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Polynomial chaos expansion in structural dynamics: Accelerating the convergence of the first two statistical moment sequences

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ABSTRACT

Polynomial chaos solution for the frequency response of linear non-proportionally damped dynamic systems has been considered. It has been observed that for lightly damped systems the convergence of the solution can be very poor in the vicinity of the deterministic resonance frequencies. To address this, Aitken's transformation and its generalizations are suggested. The proposed approach is successfully applied to the sequences defined by the first two moments of the responses, and this process significantly accelerates the polynomial chaos convergence. In particular, a 2-dof system with respectively 1 and 2 parameter uncertainties has been studied. The first two moments of the frequency response were calculated by Monte Carlo simulation, polynomial chaos expansion and Aitken's transformation of the polynomial chaos expansion. Whereas 200 polynomials are required to have a good agreement with Monte Carlo results around the deterministic eigenfrequencies, less than 50 polynomials transformed by the Aitken's method are enough. This latter result is improved if a generalization of Aitken's method (recursive Aitken's transformation, Shank's transformation) is applied. With the proposed convergence acceleration, polynomial chaos may be reconsidered as an efficient method to estimate the first two moments of a random dynamic response.

1. Introduction

The modelling of uncertain structures is a research field that has rapidly developed recently. A common objective is to estimate the statistics of the stochastic response of a structure. The oldest method is Monte Carlo simulation (MCS) [1–4], which has the drawback of very slow convergence. This problem becomes even more pronounced for structural dynamic

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problems, where response statistics is required for various time instants and frequency ranges [5,6]. For efficient computation of the response statistics, an approach to obtain some kind of response surface in the space of random parameters is necessary. There are two broad ways of obtaining response surfaces. Perturbation based methods (lower-order Taylor approximation) [7] and the Neumann expansion method, [8,9] are generally computationally efficient, however, depending on applications, may not give accurate results if the uncertainties are large. Polynomial Chaos (PC) expansion [10–12], High Dimensional Model Reduction (HDMR) [13–15] and Gaussian Process (GP) Emulators [16–18] have been developed for any level of uncertainty. Consequently these latter approximations have gained significant interest in recent years. This paper concerns Polynomial Chaos Expansion (PCE) for multi-degree of freedom damped dynamic systems with multi-parametric uncertainties.

Polynomial Chaos Expansion belong to the class of spectral methods where a Galerkin type of error minimization approach is generally employed in the space of random variables. Computation cost of the approach can be high if number of random variables are high. Several novel approaches have been developed to address the issue of computational cost, see for example [19–23]. In addition to the computational cost arising due to large random dimension, the issue of convergence of the series can be of concern in certain cases. Field and Grigoriu [24,25] studied the convergence properties of this PCE. Recently Keshavarzzadeh et al. [26] proposed a transformation approach to enhance the convergence of PCE in the context of elliptic problems.

Recent work [27,28] has showed that the convergence may be very slow when the statistics of the response is estimated by PCE for dynamic problems. Other studies [5,6] also show that care must be taken while directly extending concepts, results and computational methods based on static problems to dynamic problems. In particular, very slow convergence occurs for the response of a dynamical system submitted to harmonic forces around some critical frequencies (e.g. eigenfrequencies, critical rotating speed in rotordynamics) that are of great interest in the design of structures where large oscillations may arise. Hence the acceleration of the convergence of the statistics provided by a PCE is of great interest.

The Padé approximant [29,30] method is one of the most popular procedures to accelerate the convergence of a sequence defining a function. However, using the Padé approximant expression will produce a rational function of a random variable, which makes the analytical derivation of the moments of the response difficult. An alternative approach to tackle this problem is to work directly on the moments evaluated from the PC expansion; from the moments, it is possible to define sequences whose convergence may be accelerated. Several acceleration methods exist [29,31], such as Aitken's method, the recursive Aitken's method, and Shank's method [26]. The aim of this paper is to understand how these methods perform in the case of steady-state response of non-proportionally damped dynamic systems. A frequency-domain approach is adopted.

First the steady-state response of a random dynamical system with multiple random variables is obtained by a PCE; this is an extension to the case presented in [27], which involved a single random variable only. Then a dynamical system with a single random variable is studied to highlight the specific features due to the use of a PCE. In Section 3 Aitken's method is presented. Finally the methods developed in this paper are illustrated with two degrees of freedom uncertain linear dynamical system with one and two random variables. Both intrusive and non-intrusive methods have advantages and drawbacks. The non-intrusive approaches, such as black box methods, do not require any change in the standard structural mechanics codes, but they require a high number of function evaluations. The intrusive methods are closer to the initial problem, but require modification of standard codes. An intrusive method is used in this study to determine the PCE coefficients to address the modification of the initial problem.

2. Response of a random dynamical system

Consider an n degrees of freedom (dof) dynamical system described by its mass, damping and stiffness matrices, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. The forces acting on this system are described by force vector $\mathbf{F}(t)$, and $\mathbf{x}(t)$ denotes the response vector, which is a solution of

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{F}(t) \tag{1}$$

The stiffness matrix is assumed to be uncertain and given by

$$\mathbf{K}=\mathbf{K}(\Xi)=\mathbf{K}_0+\sum_{i=1}^{r}\xi_i\mathbf{K}_i \tag{2}$$

where $\Xi=(\xi_1,\ldots,\xi_r)$ and ξ_i is a zero-mean random variable. $\mathbf{K}_0$ is the deterministic part of the stiffness matrix and $\mathbf{K}_i$ $(i>0)$ is the stiffness matrix associated with the random variable ξ_i.

In this study the mass and damping matrices and the force vector are assumed to be deterministic. However, this is not a restriction and the following may be derived similarly with random mass and damping matrices, and a random force vector.

