Lancaster's Method of Damping Identification Revisited

Sondipon Adhikari
Department of Engineering,
University of Cambridge,
Trumpington Street
Cambridge CB2 1PZ (U.K.)
e-mail: sa225@eng.cam.ac.uk

Identification of damping is an active area of research in structural dynamics. In one of the earliest works, Lancaster [1] proposed a method to identify the viscous damping matrix from measured natural frequencies and mode shapes. His method requires the modes to be normalized in a particular way, which in turn a priori needs the very same viscous damping matrix. A method, based on the poles and residues of the measured transfer functions, has been proposed to overcome this basic difficulty associated with Lancaster's method. This approach is then extended to a class of nonviscously damped systems where the damping forces depend on the past history of the velocities via convolution integrals over some kernel functions. Suitable numerical examples are given to illustrate the modified Lancaster's method developed here. [DOI: 10.1115/1.1500742]

1 Introduction

A major reason for studying vibration in civil, mechanical and aerospace engineering is to reduce the vibration by dissipation of vibration energy or *damping*. In spite of extensive research, the knowledge regarding damping forces is least developed compared to the other forces acting on a structure. There are two basic reasons for this. First, from a theoretical point of view, it is not in general clear which state variables are relevant to determine the damping forces. This fact makes it difficult to choose what mathematical form of damping model should be used in the first place, let alone how to identify its parameters from experimental measurements. Second, from an experimental point of view, unlike the mass and stiffness properties, the damping properties can be identified only by a dynamic testing.

By far, the most common method to model damping in multiple-degree-of-freedom linear systems is to assume the so called *viscous damping*. Considerable research has proposed methods to identify a viscous damping matrix from experimental measurements (see Pilkey and Inman [2] for a recent survey). These methods can be divided into two broad categories [3]: (a) damping identification from modal testing and analysis [1,4–12], and (b) direct damping identification from the forced response measurements [3,13–18]. A detailed discussion of these methods may be found in reference [[19], Chapter 1].

The interest of the present paper lies in one of the earliest method of viscous damping identification proposed by Lancaster [1]. Suppose $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$, all $N\times N$ real matrices, are respectively the mass, stiffness and viscous damping matrices of a system. Also suppose that $\boldsymbol{\Lambda}\in\mathbb{C}^{N\times N}$ is a diagonal matrix of complex eigenvalues (λ_k) and $\mathbf{Z}\in\mathbb{C}^{N\times N}$ is the complex modal matrix whose columns are complex modes $\mathbf{z}_k$. Lancaster's result states that if the complex modes are normalized such that

$$\mathbf{z}_k^T(2\lambda_k\mathbf{M}+\mathbf{C})\mathbf{z}_k=1 \tag{1}$$

then the system matrices can be uniquely obtained from the modal data as:

$$\mathbf{M}=(\mathbf{Z}\boldsymbol{\Lambda}\mathbf{Z}^T+\mathbf{Z}^*\boldsymbol{\Lambda}^*\mathbf{Z}^{*^T})^{-1} \tag{2}$$

$$\mathbf{K}=-(\mathbf{Z}\boldsymbol{\Lambda}^{-1}\mathbf{Z}^T+\mathbf{Z}^*\boldsymbol{\Lambda}^{*^{-1}}\mathbf{Z}^{*^T})^{-1} \tag{3}$$

$$\text{and}\quad \mathbf{C}=-\mathbf{M}(\mathbf{Z}\boldsymbol{\Lambda}^2\mathbf{Z}^T+\mathbf{Z}^*\boldsymbol{\Lambda}^{*^T}\mathbf{Z}^{*^T})\mathbf{M}. \tag{4}$$

In the above equations $(\bullet)^T$ denotes matrix transpose of $(\bullet)$, and $(\bullet)^*$ denotes complex conjugate of $(\bullet)$. The advantages of Lancaster's Eqs. (2)–(4) are that, they are simple, direct, give the complete solution to the inverse problem, and also very little computational effort is required to apply them. However, there are three major problems associated with Lancaster's method. First, the complete set of modes are required in order to use Eqs. (2), (3) and (4). Second, these equations are valid only if the damping model of the structure under consideration is viscous. Third, and possibly most importantly, due to the normalization condition (1), $\mathbf{M}$ and $\mathbf{C}$ are *a priori* required, as Pilkey and Inman [2] have put it, "It is still not possible to measure normalized eigenvectors. The shortfall of this method comes in normalizing the eigenvectors, which requires knowledge of the very same damping matrix which we wish to find in the end." The main aim of this paper is to overcome this shortfall and to extend Lancaster's method to a class of nonviscously damped systems.

The method of damping identification proposed in this paper lies in between the two broad classes of damping identification methods mentioned earlier. The proposed method neither uses modal data, nor does it use direct force response measurements, but utilizes the *transfer function residues*. The use of transfer function residues provides a natural framework of avoiding the difficulties associated with normalizing the complex modes. Before going into details, dynamics of viscously damped systems are briefly discussed in Section 2. In Section 3, Lancaster's equations are reformulated in terms of the poles and the residues of the transfer functions. In Section 4, a class of nonviscously damped systems is introduced in which the damping forces are expressed in terms of the past history of the velocities via convolution integrals over suitable kernel functions. The approach developed in Section 3 is then extended to such nonviscously damped systems in Section 5. Finally Section 6 summarized the main findings of this paper.

