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Vibration insight of a nonlocal viscoelastic coupled multi-nanorod system

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ABSTRACT

Longitudinal vibration of viscoelastic multi-nanorod system (VMNS) is studied. Based on the D' Alembert's principles, nonlocal and viscoelastic constitutive relations, the system of m partial differential equations are derived which described the motion of the presented nano-system. Clamped–Clamped and Clamped–Free boundary conditions and two different chain systems, namely “Clamped-Chain” and “Free-Chain” are illustrated. The method of separations of variables and trigonometric method are utilized for solutions. The analytical expressions for critical viscoelastic parameters and asymptotic frequencies are presented. The predicted results are validated with results obtained by direct numerical simulations and results from literature. The effects of nonlocal parameter, number of nanorods, viscoelastic material constant and parameter of viscoelastic layer on the complex eigenvalue are discussed in details.

1. Introduction

Recently, growing interest in the dynamic response of nanostructures elements like a nanorods, nanobeams or nanoplates, plays an important role in the development of nanodevices. Therefore, the issue of vibration behavior of nanostructures elements has become very important from the practical point of view and it has wide application in nanotechnology. The nanodevices include biosensors (Ziegler, 2004; Sotiropoulou and Chaniotakis, 2003; Wang, 2005; Wang et al., 2003; Shen et al., 2012; Ali et al., 2009; Chowdhury et al., 2011), mass sensors (Lee et al., 2010; Mehdipour et al., 2011; Murmu and Adhikari, 2011), nanoresonators (He et al., 2005; Liu et al., 2011), gas sensors (Basu and Bhattacharyya, 2012; Llobet, 2013), nanoopto-mechanical system (Hierold et al., 2007; Lu et al., 2007) etc. Nanomaterial's such as carbon nanotubes (CNTs) (Iijima, 07 November 1991), boron nitride nanotubes (BNNTs) (Chopra et al., 18 August 1995), zinc oxide nanotubes (ZnO) (Liu and Zeng, 2009) and graphene sheet (Geim and Novoselov, 2007) are the basis material of many nanostructures and nanodevices. These nanomaterial's have extraordinarily properties resulting from their nanoscale dimensions (Guz et al., 2007; Gouadec and Colomban, 2007; Kuo et al., 2005; Dresselhaus et al., 2004; Ruoff et al., 2003). Performing controlled experiments at the nano-level is very difficult and expensive. Therefore, development appropriate mathematical models based on Eringen's continuum theory, which takes into account size effect and atomic forces is very important. By ignoring these effects in the development of mathematical models of nanoscale structures can cause completely incorrect solutions and hence erroneous designs. According to a paper (Eringen and Edelen, 1972), Eringen derived a constitutive relation in integral form, based on the assumption that the stress at the point is function of the strain at all points of the elastic body. Since then, many researchers have contributed to the development of nonlocal continuum theory and application in mathematical modeling of nanostructures.

Studying the static and dynamical behavior of elastic nanorod, nanobeam and nanoplates subject of many papers (Ansari et al., 2010; Akgöz and Civalek, 2013; Aydogdu and Filiz, 2011; Wang et al., 2006). One of the first applications of the nonlocal continuum theory in nanotechnology is the work presented by Peddieson et al. (2003). They used the nonlocal elasticity theory to develop

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nonlocal Euler–Bernoulli beam for different boundary conditions. Also, they are considering application of cantilever beam as microelectromechanical actuator. Lately, nonlocal theories for the Euler–Bernoulli, Timoshenko, Reddy and Levinson beams are derived by Reddy (Reddy, 2007) in a unique way using Hamilton's principle and nonlocal constitutive relation of Eringen. The author is obtain analytical solution of banding, vibration and buckling and showed the effect of nonlocal parameter on deflections, buckling load and natural frequencies. In the paper, presented by Reddy and Pang (Reddy and Pang, 2008), the equations of motion of Euler–Bernoulli and Timoshenko beam theories are reformulated by Eringen nonlocal theory, and then used to evaluate static bending, vibrations, and buckling response of carbon nanotubes with several boundary conditions. The influences of nonlocal parameter and aspect ratio on the natural frequency, static deflection and buckling load are considered. The small scale effect on the axial vibration of a tapered nanorod based on the nonlocal elasticity theory is studied by Danesh et al. (2012). The governing equations are solved by using the differential quadrature method for three type of boundary conditions, clamped–clamped (C–C), clamped–free (C–F) and fixed-attached spring boundary conditions. Also, it is show that the nonlocal effect plays an important role in the axial vibration of nanorods. The free vibration of double-nanorod system is investigated by Murmu and Adhikari (2010). Based on Eringen's nonlocal elasticity theory and methods of separations of variables, they obtained analytical solutions for natural frequencies for two types of boundary conditions, Clamped–-Clamped and Clamped–Free. A carbon nanotube embedded in an elastic medium was modeled by Aydogdu (2012) as a nanorod surrounded with elastic layers by using the Eringen's nonlocal elasticity theory. The author compared the longitudinal frequencies for the nonlocal and classical continuum models. Narendar and Gopalakrishnan (2011) considered the nonlocal effects in the axial wave propagation within the system of two nanorods coupled with an elastic layer. The authors studied the influence of small-scale (nonlocal) parameter and stiffness of the layer on axial wave propagation. Hsu et al. (2011) investigated the longitudinal frequencies of cracked nanobeams for different boundary conditions and using the theory of nonlocal elasticity. A wide study of the longitudinal, transversal and torsional vibration and instability was conducted by Kiani (2013) for a system of SWCNTs. Şimşek (2012) used a Galerkin approach to obtain the natural frequencies for the longitudinal vibration of axially functionally graded tapered nanorods. The author performed the analysis for nanorods with a variable cross-section, differently tapered ratios, material properties and boundary conditions. Longitudinal vibration of nanorods, which takes the nonlocal long-range interactions into account, was examined by Huang (2012). Chang (2012) considered the small-scale effects to investigate the axial vibration of elastic nanorods. The author used the differential quadrature method to solve the model equations. Filiz and Aydogdu (2010) analyzed the longitudinal vibration of carbon nanotubes with heterojunctions using the nonlocal elasticity for different lengths, diameters and chirality of heterojunctions. Karličić et al. (2015) performed a detailed analysis of the free longitudinal vibrational response of the system with two coupled viscoelastic nanorods and investigated the influence of different physical parameters on complex natural frequencies. Recently, Adhikari et al. (2013) examined the free and forced longitudinal vibration of the nonlocal nanorod by using two types of nonlocal damping models. The authors obtained the partial differential equation of motion in terms of axial displacements and then solved by analytical and finite element method. Exact analytical solutions for cut-off frequency are also obtained when the number of mode in the complex natural frequency tends to the infinity.

Damping properties appear in all nanostructures systems and help to better define suppression vibration behavior. Understanding their source is an important issue, not only for design and applications in nanoengineering practice but also to understand the inner workings of the nanomaterial's and nanostructures elements. Therefore, different technologies have been developed to investigated the damping effects on the vibration characteristics of damped or viscoelastic nanostructures (Imboden and Mohanty, 2014). Viscoelastic materials displaying both solid-like and fluid like characteristics, are common in polymeric structures. Energy dissipation or portion of energy storage from fluid-like part is irrecoverable and can be separated from energy of deformation using a complex modulus, which is represented by real and imaginary parts named storage and loss modulus, respectively. Thus, should be paid a more attention to the study of the dynamic behavior of the nanostructures with viscoelastic properties. The application of the nonlocal continuum theory to describe the internal and external damping effects in the structure elements at the nanoscale level have started recently. Lei et al. (2013a) proposed two type nonlocal dumped viscoelastic model of nanobeam based on nonlocal viscoelastic constitutive relations for vibration analysis. A transfer function methods is applied to obtain analytical solutions of free vibration for Euler–Bernoulli nanobeam with different boundary conditions. Also, the influences of material and geometric parameters on the complex eigenvalue are investigated. In the paper by Lei et al. (2013b) the dynamical behavior of nonlocal viscoelastic damped nanobeam has been investigated by using the Kelvin–Voigt viscoelastic model, velocity-dependent external damping and Timoshenko beam theory. The authors showed that nonlocal damped beams have maximum frequencies, called asymptotic frequencies, and also possess an asymptotic critical damping factor. The numerical results are presented on carbon nanotube example. In the paper by Paola et al. (2013) the dynamics of a nonlocal Timoshenko beam is presented. Nonlocal effects are modeled as long-range volume forces and moments mutually exerted by non-adjacent beam segments, that contribute to the equilibrium of any beam segment along with the classical local stress resultants. Also, model is provided with elastic and viscous long-range volume forces and moments which are linearly dependent on the product of the volumes of the interacting beam segments and on generalized measures of their relative motion, based on the pure deformation modes of the beam. The numerical results are presented for different values of nonlocal parameters. Vibration behavior of boron nitride nanotubes coupled by visco-Pasternak layer under a moving nanoparticle was proposed by Ghorbanpour Arani and Roudbari (2013) who investigated the nonlocal piezoelastic surface effect. Pouresmaeeli et al. (2013) reported on vibration characteristics of simply supported viscoelastic orthotropic nanoplates resting on viscoelastic foundation. The authors are obtained closed form solutions of complex frequencies which includes influence of nonlocal parameter and structural damping of the nanoplate and foundation. They showed that the frequency significantly decreases with increasing the structural damping.