The solution of Eq. (1) is random and may be expanded in terms of the PC basis $\{\Psi_j(\Xi): j\in\mathbb{N}\}$ [10] as

$$\mathbf{x}(t,\Xi)=\sum_{j=0}^{\infty}\mathbf{Y}_j(t)\Psi_j(\Xi) \tag{3}$$

The elements of the PC basis are obtained from an orthogonal polynomial set $\{P_j(\xi): j \in \mathbb{N}\}$, where j is the order of $P_j(\xi)$. Thus

$$\Psi_J(\Xi) = \prod_{i=1}^{r} P_{J_i}(\xi_i) \tag{4}$$

where $\sum_{i=1}^{r} J_i$ is the order of Ψ_J, J may be either a multi-index $(J_1,\ldots,J_r)$ or a single index defined from the multi-index $(J_1,\ldots,J_r)$ through a mapping; the inner product to define orthogonality is

$$\langle i,j\rangle = \int \cdots \int \{\Psi_i(\Xi)\Psi_j(\Xi)p(\Xi)\}\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_r \tag{5}$$

where $p(\Xi)$ is the joint probability density function associated with the random variables.

In the following P_j is the Hermite polynomial, H_j, and ξ_i is a normally distributed random variable. However, the following results may also be obtained for uniform random variables and $P_j = L_j$, i.e. the Legendre polynomials.

For the numerical study, Eq. (3) has to be truncated to a finite number of terms, $P+1$, which is given by $(m+r)!/(m!r!)$ where m is the chaos order. Truncating the infinite expansion, gives the approximation of $\mathbf{x}(t,\Xi)$ as

$$\mathbf{x}^P(t,\Xi) = \sum_{j=0}^{P} \mathbf{Y}_j(t)\Psi_j(\Xi) \tag{6}$$

In the following, the exponent P is dropped for the sake of simplicity.

Substituting $\mathbf{x}$ from Eq. (6) into Eq. (1), gives

$$\sum_{j=0}^{P} \Psi_j(\Xi)\left(\mathbf{M}\ddot{\mathbf{Y}}_j + \mathbf{C}\dot{\mathbf{Y}}_j + \mathbf{K}\mathbf{Y}_j\right) = \mathbf{F} \tag{7}$$

The intrusive method is applied. So from Eq. (2) and the orthogonal properties of the polynomials, Eq. (7) becomes

$$\langle 0,i,i\rangle\left(\mathbf{M}\ddot{\mathbf{Y}}_i + \mathbf{C}\dot{\mathbf{Y}}_i + \mathbf{K}_0\mathbf{Y}_i\right) + \sum_{j=0}^{P}\sum_{k=1}^{r} \langle k,i,j\rangle \mathbf{K}_k\mathbf{Y}_j = \delta_{0i}\mathbf{F} \quad \text{for } i=0\ldots P \tag{8}$$

where δ_{ij} is the Kronecker delta and

$$\langle k,i,j\rangle = \int \cdots \int \{\xi_k\Psi_i(\Xi)\Psi_j(\Xi)p(\Xi)\}\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_r \tag{9}$$

with $\xi_0 = 1$.

Define

$$\mathbf{A}_k \in \mathbb{R}^{(P+1)\times(P+1)} \quad \text{with } [\mathbf{A}_k]_{ij} = \langle k,i,j\rangle \tag{10}$$

$$\tilde{\mathbf{M}} = \mathbf{A}_0 \otimes \mathbf{M} \in \mathbb{R}^{n(P+1)\times n(P+1)} \tag{11}$$

$$\tilde{\mathbf{C}} = \mathbf{A}_0 \otimes \mathbf{C} \in \mathbb{R}^{n(P+1)\times n(P+1)} \tag{12}$$

$$\tilde{\mathbf{K}} = \sum_{k=0}^{r} \mathbf{A}_k \otimes \mathbf{K}_k \in \mathbb{R}^{n(P+1)\times n(P+1)} \tag{13}$$

$$\mathbf{Y} = [\mathbf{Y}_0^\top \ \ \mathbf{Y}_1^\top \ \ \ldots \ \ \mathbf{Y}_P^\top]^\top \in \mathbb{R}^{n(P+1)} \tag{14}$$

$$\tilde{\mathbf{F}} = [\mathbf{F}^\top \ \ \mathbf{0} \ \ \mathbf{0} \ \ \ldots \ \ \mathbf{0}]^\top \in \mathbb{R}^{n(P+1)} \tag{15}$$

where $\otimes$ denotes the Kronecker product, $[\bullet]^\top$ is the transpose of $[\bullet]$, and $[\bullet]_{ij}$ is the (i,j)th element of $[\bullet]$. Accordingly the components of the PC expansion satisfy

$$\tilde{\mathbf{M}}\ddot{\mathbf{Y}} + \tilde{\mathbf{C}}\dot{\mathbf{Y}} + \tilde{\mathbf{K}}\mathbf{Y} = \tilde{\mathbf{F}} \tag{16}$$

Hence, the PC-components are the solution of an $n(P+1)$ dof dynamical system that will be referred to as the PC-system. Thus the PCE has transformed the study of an uncertain dynamical system into the study of a deterministic dynamical system of larger size. The PC-system has resonant frequencies that will be referred to as PC-resonances. As a consequence the moments of the steady-state response to a harmonic force derived through PCE show peaks related to these PC-resonances.

The existence of PC-resonances has already been demonstrated for a dynamical system with a single random variable in [27] and two degrees of freedom. In Section 4, an example with multiple random variables will be given to illustrate the issue related to the PC-resonances and to give an approach to obtain a solution.

