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Stochastic free vibration analysis of angle-ply composite plates – A RS-HDMR approach

Sudip Dey *, Tanmoy Mukhopadhyay, Sondipon Adhikari

College of Engineering, Swansea University, Swansea SA2 8PP, United Kingdom

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ABSTRACT

This paper presents a generic random sampling-high dimensional model representations (RS-HDMR) approach for free vibration analysis of angle-ply composite plates. A metamodel is developed to express stochastic natural frequencies of the system. A global sensitivity analysis is carried out to address the influence of input random parameters on output natural frequencies. Three different types of input variables (fiber-orientation angle, elastic modulus and mass density) are varied to validate the proposed algorithm. The present approach is efficiently employed to reduce the sampling effort and computational cost when large number of input parameters is involved. The stochastic finite element approach is coupled with rotary inertia and transverse shear deformation based on Mindlin's theory. Statistical analysis is carried out to illustrate the features of the RS-HDMR and to compare its performance with full-scale Monte Carlo simulation results. The stochastic mode shapes are also depicted for a typical laminate configuration. Based on the numerical results, some new physical insights are drawn on the dynamic behavior of the system.

1. Introduction

Composite materials are extensively used for small components as well as large structures such as airplanes, ships, wind turbine blades and small bridges due to its weight sensitivity, cost-effective, high specific stiffness and can handle different strength at different directions. There has been an increasing demand for composite materials especially in aircraft industries. Because of its inherent complexity, laminated composite structures can be difficult to manufacture accurately according to its exact design specifications, resulting in undesirable uncertainties. The random structural uncertainties involve material properties, fiber parameters of the individual constituent laminae. Because of the randomness in geometry and material properties of laminated composite structures, the mass and stiffness matrices become stochastic in nature. The composite materials are generally made of fiber and matrix through an appropriate fabrication process in manufacturing with three basic steps namely tape, layup and curing. Since the mechanical properties of constituent material vary statistically, the source of uncertainties in composite material properties are produced from randomness in material properties as well as from uncertainties in the fabrication parameters. The uncertainties incurred during the layup process are due to the misalignment of ply-orientation. Typical uncertainties incurred from the curing process are intralaminate voids, incomplete curing of resin, excess resin between plies, excess matrix voids and porosity and variation in ply thickness. These variables are statistical in nature; therefore, the properties of composite materials should be quantified probabilistically. As a consequence, the behavior of composite structures shows a scatter from its average value. Traditionally, an ad hoc factor of safety is used in the design to account for the difficulty in predicting the structural behavior. However, this approach of designer may result in either an ultraconservative or an unsafe design.

In order to probabilistically assess the behavior of composite structures, these uncertainties are considered due to random variation of material properties and ply parameters. These uncertainties are referred to as the random variables which can be processed computationally through composite mechanics and probability method. Due to random input parameters, the uncertainties in the specification of mass and stiffness matrices induce the statistical variation in eigenvalues and eigenvectors, resulting in subsequently fluctuation of natural frequencies. Therefore a realistic analysis of composite laminated plates is required to arrest the volatility of natural frequencies arising from the randomness in the variation of parameters like ply-orientation angle, elastic modulus and mass density. In general, uncertainties can be

* Corresponding author.
E-mail addresses: infosudip@gmail.com, S.Dey@swansea.ac.uk (S. Dey).

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categorized into three types, namely aleatoric, epistemic and prejudicial, respectively. The first type of uncertainty is aleatoric uncertainty which is due to the inherent variability in the system parameters, for example, different cars manufactured from a single production line are not exactly the same. If enough samples are present, it is possible to characterize the variability using well established statistical methods and the probably density functions of the parameters can be obtained. The second type of uncertainty is epistemic uncertainty which is due to lack of knowledge regarding a system. This type of uncertainty is generally arisen in the modeling of complex systems, for example the problem of predicting cabin noise in helicopters. Unlike aleatoric uncertainties, it is recognized that probabilistic models are not quite suitable for epistemic uncertainties. The third type of uncertainty is prejudicial uncertainty which is similar to the first type except that the corresponding variability characterization is not available, in which case work can be directed to gain better knowledge. An example of this type of uncertainty is the use of viscous damping model in spite of knowing that the true damping model is not viscous. The total uncertainty of a system is the combination of these three types of uncertainties.

In general, Monte Carlo simulation technique is popularly utilized to generate the randomized output frequency to deal with large number of samples. Although the uncertainty in material and geometric properties can be computed by the MCS method, it is inefficient and expensive. To mitigate this lacuna, random sampling – high-dimensional model representations (RS-HDMR) is employed for quantitative model assessment and analysis tool which maps high-dimensional input–output system relationship very efficiently [1]. In this problem the actual finite element model is replaced by a response surface metamodel, making the process computationally efficient and cost effective. When input variables are randomly sampled for constructing the metamodel, a random sampling – HDMR can be constructed [2,3]. Over the years, HDMR is successfully applied in many different fields [4–6]. The effect of material uncertainty on vibration control of smart composite plate is studied by Umesh and Ganguli [7] by using polynomial chaos expansion. The dynamic stability of uncertain laminated beams subjected to subtangential loads studied by Goyal and Kapania [8] while Manan and Cooper [9] introduced probabilistic approach for design of composite wings including uncertainties. On the other hand, Fang and Springer [10] studied on the design of composite laminates by a Monte Carlo method while Kapania and Goyal [11] investigated on free vibration of unsymmetrically laminated beams. The stochastic eigenvalue problem is studied with polynomial chaos [12] and Karhunen–Loeve expansion of non-gaussian random process [13]. Of late, Chowdhury and Adhikari [14] investigated on high dimensional model representation for stochastic finite element analysis while Talha and Singh [15] investigated on stochastic perturbation-based finite element for buckling statistics of functionally graded plates with uncertain material properties in thermal environments.

In the present analysis, random samples are drawn uniformly over the entire domain ensuring good prediction capability of the constructed metamodel in the whole design space including the tail regions. The present work aims to develop an algorithm for uncertainty quantification of natural frequencies of cantilever composite plate using RS-HDMR and its comparative efficacy compared to direct Monte Carlo simulation (MCS) technique. The sensitivity analysis is carried out to cross-validate the results for constructing the metamodel and subsequently the number of sample runs required to fit the constructed metamodel is catastrophically reduced. An eight noded isoparametric plate bending element with five degrees of freedom at each node is considered in finite element formulation to study the randomized natural frequencies (see Fig. 1).

2. Governing equations

The laminated composite cantilever plate is considered with uniform thickness with the principal material axes of each layer being arbitrarily oriented with respect to mid-plane. If the mid-plane forms the x–y plane of the reference plane, then the displacements can be computed as

$$\begin{aligned} u(x,y,z) &= u^0(x,y) - z\theta_x(x,y) \\ v(x,y,z) &= v^0(x,y) - z\theta_y(x,y) \\ w(x,y,z) &= w^0(x,y) = w(x,y), \end{aligned} \tag{1}$$

where u, v and w are the displacement components in x-, y- and z-directions, respectively and u^0, v^0 and w^0 are the mid-plane displacements, and θ_x and θ_y are rotations of cross-sections along the x- and y-axes. In general, the force and moment resultants of a single lamina are obtained from stresses as

$$\begin{aligned} \{F\} &= \{N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\ Q_x\ Q_y\}^T \\ &= \int_{-t/2}^{t/2} \{\sigma_x\ \sigma_y\ \tau_{xy}\ \sigma_x z\ \sigma_y z\ \tau_{xy} z\ \tau_{xz}\ \tau_{yz}\}^T dz \end{aligned} \tag{2}$$