2 Dynamics of Viscously Damped Systems

The equations of motion describing free vibration of a viscously damped linear discrete system with N degrees of freedom can be written as

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=0. \tag{5}$$

The eigenvalue problem associated with Eq. (5) can be represented by

$$\lambda_k^2\mathbf{M}\mathbf{z}_k+\lambda_k\mathbf{C}\mathbf{z}_k+\mathbf{K}\mathbf{z}_k=0. \tag{6}$$

The eigenvalues, λ_k, are the roots of the characteristic polynomial

$$\det[s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}]=0. \tag{7}$$

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The order of the polynomial is $2N$ and the roots appear in complex conjugate pairs. For convenience, the eigenvalues are arranged as

$$\lambda_1, \lambda_2, \cdots, \lambda_N, \lambda_1^*, \lambda_2^*, \cdots, \lambda_N^*. \quad (8)$$

In this paper we assume that all the eigenvalues are distinct. Several authors, [20,21] for example, have studied nonclassically damped linear systems using the state-space methods. Following the state-space approach it may be shown [22] that each complex mode satisfies the normalization relationship

$$\mathbf{z}_k^T[2\lambda_k\mathbf{M}+\mathbf{C}]\mathbf{z}_k=\theta_k, \quad \forall k=1,\cdots,2N \quad (9)$$

for some non-zero $\theta_k \in \mathbb{C}$. Numerical values of θ_k can be selected in various ways:

(a) Choose $\theta_k=2\lambda_k$, $\forall k$. This reduces to $\mathbf{z}_k^T\mathbf{M}\mathbf{z}_k=1$, $\forall k$ when the damping is zero. This is consistent with the unity modal-mass convention often used in experimental modal analysis and finite element methods.
(b) Choose $\theta_k=1$, $\forall k$. Theoretical analysis becomes easiest with this normalization. However, as pointed out in [23,24], this normalization is inconsistent with undamped or classically damped modal theories.

Recall that Lancaster's formulation requires the normalization according to (b).

3 The Modified Lancaster's Method

3.1 Theory. In this section it is intended to reformulate Lancaster's equations in terms of the transfer function residues. The transfer function matrix of a viscously damped system has the form

$$\mathbf{H}(s)=\sum_{k=1}^{2N}\frac{\mathbf{R}_k}{s-\lambda_k} \quad (10)$$

where $\mathbf{R}_k \in \mathbb{C}^{N\times N}$ is the residue matrix corresponding to the k-th mode and $s=i\omega$ where ω denotes frequency. Because all the eigenvalues appear in complex conjugate pairs, due to Eq. (8) it is clear that

$$\lambda_{N+k}=\lambda_k^*$$
$$\text{and } \mathbf{R}_{N+k}=\mathbf{R}_k^*, \quad \text{for } 1\leq k\leq N. \quad (11)$$

In references [19,24] it was shown that the residue matrix $\mathbf{R}_k$ can be related to the corresponding mode and the normalization constant as

$$\mathbf{R}_k=\frac{\mathbf{z}_k\mathbf{z}_k^T}{\theta_k}. \quad (12)$$

In a modal testing procedure, typically, a set of transfer functions is measured by exciting a structure at some *a priori* selected grid points. The type of structures normally encountered in practice satisfy the usual check of reciprocity. This makes the matrix of transfer functions $\mathbf{H}(s)$ symmetric. The poles λ_k appearing in Eq. (10) can be related to the natural frequencies, ω_k and the damping factors, ζ_k, as

$$\lambda_k, \lambda_k^* \approx -\zeta_k\omega_k \pm i\omega_k. \quad (13)$$

Usually the damping of a structure is sufficiently light so that all modes are subcritically damped, i.e., all of them are oscillatory in nature. In this case the transfer functions of a system has "peaks" corresponding to all the modes. The natural frequencies and the damping factors can be obtained by examining each peak separately, for example using the circle fitting method [25]. Estimation of ω_k and ζ_k is likely to be good if the peaks are well separated. Once the poles are known, the residues can be obtained easily, see for example [19,26,27]. From the identified residues $\mathbf{R}_k$, the complex modes $\mathbf{z}_k$ should be obtained by using Eq. (12).

From the above discussion it clear that determination of the transfer function residues is the first step to obtain complex modes from experimental modal analysis. However, there is one difficulty in determining $\mathbf{z}_k$ from $\mathbf{R}_k$ via Eq. (12). Determination of the constants θ_k requires knowledge of the mass and damping matrices (see Eq. (9)), which are unknown in an identification problem. For this reason, in this paper it is intended to use the residues directly, thus avoiding the use of modes and bypassing this difficulty. Next, Lancaster's equations are reformulated in terms of the poles and the residues of the transfer functions.

By definition, we know that the matrix of transfer functions $\mathbf{H}(s)$ is the inverse of the *dynamic stiffness matrix* $\mathbf{D}(s)$, that is

$$\mathbf{H}(s)=\mathbf{D}^{-1}(s), \quad \text{where } \mathbf{D}(s)=s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}\in\mathbb{C}^{N\times N}. \quad (14)$$

Rewrite the expression of the dynamic stiffness matrix as

$$\mathbf{D}(s)=s^2\mathbf{M}\left[\mathbf{I}_N+\frac{\mathbf{M}^{-1}}{s}\left(\mathbf{C}+\frac{\mathbf{K}}{s}\right)\right] \quad (15)$$

where $\mathbf{I}_N$ denotes an identity matrix of size N. Taking the inverse of this equation and expanding the right-hand side one obtains

$$\mathbf{H}(s)=\mathbf{D}^{-1}(s)=\left[\mathbf{I}_N-\frac{\mathbf{M}^{-1}}{s}\left(\mathbf{C}+\frac{\mathbf{K}}{s}\right)+\left\{\frac{\mathbf{M}^{-1}}{s}\left(\mathbf{C}+\frac{\mathbf{K}}{s}\right)\right\}^2 -\cdots\right]\frac{\mathbf{M}^{-1}}{s^2}. \quad (16)$$

Equation (16) can be further simplified to obtain

$$\mathbf{H}(s)=\frac{\mathbf{M}^{-1}}{s^2}+\frac{1}{s^3}(-\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1})+\frac{1}{s^4}(\mathbf{M}^{-1}[\mathbf{C}\mathbf{M}^{-1}\mathbf{C} -\mathbf{K}]\mathbf{M}^{-1})+\cdots. \quad (17)$$

