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Nonlocal longitudinal vibration of viscoelastic coupled double-nanorod systems

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A theoretical study of the free longitudinal vibration of a nonlocal viscoelastic double-nanorod system (VDNRS) is presented in this paper. It is assumed that a light viscoelastic layer continuously couples two parallel nonlocal viscoelastic nanorods. The model is aimed at representing dynamic interactions in nanocomposite materials. The exact solution for the longitudinal vibration of a double-nanorod system is determined for two types of boundary conditions, Clamped–Clamped (C–C) and Clamped–Free (C–F). D'Alembert's principle is applied to derive the governing equations of motion in terms of the generalized displacements for a nonlocal viscoelastic constitutive equation. The solutions of a set of two homogeneous partial differential equations are obtained by using the classical Bernoulli–Fourier method. Numerical results are presented to show the effect of material length scale parameter, damping from viscoelastic constitutive equations, damping of light viscoelastic layer and boundary conditions for the free longitudinal vibration of a viscoelastic double-nanorod system.

1. Introduction

The problem of vibration of nanostructures is of considerable practical interest and has a wide application in micro and nanoscale devices such as nanosensors, nanoactuators, micro and nanoelectro-mechanical devices, nano-optomechanical systems (NOMS), drug delivery devices etc. Further, specific double-nanobeam or double-nanorod systems may be very useful in NOMS (Deotare et al., 2009; Eichenfield et al., 2009; Lin et al., 2010). Analyzing the mechanical behavior and especially resonant frequencies of such systems may provide important information for potential experiments or in design procedures of NOMS devices. Optical waveguides and cavity optomechanical systems are well-known coupled nanostructure systems where the coupling between frequency mechanical and optical modes is crucial to understand profoundly their working mechanism (Davanço et al., 2012; Ma and Povinelli, 2012; Povinelli et al., 2005). It should be noted that natural frequency as a unique property of structures increases with a decrease of length scale but a dissipation as another important characteristic of structures increases (Imboden and Mohanty, 2014). Thus, to obtain high resonant frequency systems and devices we need to include nano-scale structures which indeed should be modeled to consider both natural frequencies and dissipation as well.

There are three major methods that have been developed to simulate the dynamical behavior of nanostructures such as atomistic, atomistic–continuum and nonlocal continuum mechanics method. Unlike atomistic modeling, in continuum modeling double-nanorod is viewed as a continuous system. Nonlocal continuum models contain information about the atomic forces and internal length scale incorporate into the constitutive equations as material parameters. Therefore, nonlocal continuum theory or couple stress theory should be applied to represent a more accurate continuum model in nano-scales (Reddy and Pang, 2008; Reddy, 2010; Arash and Wang, 2012; Roque et al., 2013; Ansari and Sahmani, 2012).

Viscoelastic materials, displaying both solid-like and fluid-like characteristics, are common in polymeric structures and coatings. Therefore, any composite or complex structures with embedded polymers exhibit viscoelastic behavior during both static and dynamic loading regimes. Mathematical integer or fractional derivative type models of varying complexity can capture this behavior well, in many cases with only two or three material constants. Studying the vibrations of such a systems, in the sense of modal analysis, is significant for application in engineering. Vibration

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analysis of viscoelastic or elastic beams, rods and plates coupled with some type of elastic or viscoelastic layers is subject of many papers (Zhang et al., 2008; Rossikhin and Shitikova, 2010; Manevich and Kolakowski, 2011; Cabanska-Placzkiewicz, 2000). Interesting problem of longitudinal vibrations of a double-rod system coupled by springs and dampers is studied in the paper by Erol and Gürgöze (2004). Two rods are fixed-free supported and the forcing function is a concentrated axial sinusoidal load applied at the midpoint of the primary rod. They obtained the exact solution, by means of modal analysis, for the forced vibrations of the system and used the change of variable method to decouple the system of 2 s order partial differential equations. In the paper by Atanacković et al. (2013), forced oscillations of a rod with a body attached to its free end have been studied and described by two sets of motion equations, one of integer and the other of the fractional order. To find the solution they applied a single function of complex variable which could be defined for linear viscoelastic bodies of integer/fractional derivative type.

For viscoelastic materials, energy dissipation from the fluid-like part of the response can be separated from solid-like energy storage using a complex modulus, which is represented by real and imaginary parts named storage and loss modulus, respectively. Examples of composite structures with viscoelastic properties in nanoscale systems are recently discovered graphene–polymer composites (for more details see Kuilla et al., 2010; Srivastava et al., 2012). In most of the available articles not always nanoplates but also nanorods and nanobeams are modeled as elastic structures even they reveal viscoelastic structural damping. In the recent paper by Su et al. (2012) viscoelastic properties of the graphene oxide nanoplate have been proven by obtaining the hysteresis loops in tensile test. The authors investigated the viscoelastic properties including storage modulus, loss modulus, and damping ratio under different temperatures and frequencies. Therefore, for more accurate modeling of dynamical behavior of nanostructures it is necessary to take into account viscoelastic models as well. In the paper by Lei et al. (2013a) the dynamical behavior of nonlocal viscoelastic damped nanobeam has been investigated by using the Kelvin–Voigt viscoelastic model, velocity-dependent external damping and Timoshenko beam theory. The authors showed that nonlocal damped beams have maximum frequencies, called asymptotic frequencies, and also possess an asymptotic critical damping factor. The numerical results are presented on carbon nanotube example. Nanoplates, as a subgroup of nanostructures with a two-dimensional shape, are of great importance due to the growing application as resonators and sensors. Pouresmaeeli et al. (2013) studied the vibration characteristics of viscoelastic orthotropic nanoplate resting on viscoelastic medium which is modeled as Kelvin–Voigt foundation. The solution for the complex frequency associated with the nonlocal parameter, structural damping of the nanoplate and foundation effects is obtained in a closed-form. The obtained results are compared with other results in the literature dealing with isotropic elastic nanoplates. Ghorbanpour Arani and Roudbari (2013) gave an extensive study on vibration analysis of system with boron nitride nanotubes (BNNT) coupled with visco-Pasternak medium, with a moving nanoparticle and using piezoelectric theory and surface stress based on Euler–Bernoulli beam. The authors stated that visco-Pasternak foundation as a smart medium can be used in BNNT based systems to control the stability and vibration.

The basic concepts of the longitudinal vibration analysis of wires, rods, beams and plates can be applied to the nano-scale systems as well (Ansari et al., 2010; Akgöz and Civalek, 2013; Aydogdu and Filiz, 2011). Friswell et al. (2007) used a nonlocal viscoelastic beam model to analyze the dynamics of beams with different boundary conditions using the finite element method. The internal force of the nonlocal model is obtained as weighted average of state variables over a spatial domain via convolution integrals with spatial kernel functions that depend on a distance measure. Murmu and Adhikari (2010) presented the longitudinal vibration of a double-nanorod system. They used Eringen's nonlocal elasticity to develop the governing equations for the two rod system coupled by longitudinally directed distributed springs. Solution for nonlocal frequencies for the longitudinal vibration is obtained analytically for clamped–clamped and clamped–free boundary conditions. Numerical studies were carried out for the coupled double-carbon-nanorod system. Murmu et al. (2012) presented the vibration response of magnetically sensitive double single-walled carbon nanotube system (DSWNTS) under the influence of longitudinal magnetic field. As a result of their study the authors obtained nonlocal natural frequencies analytically and analyzed the influence of nanoscale effects and strength of longitudinal magnetic field on the synchronous and asynchronous vibration phase of the DSWNTS. Kiani (2010) presented the longitudinal vibration of tapered nanowires in the context of nonlocal continuum theory. He studied the problem for the linearly varied radii of nanowires under fixed–fixed and fixed–free boundary conditions. A perturbation technique proposed is based on the Fredholm alternative theorem. Natural frequencies, corresponding mode shapes, and phase velocities of the tapered nanowires are derived analytically up to the second-order perturbation. Kazemi-Lari et al. (2012) analyzed the non-conservative instability of cantilever carbon nanotubes (CNT) resting on viscoelastic foundation. The authors utilized three different types of viscoelastic foundation Kelvin–Voigt, Maxwell and Standard linear solid, to model the interaction between CNT and surrounding viscoelastic medium. The governing equations of motion and boundary conditions are obtained based on the nonlocal Euler–Bernoulli theory using Hamilton's principle. Gonzalez-Lopez and Fernandez-Saez (2012) studied the transverse vibrations of Euler–Bernoulli beams treated with nonlocal viscoelastic damping patches. They presented the damping behavior of the patch by spatial kernel and relaxation functions, solved the equations of motion using the method proposed by Lei et al. (2006) and analyzed influence of parameters of the model on the damped response. In the paper by Paola et al. (2013), the dynamics of a nonlocal Timoshenko beam is presented. Nonlocal effects are modeled as long-range volume forces and moments mutually exerted by non-adjacent beam segments, that contribute to the equilibrium of any beam segment along with the classical local stress resultants. Also, model is provided with elastic and viscous long-range volume forces and moments which are linearly dependent on the product of the volumes of the interacting beam segments and on generalized measures of their relative motion, based on the pure deformation modes of the beam. Numerical results were presented for a variety of nonlocal parameters. Rafiei et al. (2012) investigated vibration characteristics of non-uniform single-walled carbon nanotubes conveying fluid embedded in viscoelastic medium using nonlocal Euler–Bernoulli beam theory. Governing equations of the system are solved with the finite element method and the frequencies are obtained by solving a quadratic eigenvalue problem. The authors analyzed the effects of taper ratio, small-scale parameter and viscoelastic medium on resonant frequencies and critical steady flow velocity.