In the following the force vector is assumed to be harmonic, i.e. $\mathbf{F}(t) = \mathbf{F_0}e^{\mathrm{i}\omega t}$, and the steady-state response of the dynamical system is then $\mathbf{x}(t) = \mathbf{X}\mathrm{e}^{\mathrm{i}\omega t}$, where $\mathrm{i} = \sqrt{-1}$. $\mathbf{X}$ is the solution of

$$(-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K})\mathbf{X} = \mathbf{F_0}. \tag{17}$$

As $\mathbf{K}$ is a random matrix, $\mathbf{X}$ is a random vector, which can be described by its polynomial chaos expansion to estimate the first two moments of $\mathbf{X}$ as

$$\mathbf{X}(\omega,\xi)=\sum_{i=0}^{P}\mathbf{Y}_i(\omega)\Psi_i(\xi) \tag{18}$$

Then the components of the PC expansion satisfy

$$\left(-\omega^2\tilde{\mathbf{M}}+\mathrm{i}\omega\tilde{\mathbf{C}}+\tilde{\mathbf{K}}\right)\mathbf{Y}(\omega)=\tilde{\mathbf{F}_0} \tag{19}$$

Jacquelin et al. [27] and Sinou and Jacquelin [28] showed that hundreds of PC coefficients may be required to obtain an accurate response prediction for a single random variable: this highlighted the slow convergence of a PCE with Hermite polynomials. Consequently, it is of great interest to accelerate the convergence of a PCE.

3. Convergence acceleration of a PCE: Aitken's transformation and its generalization

The objective is to estimate the first two moments of the response from several predicted responses evaluated with a relatively low PC order. Suppose that $\mathbf{X}_n(\omega,\xi)$ denotes the approximation of $\mathbf{X}(\omega,\xi)$ with a PCE of order n, and $\mathcal{M}_{(i,n)}(\omega)$ is the corresponding ith moment, for $i=1$ or 2. The PCE may be used in several ways to calculate the first two moments of the response. First, a Monte-Carlo simulation (MCS) can be applied to Eq. (18) to estimate the probability density function and then to evaluate the statistics of the response. Such an MCS is much faster than an MCS applied directly to the dynamical equation (17). Second, the PC coefficients can provide a direct estimation of the first two moments of the response (see [10] or Eqs. (27) and (28)). Hence, for a given i, $\mathcal{M}_{(i,n)}(\omega)$ defines a sequence whose convergence can be accelerated.

Several acceleration methods exist, such as Aitken's method, the recursive Aitken's method, and Shank's method (which is a generalization of Aitken's method) [29]. These methods are all closely related to the Padé approximants.

As the methods are suitable for every moment, for the sake of simplicity $\mathcal{M}_{(i,n)}(\omega)$ will be denoted M_n. The moments of an FRF obtained with a PCE are a slowly convergent sequence especially around the resonance frequencies. Hence the objective is to accelerate the moment convergence.

3.1. Aitken's transformation

Aitken's transformation [31] nonlinearly transforms a sequence $\{M_n\}$ into a sequence $\{Z_n\}$ given by

$$Z_n=\frac{M_nM_{n+2}-(M_{n+1})^2}{M_{n+2}-2M_{n+1}+M_n} \tag{20}$$

The new sequence $\{Z_n\}$ converges toward the same limit as $\{M_n\}$ but more quickly. This method requires the determination of $\mathbf{X}_{n+1}(\omega,\xi)$ and $\mathbf{X}_{n+2}(\omega,\xi)$ and their moments, in addition to $\mathbf{X}_n(\omega,\xi)$ and its moments.

3.2. Recursive Aitken's transformation

This transformation applies Aitken's transformation several times. Hence, the sequence $\{Z_n\}$ is transformed into a new sequence $\{{}_2Z_n\}$ given by

$${}_2Z_n=\frac{Z_nZ_{n+2}-(Z_{n+1})^2}{Z_{n+2}-2Z_{n+1}+Z_n} \tag{21}$$

where $\{{}_2Z_n\}$ converges quicker than $\{Z_n\}$, which may be denoted $\{{}_1Z_n\}$.

More generally, Aitken's transformation may be applied $k+1$ times ($k\geq 1$) to obtain the sequence $\{{}_{(k+1)}Z_n\}$ as

$${}_{(k+1)}Z_n=\frac{{}_kZ_n{}_kZ_{n+2}-({}_kZ_{n+1})^2}{{}_kZ_{n+2}-2{}_kZ_{n+1}+{}_kZ_n} \tag{22}$$

To determine ${}_{(k+1)}Z_n$ requires knowledge of $\mathbf{X}_i(\omega,\xi)$ and M_i for $i=n,\ldots,(n+2(k+1))$.

3.3. Shank's transformation

Shank's transformation is an extension to Aitken's transformation, and involves more elements of the initial sequence than Aitken's transformation. Thus the kth Shank's nonlinear transformation, e_k, is

$$e_k(M_n)=\frac{\begin{vmatrix} M_n & M_{n+1} & \cdots & M_{n+k} \\ M_{n+1} & M_{n+2} & \cdots & M_{n+k+1} \\ \vdots & \vdots & \cdots & \vdots \\ M_{n+k} & M_{n+k+1} & \cdots & M_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta^2 M_n & \cdots & \Delta^2 M_{n+k-1} \\ \vdots & \cdots & \vdots \\ \Delta^2 M_{n+k-1} & \cdots & \Delta^2 M_{n+2k-2} \end{vmatrix}} \tag{23}$$

where $\Delta^2 M_n = M_n - 2M_{n+1} + M_{n+2}$. Note that when $k=1$, Shanks' transformation reduces to Aitken's transformation. The knowledge of M_i for $i=n,\ldots,(n+2k)$ is required to determine $e_k(M_n)$.