Now, express a general term of the expression of transfer function matrix given by Eq. (10) as

$$\frac{\mathbf{R}_k}{s-\lambda_k}=\left[s\left(1-\frac{\lambda_k}{s}\right)\right]^{-1}\mathbf{R}_k=\frac{1}{s}\mathbf{R}_k+\frac{1}{s^2}[\lambda_k\mathbf{R}_k]+\frac{1}{s^3}[\lambda_k^2\mathbf{R}_k] +\frac{1}{s^4}[\lambda_k^3\mathbf{R}_k]+\cdots. \quad (18)$$

Using the above expression, the transfer function matrix in Eq. (10) can be expressed as

$$\mathbf{H}(s)=\frac{1}{s}\left[\sum_{k=1}^{2N}\mathbf{R}_k\right]+\frac{1}{s^2}\left[\sum_{k=1}^{2N}\lambda_k\mathbf{R}_k\right]+\frac{1}{s^3}\left[\sum_{k=1}^{2N}\lambda_k^2\mathbf{R}_k\right] +\frac{1}{s^4}\left[\sum_{k=1}^{2N}\lambda_k^3\mathbf{R}_k\right]+\cdots. \quad (19)$$

Comparing Eqs. (17) and (19) it is clear that their right-hand sides are equal. Equating the coefficients of $1/s$, $1/s^2$, ..., $1/s^4$ in the right-hand sides of Eqs. (17) and (19) the following relationships may be obtained

$$\sum_{k=1}^{2N}\mathbf{R}_k=O \quad (20)$$

$$\sum_{k=1}^{2N}\lambda_k\mathbf{R}_k=\mathbf{M}^{-1} \quad (21)$$

$$\sum_{k=1}^{2N}\lambda_k^2\mathbf{R}_k=-\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1} \quad (22)$$

$$\text{and } \sum_{k=1}^{2N}\lambda_k^3\mathbf{R}_k=\mathbf{M}^{-1}[\mathbf{C}\mathbf{M}^{-1}\mathbf{C}-\mathbf{K}]\mathbf{M}^{-1}. \quad (23)$$

This procedure can be extended to obtain further higher order terms involving λ_k. Note that Eqs. (21) and (22) is equivalent to Lancaster's Eqs. (2) and (4). In view of Eq. (11), from Eq. (21) one has

$$2\sum_{k=1}^{N} \Re(\lambda_k \mathbf{R}_k) = \mathbf{M}^{-1}$$

$$\text{that is,} \quad \mathbf{M}^{(r)} = \frac{1}{2}\left[\sum_{k=1}^{N} \Re(\lambda_k \mathbf{R}_k)\right]^{-1} \tag{24}$$

where $\Re(\bullet)$ denote real part of $(\bullet)$ and the superscript $(\bullet)^{(r)}$ denotes the reconstructed value of $(\bullet)$. Similarly, from Eq. (22), the viscous damping matrix can be expressed as

$$\mathbf{C}^{(r)} = -2\mathbf{M}^{(r)}\left[\sum_{k=1}^{N} \Re(\lambda_k^2 \mathbf{R}_k)\right]\mathbf{M}^{(r)}. \tag{25}$$

The expression of the viscous damping matrix in Eq. (25) can also be obtained in terms of the stiffness matrix. The expression of the dynamic stiffness matrix in Eq. (14) can be rearranged as

$$\mathbf{D}(s) = \mathbf{K}[\mathbf{I}_N + s(s\mathbf{K}^{-1}\mathbf{M} + \mathbf{K}^{-1}\mathbf{C})]. \tag{26}$$

Taking the inverse of Eq. (26) and expanding the right-hand side one obtains

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s) = [\mathbf{I}_N - s(s\mathbf{K}^{-1}\mathbf{M} + \mathbf{K}^{-1}\mathbf{C}) + \{s(s\mathbf{K}^{-1}\mathbf{M} + \mathbf{K}^{-1}\mathbf{C})\}^2 - \cdots]\mathbf{K}^{-1}. \tag{27}$$

The preceding equation can be further simplified to obtain

$$\mathbf{H}(s) = \mathbf{K}^{-1} + s(-\mathbf{K}^{-1}\mathbf{C}\mathbf{K}^{-1}) + s^2(\mathbf{K}^{-1}[\mathbf{C}\mathbf{K}^{-1}\mathbf{C} - \mathbf{M}]\mathbf{K}^{-1}) + \cdots. \tag{28}$$

Further, rewrite the expression of transfer function matrix given in Eq. (10) as

$$\begin{aligned}\mathbf{H}(s) &= \sum_{k=1}^{2N} \frac{\mathbf{R}_k}{s-\lambda_k} = \sum_{k=1}^{2N} \left[-\lambda_k\left(1 - \frac{s}{\lambda_k}\right)\right]^{-1}\mathbf{R}_k \\ &= -\sum_{k=1}^{2N} \lambda_k^{-1}\mathbf{R}_k - s\left[\sum_{k=1}^{2N} \lambda_k^{-2}\mathbf{R}_k\right] \\ &\quad - s^2\left[\sum_{k=1}^{2N} \lambda_k^{-3}\mathbf{R}_k\right] - s^3\left[\sum_{k=1}^{2N} \lambda_k^{-4}\mathbf{R}_k\right] - \cdots.\end{aligned} \tag{29}$$

Equating the coefficients of s^0, s^1, s^2 in the right-hand sides of Eqs. (28) and (29) one obtains

$$\sum_{k=1}^{2N} \lambda_k^{-1}\mathbf{R}_k = -\mathbf{K}^{-1} \tag{30}$$

$$\sum_{k=1}^{2N} \lambda_k^{-2}\mathbf{R}_k = \mathbf{K}^{-1}\mathbf{C}\mathbf{K}^{-1} \tag{31}$$

$$\text{and} \quad \sum_{k=1}^{2N} \lambda_k^{-3}\mathbf{R}_k = \mathbf{K}^{-1}[\mathbf{M} - \mathbf{C}\mathbf{K}^{-1}\mathbf{C}]\mathbf{K}^{-1}. \tag{32}$$