From the literature it is evident that there is a strong scientific requirement to gain an understanding of the longitudinal vibration of structures such as complex nanorods system and the mathematical modeling of such phenomena. In the present paper, we study the free longitudinal vibration of a nonlocal viscoelastic double-nanorod system (VDNRS). It is assumed that the system under consideration is composed of two nonlocal, parallel and uniform viscoelastic nanorods of the same length coupled with light Kelvin–Voigt type viscoelastic layer. The discussions are limited to cases of two types of boundary conditions, Clamped–-Clamped (C–C) and Clamped–Free (C–F). Governing partial differential equations are obtained using D'Alembert's principle and solutions are derived by using the classical Bernoulli–Fourier method. The exact solution for the case of longitudinal vibration of VDNRS is determined. To justify the presented methodology our results are validated with the corresponding one from the paper (Murmu and Adhikari, 2010) when the nonlocal effect is taken into account and (Erol and Gürgöze, 2004) when the nonlocal effect is neglected. Dissipation effects due to the internal damping of nanorods and external damping of a light viscoelastic layer are taken into account in our model of VDNRS. Light viscoelastic layer, modeled with continuously distributed springs and dampers can represent certain polymer matrix or in case of optomechanical devices springs may represent a radiation pressure or van der Waals forces and dampers may represent some external damping effects. In difference to the references Erol and Gürgöze (2004) and Murmu and Adhikari (2010), where nanorods are considered as pure elastic and no dissipation due to the internal damping occurs, in the validation study we obtained complex natural frequency where the real part is damped natural frequency and the imaginary part is damping ratio. Such given continuum based model, which is a combination of space nonlocality and time dependent viscoelastic behavior of nanostructure, is more realistic due to the obvious presence of dissipation effects at all scales and it is convenient to obtain closed form solutions. Further, effects of nonlocal parameter, damping from viscoelastic nanorods and layer and change of boundary conditions on complex eigenvalues and displacements for the free longitudinal vibration of VDNRS are presented throughout numerical examples.

2. Problem formulation

2.1. Constitutive relations

According to the nonlocal elasticity theory, we assume that the stress at a point x is observed to be a function not only on the strain at that point x but also on strains at all other points of a body. The integral form (Wang et al., 2006; Wang and Liew, 2007; Eringen and Edelen, 1972) of nonlocal constitutive relation for a three-dimensional structure can be written as

$$\sigma_{ij}(x) = \int \alpha(|x - x'|, \tau) C_{ijkl} \varepsilon_{kl}(x') \mathrm{d}V(x'), \quad \forall x \in V, \tag{1a}$$

$$\sigma_{ij,j} = 0, \tag{1b}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1c}$$

where C_{ijkl} is the elastic modulus tensor for classical isotropic elasticity; σ_{ij} and ε_{ij} are stress and strain tensors, respectively and u_i is displacement vector. With $\alpha(|x - x'|, \tau)$ we denote the nonlocal modulus or attenuation function which incorporates nonlocal effects into the constitutive equation at the reference point x produced by local strain at the source x'. The above absolute value of difference $|x - x'|$ denotes the Euclidean metric. The parameter $\tau = (e_0 a)/l$ where l is the external characteristic length (crack length, wave length), a describes internal characteristic length (lattice parameter, granular size or distance between C–C bounds) and e_0 is a constant appropriate to each material that can be identified from atomistic simulations or by using the dispersive curve of the Born-Karman model of lattice dynamics. According to the Eringen (1983), when $\tau \to 0$ then α reverts to Dirac delta measure. In that case, internal characteristic length vanishes in the limit and classical elasticity constitutive equation is obtained. Conversely, for sufficiently small internal characteristic length when $\tau \to 1$ nonlocal elasticity theory should approximate atomic lattice dynamics. Thus, α can be obtained for each particular material by matching the dispersive curves of plane waves with those of molecular dynamics simulation or atomic lattice dynamics. Additionally, Eringen (1983) has presented the differential constitutive relation that is more convenient to find analytical solutions of static and dynamic problems in mechanics of nanostructures. For one-dimensional elastic body, the nonlocal constitutive equations in differential form are

$$\sigma_{xx} - \mu \frac{\mathrm{d}^2 \sigma_{xx}}{\mathrm{d}x^2} = E \varepsilon_{xx}, \tag{2a}$$

$$\sigma_{xz} - \mu \frac{\mathrm{d}^2 \sigma_{xz}}{\mathrm{d}x^2} = G \gamma_{xz}, \tag{2b}$$

where E and G are elastic modulus and shear modulus of the beam, respectively; $\mu = (e_0 a)^2$ is the nonlocal parameter (length scales), σ_{xx} and σ_{xz} are normal and tangential nonlocal stresses, respectively. The constitutive relation for nonlocal viscoelastic body can be obtained by combining nonlocal elasticity and viscoelasticity theory (Lei et al., 2013b). Therefore, for one-dimensional nonlocal viscoelastic solids, constitutive relations for Kelvin–Voigt viscoelastic model are given by

$$\sigma_{xx} - \mu \frac{\mathrm{d}^2 \sigma_{xx}}{\mathrm{d}x^2} = E(\varepsilon_{xx} + \tau_\mathrm{d} \dot{\varepsilon}_{xx}), \tag{3a}$$

$$\sigma_{xz} - \mu \frac{\mathrm{d}^2 \sigma_{xz}}{\mathrm{d}x^2} = G(\gamma_{xz} + \tau_\mathrm{d} \dot{\gamma}_{xz}), \tag{3b}$$

where τ_d is the viscous damping coefficient of nanorod. In continuation of this work, nonlocal viscoelastic constitutive relations (3) are used to derive dynamic equations of motion of the coupled nanorod system.

2.2. Nonlocal viscoelastic nanorod theory

Let us consider a nanorod of length L and cross-sectional area A. In this case, cross-sectional area is constant along x coordinate but in general, it could have arbitrary shapes along x coordinate. We take that material of a nanorod is viscoelastic and homogeneous. Also, we consider the free longitudinal vibration of nanorod in x-direction. An infinitesimal element of length dx is taken at a typical coordinate location x. Farther, we take that a force N is the resultant of an axial stress σ_{xx} acting internally on A, where σ_{xx} is assumed to be uniform over the cross-section. Stress resultant N varies along the length, and is also a function of time $N = N(x,t)$. In addition, the axially distributed force $\tilde{F}$ is shown, having dimensions of force per unit length of nanorod, which results from external sources, either internally or externally applied.

The equilibrium of forces in the x-direction is

$$-N + \left(N + \frac{\mathrm{d}N}{\mathrm{d}x}\mathrm{d}x\right) - \tilde{F}\,\mathrm{d}x = \ddot{u}\,\mathrm{d}m, \tag{4}$$

where dm = ρA dx is mass of infinitesimal element and u is the displacement in the x-direction. Substituting dm = ρA dx and simplifying Eq. (4) gives

$$\frac{dN}{dx} = \tilde{F} + \rho A\ddot{u}, \quad (5)$$

where N is stress resultant defined as

$$N(x,t) = \int_A \sigma_{xx}(x,t)dA = \sigma_{xx}(x,t)A. \quad (6)$$

By substituting Eq. (6) into Eq. (3a), the stress resultant for the nonlocal theory is obtained as

$$N - \mu\frac{d^2N}{dx^2} = EA\left(\frac{\partial u}{\partial x} + \tau_d\frac{\partial \dot{u}}{\partial x}\right). \quad (7)$$

The equation of motion can be expressed in terms of the displacement u for nonlocal viscoelastic constitutive relation. By substituting Eq. (5) into Eq. (7) we get the following equation of motion

$$\tilde{F} + \rho A\ddot{u} - \mu\frac{d^2}{dx^2}(\tilde{F} + \rho A\ddot{u}) = EA(\frac{d^2u}{dx^2} + \tau_d\frac{d^2\dot{u}}{dx^2}). \quad (8)$$

Next, we utilize Eq. (8) for the development of a mathematical model of VDNRS.

2.3. Mathematical model of VDNRS

For the double-nanorod system showed in Fig. 1, it is assumed that is formed of two straight, parallel, uniform, nonlocal viscoelastic nanorods, which have the same length and which are continuously joined by a light viscoelastic layer. The viscoelastic layer is modeled as a continuously distributed spring-damper system of Kelvin–Voigt type with neglected mass. To investigate free longitudinal vibration in such structures it is necessary to prepare a reliable mathematical model. Using D'Alambert's principle and the nonlocal viscoelastic constitutive relation Eq. (3a), the governing system of homogenous coupled partial differential equations for the free longitudinal vibration of VDNRS is expressed in the following forms

$$\ddot{u}_1 - \mu_1\frac{\partial^2\ddot{u}_1}{\partial x^2} - c_1^2\frac{\partial^2 u_1}{\partial x^2} - c_1^2\tau_{d1}\frac{\partial^2\dot{u}_1}{\partial x^2} + a_1^2(u_1 - u_2) + p_1^2(\dot{u}_1 - \dot{u}_2) - \mu_1 a_1^2\left(\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2}\right) - \mu_1 p_1^2\left(\frac{\partial^2\dot{u}_1}{\partial x^2} - \frac{\partial^2\dot{u}_2}{\partial x^2}\right) = 0, \quad (9a)$$

$$\ddot{u}_2 - \mu_2\frac{\partial^2\ddot{u}_2}{\partial x^2} - c_2^2\frac{\partial^2 u_2}{\partial x^2} - c_2^2\tau_{d2}\frac{\partial^2\dot{u}_2}{\partial x^2} - a_2^2(u_1 - u_2) - p_2^2(\dot{u}_1 - \dot{u}_2) + \mu_2 a_2^2\left(\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2}\right) + \mu_2 p_2^2\left(\frac{\partial^2\dot{u}_1}{\partial x^2} - \frac{\partial^2\dot{u}_2}{\partial x^2}\right) = 0, \quad (9b)$$

where u_i $(i = 1,2)$ denotes the longitudinal displacements of nanorods; $c_i = \sqrt{E_i/\rho_i}$, $(i = 1,2)$ are the velocities of propagation of longitudinal waves; $a_i = \sqrt{c/(\rho_i A_i)}$, $p_i = \sqrt{b/(\rho_i A_i)}$, $(i = 1,2)$ are notations for reduction coefficients; μ_i, $(i = 1,2)$ are nonlocal parameters; τ_{di} $(i = 1,2)$ are viscous damping coefficients of the first and the second nanorod; c and b are stiffness and damping coefficients of the light viscoelastic layer; ρ_i, A_i, E_i $(i = 1,2)$ are the mass densities, cross-sectional areas and Young's modulus of nanorods, respectively. The term $\tilde{F}$ is now the axial distributed force from light viscoelastic layer modeled as the spring-damper system

$$\tilde{F} = c(u_1 - u_2) + b(\dot{u}_1 - \dot{u}_2). \quad (10)$$

Suppose that the ends of two nanorods are Clamped–Clamped (see Fig. 1a) and Clamped–Free (see Fig. 1b), the boundary conditions are given by

Clamped–Clamped:

$$u_1(0,t) = u_1(L,t) = 0, \quad (11a)$$

$$u_2(0,t) = u_2(L,t) = 0, \quad (11b)$$

Clamped–Free:

$$u_1(0,t) = N_1(L,t) = 0, \quad (12a)$$

$$u_2(0,t) = N_2(L,t) = 0, \quad (12b)$$

where $N_i(L,t)$, $(i = 1,2)$ are stress resultants on the right ends of both nanorods.