4. Application to two degrees of freedom system

MCS and PCE will be used to evaluate the mean and the standard deviation of the response, **X**, for the discrete spring, mass and damper system shown in Fig. 1 [27]. In this example, the spring stiffnesses are equal but uncertain, i.e. $k_1 = k_2 = k$, and hence a single uncertain parameter is considered. The stiffness k is assumed to be random and given by

$$k = \overline{k}(1+\delta_K \xi) \tag{24}$$

Thus, stiffness matrix, **K**, mean stiffness matrix, $\overline{\mathbf{K}}$, mass matrix, **M** and damping matrix, **C**, are

$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}, \quad \overline{\mathbf{K}} = \overline{k}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$
$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = c\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{25}$$

The random stiffness matrix is $\mathbf{K}_1 = \delta_K \overline{\mathbf{K}}$. Tables 1 and 2 list the characteristics of the system. The response was calculated for 501 frequencies in the range of 10–35 Hz ($\Delta f = 0.05$ Hz).

The components of the PC expansion satisfy Eq. (19), given in this case by

$$\left(-\omega^2\tilde{\mathbf{M}} + \mathrm{i}\omega\tilde{\mathbf{C}} + \tilde{\mathbf{K}}_0 + \tilde{\mathbf{K}}_1\right)\mathbf{Y}(\omega) = \tilde{\mathbf{F}}_0 \tag{26}$$

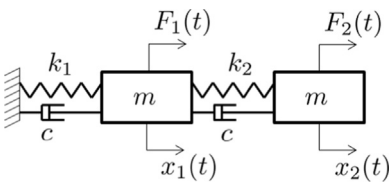

Fig. 1. A two degrees of freedom system with stochastic stiffness coefficients.

Table 1
The two degrees of freedom system characteristics.

$\overline{k}$ (N m^{-1})	m (kg)	c (N m^{-1} s^{-1})	δ_K (%)	F_{01} (N)	F_{02} (N)
15,000	1	1	5	1	0

Table 2
Modal characteristics of the deterministic system.

Eigenfrequencies f (Hz)	12.05	31.54
Damping ratio ξ (%)	0.25	0.66

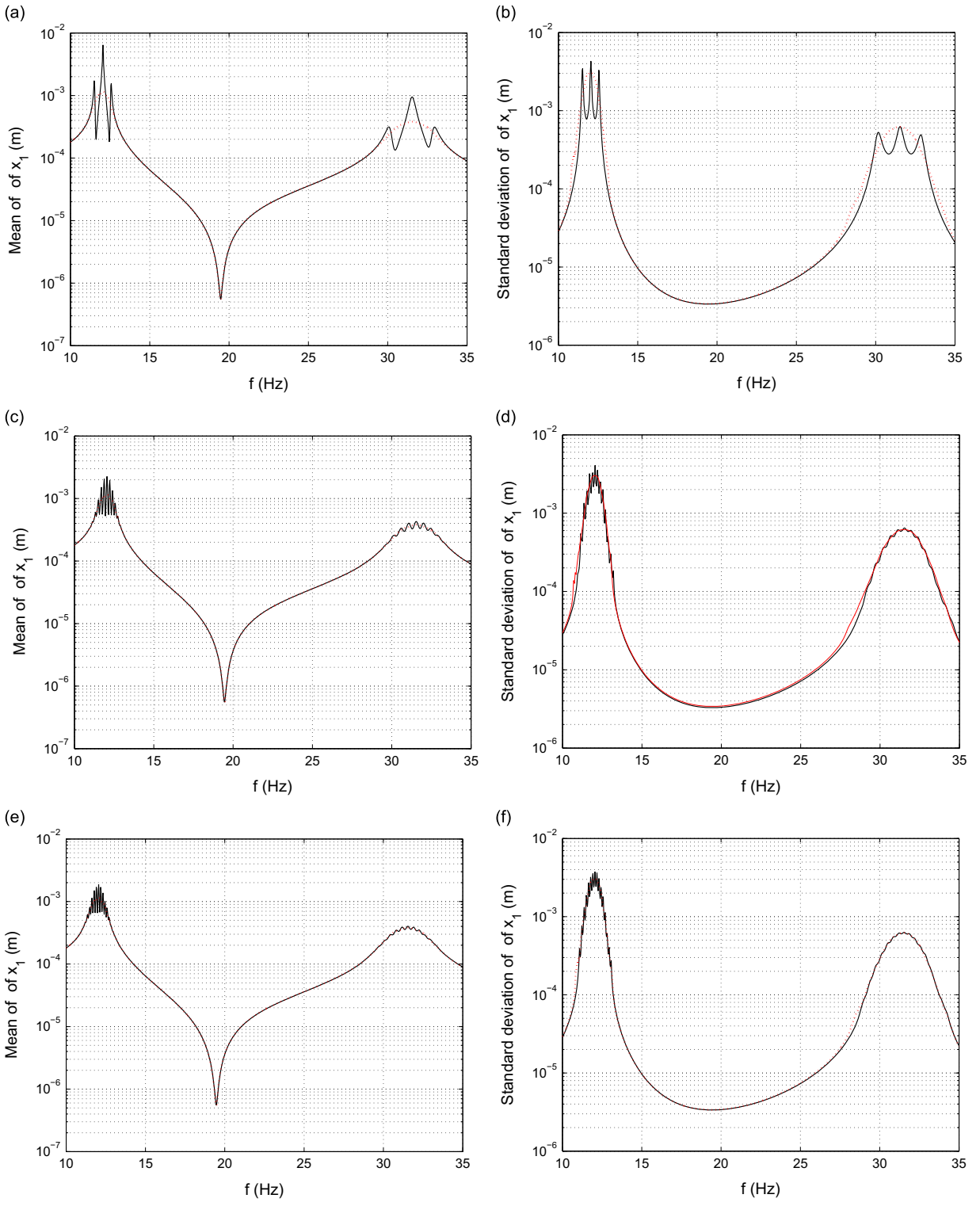

Fig. 2. The mean and standard deviation of the response of x_1 calculated by PCE (black solid lines) versus MCS (red dotted lines). (a) Mean, $P=2$, (b) standard deviation, $P=2$, (c) mean, $P=30$, (d) standard deviation, $P=30$, (e) mean, $P=50$, (f) standard deviation, $P=50$. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