By browsing the literature, the authors have found that some interesting papers about physics of multiple system of nanorods (Lao et al., 2002; Wen et al., 2003; Schulz et al., 2005). Nanorods growing from nanowire core can be viewed as multi-nanorod system Fig. 1. Mechanical modeling of those systems can be of great progress for their application and comprehension since experiments on nano-scale level cannot be well controlled. Therefore, this paper represents an extension of work Karličić et al. (2015), for systems of multiple coupled nanorods with viscoelastic properties. In the following of this work, it is presented an analytical solution of axial vibrations of a viscoelastic multi-nanorod system embedded

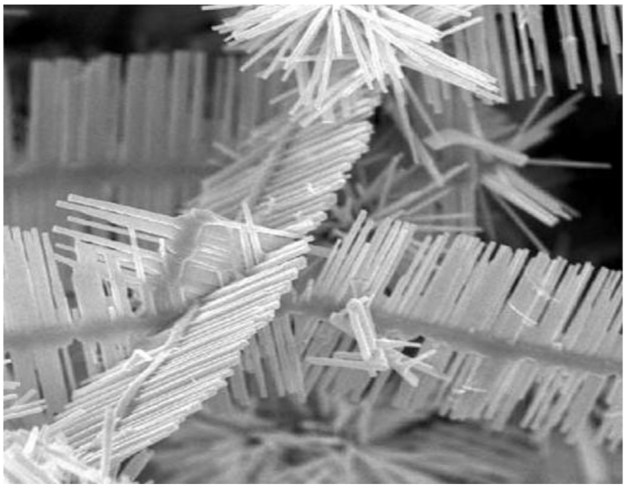

Fig. 1. ZnO side nanorods growing on central nanowire cores (Wen et al., 2003) – Physical model.

in viscoelastic medium. We assume that the system under consideration is composed of a set of m nonlocal, parallel and identical viscoelastic nanorods coupled by viscoelastic layers, with given stiffness and dumping parameters. By applying the D' Alembert's principle, nonlocal and viscoelastic constitutive relations the set of m coupled partial differential equations of motion are derived. The closed form solutions for complex eigenvalues are obtain by using method of separations of variables and trigonometric method for different number of nanorods and different boundary conditions. Also it is obtain analytical expressions for the asymptotic frequencies and critical damping factor for undamped and damped VMNS. In order to validate of present analytical research, the obtained results are compared with results obtained by numerical methods and results reported in literature. The influence of the nonlocal parameter, number of nanorods and viscoelastic material constant on the real and imaginary part of complex eigenvalues of the system is also determined through numerical experiment for boundary conditions and "Chain" systems.

The present study is very useful in designing of nanoelectromechanical devices, and it is also useable in analyzing damping effect on the dynamic excitation systems at the nanolevel.

2. Mathematical model of VMNS

2.1. Brief introduction in nonlocal constitutive relations

The basic assumption in the nonlocal elasticity theory, that the stress at a point x is observed to be a function not only on a strain at that point x but also on strains at all other points of a body. Based on this, Eringen has introduced material parameter in the constitutive relations which takes into account the size effect. The integral form (Eringen and Edelen, 1972) of nonlocal linear constitutive relation for a three-dimensional body are given as

$$\sigma_{ij}(x) = \int \alpha(|x - x'|, \tau) C_{ijkl} \varepsilon_{kl}(x') \mathrm{d}V(x'), \quad \forall x \in V, \tag{1a}$$

$$\sigma_{ij,j} = 0, \tag{1b}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1c}$$

where C_{ijkl} is the elastic modulus tensor for classical isotropic elasticity; σ_{ij} and ε_{ij} are stress and strain tensors, respectively and u_i is displacement vector. With $\alpha(|x-x'|,\tau)$ we denote the nonlocal modulus or attenuation function which incorporates nonlocal effects into the constitutive equation at the reference point x produced by local strain at the source x'. The above absolute value of difference $|x-x'|$ denotes the Euclidean metric. The parameter $\tau = (e_0 a)/l$ where l is the external characteristic length (crack length, wave length), a describes internal characteristic length (lattice parameter, granular size and distance between C–C bounds) and e_0 is a constant appropriate to each material that can be identified from atomistic simulations or by using dispersive curve of the Born–Karman model of lattice dynamics.

The main disadvantage of the integral nonlocal constitutive relations is their complexity form for using to solve particular problems in nanomechanics. According to the paper (Eringen, 1983), Eringen is presented a differential form of constitutive relations, for the one-dimensional case follows

$$\sigma_{xx} - \mu \frac{\mathrm{d}^2 \sigma_{xx}}{\mathrm{d}x^2} = E\varepsilon_{xx}, \tag{2a}$$

$$\sigma_{xz} - \mu \frac{\mathrm{d}^2 \sigma_{xz}}{\mathrm{d}x^2} = G\gamma_{xz}, \tag{2b}$$

where E and G are elastic modulus and shear modulus of the beam, respectively; $\mu = (e_0 a)^2$ is the nonlocal parameter (length scales), σ_{xx}, σ_{xz} are normal and shear nonlocal stresses, respectively, and $\varepsilon_{xx} = \partial u/\partial x$ is axial deformation. As the exact value of nonlocal parameter is scattered and depends on various parameters, in the present study, the free vibration analysis of CMNRS is carried out assuming for $e_0 a$ to be from 0 to 2 [nm]. If $e_0 a = 0$, i.e. there is no influence of non-localness (same as in macro-scale modeling) we get back to normal stress–strain relation. The constitutive relation for nonlocal viscoelastic body can be obtained by combining nonlocal elasticity and viscoelasticity theory (Lei et al., 2013b). Therefore, for one-dimensional nonlocal viscoelastic solids, constitutive relations for Kelvin–Voigt viscoelastic model are given by

$$\sigma_{xx} - \mu \frac{\mathrm{d}^2 \sigma_{xx}}{\mathrm{d}x^2} = E(\varepsilon_{xx} + \tau_d \dot{\varepsilon}_{xx}), \tag{3a}$$

$$\sigma_{xz} - \mu \frac{\mathrm{d}^2 \sigma_{xz}}{\mathrm{d}x^2} = G(\gamma_{xz} + \tau_d \dot{\gamma}_{xz}), \tag{3b}$$

where τ_d is the viscous damping coefficient of nanorod. If $\tau_d = 0$, i.e. there is no influence of internal viscosity we get back to nonlocal elastic constitutive relation. In the next section, we derived partial differential equations based on nonlocal viscoelastic constitutive relations (3a) for coupled system of nanorods.

2.2. The dynamic equations of VMNRS

Let us consider a system of m viscoelastic nanorods coupled by linear viscoelastic layer for two types of boundary conditions, Clamped–Clamped and Clamped–Free, as shown in Fig. 2. Also, on these figures shows two different cases of connections VMNRS with a fixed base, so-called chain systems. In the first case of chain system i.e. "Clamped-Chain", it is assumed that the first and last