In matrix form, the in-plane stress resultant {N}, the moment resultant {M}, and the transverse shear resultants {Q} can be expressed as

$$\{N\} = [A]\{\varepsilon^0\} + [B]\{k\} \quad \text{and} \quad \{M\} = [B]\{\varepsilon^0\} + [D]\{k\} \tag{3}$$

$$\{Q\} = [A^*]\{\gamma\} \tag{4}$$

where

$$\left[A^*_{ij}\right] = \int_{-t/2}^{t/2} \overline{Q}_{ij} dz \quad \text{where} \quad i,j = 4,5$$

$$[\overline{Q}_{ij}(\varpi)] = \begin{bmatrix} m^4 & n^4 & 2m^2n^2 & 4m^2n^2 \\ n^4 & m^4 & 2m^2n^2 & 4m^2n^2 \\ m^2n^2 & m^2n^2 & (m^4+n^4) & -4m^2n^2 \\ m^2n^2 & m^2n^2 & -2m^2n^2 & (m^2-n^2) \\ m^3n & mn^3 & (mn^3-m^3n) & 2(mn^3-m^3n) \\ mn^3 & m^3n & (m^3n-mn^3) & 2(m^3n-mn^3) \end{bmatrix} [Q_{ij}]$$

Here $m = Sin\theta(\varpi)$ and $n = Cos\theta(\varpi)$, wherein $\theta(\varpi)$ is the random fiber orientation angle. However, laminate consists of a number of laminae wherein $[Q_{ij}]$ and $[\overline{Q}_{ij}(\varpi)]$ denotes the On-axis elastic constant matrix and the off-axis elastic constant matrix, respectively. The elasticity matrix of the laminated composite plate is given by,

$$[D'(\varpi)] = \begin{bmatrix} A_{ij}(\varpi) & B_{ij}(\varpi) & 0 \\ B_{ij}(\varpi) & D_{ij}(\varpi) & 0 \\ 0 & 0 & S_{ij}(\varpi) \end{bmatrix} \tag{5}$$

where

$$[A_{ij}(\varpi), B_{ij}(\varpi), D_{ij}(\varpi)] = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [\overline{Q}_{ij}(\varpi)][1, z, z^2] dz \quad i,j = 1,2,6 \tag{6}$$

$$[S_{ij}(\varpi)] = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \alpha_s [\overline{Q}_{ij}(\varpi)]_k dz \quad i,j = 4,5 \tag{7}$$

where α_s is the shear correction factor and is assumed as 5/6. The mass per unit area is expressed as

$$P(\varpi) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho(\varpi) dz \tag{8}$$

Mass matrix is expressed as

$$[M(\varpi)] = \int_{Vol} [N][P(\varpi)][N] d(vol) \tag{9}$$

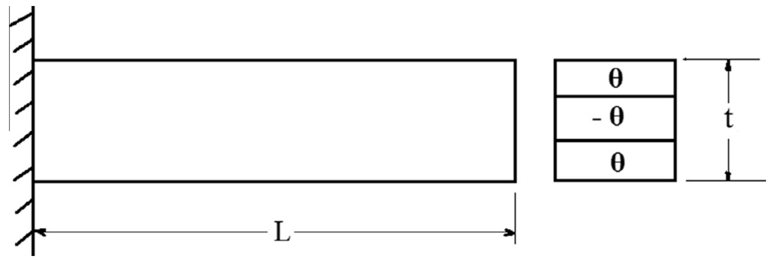

Fig. 1. Laminated composite cantilever plate.

The stiffness matrix is given by

$$[K(\varpi)] = \int_{-1}^{1}\int_{-1}^{1}[B(\varpi)]^T[D(\varpi)][B(\varpi)]d\xi d\eta \tag{10}$$

The Hamilton's principle is employed to study the dynamic nature of the composite structure. The principle used for the Lagrangian which is defined as

$$L_f = T - U - W \tag{11}$$

Where T, U and W are total kinetic energy, total strain energy and total potential of the applied load, respectively. The Hamilton's principle applicable to non-conservative system is expressed as,

$$\delta H = \int_{p_i}^{p_f}[\delta T - \delta U - \delta W]dp = 0 \tag{12}$$

Hamilton's principle [16] applied to dynamic analysis of elastic bodies states that among all admissible displacements which satisfy the specific boundary conditions, the actual solution makes the functional ($\int T + W)dp$ stationary, where T and W are the kinetic energy and the work done by conservative and non-conservative forces, respectively. For free vibration analysis (i.e., $\delta W = 0$), the stationary value is actually a minimum. In case of a dynamic problem without damping the conservative forces are the elastic forces developed within a deformed body and the non-conservative forces are the external force functions. The energy functional for Hamilton's principle is the Lagrangian (L_f) which includes kinetic energy (T) in addition to potential strain energy (U) of an elastic body. The expression for kinetic energy of an element is expressed as

$$T = \frac{1}{2}\{\dot{\delta}_e\}^T[M_e(\varpi)]\{\dot{\delta}_e\} \tag{13}$$

The potential strain energy for an element of a plate can be expressed as,

$$U = U_1 + U_2 = \frac{1}{2}\{\delta_e\}^T[K_e(\varpi)]\{\delta_e\} + \frac{1}{2}\{\delta_e\}^T[K_{\sigma e}(\varpi)]\{\delta_e\} \tag{14}$$

The Lagrange's equation of motion is given by

$$\frac{d}{dt}\left[\frac{\partial L_f}{\partial \delta_e}\right] - \left[\frac{\partial L_f}{\partial \delta_e}\right]\{F_e\} \tag{15}$$

where $\{F_e\}$ is the applied external element force vector of an element and L_f is the Lagrangian function. Substituting $L_f = T - U$, and the corresponding expressions for T and U in Lagrange's equation, the dynamic equilibrium equation for each element expressed as [17]

$$[M(\varpi)]\{\ddot{\delta}_e\} + ([K_e(\varpi)] + [K_{\sigma e}(\varpi)])\{\delta_e\} = \{F_e\} \tag{16}$$

After assembling all the element matrices and the force vectors with respect to the common global coordinates, the resulting equilibrium equation is formulated. Considering randomness of input parameters like ply-orientation angle, elastic modulus and mass density etc., the equation of motion of free vibration system with n degrees of freedom can expressed as

$$[M(\varpi)][\ddot{\delta}] + [K(\varpi)]\{\delta\} = \{F_L\} \tag{17}$$

In the above equation, $\mathbf{M}(\varpi)\epsilon\ \mathrm{R}^{n\times n}$ is the mass matrix, $[K(\varpi)]$ is the stiffness matrix wherein $[K(\varpi)] = [K_{\mathbf{e}}(\varpi)] + [K_{\sigma\mathbf{e}}(\varpi)]$ in which $\mathbf{K_e}(\varpi)\epsilon\ \mathrm{R}^{n\times n}$ is the elastic stiffness matrix, $\mathbf{K_{\sigma e}}(\varpi)\epsilon\ \mathrm{R}^{n\times n}$ is the geometric stiffness matrix (depends on initial stress distribution) while $\{\boldsymbol{\delta}\}\epsilon\ R^n$ is the vector of generalized coordinates and $\{\boldsymbol{F}_L\}\epsilon\ R^n$ is the force vector. The governing equations are derived based on Mindlin's Theory incorporating rotary inertia, transverse shear deformation. For free vibration, the random natural frequencies $[\boldsymbol{\omega}_\mathbf{n}(\varpi)]$ are determined from the standard eigenvalue problem [18] which is represented below and is solved by the QR iteration algorithm,

$$[A(\varpi)]\{\delta\} = \lambda(\varpi)\{\delta\} \tag{18}$$

where

$$[A(\varpi)] = ([K_e(\varpi)] + [K_{\sigma e}(\varpi)])^{-1}[M(\varpi)]$$

$$\lambda(\varpi) = \frac{1}{\{\omega_n(\varpi)\}^2} \tag{19}$$

3. Random sampling HDMR

Random sampling high dimensional model representation (RS-HDMR) method is employed to provide a straight forward approach in portraying the input–output mapping of a high dimensional model without requiring a large number of samples [2,3]. The mapping between the input random variables $x_1, x_2, \ldots, x_n$ (say ply orientation angle, elastic modulus, mass density) and the output variables (say random natural frequencies) $f(X) = f(x_1, x_2, \ldots, x_n)$ in the domain R^n is expressed as

$$f(X) = f_o + \sum_{i=1}^{n} f_i(x_i) + \sum_{1\leqslant i\leqslant j\leqslant n} f_{ij}(x_i, x_j) + \cdots + f_{12\ldots n}(x_1, x_2, \ldots, x_n) \tag{20}$$