This procedure can be extended to obtain further lower order terms involving λ_k. Note that Eq. (30) is equivalent to Lancaster's Eq. (3). Using the relationships in Eq. (11), equations (30) and (31) can be simplified to obtain the reconstructed value of the stiffness and damping matrices as

$$\mathbf{K}^{(r)} = -\frac{1}{2}\left[\sum_{k=1}^{N} \Re(\mathbf{R}_k/\lambda_k)\right]^{-1} \tag{33}$$

$$\text{and} \quad \mathbf{C}^{r} = 2\mathbf{K}^{(r)}\left[\sum_{k=1}^{N} \Re(\mathbf{R}_k/\lambda_k^2)\right]\mathbf{K}^{(r)}. \tag{34}$$

Thus, Eqs. (24), (25), (33) and (34) provide relationships equivalent to that of Lancaster's. Because these relationships are in terms of transfer function residues and poles, they naturally avoid the normalization problem associated with Lancaster's method. Note that the full transfer function matrix is required to apply this method. Because most structures encountered in practice satisfy the reciprocity relationship, it is sufficient to measure either the upper triangular part or the lower triangular part of the transfer function matrix. This implies that for N points in a structure, $N(N-1)$ transfer function measurements are required. However, this is not a serious drawback. With the advancement of the laser vibrometer techniques it is possible to automate the laborious task of measuring many transfer functions.

The method developed here for the identification of the system matrices is simple, direct and requires very little computational effort. Observe that there are two different expressions given for the damping matrix. If the damping of the system is truly viscous, then the results obtained from both the equations should be the same. Next, these issues are discussed using numerical examples.

3.2 Numerical Examples

3.2.1 Example 1. We consider a four-degree-of-freedom system with viscous damping. The system matrices are given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -3 & 7 & -4 & 0 \\ 0 & -4 & 7 & -3 \\ 0 & 0 & -3 & 5 \end{bmatrix} \quad \text{and}$$

$$\mathbf{C} = \begin{bmatrix} 1.0 & -0.3 & 0 & 0 \\ -0.3 & 0.7 & -0.4 & 0 \\ 0 & -0.4 & 0.7 & -0.3 \\ 0 & 0 & -0.3 & 0.5 \end{bmatrix}. \tag{35}$$

The system has four modes and all of them are subcritically damped. The poles and the residues corresponding to the four modes, obtained using the modal properties, are given in Table 1. Using these values, from Eqs. (24) and (33), the mass and stiffness matrices can be reconstructed as

$$\mathbf{M}^{(r)} = \frac{1}{2}\left[\sum_{k=1}^{N} \Re(\lambda_k \mathbf{R}_k)\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

$$\text{and} \quad \mathbf{K}^{(r)} = -\frac{1}{2}\left[\sum_{k=1}^{N} \Re(\mathbf{R}_k/\lambda_k)\right]^{-1} = \begin{bmatrix} 5 & -3 & 0 & 0 \\ -3 & 7 & -4 & 0 \\ 0 & -4 & 7 & -3 \\ 0 & 0 & -3 & 5 \end{bmatrix}. \tag{37}$$

Using the reconstructed values of the mass and stiffness matrices, from Eq. (25) or (34), the damping matrix can be calculated as

Table 1 Poles and residues of the four DOF system with viscous damping

Mode Number (k)	Pole (λ_k)	Residue Matrix ($\mathbf{R}_k$)
1	$-0.0477+0.7072i$	$\begin{bmatrix} -0.0097-0.0636i & -0.0078-0.0964i & -0.0059-0.0966i & -0.0035-0.0645i \\ -0.0078-0.0964i & -0.0013-0.1454i & 0.0016-0.1455i & 0.0018-0.0971i \\ -0.0059-0.0966i & 0.0016-0.1455i & 0.0045-0.1457i & 0.0037-0.0971i \\ -0.0035-0.0645i & 0.0018-0.0971i & 0.0037-0.0971i & 0.0029-0.0648i \end{bmatrix}$
2	$-0.2260+1.7604i$	$\begin{bmatrix} -0.0238-0.0777i & 0.0018-0.0539i & 0.0116+0.0490i & 0.0082+0.0811i \\ 0.0018-0.0539i & 0.0128-0.0335i & -0.0033+0.0332i & -0.0124+0.0526i \\ 0.0116+0.0490i & -0.0033+0.0332i & -0.0052-0.0307i & -0.0018-0.0504i \\ 0.0082+0.0811i & -0.0124+0.0526i & -0.0018-0.0504i & 0.0079-0.0812i \end{bmatrix}$
3	$-0.4137+2.4209i$	$\begin{bmatrix} -0.0086-0.1137i & 0.0287+0.0351i & -0.0208+0.0321i & 0.0093-0.0942i \\ 0.0287+0.0351i & -0.0174-0.0049i & -0.0005-0.0152i & 0.0184+0.0328i \\ -0.0208+0.0321i & -0.0005-0.0152i & 0.0120-0.0043i & -0.0216+0.0233i \\ 0.0093-0.0942i & 0.0184+0.0328i & -0.0216+0.0233i & 0.0211-0.0757i \end{bmatrix}$
4	$-0.4126+2.6565i$	$\begin{bmatrix} 0.0422-0.0192i & -0.0227+0.0284i & 0.0150-0.0359i & -0.0139+0.0494i \\ -0.0227+0.0284i & 0.0058-0.0280i & 0.0022+0.0305i & -0.0078-0.0396i \\ 0.0150-0.0359i & 0.0022+0.0305i & -0.0113-0.0308i & 0.0197+0.0384i \\ -0.0139+0.0494i & -0.0078-0.0396i & 0.0197+0.0384i & -0.0319-0.0472i \end{bmatrix}$

$$
\begin{aligned}
\mathbf{C}^{(r)} &= -2\mathbf{M}^{(r)}\left[\sum_{k=1}^{N} \Re(\lambda_k^2 \mathbf{R}_k)\right]\mathbf{M}^{(r)} \\
&= 2\mathbf{K}^{(r)}\left[\sum_{k=1}^{N} \Re(\mathbf{R}_k/\lambda_k^2)\right]\mathbf{K}^{(r)} \\
&= \begin{bmatrix} 1.0 & -0.3 & 0 & 0 \\ -0.3 & 0.7 & -0.4 & 0 \\ 0 & -0.4 & 0.7 & -0.3 \\ 0 & 0 & -0.3 & 0.5 \end{bmatrix}.
\end{aligned} \tag{38}
$$