By substituting Eq. (5) into Eq. (7) we get the stress result force N as follows

$$N = \mu\frac{d^2N}{dx^2} + EA\left(\frac{\partial u}{\partial x} + \tau_d\frac{\partial\dot{u}}{\partial x}\right) = \mu\frac{d}{dx}(\tilde{F} + \rho A\ddot{u}) + EA\left(\frac{\partial u}{\partial x} + \tau_d\frac{\partial\dot{u}}{\partial x}\right). \quad (13)$$

If we introduced Eq. (13) into boundary conditions (12), we have

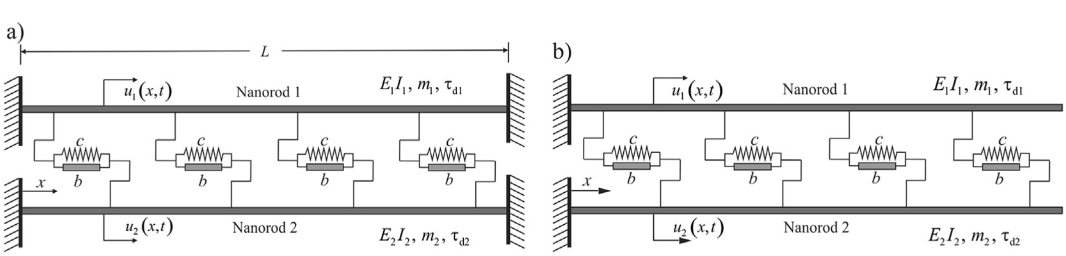

Fig. 1. The physical model of the nonlocal viscoelastic double-nanorod system for: a) Clamped–Clamped boundary conditions; b) Clamped–Free boundary conditions.

$$N_1(L,t) = \mu_1 \frac{d}{dx}[c(u_1(L,t) - u_2(L,t)) + b(\dot{u}_1(L,t) - \dot{u}_2(L,t)) + \rho_1 A_1 \ddot{u}_1(L,t)] + E_1 A_1 \left(\frac{\partial u_1(L,t)}{\partial x} + \tau_{d1} \frac{\partial \dot{u}_1(L,t)}{\partial x} \right) = 0, \tag{14a}$$

$$N_2(L,t) = \mu_2 \frac{d}{dx}[-c(u_1(L,t) - u_2(L,t)) - b(\dot{u}_1(L,t) - \dot{u}_2(L,t)) + \rho_2 A_2 \ddot{u}_2(L,t)] + E_2 A_2 \left(\frac{\partial u_2(L,t)}{\partial x} + \tau_{d2} \frac{\partial \dot{u}_2(L,t)}{\partial x} \right) = 0. \tag{14b}$$

3. Solution of coupled partial differential equations

Analytical solutions of the system of coupled partial differential equations are obtained using Bernoulli–Fourier method. Assuming time harmonic motion and using separation of variables method, the solutions of Eq. (9) and boundary conditions Eqs. (11) and (12), can be written in the form

$$u_i(x,t) = \sum_{n=1}^{\infty} X_n(x) T_{in}(t), \quad i = 1,2, \tag{15}$$

where $T_{in}(t)$ are the unknown time function, and $X_n(x)$ is corresponding mode shape function which depends on the boundary conditions of the system and it is equal for both nanorods. By substituting the general solution Eq. (15) into Eqs. (9) one gets the following relations

$$\frac{X_n''}{X_n} = \frac{\ddot{T}_{1n} + a_1^2(T_{1n} - T_{2n}) + p_1^2\left(\dot{T}_{1n} - \dot{T}_{2n}\right)}{\mu_1 \ddot{T}_{1n} + c_1^2 T_{1n} + c_1^2 \tau_{d1} \dot{T}_{1n} + \mu_1 a_1^2 (T_{1n} - T_{2n}) + \mu_1 p_1^2 \left(\dot{T}_{1n} - \dot{T}_{2n}\right)} = -\lambda_n^2, \tag{16a}$$

$$\frac{X_n''}{X_n} = \frac{\ddot{T}_{2n} - a_2^2(T_{1n} - T_{2n}) - p_2^2\left(\dot{T}_{1n} - \dot{T}_{2n}\right)}{\mu_2 \ddot{T}_{2n} + c_2^2 T_{2n} + c_2^2 \tau_{d2} \dot{T}_{2n} - \mu_2 a_2^2 (T_{1n} - T_{2n}) - \mu_2 p_2^2 \left(\dot{T}_{1n} - \dot{T}_{2n}\right)} = -\lambda_n^2. \tag{16b}$$

From Eqs. (16) one can obtain a set of ordinary differential equations for the unknown time functions T_{1n} and T_{2n}

$$d_1 \ddot{T}_{1n} + f_1 \dot{T}_{1n} + h_1 T_{1n} - g_1 T_{2n} - s_1 \dot{T}_{2n} = 0, \tag{17a}$$

$$d_2 \ddot{T}_{2n} + f_2 \dot{T}_{2n} + h_2 T_{2n} - g_2 T_{1n} - s_2 \dot{T}_{1n} = 0, \tag{17b}$$

where

$$\begin{aligned} &d_i = 1 + \lambda_n^2 \mu_i, \ f_i = p_i^2 + \lambda_n^2 c_i^2 \tau_{di} + \lambda_n^2 \mu_i p_i^2, \\ &h_i = a_i^2 + \lambda_n^2 c_i^2 + \lambda_n^2 \mu_i a_i^2, \ g_i = a_i^2 + \lambda_n^2 \mu_i a_i^2, \\ &s_i = p_i^2 + \lambda_n^2 \mu_i p_i^2, \quad i = 1,2, \end{aligned} \tag{18}$$

and one ordinary differential equation for corresponding mode shape function $X_n(x)$ of the double-nanorod system

$$X_n''(x) + \lambda_n^2 X_n(x) = 0. \tag{19}$$

The solutions of Eq. (19) can be assumed to have the following form

$$X_n(x) = A_n \cos \lambda_n x + B_n \sin \lambda_n x, \tag{20}$$

where λ_n denotes the characteristic values which are determined from the corresponding boundary conditions. Substituting Eqs. (15) and (20) into equations of boundary conditions Eqs. (11) and (12), we obtained two transcendental equations from which we can find values of λ_n.

For the *Clamped–Clamped* boundary conditions

$$\sin \lambda_n L = 0, \tag{21}$$

roots are

$$\lambda_n L = n\pi, \quad n = 1,2,\ldots,\infty, \tag{22}$$

and then we defined corresponding mode shape function $X_n(x)$ as

$$X_n(x) = \sin \frac{n\pi}{L} x. \tag{23}$$

For the *Clamped–Free* boundary conditions

$$\cos \lambda_n L = 0, \tag{24}$$

roots are

$$\lambda_n L = \frac{(2n-1)\pi}{2}, \quad n = 1,2,\ldots,\infty, \tag{25}$$

then we defined corresponding mode shape functions $X_n(x)$ as

$$X_n(x) = \sin \frac{(2n-1)\pi}{2L} x. \tag{26}$$

To solve the system of two coupled ordinary differential equation (17), we assumed solutions of the form

$$T_{1n} = A_{1n} e^{k_n t}, \quad T_{2n} = A_{2n} e^{k_n t}, \tag{27}$$

where k_n denotes eigenvalues of the system. Substituting Eq. (27) into Eq. (17), we obtain a system of homogeneous algebraic equations for the unknown constants A_{1n} and A_{2n}

$$(d_1 k_n^2 + f_1 k_n + h_1) A_{1n} - (s_1 k_n + g_1) A_{2n} = 0, \tag{28a}$$

$$(d_2 k_n^2 + f_2 k_n + h_2) A_{2n} - (s_2 k_n + g_2) A_{1n} = 0. \tag{28b}$$

Equations (28) have non-trivial solutions for the constants A_{1n} and A_{2n} when the determinant of the system is equal to zero. This yields the characteristic equation (29), which is a fourth order polynomial in k_n written as

$$k_n^4 + V_1 k_n^3 + V_2 k_n^2 + V_3 k_n + V_4 = 0, \tag{29}$$

where

$$\begin{aligned} V_1 &= \frac{f_1 d_2 + d_1 f_2}{d_1 d_2}, \quad V_2 = \frac{h_1 d_2 + d_1 h_2 + f_1 f_2 - s_1 s_2}{d_1 d_2}, \\ V_3 &= \frac{h_1 f_2 + f_1 h_2 - g_1 s_2 - s_1 g_2}{d_1 d_2}, \quad V_4 = \frac{h_1 h_2 - g_1 g_2}{d_1 d_2}, \end{aligned} \tag{30}$$

and we obtain eigenvalues of the system, as two pairs of complex conjugate roots in the form