Each term in the representation reflects the independent and cooperative contributions of the inputs upon the output. Here f_o is a constant (zeroth order) denoting the average contribution of all inputs to the output. The function $f_i(x_i)$ is a first order term giving the effect of variable x_i acting independently upon the output $f(X)$. The function $f_{ij}(x_i,x_j)$ is a second order term describing the cooperative effects of the variables x_i and x_j upon the output $f(X)$. The higher order terms reflect the cooperative effects of increasing numbers of input variables acting together to influence the output $f(X)$.The last term $f_{12\ldots n}(x_1,x_2,\ldots,x_n)$ contains any residual nth order correlated contribution of all input variables. If there is no interaction between the input variables, then only the zeroth order term and the first order terms will appear in the RS-HDMR expansion. In most of the cases a RS-HDMR expression up to second order provides satisfactory results [2]. The component functions of RS-HDMR have the following forms

$$
\begin{aligned}
f_o &= \int_{K^n} f(x)dx \\
f_i(x_i) &= \int_{K^{n-1}} f(x)dx^i - f_o \qquad (21)\\
f_{ij}(x_i, x_j) &= \int_{K^{n-2}} f(x)dx^{ij} - f_i(x_i) - f_j(x_j) - f_o
\end{aligned}
$$

Here dx^i denotes the product $dx_1 dx_2 \ldots dx_n$ without dx_i, whereas dx^{ij} stands for the same product without dx_i and dx_j. The last term $f_{12\ldots n}(x_1, x_2, \ldots, x_n)$ is evaluated from the difference between $f(X)$ and all the other component functions. RS-HDMR expansion is constructed when sample points are random in nature in a domain R^n. All the input variables are rescaled in such a way that $0 \leqslant x_i \leqslant 1$ for all i. The output response function is thus defined in the domain of an unit hypercube $K^n = \{(x_1, \ldots, x_n),\ i = 1, \ldots, n\}$. Generation of random sample points is constructed to form the RS-HDMR surrogate model. The quality of sample points governs the convergence rate of MCS which is used in formation of the component functions of RS-HDMR [19]. Quasi-random sequences (e.g., Halton sequence [20], Sobol' sequence [21], Faure sequence [22]) having low discrepancy are generally employed to generate random sample points. This ensures better uniform distribution of sample points in the input domain compared to pseudo-random sample points resulting in faster convergence. In the present study, Sobol' sequence is utilized to generate the input sample set as it exhibits better convergence than either Faure or Halton sequences [23].

The zeroth order term f_o in the RS-HDMR expression is calculated by the average value of all $f(X)$ in a set of samples. The determination of the higher order component functions involve high-dimensional integrals that may be approximately calculated by Monte Carlo integration. Because the direct determination of high-order RS-HDMR component functions by Monte Carlo integration is expensive, analytical basis functions, such as orthonormal polynomials, cubic B-spline functions, and polynomials are employed to approximate RS-HDMR component functions. With such approximations, only one set of random samples is needed to determine all RS-HDMR component functions making the sampling effort quite reasonable. Approximation using orthonormal polynomials provides the best accuracy among the different analytical basis functions [2]. Thus the first and second order component functions are expressed in the following form

$$
\begin{aligned}
f_i(x_i) &\approx \sum_{r=1}^{k} \alpha_r^i \varphi_r(x_i) \\
f_{ij}(x_i, x_j) &\approx \sum_{p=1}^{l} \sum_{q=1}^{l'} \beta_{pq}^{ij} \varphi_p(x_i) \varphi_q(x_j)
\end{aligned} \qquad (22)
$$

where k, l, l' represent the order of the polynomial expansion, α_r^i and β_{pq}^{ij} are the constant coefficients which are determined by a minimization process and MC integration. For a sample set $X^{(s)} = \left(x_1^{(s)}, x_2^{(s)}, \ldots, x_n^{(s)}\right)$, $s = 1, 2, \ldots, N$ these constant coefficients are evaluated as

$$
\begin{aligned}
\alpha_r^i &\approx \frac{1}{N} \sum_{S=1}^{N} f(X^{(s)}) \varphi_r\left(x_i^{(s)}\right) \\
\beta_{pq}^{ij} &\approx \frac{1}{N} \sum_{S=1}^{N} f(X^{(s)}) \varphi_p\left(x_i^{(s)}\right) \varphi_q\left(x_j^{(s)}\right)
\end{aligned} \qquad (23)
$$

$\varphi_r(x_i), \varphi_p(x_i)$ and $\varphi_q(x_j)$ are the orthonormal basis functions. An orthonormal basis function $\varphi_k(x)$ has the following properties in a domain $[a, b]$

$$
\begin{aligned}
&\int_a^b \varphi_k(x)dx = 0 \quad k = 1, 2, \ldots \quad \text{(i.e. zero mean)} \\
&\int_a^b \varphi_k^2(x)dx = 1 \quad k = 1, 2, \ldots \quad \text{(i.e. unit norm)} \\
&\int_a^b \varphi_k(x)\varphi_l(x)dx = 0 \quad k \neq l \quad \text{(i.e. mutually orthogonal)}
\end{aligned} \qquad (24)
$$

From above condition, the orthonormal polynomials are constructed in the domain $[0, 1]$ as

$$
\begin{aligned}
\varphi_1(x) &= \sqrt{3}(2x - 1) \\
\varphi_2(x) &= 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) \\
\varphi_3(x) &= 20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right) \\
&\vdots
\end{aligned} \qquad (25)
$$

The final equation of RS-HDMR up to second order component functions are expressed as

$$
f(X) = f_o + \sum_{i=1}^{n} \sum_{r=1}^{k} \alpha_r^i \varphi_r(x_i) + \sum_{1 \leqslant i \leqslant j \leqslant n} \sum_{p=1}^{l} \sum_{q=1}^{l'} \beta_{pq}^{ij} \varphi_p(x_i) \varphi_q(x_j) \qquad (26)
$$

The accuracy of RS-HDMR expansion is controlled by the error of Monte Carlo integration for calculating the expansion coefficients α and β. Variance reduction method is utilized to improve the accuracy of Monte Carlo integration without increasing the sample size. Correlation method [24] and ratio control variate method [25] are successfully applied for this purpose. In both cases the determination of the expansion coefficients is an iterative process and requires an analytical reference function which has to be similar to $f(X)$. A truncated RS-HDMR expansion is used as a reference function by calculating its expansion coefficients using direct Monte Carlo integration. As the HDMR component functions are independent, the order of the polynomial approximation is chosen separately for each component function to improve the accuracy of the final surrogate model. For highly nonlinear input–output relationship, higher-order polynomials may be used. But unnecessary use of higher order polynomials may lead to poor approximation, especially if a small sample size (N) is used to form the metamodel. The reason is attributed to the fact that the higher order polynomials have more number of terms than lower-order polynomials, and each term has its own Monte Carlo integration error. If the input–output relationship of a component function is linear, then it is found sufficient to use first-order polynomial function. If the contribution of any RS-HDMR component function to the overall function value is close to zero, then it is excluded from the RS-HDMR expansion function completely. A computationally efficient optimization technique based on least square method is developed to choose the best polynomial order for each of the component functions [26]. The idea behind this optimization technique is to calculate the sum of the square errors using the results of the full model runs of sample size (N) and the approximation of the component functions by either first, second or third-order polynomials or excluding the component function. The best approximation order for the corresponding component function is indicated by the smallest sum of square error. Further, a threshold criterion is introduced [27] to exclude ineffective terms from the RS-HDMR expansion for the systems having high number of input parameters.