Equations (36), (37) and (38) clearly show that the system matrices are reconstructed exactly. Note that only the poles and their associated residues are used to identify the system matrices. In a similar study, Pilkey and Inman [9] have used the original Lancaster's Eqs. (2), (3) and (4), to identify the viscous damping matrix of a system from a full set of modal data. Their method was based on an iterative approach and requires the mass matrix of the system. The proposed method neither requires the mass matrix nor does it use an iterative approach. This demonstrates the strength of the modified Lancaster's equations (24), (25), (33) and (34).

3.2.2 Example 2. In the last example, the exact values of the poles and the residues are used for identification of the system matrices. The exact reconstructed values of the system matrices tell very little other than verifying the correctness of the mathematical expressions developed so far in this paper. In practice, due to the presence of noise or random errors, the measured transfer functions become noisy. This in turn makes the poles and the residues erroneous. The purpose of this example is to analyze the effects of random errors in the measured data. A system consisting of a linear array of spring-mass oscillators and dampers is considered for this purpose. Figure 1 shows the model system. N masses, each with mass m_u, are connected by springs of stiffness value k_u. The mass matrix of the system has the form $\mathbf{M}=m_u\mathbf{I}_N$ where $\mathbf{I}_N$ is the $N\times N$ identity matrix. The stiffness matrix of the system is

$$
\mathbf{K}=k_u\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}. \tag{39}
$$

Certain of the masses of the system shown in Fig. 1 have viscous dampers connecting them to the ground. The damping matrix can be expressed as $\mathbf{C}=c_u\overline{\mathbf{I}}$ where c_u is the viscous damping constant and $\overline{\mathbf{I}}$ is a block identity matrix which is non-zero only between the n_1-th and n_2-th entries along the diagonal, so that n_1 denotes the first damped mass and n_2 the last one. For the numerical calculations, we have considered a thirty-degree-of-freedom system so that $N=30$. Values of the mass and stiffness associated with each unit are assumed to be the same with numerical values of $m_u=1$ kg, $k_u=10$ N/m. The start and end positions of the dampers are assumed to be $n_1=5$ and $n_2=15$ with values for each unit of $c_u=0.5$ Nm/s.

In order to simulate the effect of noise, we perturb the poles and the residues by adding zero-mean Gaussian random noise to them. Numerical experiments have been performed by adding different levels of noise to the following four quantities:

1. Real parts of complex eigenvalues λ_R.
2. Imaginary parts of complex eigenvalues λ_I.
3. Real parts of residues $\mathbf{R}_{k_R}$.
4. Imaginary parts of residues $\mathbf{R}_{k_I}$.

Levels of noise associated with the above quantities, denoted by $\tilde{\lambda}_R$, $\tilde{\lambda}_I$, $\tilde{\mathbf{R}}_{k_R}$ and $\tilde{\mathbf{R}}_{k_I}$, are expressed as a percentage of their corresponding original values. In practice we hope to obtain the natural frequencies and damping factors, i.e., the poles, with good accuracy. So, in what follows next, we assume $\tilde{\lambda}_R=\tilde{\lambda}_I=2\%$ for all the modes. The following cases are considered regarding the noise levels $\tilde{\mathbf{R}}_{k_R}$ and $\tilde{\mathbf{R}}_{k_I}$ for all k:

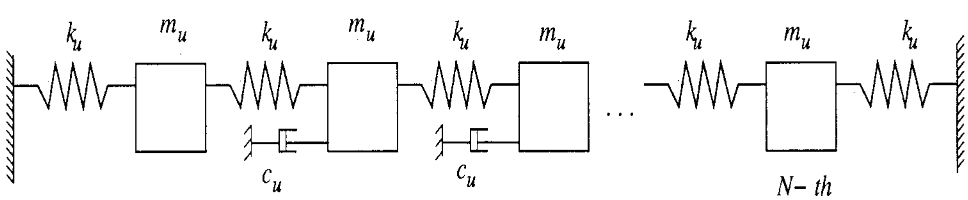

Fig. 1 Linear array of N spring-mass oscillators, $N=30$, $m_u=1$ kg, $k_u=10$ N/m and $c_u=0.5$ Nm/s.