$$k_{1/2n} = -\tilde{\delta}_{1n} \mp \mathrm{i}\tilde{p}_{1n}, \quad k_{3/4n} = -\tilde{\delta}_{2n} \mp \mathrm{i}\tilde{p}_{2n}, \tag{31}$$

where $\tilde{\delta}_{1n}$ and $\tilde{\delta}_{2n}$ are real and $\tilde{p}_{1n}$ and $\tilde{p}_{2n}$ are imaginary parts of the corresponding pair of roots of characteristic equation. For each of the eigenvalues, the associated amplitude ratios of vibration modes of two nanorods are given as

$$C_{(s)n} = \frac{A_{1n}^{(s)}}{K_{21n}^{(s)}} = \frac{A_{2n}^{(s)}}{K_{22n}^{(s)}}, \quad s = 1,2,3,4, \tag{32}$$

where $K_{21n}^{(s)} = s_1 k_{(s)n} + g_1$ and $K_{22n}^{(s)} = d_1 k_n^2 + f_1 k_n + h_1$ are cofactors of the system (28) and $C_{(s)n}$ are unknown complex conjugate constants which writes as

$$C_{(1/2)n} = \tilde{a}_n \mp \mathrm{i}\tilde{b}_n, \quad C_{(3/4)n} = \tilde{c}_n \pm \mathrm{i}\tilde{d}_n, \tag{33}$$

where $\tilde{a}_n$, $\tilde{b}_n$, $\tilde{c}_n$ and $\tilde{d}_n$ are unknown constants which will be determined in the following.

Thus, the general solutions (27) may be written as

$$T_{1n} = A_{1n}^{(1)} e^{k_{(1)n}t} + A_{1n}^{(2)} e^{k_{(2)n}t} + A_{1n}^{(3)} e^{k_{(3)n}t} + A_{1n}^{(4)} e^{k_{(4)n}t}, \tag{34a}$$

$$T_{2n} = A_{2n}^{(1)} e^{k_{(1)n}t} + A_{2n}^{(2)} e^{k_{(2)n}t} + A_{2n}^{(3)} e^{k_{(3)n}t} + A_{2n}^{(4)} e^{k_{(4)n}t}, \tag{34b}$$

where amplitudes $A_{1n}^{(s)}$ and $A_{2n}^{(s)}$ are defined as

$$\begin{aligned} A_{1n}^{(1/2)} &= K_{21n}^{(1/2)} \cdot C_{(1/2)n} = \tilde{u}_{1n} \pm \mathrm{i}\tilde{v}_{1n}, \\ A_{1n}^{(3/4)} &= K_{21n}^{(3/4)} \cdot C_{(3/4)n} = \tilde{u}_{2n} \pm \mathrm{i}\tilde{v}_{2n}, \end{aligned} \tag{35a}$$

$$\begin{aligned} A_{2n}^{(1/2)} &= K_{22n}^{(1/2)} \cdot C_{(1/2)n} = \tilde{r}_{1n} \pm \mathrm{i}\tilde{s}_{1n}, \\ A_{1n}^{(3/4)} &= K_{21n}^{(3/4)} \cdot C_{(3/4)n} = \tilde{r}_{2n} \pm \mathrm{i}\tilde{s}_{2n}. \end{aligned} \tag{35b}$$

Introducing Eqs. (31) and (35) into Eqs. (34), general solutions of time functions can be written as

$$\begin{aligned} T_{1n}(t) = {} & 2e^{-\tilde{\delta}_{1n}t}(\tilde{u}_{1n} \cos \tilde{p}_{1n}t + \tilde{v}_{1n} \sin \tilde{p}_{1n}t) \\ & + 2e^{-\tilde{\delta}_{2n}t}(\tilde{u}_{2n} \cos \tilde{p}_{2n}t + \tilde{v}_{2n} \sin \tilde{p}_{2n}t), \end{aligned} \tag{36a}$$

$$\begin{aligned} T_{2n}(t) = {} & 2e^{-\tilde{\delta}_{1n}t}(\tilde{r}_{1n} \cos \tilde{p}_{1n}t + \tilde{s}_{1n} \sin \tilde{p}_{1n}t) \\ & + 2e^{-\tilde{\delta}_{2n}t}(\tilde{r}_{2n} \cos \tilde{p}_{2n}t + \tilde{s}_{2n} \sin \tilde{p}_{2n}t), \end{aligned} \tag{36b}$$

where

$$\begin{aligned} &\tilde{u}_{1n} = (-s_1\tilde{\delta}_{1n} + g_1)\tilde{a}_n - s_1\tilde{p}_{1n}\tilde{b}_n, \ \tilde{v}_{1n} = -s_1\tilde{p}_{1n}\tilde{a}_n + (s_1\tilde{\delta}_{1n} - g_1)\tilde{b}_n, \\ &\tilde{u}_{2n} = (-s_1\tilde{\delta}_{2n} + g_1)\tilde{c}_n - s_1\tilde{p}_{2n}\tilde{d}_n, \ \tilde{v}_{2n} = -s_1\tilde{p}_{2n}\tilde{c}_n + (s_1\tilde{\delta}_{2n} - g_1)\tilde{d}_n, \\ &\tilde{r}_{1n} = (d_1\tilde{\delta}_{1n}^2 - d_1\tilde{p}_{1n}^2 - f_1\tilde{\delta}_{1n} + h_1)\tilde{a}_n + (2d_1\tilde{\delta}_{1n}\tilde{p}_{1n} - f_1\tilde{p}_{1n})\tilde{b}_n, \\ &\tilde{s}_{1n} = (2d_1\tilde{\delta}_{1n}\tilde{p}_{1n} - f_1\tilde{p}_{1n})\tilde{a}_n - (d_1\tilde{\delta}_{1n}^2 - d_1\tilde{p}_{1n}^2 - f_1\tilde{\delta}_{1n} + h_1)\tilde{b}_n, \\ &\tilde{r}_{2n} = (d_1\tilde{\delta}_{2n}^2 - d_1\tilde{p}_{2n}^2 - f_1\tilde{\delta}_{2n} + h_1)\tilde{c}_n + (2d_1\tilde{\delta}_{2n}\tilde{p}_{2n} - f_1\tilde{p}_{2n})\tilde{d}_n, \\ &\tilde{s}_{2n} = (2d_1\tilde{\delta}_{2n}\tilde{p}_{2n} - f_1\tilde{p}_{2n})\tilde{c}_n - (d_1\tilde{\delta}_{2n}^2 - d_1\tilde{p}_{2n}^2 - f_1\tilde{\delta}_{2n} + h_1)\tilde{d}_n. \end{aligned} \tag{37}$$

Then, the longitudinal vibrations of VDNRS can be described as

$$\begin{aligned} u_1(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n x \Big[& 2e^{-\tilde{\delta}_{1n}t}(\tilde{u}_{1n} \cos \tilde{p}_{1n}t + \tilde{v}_{1n} \sin \tilde{p}_{1n}t) \\ & + 2e^{-\tilde{\delta}_{2n}t}(\tilde{u}_{2n} \cos \tilde{p}_{2n}t + \tilde{v}_{2n} \sin \tilde{p}_{2n}t) \Big], \end{aligned} \tag{38a}$$

$$\begin{aligned} u_2(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n x \Big[& 2e^{-\tilde{\delta}_{1n}t}(\tilde{r}_{1n} \cos \tilde{p}_{1n}t + \tilde{s}_{1n} \sin \tilde{p}_{1n}t) \\ & + 2e^{-\tilde{\delta}_{2n}t}(\tilde{r}_{2n} \cos \tilde{p}_{2n}t + \tilde{s}_{2n} \sin \tilde{p}_{2n}t) \Big]. \end{aligned} \tag{38b}$$

The initial conditions of the theory are involved specifying the values of the displacements $u_i(x,t)$ and their first derivatives $\dot{u}_i(x,t)$ with respect to time at $t = 0$ are

$$\tilde{f}_1(x) = u_1(x,t)\Big|_{t=0} = \sum_{n=1}^{\infty} X_n(x) T_{1n}(0), \tag{39a}$$

$$\tilde{f}_2(x) = u_2(x,t)\Big|_{t=0} = \sum_{n=1}^{\infty} X_n(x) T_{2n}(0), \tag{39b}$$

and

$$\tilde{g}_1(x) = \dot{u}_1(x,t)|_{t=0} = \sum_{n=1}^{\infty} X_n(x) \dot{T}_{1n}(0), \tag{40a}$$

$$\tilde{g}_2(x) = \dot{u}_2(x,t)|_{t=0} = \sum_{n=1}^{\infty} X_n(x) \dot{T}_{2n}(0). \tag{40b}$$

On the basis of the orthogonality properties of mode shape functions, the unknown constants $\tilde{a}_n$, $\tilde{b}_n$, $\tilde{c}_n$ and $\tilde{d}_n$ can be determined from assumed initial conditions (39) and (40). In this case, the classical orthogonality conditions are applied as

$$\int_0^L X_n(x) X_m(x) \mathrm{d}x = \int_0^L \sin(\lambda_n x) \sin(\lambda_m x) \mathrm{d}x = \begin{cases} \frac{L}{2}, & n = m \\ 0, & n \neq m \end{cases} \tag{41}$$