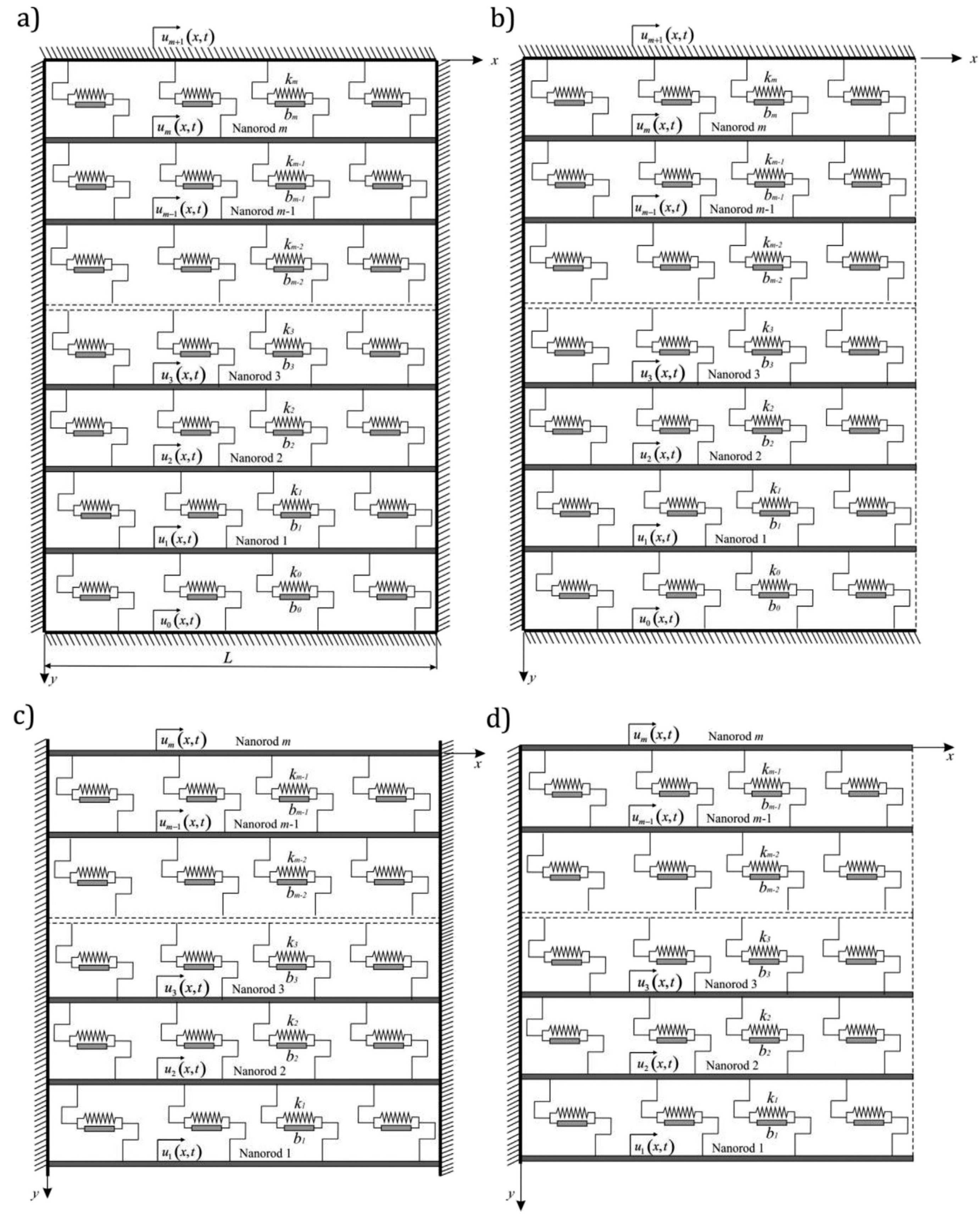

Fig. 2. The VMNRS for different boundary conditions: a) Clamped-Chain system with C–C boundary conditions, b) Clamped-Chain system with C–F boundary conditions, c) Free-Chain system with C–C boundary conditions, d) Free-Chain system with C–F boundary conditions – Mechanical model.

nanorod are connected with fixed base by viscoelastic layers of stiffness k_0 and k_m and damping b_0 and b_m (Fig. 2a) and b)).

For the second case of chain system or "Free-Chain", it is assumed that the first and last nanorods without connection with fixed base i.e. parameters of the first and last viscoelastic layer are equal to zero ($k_0 = k_m = 0$ and $b_0 = b_m = 0$) (Fig. 2c) and d)). The other nanorods are also joined by an axially distributed viscoelastic layers with stiffness and damping per length denote as $k_1 = k_2 = \ldots = k_i = \ldots = k_{m-1} = k$ and $b_1 = b_2 = \ldots = b_i = \ldots = b_{m-1} = b$, respectively. The system of m nanorods is referred to as nanorod 1, nanorod 2 and so on to m-th nanorod. Also, it should be noted that the all nanorods are made of same viscoelastic materials with coefficient of internal damping τ_d, elastic modulus E, mass density ρ, uniform cross-section of area A and length L. The axial displacement of the i-th nanorods is $u_i(x,t)$.

Consider now the axial motion of an infinitesimal element of i-th nanorod of VMNRS, as shown in Fig. 3. Based on D' Alembert's principle, summing all forces in the x-direction gives equation of motion in following form

$$\frac{\mathrm{d}N_i}{\mathrm{d}x}\mathrm{d}x + F_i\mathrm{d}x - F_{i-1}\mathrm{d}x = \ddot{u}_i\mathrm{d}m, \tag{4}$$

where F_i and F_{i-1} are external forces which results from viscoelastic layers; $\mathrm{d}m$ is mass of the infinitesimal element; $N_i(x,t)$ is the stress resultant, defined as

$$F_i = k_i(u_{i+1} - u_i) + b_i(\dot{u}_{i+1} - \dot{u}_i), \quad F_{i-1} = k_{i-1}(u_i - u_{i-1}) + b_{i-1}(\dot{u}_i - \dot{u}_{i-1}), \tag{5a}$$

$$\mathrm{d}m = \rho A\mathrm{d}x, \tag{5b}$$

$$N_i(x,t) = \int_A \sigma_{xx}(x,t)\mathrm{d}A. \tag{5c}$$

By substituting Eq. (5c) into the Eq. (3a), using that result and Eqs. (5a), (5b) and Eq. (4), we get the following equation of motion expressed in terms of the displacement $u_i(x,t)$ for axial vibration of the i-th nanorod

$$\begin{aligned}&\overline{m}\ddot{u}_i - \overline{e}\left(\frac{\mathrm{d}^2u_i}{\mathrm{d}x^2} + \tau_d\frac{\mathrm{d}^2\dot{u}_i}{\mathrm{d}x^2}\right) + k_i(u_i - u_{i+1}) + b_i(\dot{u}_i - \dot{u}_{i+1}) + k_{i-1}(u_i - u_{i-1}) + b_{i-1}(\dot{u}_i - \dot{u}_{i-1})\\ &= \mu\frac{\mathrm{d}^2}{\mathrm{d}x^2}[\overline{m}\ddot{u}_i + k_i(u_i - u_{i+1}) + b_i(\dot{u}_i - \dot{u}_{i+1}) + k_{i-1}(u_i - u_{i-1}) + b_{i-1}(\dot{u}_i - \dot{u}_{i-1})], \quad i = 1,2,\ldots,m\end{aligned} \tag{6}$$

and

$$\mathrm{EA} = \overline{e} = \text{constant}, \tag{7a}$$

$$\rho A = \overline{m} = \text{constant}, \tag{7b}$$

where $\overline{e}$ and $\overline{m}$ denotes axial rigidity and mass per unit length, respectively.

Setting the axial displacement $u_0(x,t)$ and $u_{m+1}(x,t)$ to zero in Eq. (6), we can obtain equations of motion for the "Clamped-Chain" system, where $k_0 = k_m = k$ and$b_0 = b_m = b$, (Fig. 2a) and b)) as follows

$$\begin{aligned}&\overline{m}\ddot{u}_1 - \overline{e}\left(\frac{\mathrm{d}^2u_1}{\mathrm{d}x^2} + \tau_d\frac{\mathrm{d}^2\dot{u}_1}{\mathrm{d}x^2}\right) + k(u_1 - u_2) + b(\dot{u}_1 - \dot{u}_2) + ku_1 + b\dot{u}_1\\ &= \mu\frac{\mathrm{d}^2}{\mathrm{d}x^2}[\overline{m}\ddot{u}_1 + k(u_1 - u_2) + b(\dot{u}_1 - \dot{u}_2) + ku_1 + b\dot{u}_1],\end{aligned} \tag{8a}$$

$$\begin{aligned}&\overline{m}\ddot{u}_i - \overline{e}\left(\frac{\mathrm{d}^2u_i}{\mathrm{d}x^2} + \tau_d\frac{\mathrm{d}^2\dot{u}_i}{\mathrm{d}x^2}\right) + k(u_i - u_{i+1}) + b(\dot{u}_i - \dot{u}_{i+1}) + k(u_i - u_{i-1}) + b(\dot{u}_i - \dot{u}_{i-1})\\ &= \mu\frac{d^2}{dx^2}[\overline{m}\ddot{u}_i + k(u_i - u_{i+1}) + b(\dot{u}_i - \dot{u}_{i+1}) + k(u_i - u_{i-1}) + b(\dot{u}_i - \dot{u}_{i-1})], \quad i = 2,\ldots,m-1\end{aligned} \tag{8b}$$

$$\begin{aligned}&\overline{m}\ddot{u}_m - \overline{e}\left(\frac{\mathrm{d}^2u_m}{\mathrm{d}x^2} + \tau_d\frac{\mathrm{d}^2\dot{u}_m}{\mathrm{d}x^2}\right) + ku_m + b\dot{u}_m + k(u_m - u_{m-1}) + b(\dot{u}_m - \dot{u}_{m-1})\\ &= \mu\frac{\mathrm{d}^2}{\mathrm{d}x^2}[\overline{m}\ddot{u}_i + ku_m + b\dot{u}_m + k(u_m - u_{m-1}) + b(\dot{u}_m - \dot{u}_{m-1})],\end{aligned} \tag{8c}$$

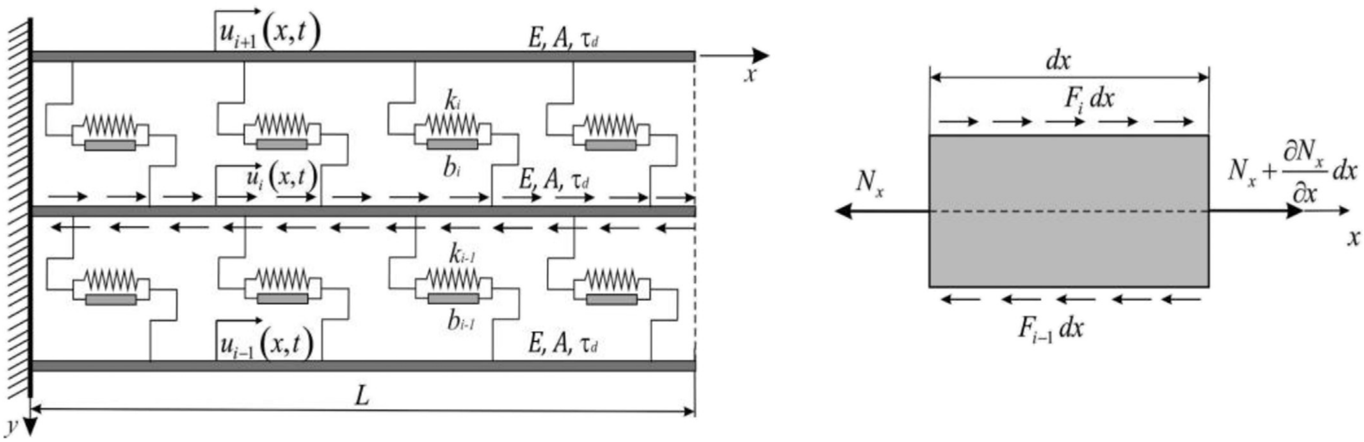

Fig. 3. The *i-th* nanorod (left) of VMNRS and differential element of the *i-th* nanorod (right).