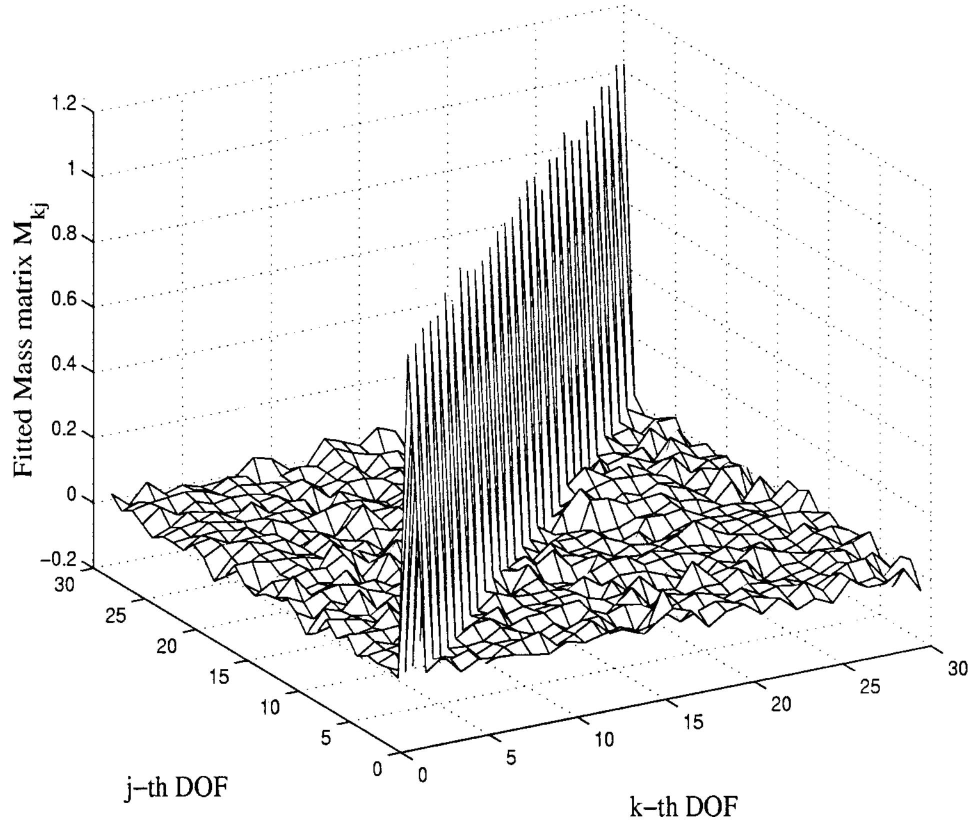

Fig. 2 Identified mass matrix for noise case (*a*).

(a) $\tilde{\mathbf{R}}_{k_R}=0\%$ and $\tilde{\mathbf{R}}_{k_I}=10\%$
(b) $\tilde{\mathbf{R}}_{k_R}=10\%$ and $\tilde{\mathbf{R}}_{k_I}=0\%$
(c) $\tilde{\mathbf{R}}_{k_R}=5\%$ and $\tilde{\mathbf{R}}_{k_I}=5\%$.

Figures 2 and 3 show the identified mass matrix and the viscous damping matrix corresponding to the noise cases (a). The fitted viscous damping matrix in Fig. 3 is obtained using the reconstructed mass matrix from Eq. (25). The high portion of this diagram corresponds to the position of the dampers. Both these matrices are reasonably close to their corresponding exact values. For this set of data, application of Eqs. (33) and (34) to reconstruct the stiffness and damping matrices produces results (not shown here) which are very far from the true values. This demonstrates that the identification procedure of the stiffness matrix and the viscous damping matrix using Eqs. (33) and (34) is sensitive to errors in the imaginary parts of the transfer function residues.

Now consider the noise case (b). It is observed that (result not shown) the mass matrix can be reconstructed with very good accuracy using Eq. (24). The identified damping matrix, obtained from Eq. (25), using the reconstructed mass matrix is shown in Fig. 4. Observe that the identified viscous damping matrix is less accurate compared to that for the noise case (a), as shown in Fig. 3. The identified stiffness matrix obtained using Eq. (33) is shown in Fig. 5. This result is much better than that for the noise case (a). However, the identified viscous damping matrix, obtained using Eq. (34) is very far from its true value, as in noise case (a). Numerical experiments were conducted using small values (1%–2%) of $\tilde{\mathbf{R}}_{k_I}$. In general, it was observed that the identified stiffness matrix obtained using Eq. (33) is very sensitive to errors in the imaginary parts of the transfer function residues.

For the noise case (c), again, the identified mass matrix turns out to be close to its exact value. The identified damping matrix using Eq. (25), shown in Fig. 6, is also reasonably close to its exact value. Again, like the noise case (a), the identified stiffness and damping matrices using Eqs. (33) and (34) produce unacceptable results.

From the above results we conclude that the presence of noise affects the identified matrices obtained using Eq. (33) and (34) to a great extent. Thus, for all practical cases Eqs. (24) and (25) should be used to reconstruct the mass matrix and the viscous damping matrix. Next, we extend the current studies to a class of nonviscously damped systems.

4 Dynamics of Nonviscously Damped Systems

Possibly the most general way to model damping within the linear range is to use nonviscous damping models in which the damping forces depend on the past history of motion via convolution integrals over some kernel functions. The equations of motion describing free vibration of an *N*-degree-of-freedom linear system with such nonviscous damping can be expressed by

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\int_0^t \mathcal{G}(t-\tau)\dot{\mathbf{q}}(\tau)\,d\tau+\mathbf{K}\mathbf{q}(t)=0. \qquad (40)$$

The kernel functions $\mathcal{G}(t)\in\mathbb{R}^{N\times N}$, or others closely related to them, are described under many different names in the literature of different subjects: for example, retardation functions, heredity functions, after-effect functions, relaxation functions etc. This model was originally introduced by Biot [28]. In the special case when $\mathcal{G}(t-\tau)=\mathbf{C}\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, Eq. (40) reduces to the case of viscous damping. The damping model of this kind is a further generalization of the familiar viscous damping.

