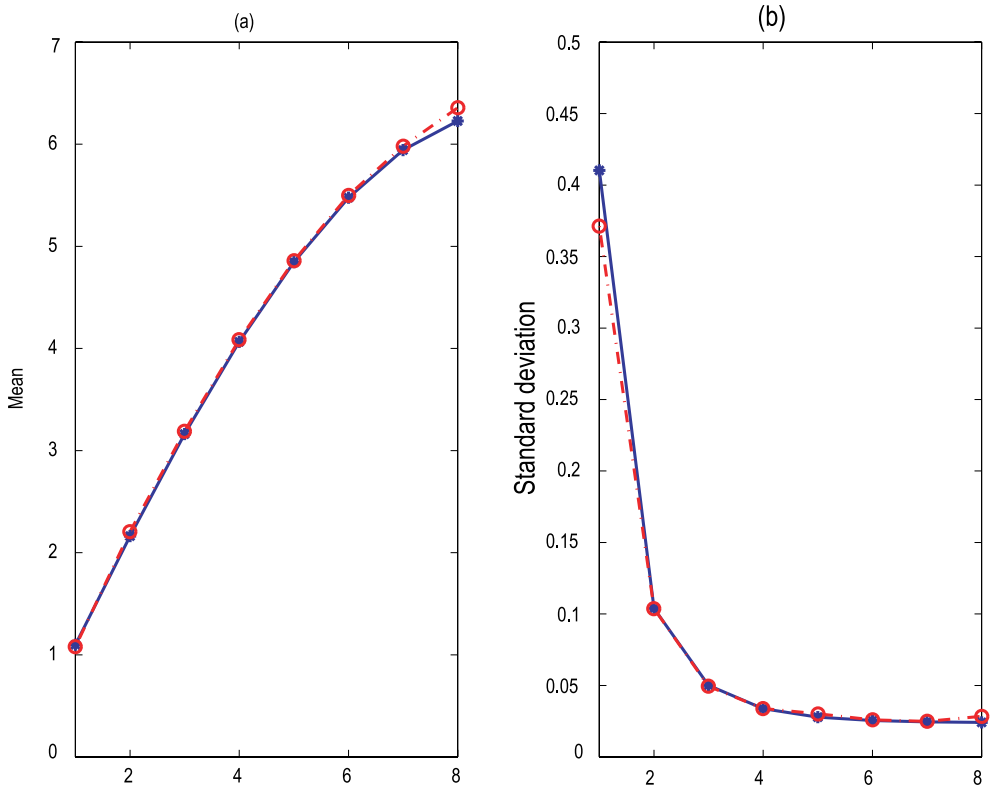

Fig. 2 (a) Absolute value of mean of complex natural frequencies. (b) Standard deviation of complex natural frequencies. X-axis: Mode number; '—' analytical; '-.-.-' MCS

Figure 2(a) shows the absolute values of the mean of the eigenvalues corresponding to the eight modes of the system. Because the random variables describing the system properties are assumed to be Gaussian, the mean values are the same as the nominal values (see equation (24)). In the same figure, the mean values obtained from the proposed theory are compared with the results obtained from an independent Monte Carlo simulation (MCS) using 500 samples. Observe that both the curves follow each other very closely. Figure 2(b) compares the standard deviation of the complex natural frequencies obtained from (28) with that obtained from the MCS. Again, observe that the results obtained from the formulation developed in this paper match those of the numerical simulations with excellent accuracy.

Now we turn our attention to the complex modes of the system. Figure 3 shows the mean value of the real parts of the eight complex modes. Analytical results and those obtained from MCS are compared in this figure. Complex modes are calculated using Adhikari's method[1] and normalized according to (12) with $\gamma_j = 1/2s_j$. The standard deviation of the modes obtained using (32) with $j = k$, and those calculated from MSC are shown in Fig. 4. From these two figures, observe that the mean and standard deviation of the complex modes obtained from the analytical method proposed in this paper match excellently with the corresponding results obtained from MCS. It is useful to understand the results in the light of amount of the damping present in the system. The Q-factors, defined as $Q_j = \Im(s_j)/2\Re(s_j)$, for the eight modes corresponding to the nominal system are obtained as 12.8162, 35.9982, 63.2519,