Introduction of Eqs. (38) into initial conditions (39) and (40), and using orthogonality conditions (41), we obtain the system of algebraic equations in the following form

$$\overline{k}_1\tilde{a}_n+\overline{l}_1\tilde{b}_n+\overline{k}_2\tilde{c}_n+\overline{l}_2\tilde{d}_n=\frac{1}{2}\tilde{A}, \tag{42a}$$

$$\overline{h}_1\tilde{a}_n+\overline{g}_1\tilde{b}_n+\overline{h}_2\tilde{c}_n+\overline{g}_2\tilde{d}_n=\frac{1}{2}\tilde{B}, \tag{42b}$$

$$\begin{aligned}&(-\tilde{\delta}_{1n}\overline{k}_1+\tilde{p}_{1n}\overline{l}_1)\tilde{a}_n+(-\tilde{\delta}_{1n}\overline{l}_1-\tilde{p}_{1n}\overline{k}_1)\tilde{b}_n+(-\tilde{\delta}_{2n}\overline{k}_2+\tilde{p}_{2n}\overline{l}_2)\tilde{c}_n\\&+(-\tilde{\delta}_{2n}\overline{l}_2-\tilde{p}_{2n}\overline{k}_2)\tilde{d}_n=\frac{1}{2}\tilde{C},\end{aligned} \tag{42c}$$

$$\begin{aligned}&(-\tilde{\delta}_{1n}\overline{h}_1+\tilde{p}_{1n}\overline{g}_1)\tilde{a}_n+(-\tilde{\delta}_{1n}\overline{g}_1-\tilde{p}_{1n}\overline{h}_1)\tilde{b}_n+(-\tilde{\delta}_{2n}\overline{h}_2+\tilde{p}_{2n}\overline{g}_2)\tilde{c}_n\\&+(-\tilde{\delta}_{2n}\overline{g}_2-\tilde{p}_{2n}\overline{h}_2)\tilde{d}_n=\frac{1}{2}\tilde{D},\end{aligned} \tag{42d}$$

where

$$\begin{aligned}&\overline{k}_1=-s_1\tilde{\delta}_{1n}+g_1, && \overline{l}_1=-s_1\tilde{p}_{1n},\\&\overline{k}_2=-s_1\tilde{\delta}_{2n}+g_1, && \overline{l}_2=-s_1\tilde{p}_{2n},\\&\overline{h}_1=d_1\tilde{\delta}_{1n}^2-d_1\tilde{p}_{1n}^2-f_1\tilde{\delta}_{1n}+h_1, && \overline{g}_1=2d_1\tilde{\delta}_{1n}\tilde{p}_{1n}-f_1\tilde{p}_{1n},\\&\overline{h}_2=d_1\tilde{\delta}_{2n}^2-d_1\tilde{p}_{2n}^2-f_1\tilde{\delta}_{2n}+h_1, && \overline{g}_2=2d_1\tilde{\delta}_{2n}\tilde{p}_{2n}-f_1\tilde{p}_{2n},\end{aligned} \tag{43}$$

and

$$\begin{aligned}&\tilde{A}=\frac{2}{L}\int_0^L\tilde{f}_1(x)\sin(\lambda_n x)\mathrm{d}x,\quad \tilde{B}=\frac{2}{L}\int_0^L\tilde{f}_2(x)\sin(\lambda_n x)\mathrm{d}x,\\&\tilde{C}=\frac{2}{L}\int_0^L\tilde{g}_1(x)\sin(\lambda_n x)\mathrm{d}x,\quad \tilde{D}=\frac{2}{L}\int_0^L\tilde{g}_2(x)\sin(\lambda_n x)\mathrm{d}x.\end{aligned} \tag{44}$$

Solving the system of algebraic equations (42), we obtain required unknown constants$\tilde{a}_n$, $\tilde{b}_n$, $\tilde{c}_n$ and $\tilde{d}_n$.

4. Numerical results and discussion

In this section, validation of the obtained results is given by comparison with the corresponding results in the literature. Numerical experiments are performed for the longitudinal vibration of a viscoelastic double nanorod system (VDNRS). Two different types of numerical analysis are used. In the first type of analysis, the influence of the nonlocal parameter and the nanorod damping coefficient on the first and the third eigenvalue is investigated. In the second type of analysis, the effect of nonlocal parameter, nanorod damping coefficient and layers damping coefficient on midpoint displacement solutions for both nanorods is examined. In this study, numerical analysis in all cases is employed for two types of boundary conditions, clamped–clamped (C–C) and clamped–free (C–F) (see Fig. 1).

4.1. Validation of the proposed model

In order to justify this model, we derive the expressions for natural frequencies of the viscoelastic double-nanorod system coupled by a light viscoelastic layer and compare with natural frequency derived by Erol and Gürgöze (2004) for local elasticity case and Murmu and Adhikari (2010) for the nonlocal elasticity case. For this purpose, we assume that the geometric and material properties of both nanorods are equal and introduce the following constants suggested in Erol and Gürgöze (2004)

$$E_1A_1=E_2A_2=\overline{e}=\text{constant}, \tag{45a}$$

$$\rho_1A_1=\rho_2A_2=m=\text{constant}, \tag{45b}$$

where $\overline{e}$ and m denote axial rigidity and mass per unit length, respectively. Substituting Eqs. (45) into Eqs. (9) yields the following expressions:

$$m\ddot{u}_1-\mu m\frac{\partial^2\ddot{u}_1}{\partial x^2}-\overline{e}\frac{\partial^2 u_1}{\partial x^2}-\overline{e}\tau_{\mathrm{d}}\frac{\partial^2\dot{u}_1}{\partial x^2}+c(u_1-u_2)+b(\dot{u}_1-\dot{u}_2)-\mu\frac{\partial^2}{\partial x^2}[c(u_1-u_2)+b(\dot{u}_1-\dot{u}_2)]=0, \tag{46a}$$

$$m\ddot{u}_2-\mu m\frac{\partial^2\ddot{u}_2}{\partial x^2}-\overline{e}\frac{\partial^2 u_2}{\partial x^2}-\overline{e}\tau_{\mathrm{d}}\frac{\partial^2\dot{u}_2}{\partial x^2}-c(u_1-u_2)-b(\dot{u}_1-\dot{u}_2)+\mu\frac{\partial^2}{\partial x^2}[c(u_1-u_2)+b(\dot{u}_1-\dot{u}_2)]=0. \tag{46b}$$

Homogeneous partial differential equations (46) can be solved by assuming the solutions of the form

$$u_1(x,t)=X_1(x)e^{\mathrm{i}\omega t},\quad u_2(x,t)=X_2(x)e^{\mathrm{i}\omega t}, \tag{47}$$

where ω is the natural frequency and $X_1(x)$ and $X_2(x)$ are the corresponding mode shapes of viscoelastic double-nanorod system. Substituting Eqs. (47) into Eqs. (46) we obtain

$$\frac{\partial^2X_1}{\partial x^2}\left[\mu m\omega^2-\mu c-\overline{e}-\mathrm{i}(\mu\omega b+\overline{e}\tau_{\mathrm{d}}\omega)\right]+X_1(-m\omega^2+c+\mathrm{i}\omega b)+\frac{\partial^2X_2}{\partial x^2}(\mu c+\mathrm{i}\mu\omega b)-X_2(c+\mathrm{i}\omega b)=0, \tag{48a}$$

$$\frac{\partial^2X_2}{\partial x^2}\left[\mu m\omega^2-\mu c-\overline{e}-\mathrm{i}(\mu\omega b+\overline{e}\tau_{\mathrm{d}}\omega)\right]+X_2(-m\omega^2+c+\mathrm{i}\omega b)+\frac{\partial^2X_1}{\partial x^2}(\mu c+\mathrm{i}\mu\omega b)-X_1(c+\mathrm{i}\omega b)=0. \tag{48b}$$

Now we assume that the viscoelastic coupled double-nanorod system executes two kinds of free vibration motions, synchronous vibrations, where $X_1(x) = X_2(x) = X(x)$ with lower frequency ω_1^D and asynchronous vibrations, where $X_1(x) = -X_2(x) = X(x)$ with higher frequencyω_2^D where $\omega_1^D < \omega_2^D$.

For the case of synchronous vibrations, we get

$$\frac{\partial^2 X(x)}{\partial x^2} + \lambda_n^2 X(x) = 0, \tag{49}$$

where

$$\lambda_n^2 = \frac{m(\omega_1^D)^2}{\bar{e} + i\bar{e}\tau_d\omega_1^D - \mu m(\omega_1^D)^2}, \tag{50}$$

and λ_n is given for the corresponding boundary conditions (22) and Eq. (25).

For the case of asynchronous vibrations, we have

$$\lambda_n^2 = \frac{m(\omega_2^D)^2 - 2c - i2\omega_2^D b}{\bar{e} + 2\mu c - \mu m(\omega_2^D)^2 + i(2\mu\omega_2^D b + \bar{e}\tau_d\omega_2^D)}. \tag{51}$$

Now, we can express the natural frequency from Eqs. (50) and (51) as

$$\omega_{1n(1/2)}^D = \mp\sqrt{\frac{\bar{e}\lambda_n^2}{m(1+\mu\lambda_n^2)} - \frac{(\bar{e}\tau_d\lambda_n^2)^2}{4m^2(1+\mu\lambda_n^2)^2}} + i\frac{\bar{e}\tau_d\lambda_n^2}{2m(1+\mu\lambda_n^2)}, \tag{52}$$

$$\omega_{2n(1/2)}^D = \mp\sqrt{\frac{\bar{e}\lambda_n^2 + (1+\mu\lambda_n^2)2c}{m(1+\mu\lambda_n^2)} - \frac{\left[2b(1+\mu\lambda_n^2) + \bar{e}\tau_d\lambda_n^2\right]^2}{4m^2(1+\mu\lambda_n^2)^2}} + i\frac{2b(1+\mu\lambda_n^2) + \bar{e}\tau_d\lambda_n^2}{2m(1+\mu\lambda_n^2)}. \tag{53}$$

The natural frequency of a nonlocal elastic double-nanorod system proposed in Murmu and Adhikari (2010) can be obtained from Eqs. (52) and (53) by setting the viscoelastic dumping parameter τ_d and dumping coefficient from viscoelastic layer b equal to zero.