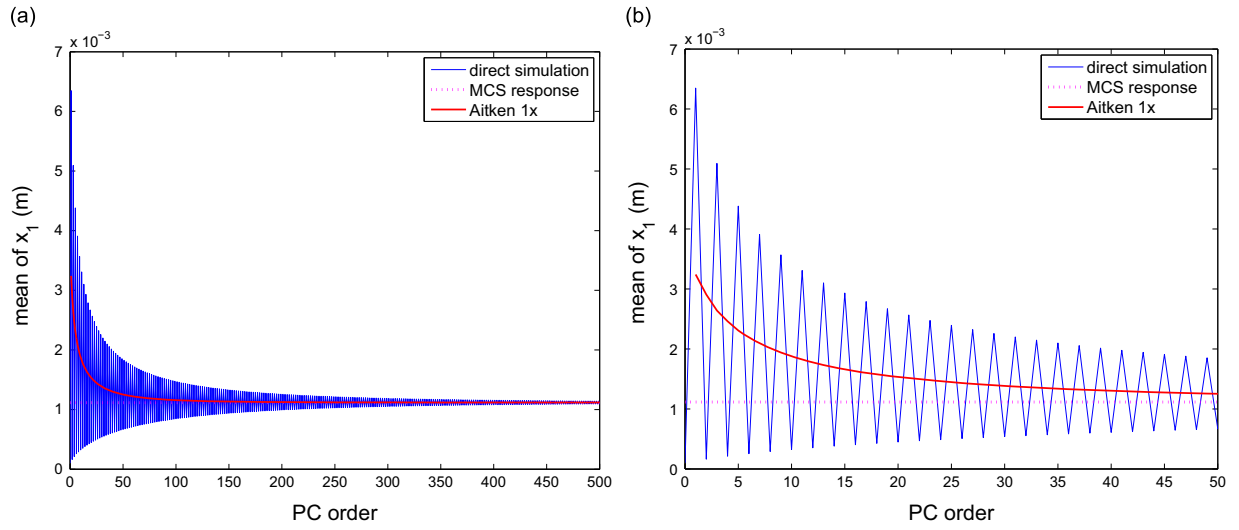

Fig. 3. The mean of x_1 at 12.05 Hz predicted by PCE (thin blue solid line), MCS (magenta dashed line) and Aiken's method (red line). For a given abscissa, the ordinate provides the maximum PC order required to derive the response. (a) PC order up to 500, (b) PC order up to 50. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

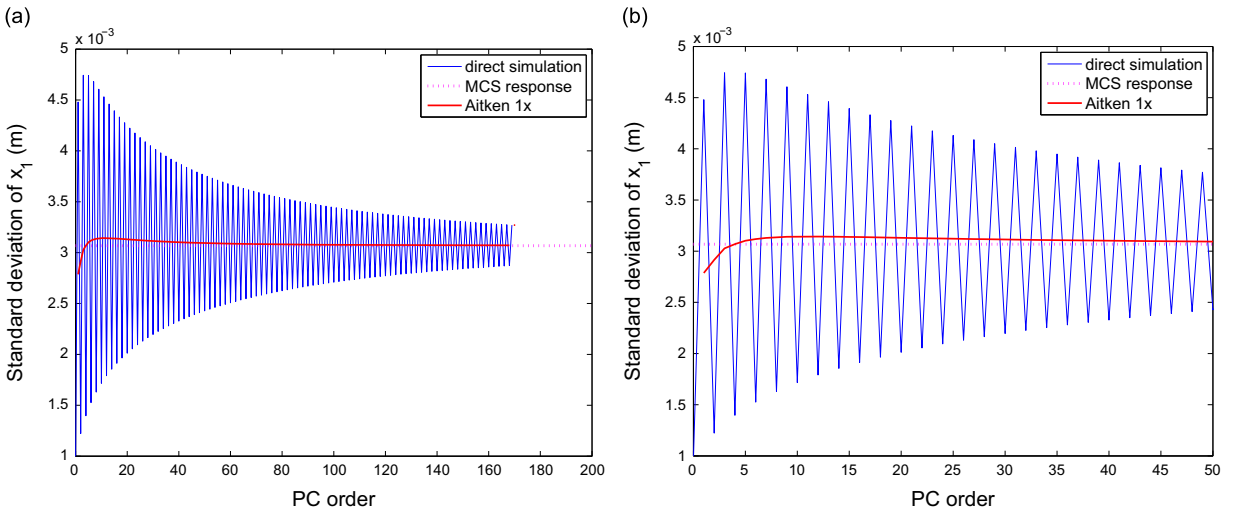

Fig. 4. The standard deviation of x_1 at 12.05 Hz predicted by PCE (thin blue solid line), MCS (magenta dashed line) and Aiken's method (red line). For a given abscissa, the ordinate provides the maximum PC order required to derive the response. (a) PC order up to 500, (b) PC order up to 50. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

The MCS results were obtained with 10,000 samples of the random variable ξ. These results are very close to those obtained with 4000 samples, and so we can conclude that convergence is achieved. These results will be the reference results. Fig. 2 shows the first two moments of x_1 (mean, $\overline{x}_1$, and standard deviation, σ_1) calculated from the PC components (see the red dotted line)

$$\overline{x}_1 = \{\mathbf{Y}_0\}_1 \tag{27}$$

$$\sigma_1^2 = \sum_{j=1}^{P} j!\left(\{\mathbf{Y}_j\}_1\right)^2 \tag{28}$$

where $\{\mathbf{Y}_j\}_1$ denotes the first element of $\mathbf{Y}_j$.