The eigenvalue problem associated with Eq. (40) can be defined by taking the Laplace transform as

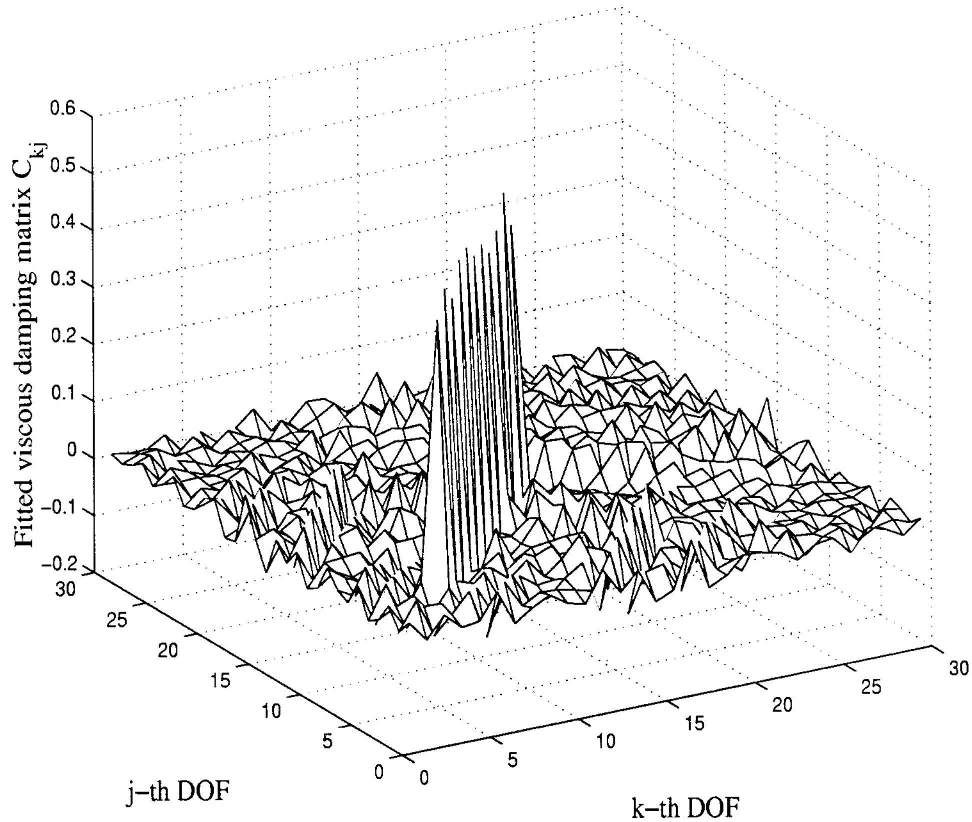

Fig. 3 Identified viscous damping matrix for noise case (*a*).

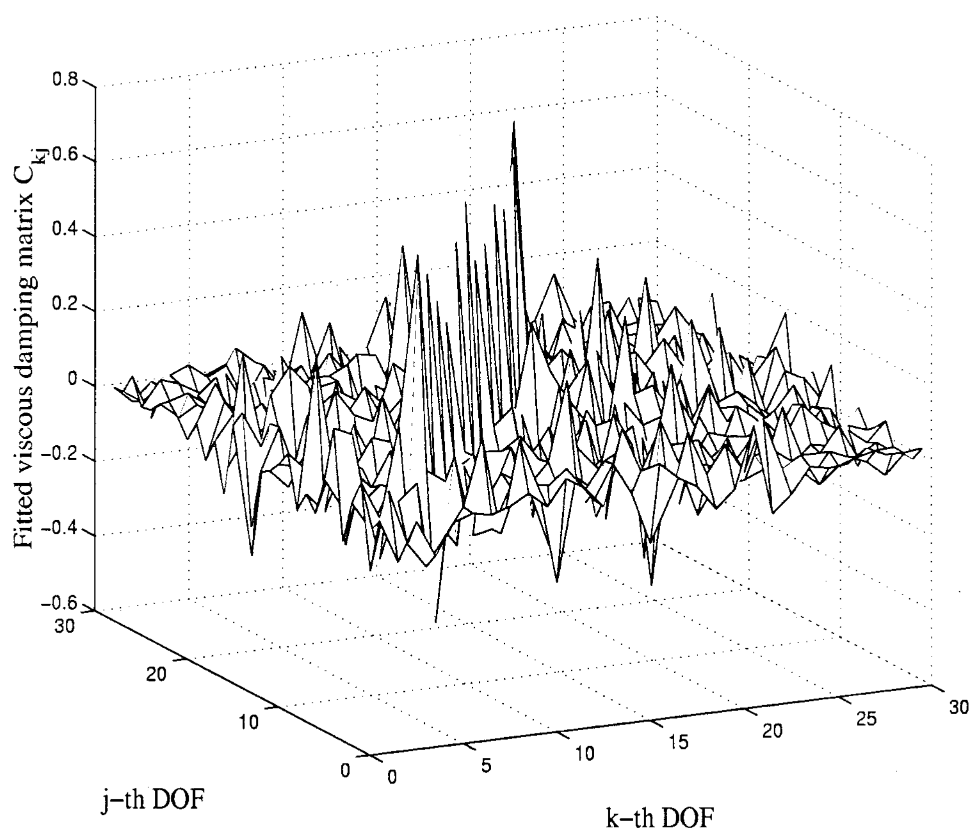

Fig. 4 Identified viscous damping matrix for noise case (*b*).