4. Stochastic approach using RS-HDMR

The stochasticity in material properties of laminated composite plates, such as elastic modulus, mass density, Poisson's ratio and geometric properties such as ply-orientation angle, thickness,

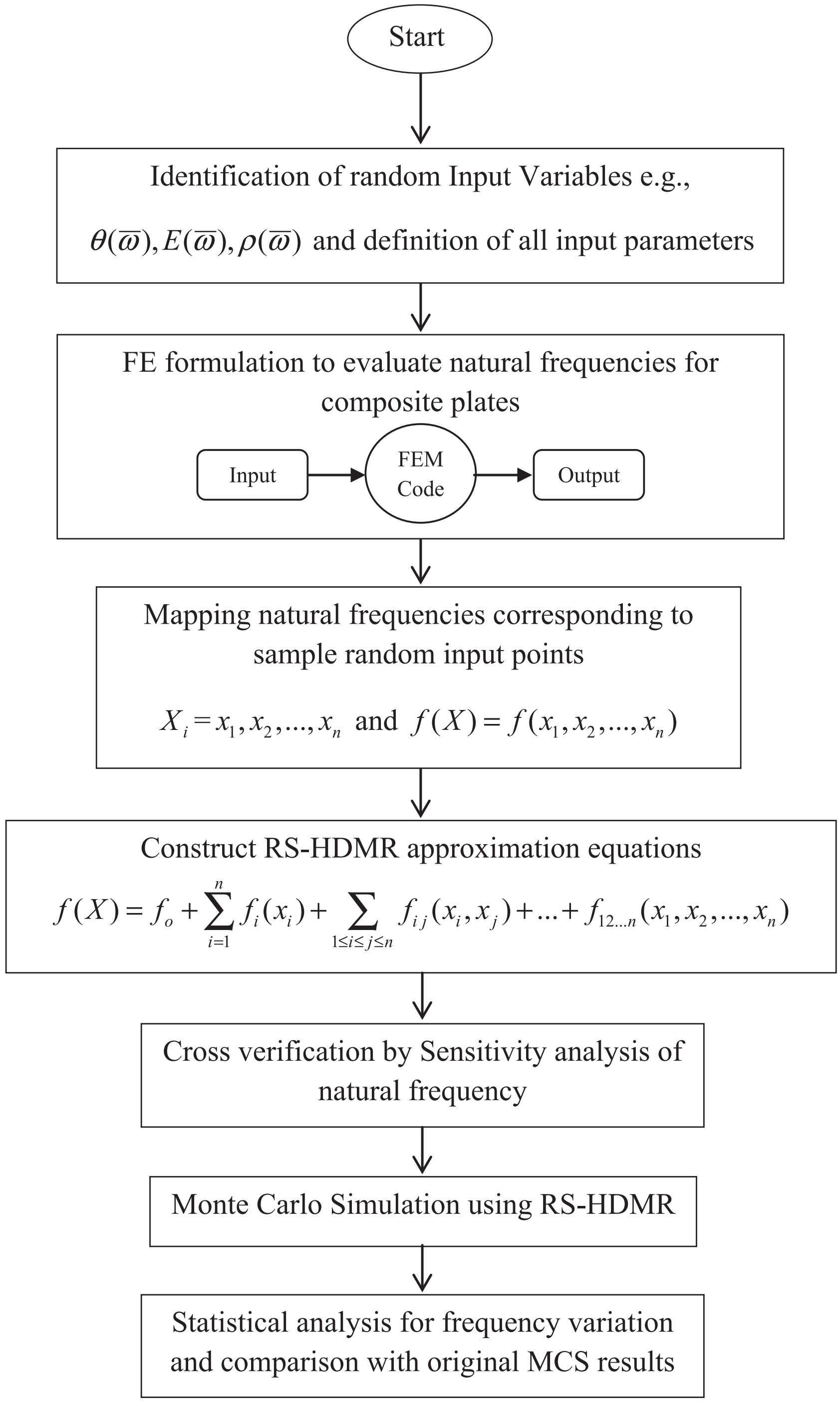

Fig. 2. Flowchart of stochastic free vibration analysis using RS-HDMR.

section inertia as input parameters is considered depending on boundary condition for free vibration analysis of composite plates. In the present study, frequency domain feature (first three natural frequencies) is considered as output. It is assumed that ±10% variation in randomness of symmetrically distributed input parameters for angle-ply composite cantilever plate is considered. Therefore normal distribution is considered as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad (27)$$

where σ^2 is the variance of the random variable; μ is the mean of the random variable. For example, the random variables considered in fiber orientation angle as independent in each layer of laminate and are denoted as

$$\begin{aligned} &\theta(\overline{\varpi}) = \{\theta_1\ \theta_2\ \theta_3\ \ldots\theta_i\ldots\theta_n\} \text{ (for only variation in ply-orientation angle)} \quad (28)\\ &E_1(\overline{\varpi}) = \{E_{1a}\ E_{1b}\ E_{1c}\ldots E_{1i}\ldots E_{1n}\} \text{ (for only variation in elastic modulus)}\\ &\rho(\overline{\varpi}) = \{\rho_1\rho_2\rho_3\ldots\rho_i\ldots\rho_n\} \text{ (for only variation in mass density)}\\ &g\{\theta(\overline{\varpi}), E_1(\overline{\varpi}), \rho(\overline{\varpi})\} = \{\Phi_1(\theta_1\theta_2\ldots\theta_i\ldots\theta_n), \Phi_2(E_{1a}E_{1b}\ldots E_{1i}\ldots E_{1n}),\\ &\quad \Phi_3(\rho_1\rho_2\ldots\rho_i\ldots\rho_n)\} \text{ (for combined variation in ply orientation angle,}\\ &\quad \text{elastic modulus and mass density)} \end{aligned}$$

where θ_i, E_{1i} and ρ_i are the ply orientation angle, elastic modulus and mass density, respectively and n denotes the number of layer in the laminate.