For the case of "Free-Chain" system (Fig. 2c) and d)) we can obtain equations of motion, by setting $k_0 = k_m = 0$ and $b_0 = b_m = 0$, in Eq. (6), yields

$$\overline{m}\ddot{u}_1 - \overline{e}\left(\frac{d^2u_1}{dx^2} + \tau_d\frac{d^2\dot{u}_1}{dx^2}\right) + k(u_1 - u_2) + b(\dot{u}_1 - \dot{u}_2) = \mu\frac{d^2}{dx^2}[\overline{m}\ddot{u}_1 + k(u_1 - u_2) + b(\dot{u}_1 - \dot{u}_2)], \tag{9a}$$

$$\overline{m}\ddot{u}_i - \overline{e}\left(\frac{d^2u_i}{dx^2} + \tau_d\frac{d^2\dot{u}_i}{dx^2}\right) + k(u_i - u_{i+1}) + b(\dot{u}_i - \dot{u}_{i+1}) + k(u_i - u_{i-1}) + b(\dot{u}_i - \dot{u}_{i-1}) = \mu\frac{d^2}{dx^2}[\overline{m}\ddot{u}_i + k(u_i - u_{i+1}) + b(\dot{u}_i - \dot{u}_{i+1}) + k(u_i - u_{i-1}) + b(\dot{u}_i - \dot{u}_{i-1})], \quad i = 2, \ldots, m-1, \tag{9b}$$

$$\overline{m}\ddot{u}_m - \overline{e}\left(\frac{d^2u_m}{dx^2} + \tau_d\frac{d^2\dot{u}_m}{dx^2}\right) + k(u_m - u_{m-1}) + b(\dot{u}_m - \dot{u}_{m-1}) = \mu\frac{d^2}{dx^2}[\overline{m}\ddot{u}_i + k(u_m - u_{m-1}) + b(\dot{u}_m - \dot{u}_{m-1})] \tag{9c}$$

The boundary conditions for Clamped–Clamped (see Fig. 2a) and c)) and Clamped–Free (see Fig. 2b) and d)), may be written as follows

Clamped–Clamped:

$$u_i(0,t) = u_i(L,t) = 0, \tag{10}$$

Clamped–Free:

$$u_i(0,t) = N_i(L,t) = 0, \tag{11}$$

where $N_i(L,t)$, $(i = 1,2,..,m)$ are stress resultants on the right side of a set of m nanorods.

By substituting Eqs. (4) and (5) into Eq. (3a) we get the stress resultant force N_i as follows

$$N_i(x,t) = \mu\frac{d^2N_i}{dx^2} + \overline{e}\left(\frac{du_i}{dx} + \tau_d\frac{d\dot{u}_i}{dx}\right) = \mu\frac{d}{dx}[\overline{m}\ddot{u}_i + k(u_i - u_{i+1}) + b(\dot{u}_i - \dot{u}_{i+1}) + k(u_i - u_{i-1}) + b(\dot{u}_i - \dot{u}_{i-1})] + \overline{e}\left(\frac{du_i}{dx} + \tau_d\frac{d\dot{u}_i}{dx}\right). \tag{12}$$

Introducing Eq. (13) into the boundary conditions (12), yields

$$N_i(L,t) = \mu\frac{d}{dx}[\overline{m}\ddot{u}_i(L,t) + k(u_i(L,t) - u_{i+1}(L,t)) + b(\dot{u}_i(L,t) - \dot{u}_{i+1}(L,t)) + k(u_i(L,t) - u_{i-1}(L,t)) + b(\dot{u}_i(L,t) - \dot{u}_{i-1}(L,t))] + \overline{e}\left(\frac{du_i(L,t)}{dx} + \tau_d\frac{d\dot{u}_i(L,t)}{dx}\right) = 0. \tag{13}$$

3. Solution of the problem

3.1. Complex natural frequencies

Using the method of separation of variables the general solutions of dynamics equation of motion Eq. (6) and boundary conditions Eqs. (10) and (11) are taken in the form

$$u_i(x,t) = \sum_{n=1}^{\infty} U_{in} \sin\ \alpha_n x e^{i\omega_n t}, \tag{14}$$

where for *Clamped–Clamped* boundary conditions, (Murmu and Adhikari, 2010) and (Graff, 1975, pp. 91), we have

$$\alpha_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots, \infty, \tag{15}$$

for *Clamped–Free* boundary conditions (Murmu and Adhikari, 2010; Graff, 1975), we have

$$\alpha_n = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, \ldots, \infty, \tag{16}$$

where $i = -1$, U_{in} is the amplitude and ω_n is natural frequency in n-th mode of vibration.

By substituting general solutions (14) into the Eq. (6), we obtain system of m algebraic equations as

$$-v_{i-1n}U_{i-1n} + S_{in}U_{in} - v_{in}U_{i+1n} = 0, \quad i = 1, 2, 3, \ldots, m, \tag{17}$$

where

$$S_{in} = \alpha_n^2\overline{e}(1 + i\omega_n\tau_d) - \overline{m}\omega_n^2\left(1 + \mu\alpha_n^2\right) + v_{in} + v_{i-1n}, \tag{18a}$$

$$v_{in} = k_i\left(1 + \mu\alpha_n^2\right) + i\omega_n b_i\left(1 + \mu\alpha_n^2\right), \tag{18b}$$

$$v_{i-1n} = k_{i-1}\left(1 + \mu\alpha_n^2\right) + i\omega_n b_{i-1}\left(1 + \mu\alpha_n^2\right). \tag{18c}$$

For a homogenous system of algebraic equations (17), analytical expressions for complex eigenvalues of VMNRS can be determined using the trigonometric method (Rašković, 1953, 1963, 1957; Stojanović et al., 2013). According to the paper by Rašković (1963), solution of the *i-th* algebraic equation is assumed in the following form

$$U_{in} = N\cos(i\,\varphi) + M\sin(i\varphi), \quad i = 1, 2, 3, \ldots, m. \tag{19}$$

where unknown parameter φ which will be determined in the following. Introducing Eq. (19) into the *i-th* algebraic equation of system (17), and assuming that the all nanorods have the same material and geometrical properties and connected with the same viscoelastic layers, we get two trigonometric equations

$$N\{-v_n\cos[(i-1)\varphi] + S_n\cos(i\varphi) - v_n\cos[(i+1)\varphi]\} = 0, \quad i = 2, 3, \ldots, m-1, \tag{20a}$$

$$M\{-v_n\sin[(i-1)\varphi] + S_n\sin(i\varphi) - v_n\sin[(i+1)\varphi]\} = 0, \quad i = 2, 3, \ldots, m-1, \tag{20b}$$

where

$$S_n = \alpha_n^2\overline{e}(1 + i\omega_n\tau_d) - \overline{m}\omega_n^2\left(1 + \mu\alpha_n^2\right) + 2v_n, \tag{21a}$$

$$v_n = k\left(1 + \mu\alpha_n^2\right) + i\omega_n b\left(1 + \mu\alpha_n^2\right), \tag{21b}$$

under the condition that the constants M and N are not simultaneously equal to zero.

After some algebra, Eq. (20) can be written as

$$(S_n - 2v_n\cos\varphi)N\cos(i\varphi) = 0, \tag{22a}$$

$$(S_n - 2v_n\cos\varphi)M\sin(i\varphi) = 0. \tag{22b}$$

From Eqs. (20) and (22), we can conclude that, $N \neq 0$ and $cos(i\varphi) \neq 0$ or $M \neq 0$ and $sin(i\varphi) \neq 0$ in order the system had an oscillatory behavior, for i = 2,3,…,m−1. Now it gets frequency equation

$$S_n = 2v_n\cos\varphi. \tag{23}$$

Introducing Eq. (21) into Eq. (23), we get the frequency equation in the following form

$$\begin{aligned} -\overline{m}\lambda_n\omega_n^2 + i\left[\alpha_n^2\overline{e}\tau_d + 2b\lambda_n(1 - \cos\varphi)\right]\omega_n \\ + \left[\alpha_n^2\overline{e} + 2k\lambda_n(1 - \cos\varphi)\right] = 0, \end{aligned} \tag{24}$$

where $\lambda_n = (1 + \mu\alpha_n^2)$.