In order to validate the accuracy of obtained numerical results, we compare natural frequencies of the longitudinal vibration of VDNRS with the results presented by Erol and Gürgöze (2004) and Murmu and Adhikari (2010). The following values are used for the comparative study: $L = 1$ [nm], $m = 10^{-9}$ [kg/m], $\bar{e} = 10$ [nN], $e_0a = 0{-}2$ [nm], stiffness coefficient $K = 8$ N/nm, and two different values of nanorod damping coefficient $\tau_d = 0.001$ [ns] and 0.004 [ns]. Similar numerical values were adopted in Erol and Gürgöze (2004) and Murmu and Adhikari (2010) for the classical viscoelastically coupled double rod system and for the nonlocal elastic double-nanorod system (DNRS), respectively. The compared natural frequencies are given in Table 1 for the C–F boundary conditions. The magnitudes in Table 1 are 2π times in GHz so as to be in-line with Erol and Gürgöze (2004). As it can be seen, the real part of the first four eigenvalues from the present study overlaps with the natural frequencies obtained for e_0a=0 [nm] in the classical case (Erol and Gürgöze, 2004) and for $e_0a > 0$ in the nonlocal elasticity case (Murmu and Adhikari, 2010). It is obvious that for the VDNRS natural frequencies decrease with increase of the nonlocal parameter which is also stated for the DNRS in Murmu and Adhikari (2010).

By searching the literature, the authors have found that a work analyzing the free longitudinal vibration of VDNRS using the experimental or MD simulation approach hasn't been yet published. From the results above, it is obvious that VDNRS has two characteristic complex natural frequencies, the lower and the higher one. In a multiple coupled nanostructures systems, the lowest frequency is the so called fundamental frequency and it is independent of the influence of other nanostructures and coupling conditions in the system (Karličić et al., 2014). For the coupling between nanorods as those in Fig. 1, this fundamental frequency is equivalent to the natural frequency of a single nanorod that can be used to validate our results for natural frequency ω_{1n}^D with the results obtained for longitudinally vibrating single nanorod via molecular dynamics simulation in Cao et al. (2006), Murmu and Adhikari (2011). In that purpose, we neglect the dissipation effects in ω_{1n}^D and imaginary part of complex value, where the remaining part in the form $f = \omega_{1n}^D/2\pi$ represents a natural resonant frequency as given in Cao et al. (2006). The following dimensions and values of parameters are used to obtain the results given in Table 2: $\rho = 9517$ [kg/m^3], $E = 6.85$ [TPa], $L = 12.2$ [nm], $n = 1$ and $L = 12.2$ [nm]. The results are obtained for clamped–free boundary conditions and different values of e_0a and compared with the results for natural resonant frequency of single-walled carbon nanotube (SWCNT) (5, 5) obtained in Cao et al. (2006) by molecular dynamics simulation. As it can be noticed, in local elasticity case when $e_0a = 0$ natural frequency is over-predicted and a better

Table 1
Comparison of first four natural frequencies (GHz) of the double-nanorod system for C–F boundary conditions and values of viscoelastic constant τ_d and nonlocal parameter e_0a.

C–F		$e_0a = 0$ nm	$e_0a = 0.5$ nm	$e_0a = 1$ nm	$e_0a = 1.5$ nm	$e_0a = 2$ nm
		Erol and Gürgöze (2004)	Murmu and Adhikari (2010)			
$\tau_d = 0$ ns	1	1.5708	1.2353	0.8436	0.6137	0.4764
$b = 0$ N ns/nm	2	4.2974	4.1864	4.088	4.0468	4.0283
$k = 8$ N/nm	3	47124	1.8411	0.9782	0.6601	0.4972
	4	6.1812	4.4033	4.1179	4.0541	4.0308
		Viscoelastic double nanorod system coupled by light viscoelastic layer – presented analysis				
$\tau_d = 0.001$ ns	1	1.5707 + 0.0012i	1.2353 + 0.00076i	0.8435 + 0.00035i	0.6136 + 0.00018i	0.4764 + 0.0001i
$b = 0.01$ N ns/nm	2	4.2973 + 0.0112i	4.1863 + 0.0107i	4.0879 + 0.01035i	4.0467 + 0.01018i	4.0282 + 0.0101i
$k = 8$ N/nm	3	4.7123 + 0.0111i	1.8410 + 0.00169i	0.9782 + 0.00047i	0.6601 + 0.00021i	0.4972 + 0.0001i
	4	6.1811 + 0.0211i	4.4033 + 0.01169i	4.1178 + 0.01047i	4.0541 + 0.01021i	4.0307 + 0.0101i
$\tau_d = 0.004$ ns	1	1.5707 + 0.0049i	1.2353 + 0.003i	0.8435 + 0.0014i	0.6136 + 0.00075i	0.4764 + 0.0004i
$b = 0.01$ N ns/nm	2	4.2973 + 0.0149i	4.1863 + 0.013i	4.0879 + 0.0114i	4.0467 + 0.01075i	4.0282 + 0.0105i
$k = 8$ N/nm	3	4.7121 + 0.0444i	1.8410 + 0.0067i	0.9782 + 0.0019i	0.6601 + 0.00087i	0.4972 + 0.0004i
	4	6.1809 + 0.0544i	4.4033 + 0.0167i	4.1178 + 0.0119i	4.05408 + 0.0108i	4.0307 + 0.0105i

Table 2
Comparison of the lower nonlocal natural frequency $f = \omega_{1n}^{D}/2\pi$ of DNRS given in (THz) with the results obtained with MD simulation for armchair SWCNT (5, 5).

Boundary conditions C–F		Nonlocal parameter					MD simulation (Cao et al., 2006)
		$e_0a = 0$ nm	$e_0a = 0.5$ nm	$e_0a = 1$ nm	$e_0a = 1.5$ nm	$e_0a = 2$ nm	
$\tau_d = 0$ ns $b = 0$ N ns/nm Eq. (52)	f	0.549763	0.548627	0.545262	0.539788	0.532395	0.544

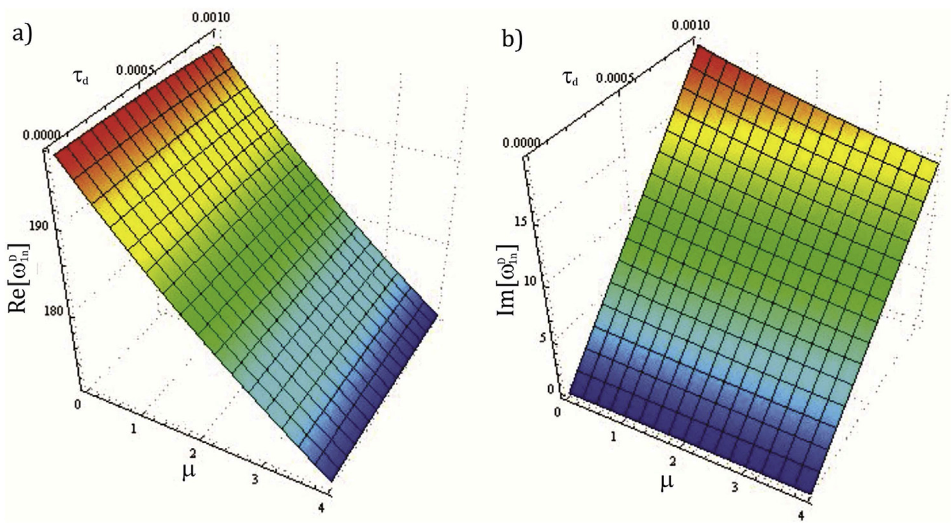

Fig. 2. The real and imaginary parts of the first higher eigenvalue of VDNRS, C–C boundary conditions.

agreement is achieved when the nonlocal effect is considered by increasing the value of e_0a towards unit value. Since the value of internal length scale parameter a is fixed for C–C bonds in SWCNT, to fit the results for natural frequency it is necessary to optimize the value of parameter e_0 for each problem of boundary conditions and geometrical properties of nanostructures. Since the molecular dynamics study is computationally prohibitive for nano-scale systems with large number of atoms such as VDNRS, nonlocal theory should be used to obtain satisfying results for more complex systems as shown in Table 2.

4.2. Numerical results

In the following numerical examples, we first analyzed the change in eigenvalues of VDNRS for change of the nonlocal parameter and damping coefficient of nanorods for the first and the third mode of vibration. Afterward, we considered the midpoint displacements in the first mode of vibration for change of the nonlocal parameter, damping coefficient of nanorods and damping coefficient of the viscoelastic layer. We used following arbitrary values: diameter of nanorod $d_1 = d_2 = d = 1.1$ [nm], length $L = 10d$,

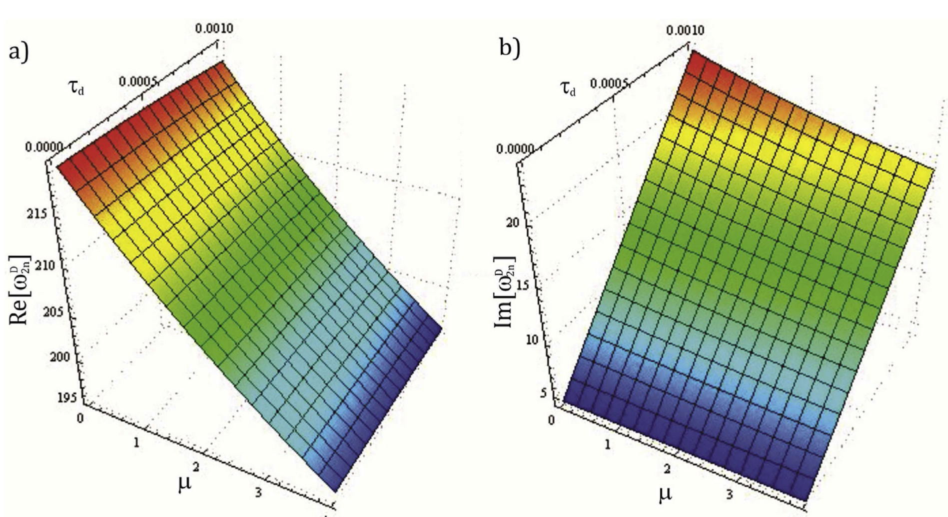

Fig. 3. The real and imaginary parts of the first lower eigenvalue of VDNRS, C–C boundary conditions.