The insignificant input features are screened out and those are not considered in the model formation. For parameter screening purpose global sensitivity analysis based on RS-HDMR is employed in this study. The orthogonal relationship between the component functions of RS-HDMR expression implies that the component functions are independent and contribute their effects independently to the overall output response suggesting that each individual component function has a direct statistical correlation with the output. So the sensitivity of each component function is determined by calculating the total and partial variances [4].The total variance (D) is expressed as

$$D = \int_{K^n} f^2(X)dX - f_o^2 \approx \frac{1}{N}\sum_{S=1}^{N} f^2(X^{(s)}) - f_o^2 \tag{29}$$

(Using MC integration for a set of N samples $X^{(s)} = \left(x_1^{(s)}, x_2^{(s)}, \ldots, x_n^{(s)}\right)$, $s = 1, 2, \ldots, N$)

The partial variances obtained using the properties of orthonormal polynomials are expressed as follows, (see Fig. 2)

$$\begin{aligned} D_i &= \int_0^1 f_i^2(x_i)dx_i \approx \int_0^1 \left[\sum_{r=1}^{k} \alpha_r^i \varphi_r(x_i)\right]^2 dx_i = \sum_{r=1}^{k} (\alpha_r^i)^2 \\ D_{ij} &= \int_0^1 \int_0^1 f_{ij}^2(x_i, x_j)dx_i dx_j = \int_0^1 \int_0^1 \left[\sum_{p=1}^{l}\sum_{q=1}^{l'} \beta_{pq}^{ij} \varphi_p(x_i)\varphi_q(x_j)\right] dx_i dx_j \\ &= \sum_{p=1}^{l}\sum_{q=1}^{l'} \left(\beta_{pq}^{ij}\right)^2 \end{aligned} \tag{30}$$

The sensitivity indices are expressed as

$$\begin{aligned} S_{i_1,\ldots i_s} &= \frac{D_{i_1,\ldots i_s}}{D}, 1 \leqslant i_1 \leqslant i_2 \ldots \leqslant i_s \leqslant n \\ \text{So that}, &\sum_{i=1}^{n} S_i + \sum_{1 \leqslant i<j \leqslant n}^{n} S_{ij} + \cdots + S_{1,2,\ldots,n} = 1 \end{aligned} \tag{31}$$

In present study, the metamodels are formed to obtain the responses in terms of input parameters using RS-HDMR. The present RS-HDMR models are constructed using numerical data for first three natural frequencies. FORTRAN and MATLAB are utilized for the present study. The constructed metamodel is checked for the coefficient of determination (R^2) value which is found close to one as expected. The coefficient of determination (R^2) is expressed as,

$$R^2 = \frac{SS_R}{SS_T} = 1 - \frac{SS_E}{SS_T} (0 \leqslant R^2 \leqslant 1) \tag{32}$$

where, $SS_T = SS_E + SS_R$ is the total sum of square,SS_R and SS_E are the regression sum of squares and the residual sum of squares, respectively. The quality of the surrogate model obtained using RS-HDMR can also be determined by Relative Error (RE), computed as

$$RE(\%) = \frac{|F - F'|}{F} \times 100 \tag{33}$$

where F is the actual response and F' is the approximated response using RS-HDMR approach.

5. Results and discussion

In the present study, random natural frequencies for three layered graphite–epoxy angle-ply laminated composite cantilever plate having square plan form ($L/b = 1$) and length to thickness ratio $L/t = 250$ are obtained corresponding to different types of laminates. Material properties of graphite–epoxy composite [28] considered with mean value as $E_1 = 138.0$ GPa, $E_2 = 8.96$ GPa, $\nu = 0.3$, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 2.84$ GPa. For full scale MCS, number of original FE analysis is same as the sampling size. A typical discretization of (6×6) mesh on plan area with 36 elements 133 nodes with natural coordinates of an isoparametric quadratic plate bending element are considered for the present FEM. In general for complex composite structures, the performance function is not available as an explicit function of the random design variables. The performance or response in terms of natural frequencies of the composite structure can only be evaluated numerically at the end of a structural analysis procedure such as the finite element method which is often time-consuming. The present RS-HDMR methodology is employed to find a predictive and representative metamodel equation relating each natural frequency to a number of input variables. These metamodel equations are used to determine the first three natural frequencies corresponding to given values of input variables, instead of having to repeatedly run the time-consuming deterministic finite element analysis. The response surface thus represents the result (or output) of the structural analysis encompassing (in theory) every reasonable combination of all input variables. From this, thousands of combinations of all design variables can be created (via simulation) and performed a pseudo analysis for each variable set, by simply adopting the corresponding surface values. The probability density function (PDF) and cumulative distribution function (CDF) are plotted as the benchmark of bottom line results. Due to paucity of space, only a few important representative results are furnished.

5.1. Validation

In the present analysis, convergence study is carried out to determine the coefficient of determination R^2 (second order) of the RS-HDMR expansions for different sample sizes as furnished in Table 1. The comparative study depicts an excellent agreement with the previously published results and hence it demonstrates the capability of the computer codes developed and insures the accuracy of analyses. Table 2 presents the non-dimensional fundamental frequencies of graphite–epoxy composite twisted plates with different ply-orientation angle [29]. Convergence studies are performed using uniform mesh division of (6×6) and (8×8) and the results are found to be nearly equal, with the difference being around 1% and the results also corroborate monotonic

Table 1
Convergence study for coefficient of determination R^2 (second order) of the RS-HDMR expansions with different sample sizes for variation of only ply-orientation angle of graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate, considering $E_1 = 138$ GPa, $E_2 = 8.9$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 2.84$ GPa, $t = 0.004$ m, $\nu = 0.3$.

Frequency	Sample size				
	32	64	128	256	512
FF	65.68	93.48	99.60	99.95	99.96
SF	69.67	93.74	99.47	96.38	97.81
TF	66.44	97.85	99.40	98.86	99.61

FF – fundamental natural frequency, SF – second natural frequency, TF – third natural frequency

Table 2
Non-dimensional fundamental natural frequencies (deterministic mean value) [$\omega = \omega_n L^2 \sqrt{(\rho/E_1 t^2)}$] of three layered [$\theta$, $-\theta$, θ] graphite- epoxy twisted plates, $L/b = 1$, $b/t = 20$, $\psi = 30°$.

Fiber orientation angle, θ	Present FEM (6×6)	Present FEM (8×8)	Qatu and Leissa [29]
15°	0.8618	0.8591	0.8759
30°	0.6790	0.6752	0.6923
45°	0.4732	0.4698	0.4831
60°	0.3234	0.3194	0.3283

Table 3
Comparative study between MCS (10,000 samples) and RS-HDMR (128 samples) for maximum values, minimum values and percentage of difference for first three natural frequencies obtained due to individual stochasticity in ply-orientation angle, elastic modulus and mass density for three layered graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate considering E_1 = 138 GPa, E_2 = 8.9 GPa, $G_{12} = G_{13}$ = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

Parameter	Fundamental frequency			Second natural frequency			Third natural frequency		
	$\theta(\overline{\omega})$	$E_1(\overline{\omega})$	$\rho(\overline{\omega})$	$\theta(\overline{\omega})$	$E_1(\overline{\omega})$	$\rho(\overline{\omega})$	$\theta(\overline{\omega})$	$E_1(\overline{\omega})$	$\rho(\overline{\omega})$
Max. (original MCS)	4.4286	3.9607	2.7999	11.8991	12.0184	12.5486	27.0737	24.6465	26.1869
Max. (RS-HDMR)	4.4214	3.9525	2.7911	11.8725	11.8496	12.3447	27.4348	24.6123	25.7615
Difference (%)	0.16%	0.21%	0.32%	0.22%	1.40%	1.63%	−1.33%	0.14%	1.62%
Min. (original MCS)	3.2100	3.6759	2.7000	10.7492	10.8664	10.5673	20.1827	23.0173	22.0367
Min. (RS-HDMR)	3.1971	3.6818	2.7095	10.7473	11.0184	10.5247	20.0920	23.1700	21.9631
Difference (%)	0.14%	−0.16%	−0.35%	0.02%	−1.40%	0.40%	0.45%	−0.66%	0.33%