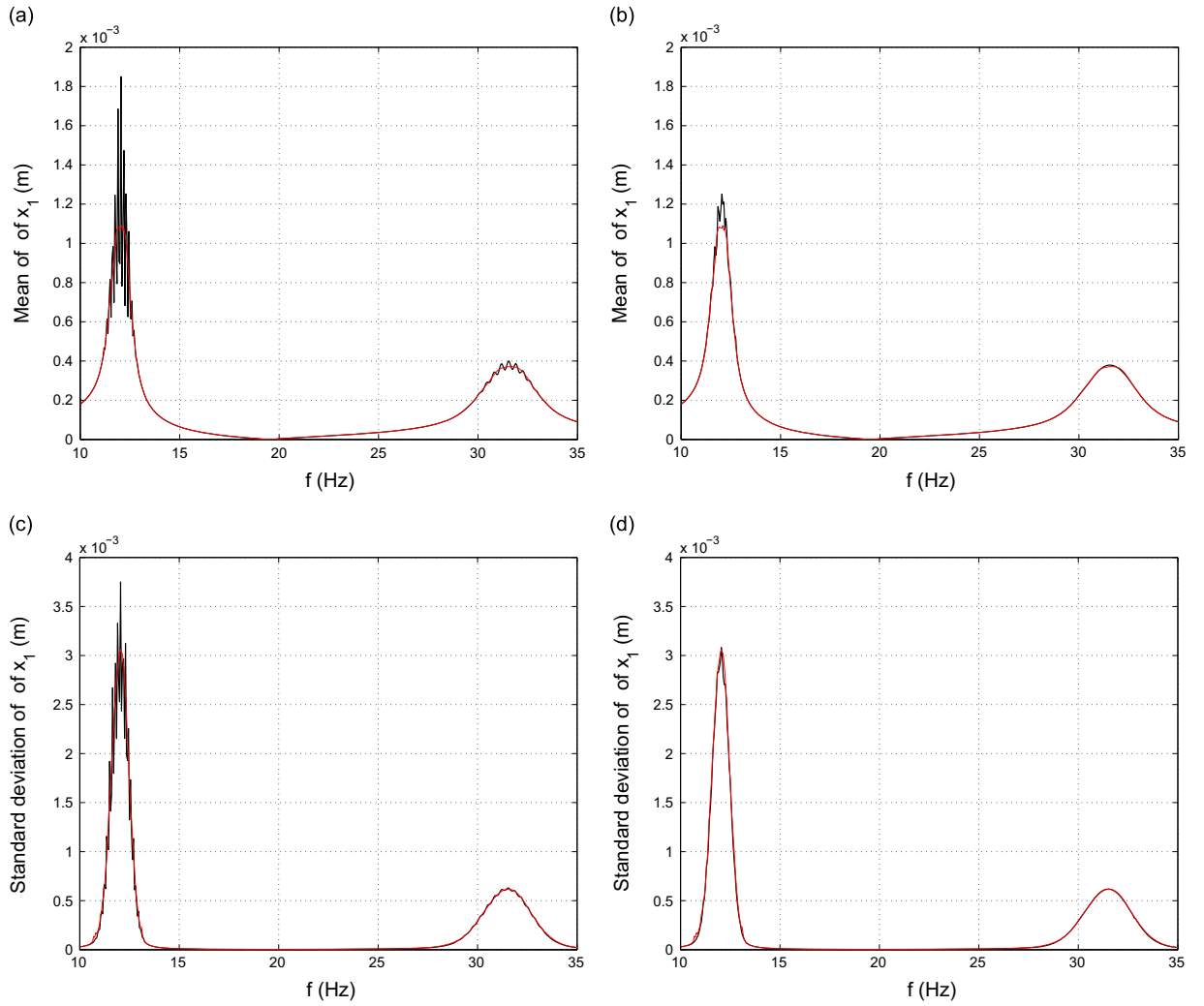

Fig. 5. Statistics of x_1 for MCS (red line), PCE with $P=50$ and Aitken's transformation of the PCE. (a) Mean, PCE (black line), (b) mean, Aitken's method (black line), (c) standard deviation, PCE (black line), (d) standard deviation, Aitken's method (black line). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

4.1. PCE results

The PC expansion is calculated for three PC orders: 2, 30, and 50. The results are plotted in Fig. 2, and they are in a perfect agreement with the MCS results, except around the resonances. Around the resonances some spurious oscillations arise, the curves oscillate more as P increases, and the oscillations are located around the deterministic eigenfrequencies given in Table 2. This illustrates the theoretical development in Section 2; more details on these spurious resonances can be found in [27]. At first sight, a fast convergence occurs even around the deterministic eigenfrequencies, and the oscillations seem to vanish for $P=50$. However, this is due to the logarithmic scale. Fig. 3 shows the mean response predicted at 12.05 Hz (the first natural frequency of the deterministic system) on a linear scale for a PC order range from 1 to 500 (thin black solid line). The response evaluated with MCS is also plotted in Fig. 3 (magenta dashed line); the PC expansion overestimates the response by 472 percent, 106 percent and 63 percent for $P=2$, $P=30$ and $P=50$ respectively, compared to the MCS response. Thus even for a relatively high PC order, the results are not very accurate, especially if the objective is to use such a response to design a structure. For example, a PC order of about 200 is required to have a discrepancy of 10 percent. The same results can be observed at the second resonant frequency, although the convergence is faster due to the higher damping ratio of the second eigenmode.