Now we can express the complex natural frequency from quadratic equation (24), in the following form

$$\omega_{n(1/2)} = i\left[\frac{\alpha_n^2\overline{e}\tau_d + 2b\lambda_n(1 - \cos\varphi)}{2\overline{m}\lambda_n}\right] \pm \frac{\sqrt{4\overline{m}\lambda_n\left[\alpha_n^2\overline{e} + 2k\lambda_n(1 - \cos\varphi)\right] - \left[\alpha_n^2\overline{e}\tau_d + 2b\lambda_n(1 - \cos\varphi)\right]^2}}{2\overline{m}\lambda_n} \tag{25}$$

Equation (25) is a general solution of the complex natural frequencies of VMNRS for m coupled nanorods.

3.2. Chain systems of VMNRS

In this subsection we will discusses about the upper and lower boundary condition for both chain system, “Clamped-Chain” and “Free-Chain”, and method for determination of the unknown φ. It is also assumed that all four cases of VMNRS shown Fig. 2, composed of m identical nanorods with same material and geometrical properties and coupled by viscoelastic layers with same characteristics (stiffness and damping coefficients). Introducing assumed solutions Eq. (14) into a set of m partial differential equations for

"Clamped-Chain" Eq. (8) and "Free-Chain" Eq. (9), we obtain two homogenous systems of algebraic equations in matrix form as

for *Clamped-Chain system*:

$$\begin{bmatrix} S_n & -\upsilon_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\upsilon_n & S_n & -\upsilon_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & S_n & -\upsilon_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\upsilon_n & S_n & -\upsilon_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\upsilon_n & S_n & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -\upsilon_n & S_n & -\upsilon_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\upsilon_n & S_n \end{bmatrix} \begin{bmatrix} U_{1n} \\ U_{2n} \\ U_{3n} \\ \dots \\ U_{i-1n} \\ U_{in} \\ U_{i+1n} \\ \dots \\ U_{m-2n} \\ U_{m-1n} \\ U_{mn} \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{26}$$

for *Free-Chain system*:

$$\begin{bmatrix} S_n-\upsilon_n & -\upsilon_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\upsilon_n & S_n & -\upsilon_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & S_n & -\upsilon_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\upsilon_n & S_n & -\upsilon_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\upsilon_n & S_n & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -\upsilon_n & S_n & -\upsilon_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\upsilon_n & S_n-\upsilon_n \end{bmatrix} \begin{bmatrix} U_{1n} \\ U_{2n} \\ U_{3n} \\ \dots \\ U_{i-1n} \\ U_{in} \\ U_{i+1n} \\ \dots \\ U_{m-2n} \\ U_{m-1n} \\ U_{mn} \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{27}$$

where expressions for S_n and υ_n are defined in Eq. (21).

Now we determine the unknown parameter φ, from the upper and lower boundary conditions, i.e. equation (19) must satisfy the first and the last equation of system of algebraic equations Eq. (26) for "*Clamped-Chain*" and Eq. (27) for "*Free-Chain*".

Let us first consider the case of "*Clamped-Chain*" system. If introducing expressions $U_{1n} = N\cos\,\varphi + M\,\sin\,\varphi$ and $U_{2n} = N\cos(2\varphi) + M\,\sin(2\varphi)$ into first equation and $U_{m-1n} = N\cos[(m-1)\varphi] + M\,\sin[(m-1)\varphi]$ and $U_{m-1n} = N\cos(m\varphi) + M\,\sin(m\varphi)$ into last equation of the system (26), we obtain system of algebraic equations

$$N[S_n\cos\,\varphi - \upsilon_n\cos(2\varphi)] + M[S_n\sin\,\varphi - \upsilon_n\sin(2\varphi)] = 0, \tag{28a}$$

$$N[S_n\cos(m\varphi) - \upsilon_n\cos[(m-1)\varphi]] + M[S_n\sin(m\varphi) - \upsilon_n\sin[(m-1)\varphi]] = 0. \tag{28b}$$

Non-trivial solutions for the constants N and M can be obtained, which yields the following trigonometric equation:

$$\begin{vmatrix} 1 & 0 \\ cos[(m+1)\varphi] & sin[(m+1)\varphi] \end{vmatrix} = 0 \quad\Rightarrow\quad sin[(m+1)\varphi] = 0, \tag{29}$$

from which we obtain solutions for unknown φ as

$$\varphi_{cc,s} = \frac{s\pi}{m+1}, \quad s = 1, 2, \dots, m. \tag{30}$$

where $\varphi_{cc,s}$ is unknown parameter φ for "Clamped-Chain" system.

Using the procedures described for previously case, we again introducing the assumed solution $U_{1n} = N\cos\,\varphi + M\,\sin\,\varphi$ and $U_{2n} = N\cos\,(2\varphi) + M\,\sin\,(2\varphi)$ into first equation and $U_{m-1n} = N\cos[(m-1)\varphi] + M\,\sin[(m-1)\varphi]$ and $U_{m-1n} = N\cos(m\varphi) + M\,\sin(m\varphi)$ into last equation of the system (27), after some algebra we obtain again system of algebraic equations in following form

$$N[(S_n-\upsilon_n)\cos\,\varphi - \upsilon_n\cos(2\varphi)] + M[(S_n-\upsilon_n)\sin\,\varphi - \upsilon_n\sin(2\varphi)] = 0, \tag{31a}$$

$$N[(S_n-\upsilon_n)\cos(m\varphi) - \upsilon_n\cos[(m-1)\varphi]] + M[(S_n-\upsilon_n)\sin(m\varphi) - \upsilon_n\sin[(m-1)\varphi]] = 0. \tag{31b}$$

Non-trivial solutions for the constants N and M can be obtained, which yields trigonometric equation in the following form

$$\begin{vmatrix} 1-\cos\,\varphi & -\sin\,\varphi \\ \cos[(m+1)\varphi] - \cos(m\varphi) & \sin[(m+1)\varphi] - \sin(m\varphi) \end{vmatrix} = 0 \Rightarrow \sin(m\varphi) = 0, \tag{32}$$

from which we obtain solutions for unknown φ as

$$\varphi_{fc,s} = \frac{s\pi}{m}, \quad s = 0, 1, \dots, m-1. \tag{33}$$

where $\varphi_{fc,s}$ is unknown parameter φ for "Free-Chain" system.

Now substituting expression for $\varphi_{cc,s}$ into Eq. (25) and $\varphi_{fc,s}$ into Eq. (25), we obtain expressions for complex natural frequencies for both cases, respectively,

"*Clamped-Chain*" system

$$\omega_{cc,n(1/2)} = i\left[\frac{\alpha_n^2\overline{e}\tau_d + 2b\lambda_n(1-\cos\ \varphi_{cc,s})}{2\overline{m}\lambda_n}\right] \pm \frac{\sqrt{4\overline{m}\lambda_n\left[\alpha_n^2\overline{e} + 2k\lambda_n(1-\cos\ \varphi_{cc,s})\right] - \left[\alpha_n^2\overline{e}\tau_d + 2b\lambda_n(1-\cos\ \varphi_{cc,s})\right]^2}}{2\overline{m}\lambda_n}, \tag{34}$$

and "*Free-Chain*" system

$$\omega_{fc,n(1/2)} = i\left[\frac{\alpha_n^2\overline{e}\tau_d + 2b\lambda_n\left(1-\cos\ \varphi_{fc,s}\right)}{2\overline{m}\lambda_n}\right] \pm \frac{\sqrt{4\overline{m}\lambda_n\left[\alpha_n^2\overline{e} + 2k\lambda_n\left(1-\cos\ \varphi_{fc,s}\right)\right] - \left[\alpha_n^2\overline{e}\tau_d + 2b\lambda_n\left(1-\cos\ \varphi_{fc,s}\right)\right]^2}}{2\overline{m}\lambda_n}. \tag{35}$$

It should be noted that real part of the complex eigenvalues represents the natural frequency of the system while the imaginary part represents dumping of the system.

3.3. Asymptotic analysis

Suppose that the number of mode tends to infinity i.e. $n \to \infty$ introducing in frequency equation (24), an asymptotic equation for the complex natural frequencies can be obtain as

$$\omega_{n\to\infty(1/2)} = i\left[\frac{\overline{e}\tau_d + 2b\mu(1-\cos\ \varphi)}{2\overline{m}\mu}\right] \pm \frac{\sqrt{4\overline{m}\mu[\overline{e} + 2k\mu(1-\cos\ \varphi)] - [\overline{e}\tau_d + 2b\mu(1-\cos\ \varphi)]^2}}{2\overline{m}\mu}, \tag{36}$$

where φ can take the values for both chain systems Eq. (30) or Eq. (33). Expression (36) is representing complex natural frequencies which are independent of the boundary conditions of the system.

In the next case we consider asymptotic equation for the complex natural frequencies when a number of nanorods tend to infinity, i.e. $m \to \infty$ introducing in frequency equations Eq. (30) or Eq. (33), which implies

$$\omega_{m\to\infty n(1/2)} = i\frac{\alpha_n^2\overline{e}\tau_d}{2\overline{m}\lambda_n} \pm \frac{\sqrt{4\overline{m}\lambda_n\alpha_n^2\overline{e} - \left(\alpha_n^2\overline{e}\tau_d\right)^2}}{2\overline{m}\lambda_n}. \tag{37}$$

From Eq. (37), we can concluded that the asymptotic complex natural frequencies are independent of influence of viscoelastic layers, and it is also represent a lowest complex natural frequency of the system for both chain system, "Clamped-Chain" and "Free-Chain".