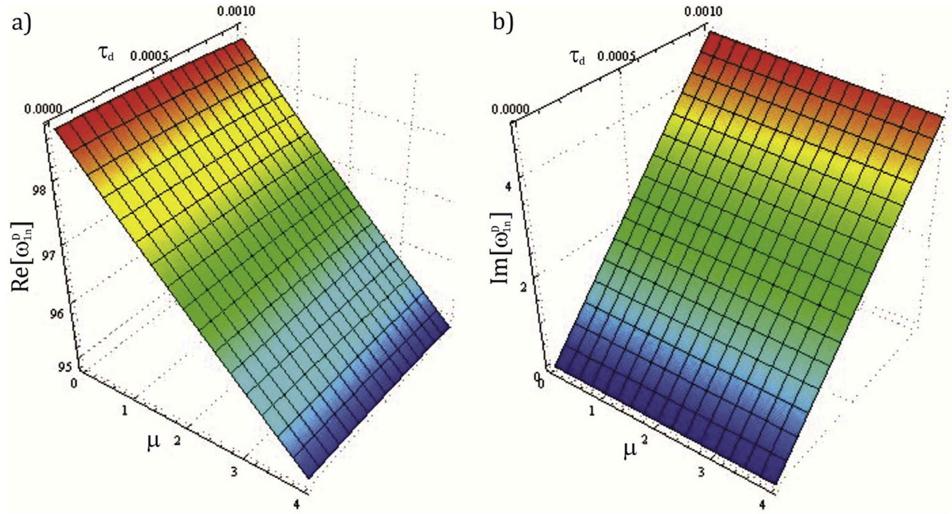

Fig. 4. The real and imaginary parts of the first higher eigenvalue of VDNRS, C–F boundary conditions.

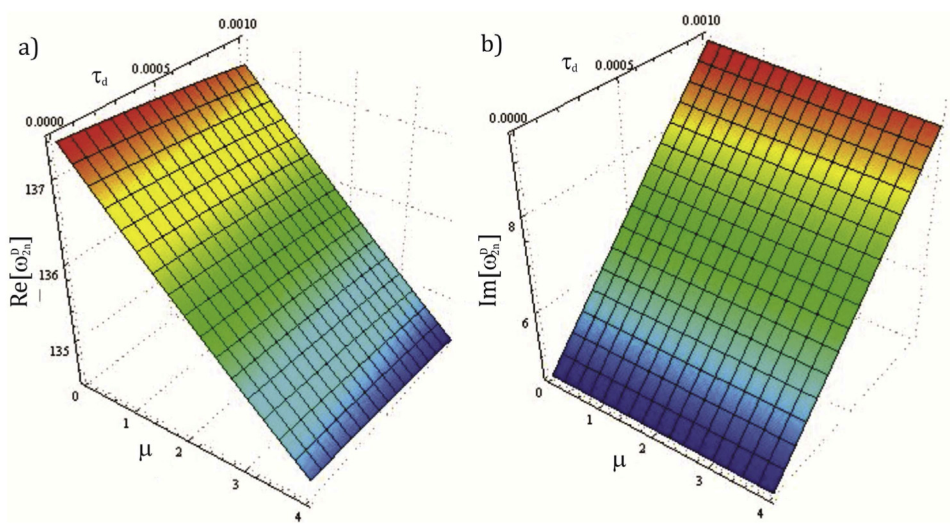

Fig. 5. The real and imaginary parts of the first lower eigenvalue of VDNRS, C–F boundary conditions.

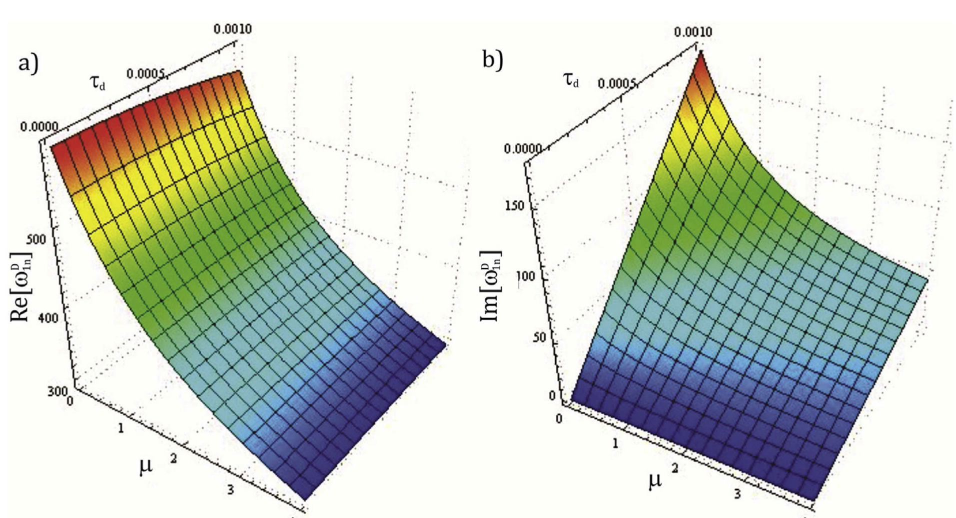

Fig. 6. The real and imaginary parts of the third higher eigenvalue of VDNRS, C–C boundary conditions.

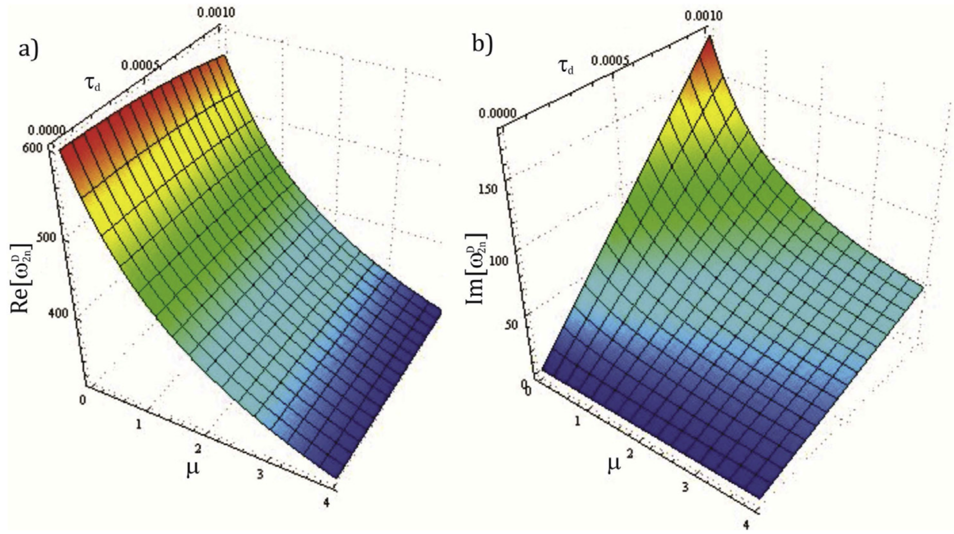

Fig. 7. The real and imaginary parts of the third lower eigenvalue of VDNRS, C–C boundary conditions.

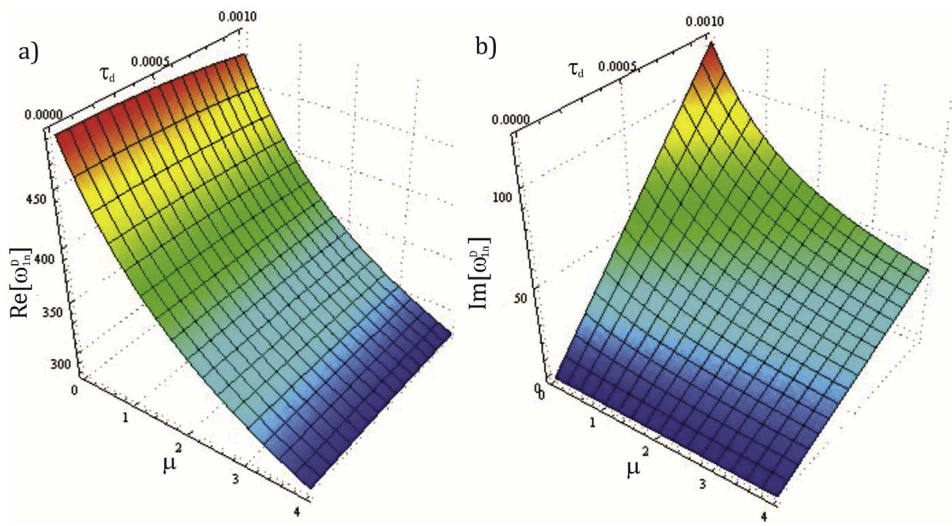

Fig. 8. The real and imaginary parts of the third higher eigenvalue of VDNRS, C–F boundary conditions.

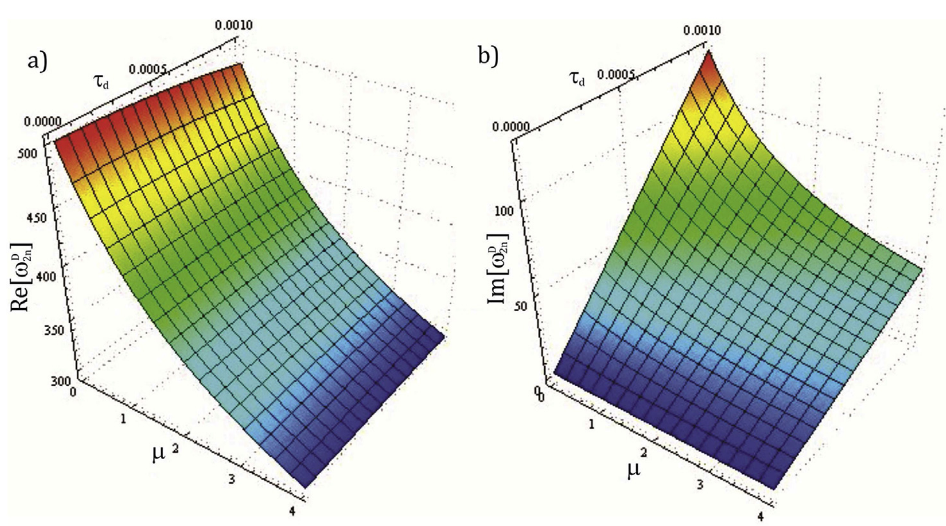

Fig. 9. The real and imaginary parts of the third lower eigenvalue of VDNRS, C–F boundary conditions.