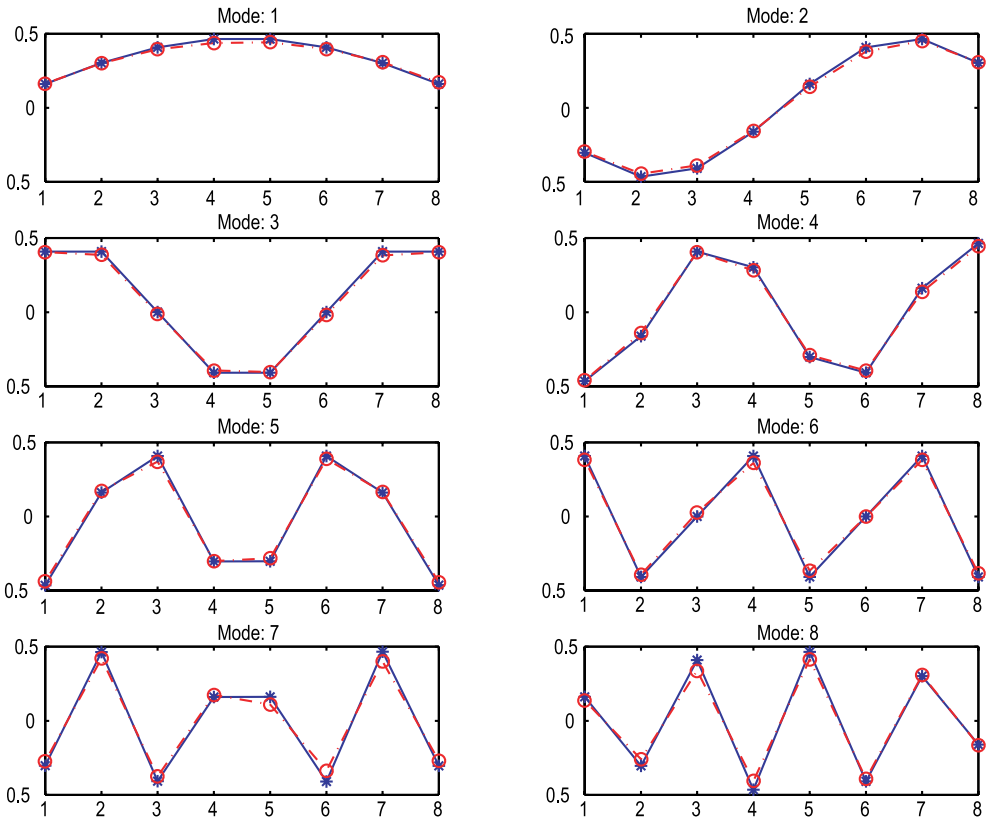

Fig. 3 Real part of mean of the complex modes, X-axis DOF; '—' analytical; '-.- -' MCS

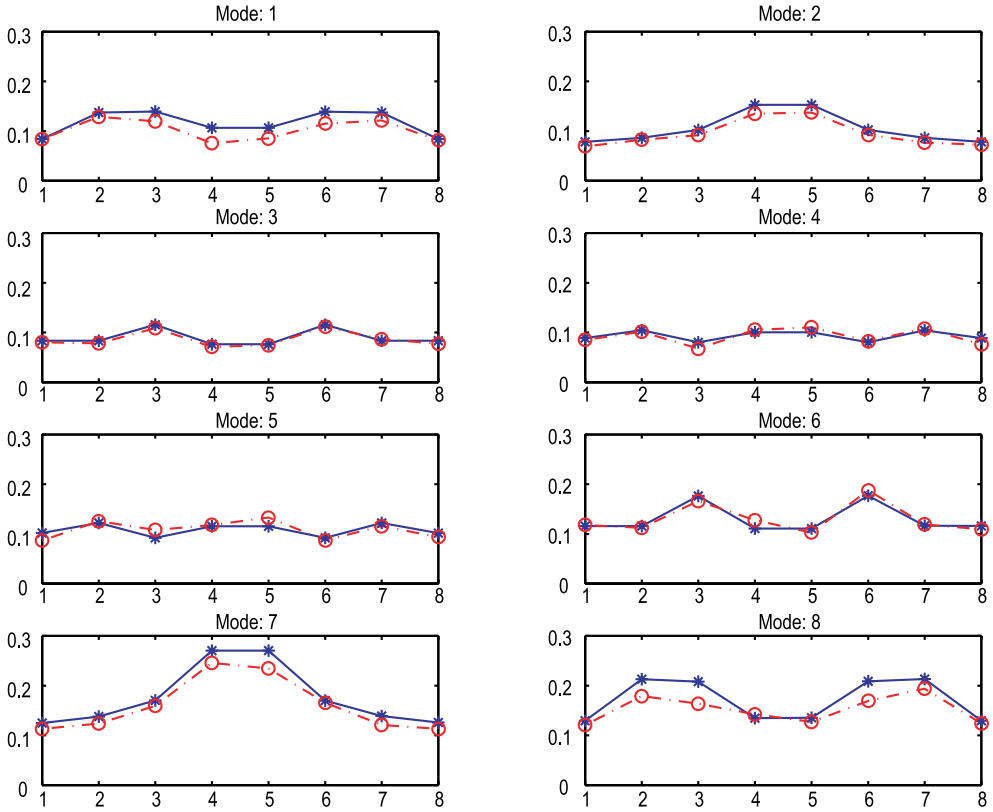

Fig. 4 Standard deviation of the complex modes. X-axis-DOF; '—' analytical; '-.-.-' MCS

74.8700, 89.2225, 109.5776, 98.9754, and 72.7037 respectively. These low values of Q-factor indicate that the damping of the system is high. Thus, the method developed here can be applied to systems with high non-proportional damping.

In the simulation study, it was observed that for some cases the system matrices become *negative* definite, which is in a way non-physical. This fact arises due to the non-bounded nature of the Gaussian random variables. For the simulation results shown in Figs. 2 to 4, we have removed the samples corresponding to such non-physical cases. Although the probability of such occurrence is low because the randomness is assumed to be small, nevertheless, it indicates that, in general, Gaussian random variables are not physically realistic for modelling randomness of the system properties. To avoid this problem, a (bounded) non-Gaussian model should be used.

6 Conclusions

An approach has been proposed to obtain the statistics of complex eigenvalues and eigenvectors of non-proportionally damped linear stochastic systems. The proposed method does not require conversion of

the equations of motion into the first-order form. It is assumed that the randomness is small so that the first-order perturbation method can be applied. Attention is restricted to the second-order statistics of the complex eigensolutions. The mean of the complex eigensolutions turned out to be the same as the corresponding deterministic values. The covariance matrices of the complex eigensolutions are expressed in terms of the covariance matrices of the system properties and derivatives of the eigensolutions with respect to the system parameters. The proposed method was applied to a non-proportionally damped random eight-degree-of-freedom system and the results were verified by an independent Monte Carlo simulation. It was shown that the method works well even when the damping is quite high.

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