Now, consider the case when the number of modes and the number of nanorods tends to the infinity, i.e. introducing $m \to \infty$ into the Eq. (36), we get the asymptotic complex natural frequency as

$$\omega_{n\to\infty,m\to\infty(1/2)} = i\frac{\overline{e}\tau_d}{2\overline{m}\mu} \pm \frac{\sqrt{4\overline{m}\mu\overline{e} - (\overline{e}\tau_d)^2}}{2\overline{m}\mu}. \tag{38}$$

Expression (38) is representing the fundamental complex natural frequency of the system when the number of nanorods and the number of modes tends to the infinite, and also is the same for both boundary conditions (Clamped–Clamped and Clamped–Free) and both chain systems ("Clamped-Chain" and "Free-Chain").

The critical damping rations are obtained by setting the natural frequencies to zero, from Eqs. 36–38 may be written as

$$\tau_{cr,n\to\infty} = \frac{\sqrt{4\overline{m}\mu[\overline{e} + 2k\mu(1-\cos\ \varphi)]} - 2b\mu(1-\cos\ \varphi)}{\overline{e}}, \tag{39a}$$

$$\tau_{cr,m\to\infty} = \frac{\sqrt{4\overline{m}\lambda_n\alpha_n^2\overline{e}}}{\alpha_n^2\overline{e}}, \tag{39b}$$

$$\tau_{\substack{cr,m\to\infty\\ n\to\infty}} = \frac{\sqrt{4\overline{m}\mu\overline{e}}}{\overline{e}}. \tag{39c}$$

The obtained analytical expression of the critical damping rations is function of the material parameters of nanorods and independent of boundary condition and chain systems, for the case when the number of nanorods and the number of modes tends to the infinite.

4. Numerical results and discussion

The analytical model for VMNRS presented here is the generalized theory, which includes the damping effect of the system, and therefore represents a more realistic case. This model can be applied for the axial vibration analysis of m coupled carbon nanotubes system, ZnO nanorods system and it is also useful for the study of vibration behavior of multiple-nanobeam and nanoplates system for nanoresonator application. The first part of this section is relates to the comparison of the results obtained by applying trigonometric method with available data in the literature (Murmu and Adhikari, 2010) for special case of VMNRS when $m = 2$, and also with the results obtained by using

Table 1
Validation of first four complex eigenvalues of the "*Free-Chain*" VMNRS for C–F boundary conditions and different values of viscoelastic constant τ_d and nonlocal parameter e_0a.

C–F		$e_0a = 0$ nm	$e_0a = 0.5$ nm	$e_0a = 1$ nm	$e_0a = 1.5$ nm	$e_0a = 2$ nm
Murmu and Adhikari (2010)						
$\tau_d = 0$ ns	1	1.5708	1.2353	0.8436	0.6137	0.4764
$b = 0$ Nns/nm	2	4.2974	4.1864	4.0880	4.0468	4.0283
$k = 8$ N/nm	3	4.7124	1.8411	0.9782	0.6601	0.4972
	4	6.1812	4.4033	4.1179	4.0541	4.0308
"Free-Chain" VMNRS – Trigonometric method						
$\tau_d = 0.001$ ns	1	1.5708 + 0.0012i	1.2353 + 0.0007i	0.8435 + 0.0003i	0.6136 + 0.0001i	0.4764 + 0.0001i
$b = 0.01$ Nns/nm	2	4.2973 + 0.0112i	4.1864 + 0.0107i	4.0879 + 0.0103i	4.0467 + 0.0101i	4.0282 + 0.0101i
$k = 8$ N/nm	3	4.7123 + 0.0111i	1.8410 + 0.0016i	0.9782 + 0.0004i	0.6600 + 0.0002i	0.4972 + 0.0001i
$m = 2$	4	6.1811 + 0.0211i	4.4033 + 0.0116i	4.1178 + 0.0104i	4.0540 + 0.0102i	4.0307 + 0.0101i
$\tau_d = 0.004$ ns	1	1.5707 + 0.0049i	1.2353 + 0.0030i	0.8435 + 0.0014i	0.6136 + 0.0007i	0.4764 + 0.0004i
$b = 0.01$ Nns/nm	2	4.2973 + 0.0149i	4.1863 + 0.0130i	4.0879 + 0.0114i	4.0467 + 0.0107i	4.0282 + 0.0104i
$k = 8$ N/nm	3	4.7121 + 0.0444i	1.8410 + 0.0067i	0.9782 + 0.0019i	0.6600 + 0.0008i	0.4972 + 0.0004i
$m = 2$	4	6.1809 + 0.0544i	4.4033 + 0.0167i	4.1178 + 0.0119i	4.0540 + 0.0108i	4.0307 + 0.0104i

the numerical methods for the general case of VMNRS when $m > 2$. The following parameter has been considered in the comparative analysis proposed in ref. Murmu and Adhikari (2010): $L = 1$ [nm], $\overline{m} = 10^{-9}$ [kg/m], $e_0a = 0$–2 [nm], stiffness coefficient $K = 8$ N/nm, and two different values of viscoelastic parameters $\tau_d = 0.001$ [ns] and 0.004 [ns]. The complex eigenvalue of two coupled viscoelastic nanorods system obtained by trigonometric method for different values of nonlocal and viscoelastic parameters are presented in Table 1. This table illustrate that the both parts of complex eigenvalues i.e. natural frequency and damping of the system decrease with increase of nonlocal parameter. Also, it can be concluded that the increasing of viscoelastic parameters have very small effect on the natural frequency, but on the system damping has a significant influence. The present results for natural frequencies are in excellent agreement with work proposed by Murmu and Adhikari (2010). In order to confirm the accuracy of the trigonometric method for general case of VMNRS, the obtained results for the complex eigenvalue will be compared with the results obtained by using the numerical methods. These results are presented in Table 2. For this case, we consider the system which consists of three, five and ten ($m = 3, 5, 10$) identical nanorods coupled by viscoelastic layers. The analytical solutions obtained from Eqs. (34) and (35) are validated with results obtained by numerical solutions of homogenous system of algebraic equations given in Eqs. (26) and (27), for both boundary conditions. Based on the presented results we can concluded that the values of complex eigenvalue obtained by analytical and numerical methods in excellent agreement.

However, for a more general and detailed vibration analysis of VMNRS, a system of m coupled viscoelastic single walled carbon nanotube is used as an example in the second part of this section. Also, the influence of nonlocal parameter, the number of nanotubes, the number of mode and viscoelastic parameter on the complex eigenvalues for both boundary conditions and both chain system are analyzed numerically. The following values are used for the numerical study: diameter of nanorod $d = 1.1$ [nm], length $L = 10$ d, Young's modulus $E = 1.1$ [TPa], the mass density $\rho = 2300$ [kg/m^3], parameters of viscoelastic layers, stiffness $k = 10$ [N/nm] and damping $b = 0.01$ [Nns/nm]. The parameter values of Kelvin–Voigt damping coefficient τ_d and nonlocal parameter μ are given in follow.

In order to demonstrate the effects of the number of nanorods and number of mode on the real and imaginary parts of complex eigenvalues of the VMNRS for both boundary conditions, surface have been plotted as function of these two parameters $n \in [1, 50]$ and $m \in [2, 50]$ shown on Figs. 4 and 5. From these figures we can notice that the effects of the above mentioned parameters on the real and imaginary part of the complex values are very similar for all cases. In general, the influence of increasing the number of mode n causing an increase in the real and imaginary parts, but increases of the number of nanorods m in the VMNRS causes a reduction of their values. However, it was found that, when the number of modes and the number of nanorods tends to the larger values ($n \to \infty$, $m \to \infty$), this leads to the asymptotic values of the real and imaginary part of the complex eigenvalues, whence we can conclude that it is independent of the number of nanorods, boundary conditions and "Chain" systems. It should be noted that asymptotic values of the real part of complex eigenvalue represents a finite value the natural frequency, beyond which vibration of the system is impossible.

The real and imaginary parts of complex eigenvalues or natural frequencies and damping ration of a VMNRS are shown as a function of the nonlocal parameter in Figs. 6–9, for different values of viscoelastic parameter $\tau_d = 0.001$ and 0.003 [ns] and number of nanorods $m = 2, 5, 15$. The presented results suggest that the both parts of the complex eigenvalue significantly influenced by nonlocal parameter μ for Clamped–Clamped boundary conditions, but influence are reduced for Clamped–-Free boundary conditions. Comparison of the results obtained for natural frequencies and damping rations by variation of the viscoelastic parameter τ_d, it is obvious that the influence on the dumping ration is very a significant, but on the natural

Table 2
Validation of complex eigenvalues of the VMNRS when number of nanorods is greater than two ($m > 2$), for both boundary conditions and chain systems.