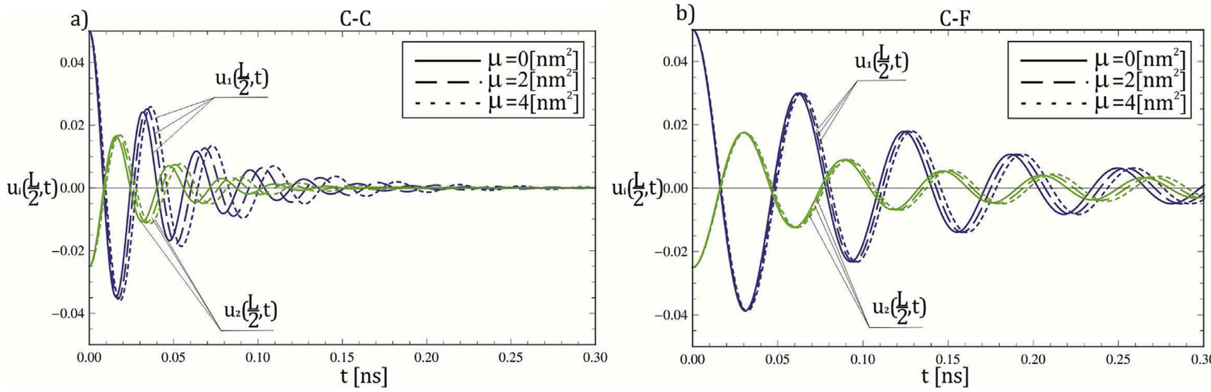

Fig. 10. Displacements $u_1(x,t)$ and $u_2(x,t)$ in the first mode and for different values of μ: a) C–C boundary condition case; b) C–F boundary condition case.

Young's modulus $E_1 = E_2 = E = 1.1$ [TPa] and the mass density $\rho_1 = \rho_2 = \rho = 2300$ [kg/m^3]. The parameter values of Kelvin–Voigt damping coefficient τ_d, nonlocal parameter μ and parameters of viscoelastic layer are given in follow.

The complex eigenvalues plotted in Figs. 2–9 are evaluated from Eqs. (52) and (53) as functions of damping coefficient τ_d from constitutive relations and nonlocal parameter μ. Analyzing the effect of nanorod damping coefficient τ_d and nonlocal parameter μ on complex eigenvalues of the VDNRS the following conclusions can be drawn similar for both boundary conditions. In general, nanorod damping coefficient τ_d does not influence the real part of complex eigenvalue significantly. This is to be expected since the real part represents the natural frequency of the system while the imaginary part represents damping of the system. The influence of the nanorod damping coefficient τ_d on the imaginary part is quite linear for all three cases of the complex eigenvalue while its effect on real part is negligible. Also, it can be concluded that increase of the nonlocal parameter μ reduces the influence of the nanorod damping coefficient τ_d in the imaginary part of complex eigenvalue. However, for the first eigenvalue this effect is less pronounced for the C–F boundary conditions then for the C–C one. Furthermore, it is obvious that increases of the nonlocal parameter μ decrease the real part of complex eigenvalue significantly and therefore decrease the natural frequency of the system.

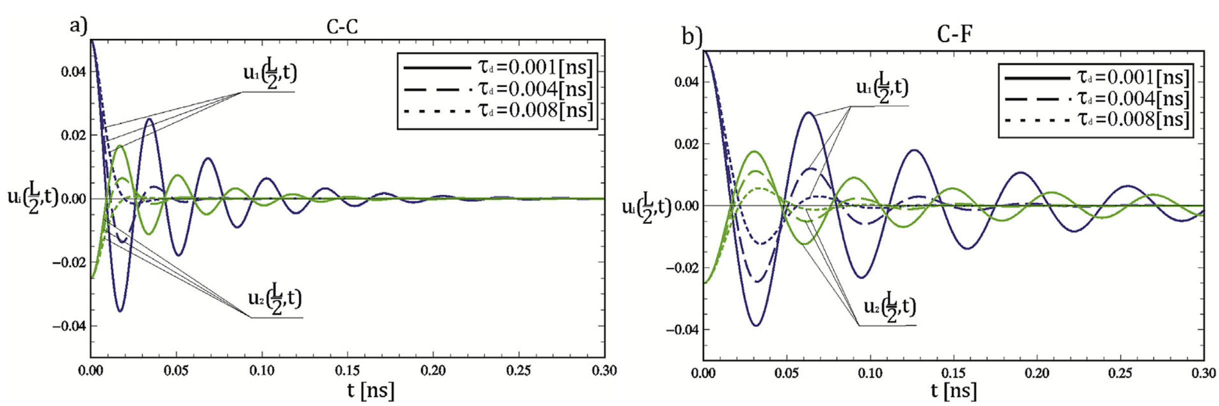

Fig. 11. Displacements $u_1(x,t)$ and $u_2(x,t)$ in the first mode and for different values of τ_d: a) C–C boundary condition case; b) C–F boundary condition case.

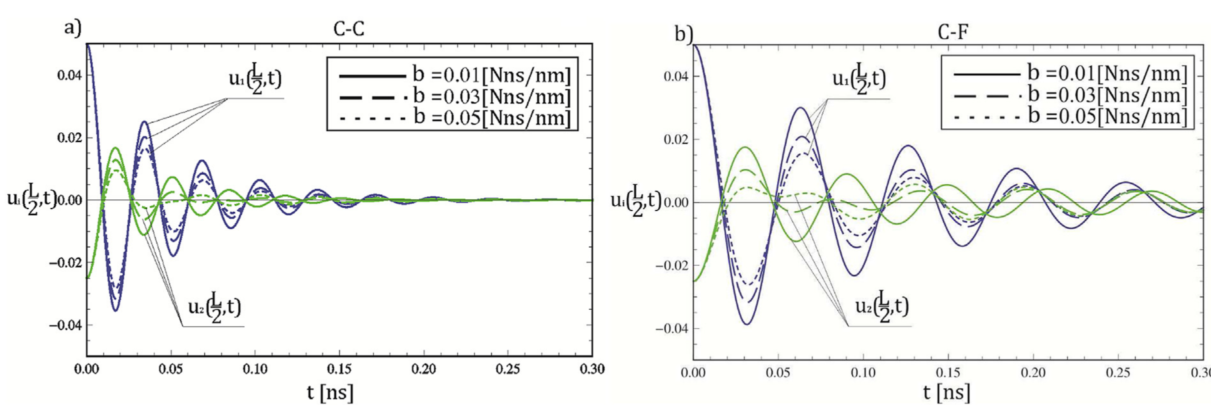

Fig. 12. Displacements $u_1(x,t)$ and $u_2(x,t)$ in the first mode and for different values of b: a) C–C boundary condition case; b) C–F boundary condition case.

Numerical simulations in Figs. 10–12 are performed for different initial time displacements of the nanorod 1 and nanorod 2. From Fig. 10a and b, it is observed that the nonlocal displacement solution curves of midpoints of both nanorods overlaps at the beginning time for different values of nonlocal parameter. It can be noticed that for the case without considered nonlocal parameter $\mu = 0$ the vibration decays faster and increase of the value of nonlocal parameter μ leads to delayed damping of the longitudinal vibration of VDNRS. The effect of nonlocal parameter on damping properties of VDNRS is similar for both cases of boundary conditions. However, for the C–F case damping is less pronounced since the free end conditions increase the damping time significantly compared to the C–C case. These effects can be related to the previous statement for the eigenvalues in the first mode of vibration. Changes in nonlocal parameter reduce the influence of the damping coefficient on imaginary (damping) part of the eigenvalue (see Figs. 2–5) and therefore damping of the system vibration is delayed compared to the cases with lower values of nonlocal parameter (see Fig. 10a and b). In Fig. 11a and b it is apparent that the midpoint vibrations of nanorods are strongly influenced by the damping coefficient τ_d. Therefore, increase of the τ_d results in faster decay of the longitudinal vibration of VDNRS. To show the effect of the damping coefficient of the layer on the longitudinal vibration of VDNRS, three different values of the damping coefficient have been plotted and shown in Fig. 12a and b. Result shows that increase of the damping coefficient b causes faster decaying of the system vibration. However, VDNRS is less sensitive to changes in layers damping coefficient than on the changes in nanorods damping coefficient.

5. Conclusions

For the purpose of modeling of the viscoelastic double-nanorod system (VDNRS), Eringen's nonlocal theory is used. Models based on this theory take small-scale effects into account by using the so called nonlocal parameters. The nonlocal frequencies and displacements for Clamped–Clamped and Clamped–Free longitudinally vibrating VDNRS are obtained via exact analytical method. The numerical results for natural frequencies are in line with the results presented in Erol and Gürgöze (2004), Murmu and Adhikari (2010) and Cao et al. (2006). It is found that the increase of nonlocal parameter reduces the influence of nanorods damping coefficient in the imaginary part of the eigenvalues and results in delayed vibration damping of VDNRS. It can be concluded that VDNRS is more sensitive on change of nanorods damping coefficient than change of layers damping coefficient. Nevertheless, viscoelastic properties of the layer can be important in nano-engineering applications to ensure stability and to control vibrations of the nano-scale systems. Closed-form analytical results derived in the paper provide physical insight as well as computational advantage. The expressions derived here can be used as benchmarks for the verification of general numerical approaches, such as the finite element method. This study may be helpful while investigating multiple-nanorod system models as well as in the next-generation polymer nanocomposites.

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