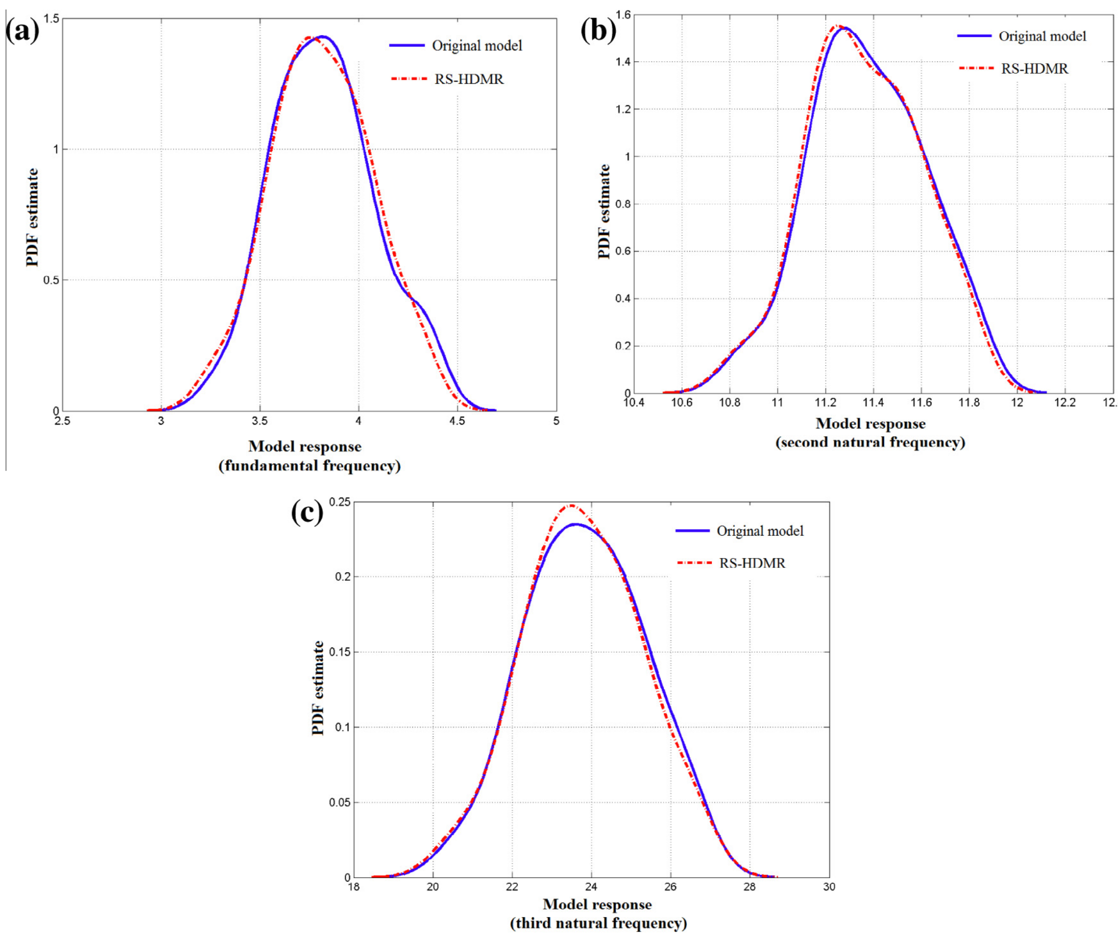

Fig. 3. Probability distribution function (PDF) with respect to model response of first three natural frequencies for variation of ply-orientation angle of graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate, considering E_1 = 138 GPa, E_2 = 8.9 GPa, $G_{12} = G_{13}$ = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

downward convergence. The differences between the results by Qatu and Leissa [29] and the present FEM approach can be attributed to consideration of transverse shear deformation and rotary inertia in the present FEM approach and also to the fact that Ritz method overestimates the structural stiffness of the composite plates. Moreover, increasing the size of matrix because of higher mesh size increases the ill-conditioning of the numerical eigenvalue problem. Hence, the lower mesh size (6×6) is employed in the present analysis due to computational efficiency. In present RS-HDMR metamodel analysis, a representative sample size of 128 is considered using the Sobol' sequence for individual random variation of ply-orientation angle, longitudinal elastic modulus and mass density, respectively. As the number of input parameter is increased for combined random variation of ply-orientation angle, longitudinal elastic modulus and mass density, the subsequent enhanced sample size of 256 is adopted to meet the convergence criteria. While evaluating the statistics of responses through full scale Monte Carlo simulation (MCS), computational time is exorbitantly high because it involves number of repeated FE analysis. However, in the present method, MCS is conducted in conjunction with RS-HDMR-based metamodel. Here, although the same sampling size as in direct MCS is considered, the number of FE analysis is much less compared to original MCS and is equal to number representative sample required to construct the RS-HDMR metamodel. A representative RS-HDMR equation is formed on which the full sample size of direct MCS is conducted. Hence, the computational time and effort expressed in terms of finite element calculation is reduced compared to full scale direct MCS. Hence, in order to save

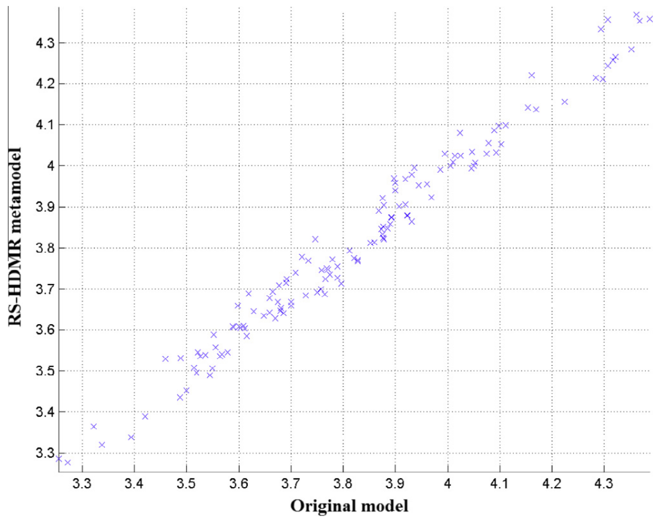

Fig. 4. Scatter plot for RS-HDMR metamodel with respect to original model of fundamental natural frequencies for variation of ply-orientation angle of graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate, considering E_1 = 138 GPa, E_2 = 8.9 GPa, G_{12} = G_{13} = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

computational time, the present constructed RS-HDMR metamodel methodology is employed instead of traditional Monte Carlo simulation. The computational time required in the present RS-HDMR approach is observed to be around (1/80) times (for individual variation of inputs) and (1/40) times (for combined variation of inputs) of direct Monte Carlo simulation. This provides an efficient affordable way for computationally simulating the uncertainties in natural frequency. The sensitivity of material or geometric property to each random variable is also qualified in the present metamodel context. All the cumulative distribution function of the respective stochastic input parameters at ply and laminate levels are compared with the results predicted by a Monte Carlo simulation with 10,000 samples wherein a good agreement is observed in all cases.

5.2. Statistical analysis

The random input values of ply orientation angle, $\theta(\overline{\omega})$, elastic modulus in longitudinal direction, $E_1(\overline{\omega})$ and mass density, $\rho(\overline{\omega})$ for each layer of the composite laminate are scaled randomly in the range of 0 to 1, having the lower and the upper limit as ±10% variation with respective mean values. The RS-HDMR metamodels are formed to generate first three natural frequencies. The lower and upper bounds of the input random data set is considered as 0 and 1, respectively. Metamodels are developed using first and second order component functions with orthonormal polynomials up to third order. Table 3 presents the comparative study between MCS and RS-HDMR for maximum values, minimum values and percentage of difference for first three natural frequencies obtained due to individual stochasticity in ply-orientation angle, elastic modulus and mass density for three layered graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate. Fig. 4 depicts a sample scatter plot describing the relationship between the original FE model and the constructed RS-HDMR metamodel for natural frequencies. The coefficient of determination (R^2) values and relative error in percentage (1%, 5% and 10%) values corresponding to each metamodel are found to be around 100%, indicating high accuracy of the fitted models. Fig. 3 shows a sample comparison of the probability density functions (PDF) for both original model and RS-HDMR metamodel. The low scatterness of the points around the diagonal line in Fig. 4 and the low deviation between the pdf estimations of original and RS-HDMR responses in Fig. 3 corroborates that RS-HDMR metamodels are formed with accuracy. These two plots are checked and are found in good agreement ensuring the efficiency and accuracy of the constructed metamodel.