$s = 1, n = 1$		Trigonometric method			Numerical method		
		$m = 3$	$m = 5$	$m = 10$	$m = 3$	$m = 5$	$m = 10$
C–C	Clamped Chain	2.22038 + 0.00305i	1.54513 + 0.00146i	0.94442 + 0.00052i	2.22038 + 0.00305i	1.54513 + 0.00146i	0.94442 + 0.00052i
	Free Chain	2.8712 + 0.00512i	1.81647 + 0.00203i	1.01337 + 0.00061i	2.8712 + 0.00512i	1.81647 + 0.00203i	1.01337 + 0.00061i
C–F	Clamped Chain	2.21659 + 0.00304i	1.53967 + 0.00145i	0.93547 + 0.00051i	2.21659 + 0.00304i	1.53967 + 0.00145i	0.93547 + 0.00051i
	Free Chain	2.86827 + 0.00511i	1.81183 + 0.00202i	1.00504 + 0.00060i	2.86827 + 0.00511i	1.81183 + 0.00202i	1.00504 + 0.00060i

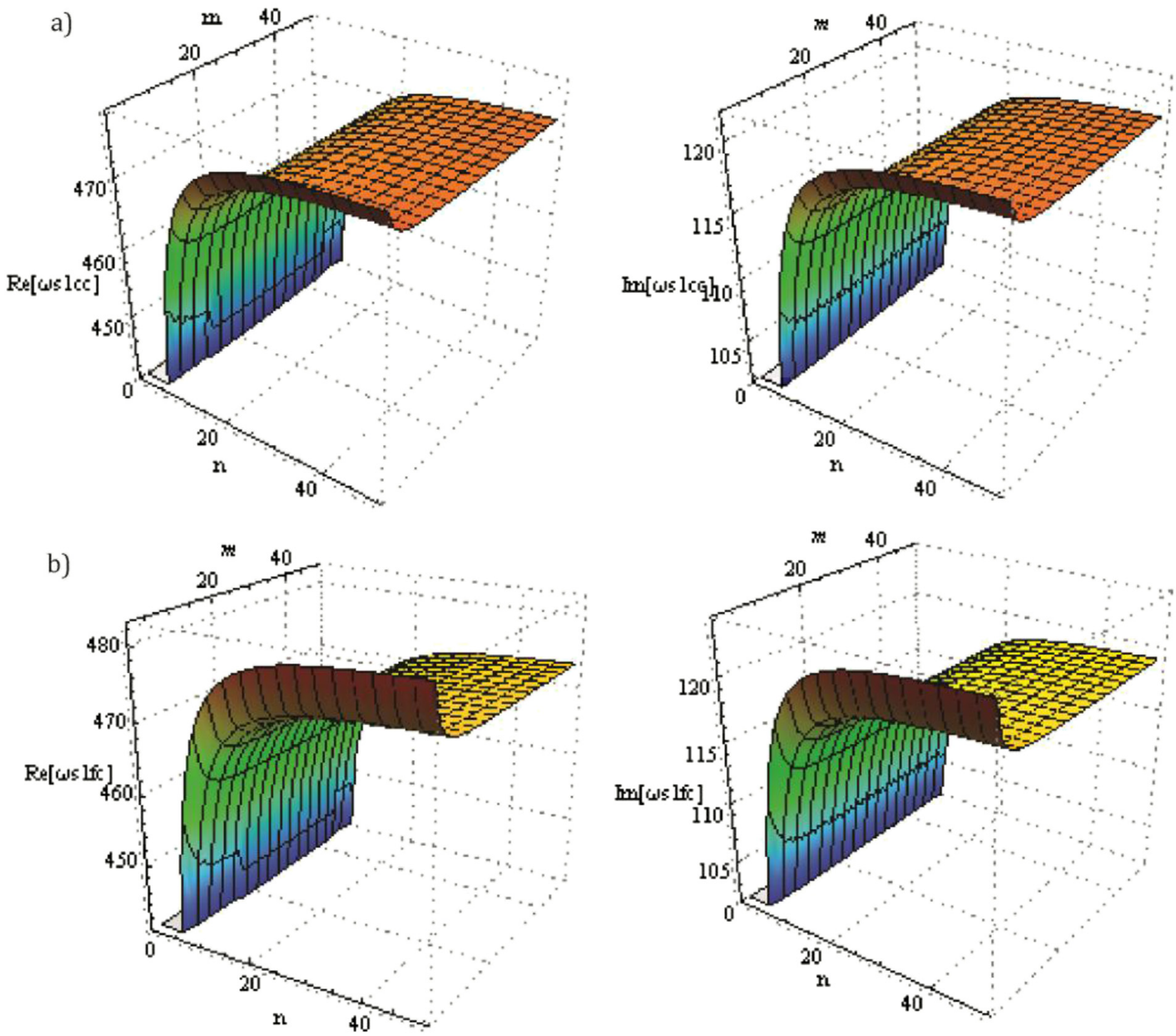

Fig. 4. The real and imaginary parts of the complex eigenvalues for Clamped–Clamped boundary conditions: a) "Clamped-Chain" system, b) "Free-Chain" system, for $s = 1$, $\mu = 2$ [nm^2], $\tau_d = 0.001$ [ns].

frequencies can be see much smaller influence. From this figures we can note a quite linear influence on the imaginary parts of complex eigenvalue. As expected, the both parts of complex eigenvalues are very sensitive to variation of the number of nanorods. The results show that increases of the number of nanorods in the system causes a reduction in both, natural frequencies and damping ratio. This is valid for all presented cases. To see the effect of the boundary conditions on natural frequencies and damping ratio for the same "Chain" systems, we can consider results presented on two figures Figs. 6 and 8 or Figs. 7 and 9. It can be observed that the both parts of complex eigenvalues have a much higher values for C–C boundary conditions than for the C–F one. Moreover, the natural frequencies for C–C boundary conditions more sensitive to the influence of the viscoelastic parameter τ_d.

5. Conclusion

This work presents an analytical and numerical investigation of the vibration behavior of the viscoelastic coupled multi-nanorods system. The set of m partial differential equations of motion are obtained based on D' Alembert's principle and nonlocal Kelvin–Voigt viscoelastic constitutive relations for both boundary conditions and both "Chain" systems. The closed form solutions of complex eigenvalues are derived in a unique manner by combining two analytical methods, namely, separations of variables and trigonometric method. We have derived analytical expressions of the asymptotic values of natural frequency and damping ratio of VMNRS, when the number of nanorods and the number of modes tends to the infinite. It is found that the both asymptotic values depend only on the material characteristics of VMNRS. In order to demonstrate the accuracy of proposed trigonometric method, we compared the analytical results with results obtained by numerical methods for the general case of VMNRS. It is shown that the application of these analytical methods provides excellent agreement when compared with the numerical results. We have also validated the analytical results obtained for the special case of VMNRS when $m = 2$, with the results found in the literature. In the presented numerical analyses, the influence of number of nanorods, number of mode, nonlocal parameter and viscoelastic constant on the both parts of complex eigenvalues are investigated. From these observations we can conclude that the real parts or a

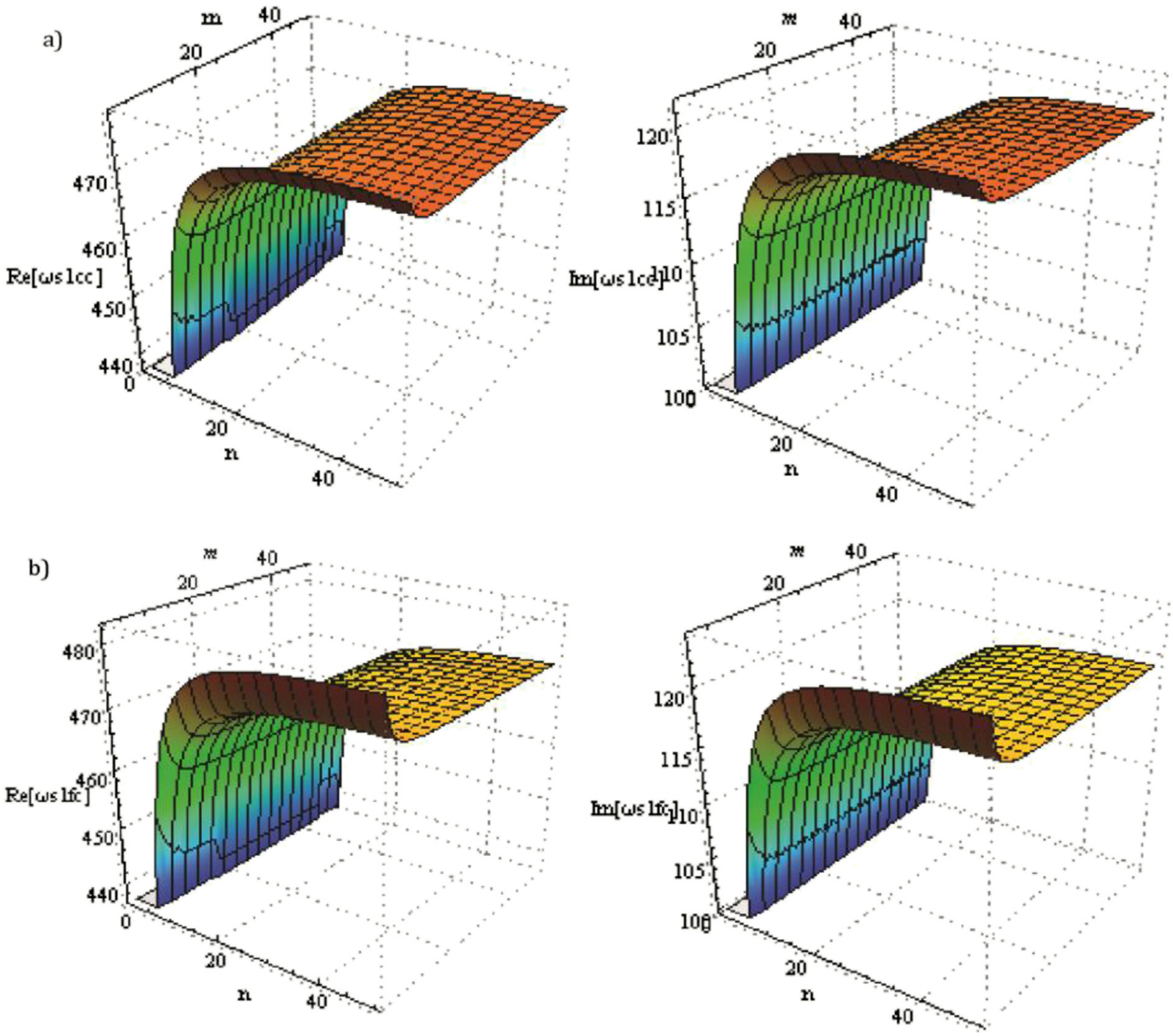

Fig. 5. The real and imaginary parts of the complex eigenvalues for Clamped–Free boundary conditions: a) "Clamped-Chain" system, b) "Free-Chain" system, for $s = 1$, $\mu = 2$ [nm^2], $\tau_d = 0.001$ [ns].