A similar study was performed on the standard deviation of x_1 and led to similar results (see Fig. 4). Beyond $P=170$ it is not possible to calculate the standard deviation due to numerical problems with factorial number evaluation. The discrepancy between the direct PC response and the MCS response is approximately 6.5 percent for $P=170$. This example

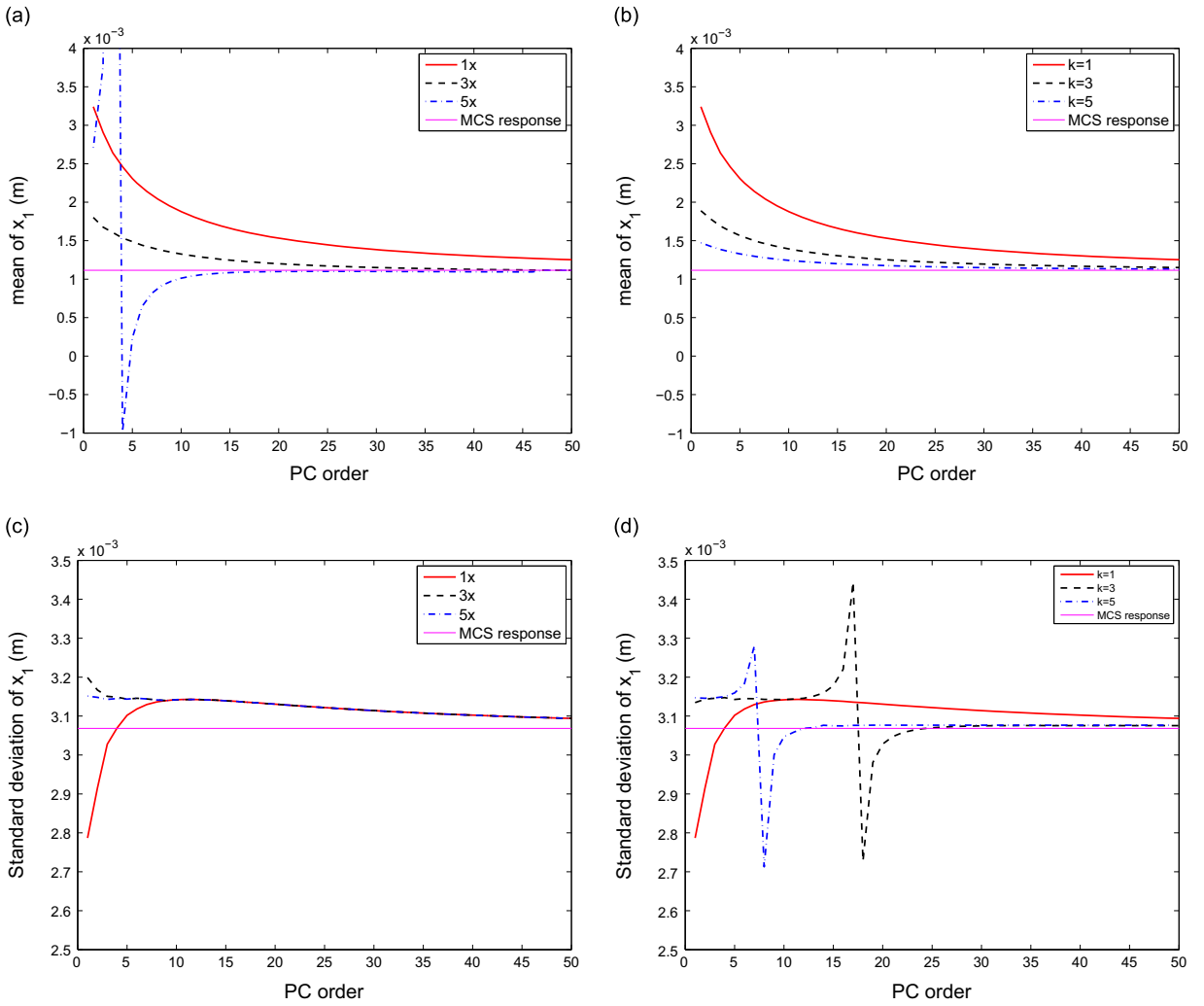

Fig. 6. The mean and standard deviation of x_1 using the recursive Aitken method and Shank's method. The MCS results are given by the magenta line. For a given abscissa, the ordinate provides the maximum PC order required to derive the response. (a) Mean, Aitken's method, (b) mean, Shank's method, (c) standard deviation, Aitken's method, (d) standard deviation, Shank's method. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

shows that the PC expansion is not efficient around the deterministic eigenfrequencies, and very high orders of PCE must be used to obtain consistent results.

4.2. Convergence acceleration

Aitken's method was applied to accelerate convergence of the mean responses. Z_P requires knowledge of response at PC orders of P, $P+1$ and $P+2$. To assess the efficiency of this method, Z_P is also plotted in Fig. 3 for $P=2$ to $P=500$ (thick red solid line); Z_P is an estimate of the mean response of x_1 excited at the first eigenfrequency. Clearly Z_P converges much faster than the PCE sequence to the MCS response. For example Z_{50} is as good as the PCE response for $P=200$; the discrepancy with MCS is approximately 10 percent. Fig. 3 shows that the convergence is almost achieved for Z_{200} as the discrepancy with the MCS response is lower than 1 percent.

This feature is also observed for the standard deviation. Fig. 4(b) demonstrates the efficiency of Aitken's algorithm as Z_{20} estimates the standard deviation of the MCS response within 2 percent. The convergence is much faster for the standard deviation than for the mean.