A comparative study between MCS and RS-HDMR for maximum values, minimum values, mean and standard deviation for first three natural frequencies obtained due to combined stochasticity

Table 4
Comparative study between MCS (10,000 samples) and RS-HDMR (256 samples) for maximum values, minimum values, mean and standard deviation for first three natural frequencies obtained due to combined stochasticity in ply-orientation angle, elastic modulus and mass density for three layered graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate considering E_1 = 138 GPa, E_2 = 8.9 GPa, G_{12} = G_{13} = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

Parameter	Fundamental frequency			Second natural frequency			Third natural frequency		
	MCS	RS-HDMR	Difference (%)	MCS	RS-HDMR	% Deviation	MCS	RS-HDMR	Difference (%)
Max. value	4.8190	4.6896	2.69	12.9625	12.7989	1.26%	29.1570	28.5391	2.12
Min. value	2.9811	2.9248	1.89	9.9008	9.9831	−0.83%	18.0941	18.5681	−2.62
Mean value	3.8232	3.8270	−0.10	11.3762	11.3669	0.08%	2.3966	2.3847	0.50
Standard deviation	0.2839	0.2828	0.39	0.4525	0.4257	5.92%	1.6959	1.6405	3.27

Table 5
Maximum values, Minimum values, Means and standard deviations of first three natural frequencies obtained by RS-HDMR (128 samples) due to individual stochasticity in ply-orientation angle [$\theta(\overline{\omega})$], elastic modulus [$E_1(\overline{\omega})$] and mass density [$\rho(\overline{\omega})$] for three layered graphite–epoxy angle-ply composite cantilever plate considering E_1 = 138 GPa, E_2 = 8.9 GPa, G_{12} = G_{13} = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

Laminate configuration	Parameter	Fundamental frequency			Second natural frequency			Third natural frequency		
		$\theta(\overline{\omega})$	$E_1(\overline{\omega})$	$\rho(\overline{\omega})$	$\theta(\overline{\omega})$	$E_1(\overline{\omega})$	$\rho(\overline{\omega})$	$\theta(\overline{\omega})$	$E_1(\overline{\omega})$	$\rho(\overline{\omega})$
30°/−30°/30°	Max. (RS-HDMR)	5.9885	5.6770	5.8476	12.1026	12.5724	12.9808	26.7072	26.8684	28.3043
	Min. (RS-HDMR)	4.8170	5.0907	4.9677	11.2000	11.3373	11.0273	25.3481	25.3242	24.0445
	Mean	5.4291	5.4155	5.4224	11.5638	12.0255	12.0369	26.0758	26.2469	26.2461
	Standard deviation	0.2374	0.1047	0.1526	0.3593	0.2174	0.3388	0.2421	0.2743	0.7389
60°/−60°/60°	Max (RS-HDMR)	3.2987	2.7911	2.9802	11.0742	10.2016	10.6604	20.9504	17.4650	18.7011
	Min (RS-HDMR)	2.3865	2.7095	2.5284	8.3658	9.4612	9.0450	14.8491	17.0566	15.8671
	Mean	2.7741	2.7662	2.7673	9.7640	9.8859	9.8994	17.4308	17.3604	17.3661
	Standard deviation	0.1856	0.0141	0.0782	0.4622	0.1300	0.2799	1.2425	0.0724	0.4910

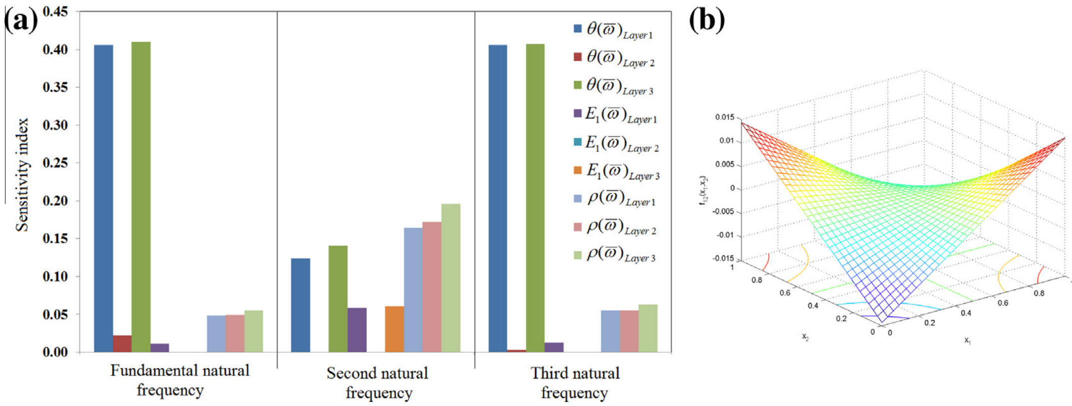

Fig. 5. (a) Sensitivity index and (b) 3D plot for second order component function $f_{12}(x_1,x_2)$ of the response f_1 with x_1 (ply-orientation angle) and x_2 (elastic modulus) with respect to first three natural frequencies indicating for combined variation (10,000 samples) of ply-orientation angle, elastic modulus and mass density for graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate, considering E_1 = 138 GPa, E_2 = 8.9 GPa, G_{12} = G_{13} = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