natural frequency was significantly influenced by nonlocal parameter while on the damping ration have smaller influence for both boundary conditions. Also, it can be noted that the increase in viscoelastic parameter cause increase in the damping ratio for all considered cases. Regarding the influence of the number of nanorods and the number of modes on the complex eigenvalues, it was found that when the values of the both parameters tend to the infinite their influence are vanishes.

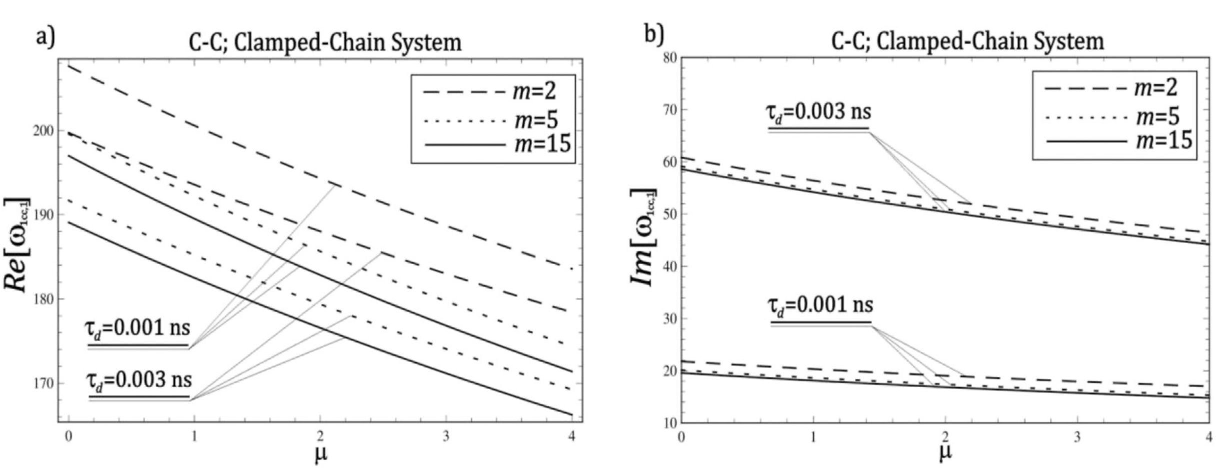

Fig. 6. The a) real and b) imaginary part of the complex eigenvalues for Clamped–Clamped boundary conditions and "Clamped-Chain" system.

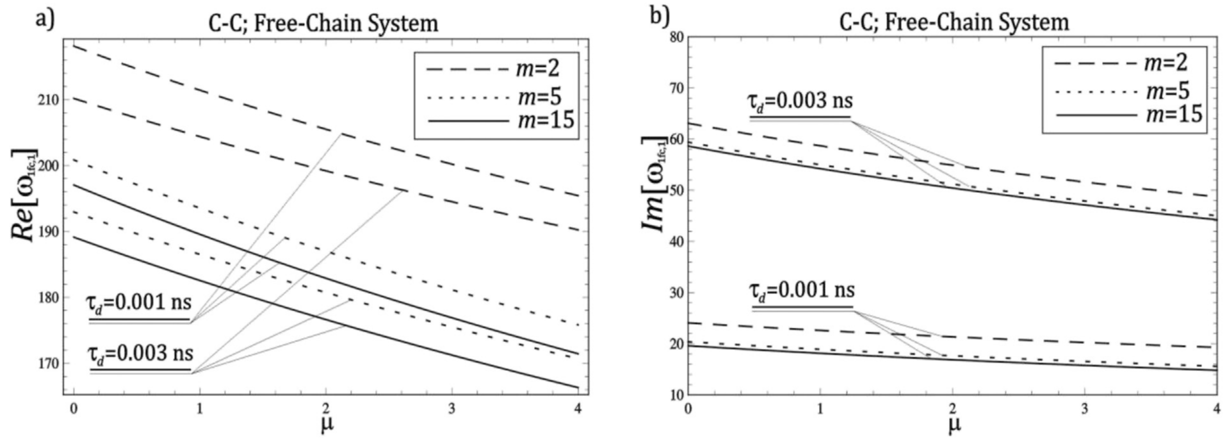

Fig. 7. The a) real and b) imaginary part of the complex eigenvalues for Clamped–Clamped boundary conditions and "Free-Chain" system.

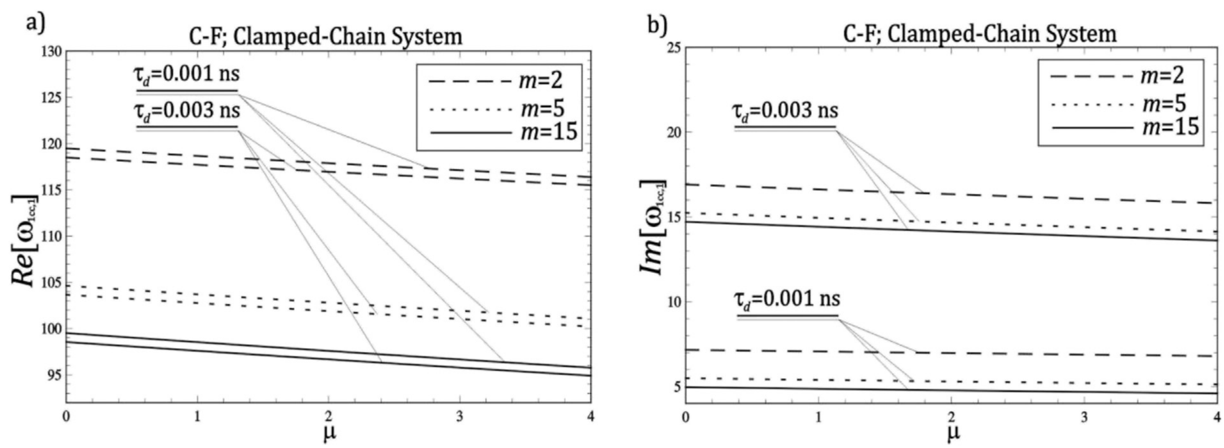

Fig. 8. The a) real and b) imaginary part of the complex eigenvalues for Clamped–Free boundary conditions and "Clamped-Chain" system.

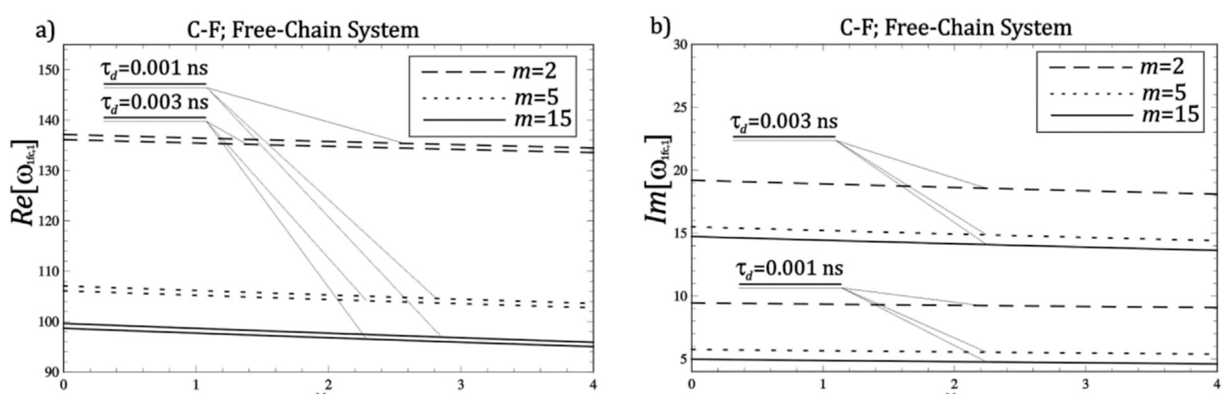

Fig. 9. The a) real and b) imaginary part of the complex eigenvalues for Clamped–Free boundary conditions and "Free-Chain" system.

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