To highlight the efficiency of Aitken's transformation, the mean and standard deviation of x_1 are plotted in Fig. 5 on a linear scale for a range of frequencies instead of the usual logarithmic scale. Fig. 5a and c shows exactly the same quantities as Fig. 2e and f: in particular the PC order is the same ($P=50$); the discrepancies between PCE and MCS are highlighted with the linear scale. Fig. 5 shows that the MCS is very well estimated using Aitken's transformation for both the mean and the

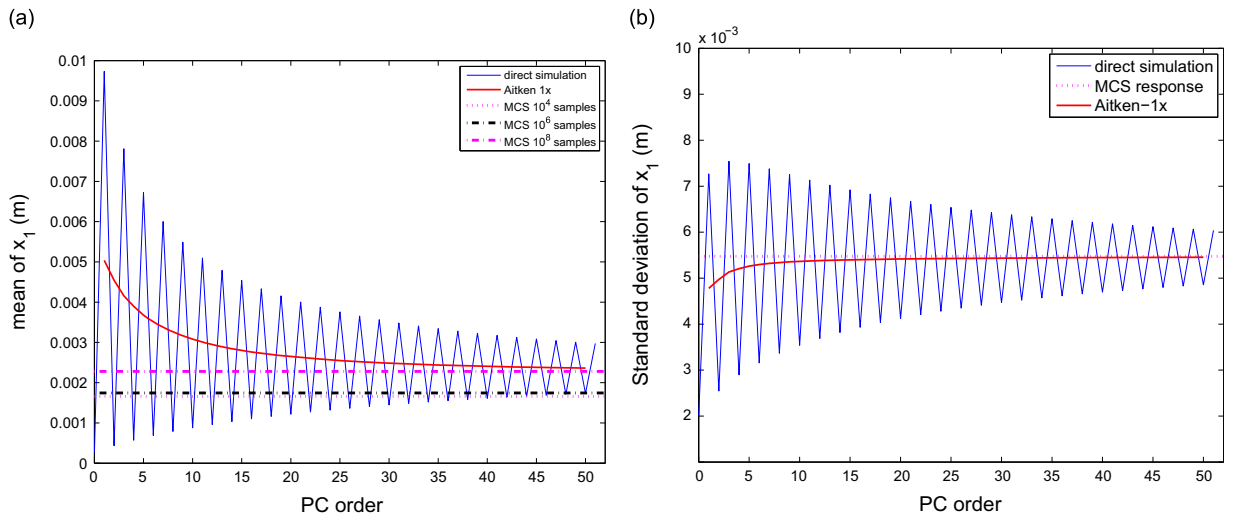

Fig. 7. Statistics of x_1 for MCS, the PCE and Aitken's method at 12.05 Hz for two dof system with two uncertain stiffnesses. For a given abscissa, the ordinate provides the maximum PC order required to derive the response. (a) Mean, (b) standard deviation.

standard deviation, although the mean is slightly overestimated just around the deterministic eigenfrequencies. Note that a PC order of 50 was used, and hence Aitken's transformation used PC results for orders from 48 to 50.

The recursive Aitken transformation was also applied. The results for the mean of x_1 are given in Fig. 6a, where the transformation was applied one, three and five times. The recursive approach significantly improves the results, and the discrepancy between the MCS result and ${}_5Z_n$ is lower than 1.5 percent for $n=20$. The same conclusion applies to Shank's transformation (see Fig. 6b). Shank's transformation appears to be less efficient than the recursive Aitken transformation as the discrepancy between the MCS result and $e_5(M_n)$ is approximately 5.4 percent for $n=20$. Similar results were obtained for the standard deviation (Fig. 6(c) and (d)) except for that, in this case, Shank's transformation is more efficient than the recursive Aitken transformation. The discrepancy between ${}_5Z_n$ and the MCS response is approximately 2 percent whereas the discrepancy between the MCS result and $e_5(M_n)$ is approximately 0.3 percent for $n=20$. Fig. 6(a) and (d) shows a non-smooth behavior. In fact this is a consequence of the very good convergence acceleration rate. The non-smooth behavior occurs when consecutive terms in the accelerated sequences are very close, and then Eqs. (22) and (23) have a very small denominator that leads to the non-smooth behavior.

4.3. Multiple random parameters

It is important to demonstrate the behavior of systems with multiple random parameters is similar to the one described above. The results for a system with two uncertain parameters are presented. The two degrees of freedom system of Fig. 1 are still considered, but the stiffnesses k_1 and k_2 are now assumed to be random and independent. Hence

$$k_1 = \overline{k}(1+\delta_K \xi_1) \tag{29}$$

$$k_2 = \overline{k}(1+\delta_K \xi_2) \tag{30}$$

where ξ_1 and ξ_2 are two independent normally distributed random variables.

Thus, according to Eq. (2), the mean stiffness matrix, $\mathbf{K}_0$, and the random stiffness matrices, $\mathbf{K}_1$ and $\mathbf{K}_2$, are

$$\mathbf{K}_0 = \overline{k}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{K}_1 = \delta_K \overline{k}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \delta_K \overline{k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{31}$$

Similar to Figs. 3 and 4, Fig. 7 gives the mean and the standard deviation of the response. The MCS required 10^8 runs to converge; Fig. 7(a) also shows MCS results with 10^4 and 10^6 samples. Latin Hypercube Sampling was also tested with 10^4 samples, and the results was the same as the converged MCS results but required much less computational cost. Again, Aitken's method proved to be very efficient and to quickly reach the converged MCS value. In fact all the conclusions made for the convergence acceleration methods for a single random variable, can be made for multiple random variables.

5. Conclusion

The PCE is a very efficient way to estimate the dynamic response of a system, except around the deterministic eigenfrequencies, where a very high PC order is required. Numerical issues arise in the calculation of the PCE coefficients for a high PC order, where factorial numbers exceed the computer precision and the systems involve a high number of

equations. These issues have been solved in [27,28] but the PCE is still not very efficient around the deterministic eigenfrequencies, where precision is often required.

Aitken's transformation and its generalization were successfully applied to the sequences defined by the first two moments of the responses, and this process significantly accelerates their convergence. As a consequence, the PCE may be considered as an efficient method to estimate the first two moments of a random dynamic response when associated with a convergence accelerator. This paper focuses mainly on a single random variable, but the same procedure can also be applied to multiple random variables with successive PC-orders, as demonstrated with the final example with two random parameters.

The mean and the standard deviation may be efficiently derived through an intrusive PCE. However, these first two moments are not sufficient to provide all the statistics of the response distribution; the probability density function must be derived. Ongoing research will determine the probability density function of the response.

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