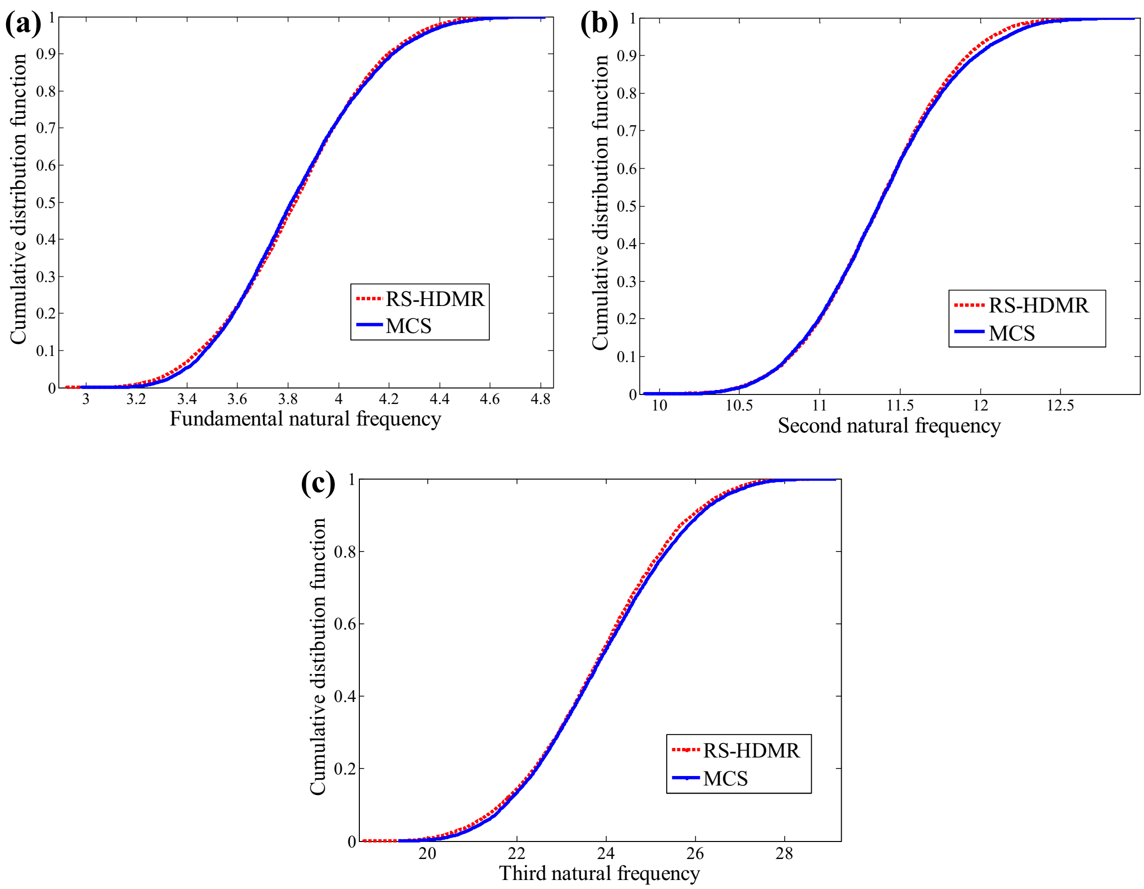

Fig. 6. (a–c) Cumulative distribution function obtained by RS-HDMR (10,000 samples) with respect to first three natural frequencies indicating for combined variation of ply-orientation angle, elastic modulus and mass density for graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate, considering E_1 = 138 GPa, E_2 = 8.9 GPa, G_{12} = G_{13} = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

in ply-orientation angle, elastic modulus and mass density for three layered graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate is furnished in Table 4. The parametric study carried out for maximum values, minimum values, means and standard deviations of first three natural frequencies obtained by RS-HDMR due to individual stochasticity in ply-orientation angle [$\theta(\bar{\varpi})$], elastic modulus [$E_1(\bar{\varpi})$] and mass density [$\rho(\bar{\varpi})$] for three layered graphite–epoxy angle-ply composite cantilever plate as depicted in Table 5. Global sensitivity analysis using RS-HDMR is performed for significant input parameter screening. The sensitivity indices for each input parameter (including the interaction effects) corresponding to different output responses are shown in Fig. 3. For combined variation of inputs, the sensitivity indices for each input parameter is presented in Fig. 5(a), while a typical interaction effect between ply-orientation angle and longitudinal elastic modulus corresponding to second order component function of the fundamental frequency is furnished in Fig. 5(b). In combined stochastic approach, the ply-orientation [$\theta(\bar{\varpi})$], longitudinal elastic

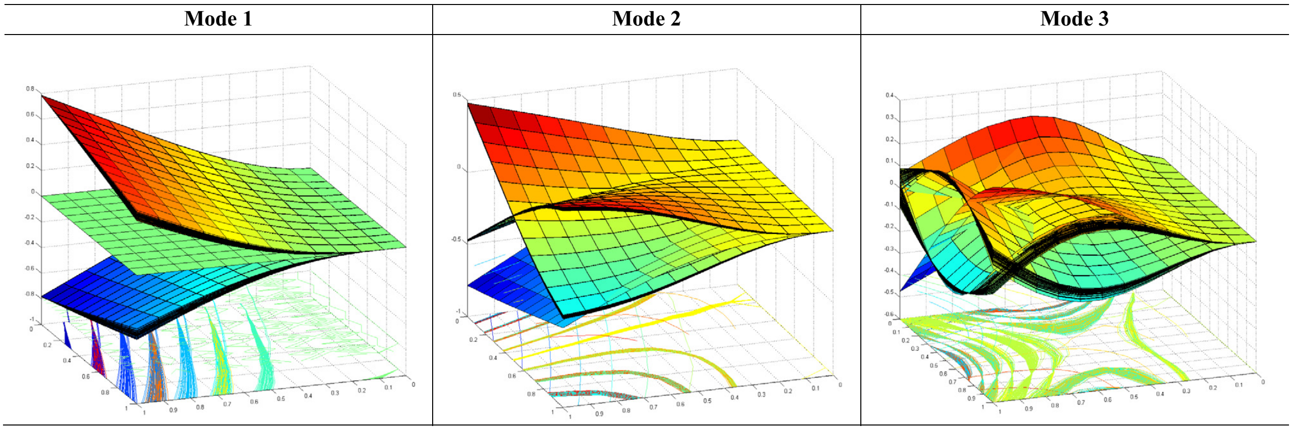

Fig. 7. Effect on modeshapes of first three modes due to combined stochasticity in ply-orientation angle, elastic modulus and mass density for three layered graphite–epoxy angle-ply (45°/−45°/45°) composite cantilever plate considering E_1 = 138 GPa, E_2 = 8.9 GPa, G_{12} = G_{13} = 7.1 GPa, G_{23} = 2.84 GPa, ρ = 3202 kg/m^3, t = 0.004 m, ν = 0.3.

modulus $[E_1(\overline{\omega})]$ and mass density $[\rho(\overline{\omega})]$ are considered as the significant input parameters while the output parameter is taken as first three natural frequencies. From Fig. 5(a) it is observed that ply-orientation angles of two outer surface layers has predominant effect on sensitivity of tested natural frequencies while null sensitivity effect is identified due to random variation for elastic modulus of the middle layer of the laminate. The sensitivity index obtained of material and geometric properties to each random variable at the constituent and ply levels is used as a guide to increase the structural reliability or to reduce the cost. Furthermore, the probabilistic structural analysis and risk assessment is performed once the uncertain laminate properties are computationally simulated. The sensitivity of different random variables for the laminate frequencies of different modes may be different. A comparative study between original MCS method and present RS-HDMR method using 10,000 sample run is carried out to develop the cumulative distribution function plot as furnished in Fig. 6(a–c). The predicted cumulative distribution functions is presented as the upper and lower bounds of natural frequencies. The representative stochastic mode shapes for first three natural frequencies are furnished in Fig. 7 wherein the first spanwise bending mode is observed corresponding to its fundamental frequency while the dominance of torsion mode is found for second and third natural frequencies, respectively.

6. Conclusions

In the present study, a bottom up approach is considered for the propagation of uncertainty in the natural frequencies of angle-ply composite cantilever plates. The feasibility of applying metamodel based approach using RS-HDMR is portrayed in the realm of stochastic analysis. Although the same sampling size as in direct MCS (sample size of 10,000) is considered, the number of FE analysis is much less compared to original MCS and is equal to number representative sample (sample size of 128) required to construct the RS-HDMR metamodel. It is observed that RS-HDMR method can handle the large number of input parameters. The metamodel formed from a small set of samples is found to establish the accuracy and computational efficacy. As the number of sample is increased for formation of RS-HDMR metamodel, the accuracy of results is found to improve irrespective of input parameter. The results obtained as probability density function and cumulative distribution function employing RS-HDMR metamodels are compared with the results from direct Monte Carlo Simulation. The key findings of this study include:

1. The maximum effect on sensitivity is observed for ply-orientation angle while the least sensitivity effect is identified for elastic modulus corresponding to first three natural frequencies.
2. From global sensitivity analysis of the combined input variation case, the ply-orientation angles of the two outer surface layers are found to predominantly effect on all the tested natural frequencies.
3. Interestingly, null sensitivity effect is identified for elastic modulus of the middle layer for combined input variation case.
4. Mass density is found to hold the maximum sensitivity for the second natural frequency among the three stochastic input parameters.
5. The first spanwise bending mode is observed corresponding to its fundamental frequency while the dominance of torsion mode is found for second and third natural frequencies, respectively.

The future investigation will be carried out to interrogate whether the above conclusions hold true for more complex system.

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