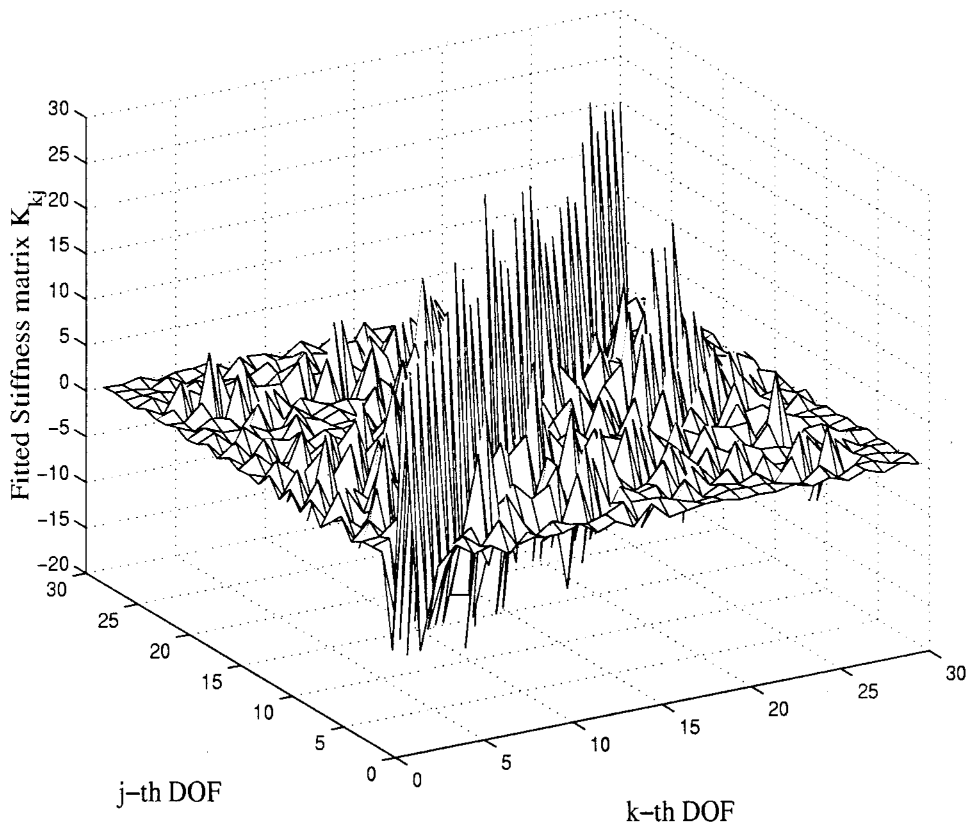

Fig. 5 Identified stiffness matrix for noise case (*b*).

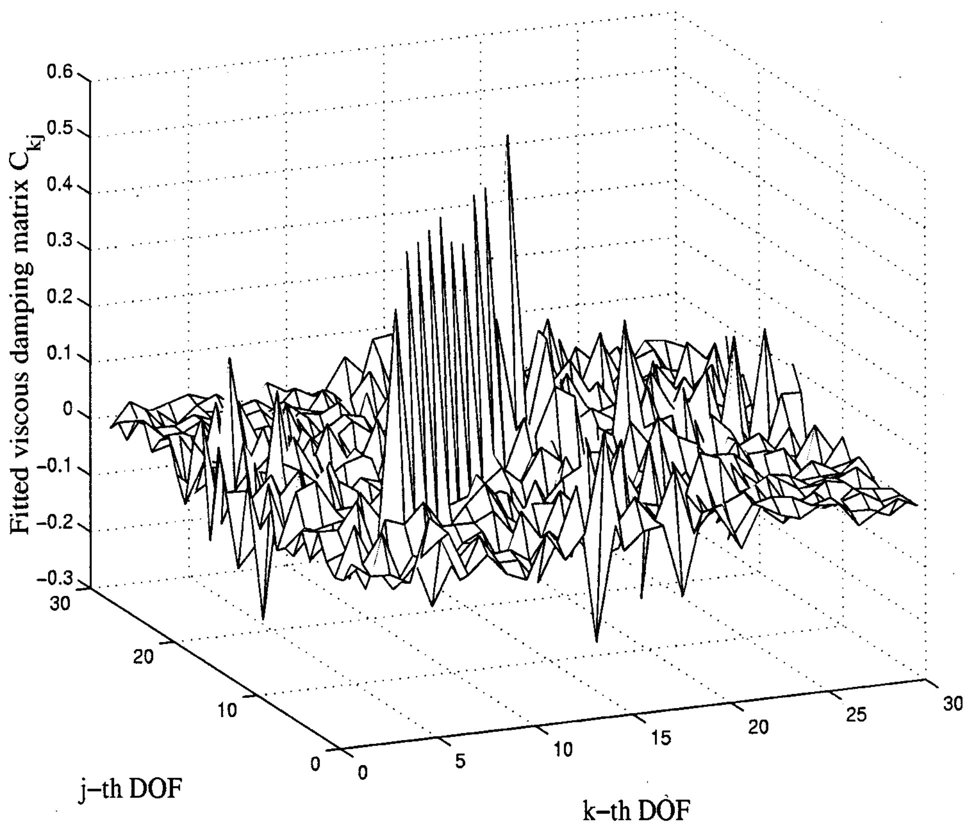

Fig. 6 Identified viscous damping matrix for noise case (*c*).

$$\lambda_k^2 \mathbf{M}\mathbf{z}_k + \lambda_k \mathbf{G}(\lambda_k)\mathbf{z}_k + \mathbf{K}\mathbf{z}_k = 0, \tag{41}$$

where $\mathbf{G}(s)$ is the Laplace transform of $\mathcal{G}(t)$. We do not assume any specific functional form of $\mathbf{G}(s)$ but assume that $\mathbf{G}(s)$ has finite poles and $|G_{jk}(s)| < \infty$ when $s \to \infty$. This in turn implies that the elements of $\mathbf{G}(s)$ are at the most of order $1/s$ in s or constant, as in the case of viscous damping. The eigenvalue problem of the form (41) has been discussed by Adhikari [29]. Here we briefly outline the main features.

The eigenvalues, λ_k, associated with Eq. (41) are roots of the characteristic equation

$$\det[s^2 \mathbf{M} + s\mathbf{G}(s) + \mathbf{K}] = 0. \tag{42}$$

Suppose the order of the characteristic equation is m. In general m is more than $2N$, that is $m = 2N + p$; $p \geqslant 0$. Thus, although the system has N degrees of freedom, the number of eigenvalues is more than $2N$. This is a major difference between nonviscously damped systems and viscously damped systems where the number of eigenvalues is exactly $2N$, including any multiplicities. We restrict our attention to a special case when, among the m eigenvalues, only $2N$ appear in complex conjugate pairs. For convenience, arrange the eigenvalues as

$$\lambda_1, \lambda_2, \cdots, \lambda_N, \lambda_1^*, \lambda_2^*, \cdots, \lambda_N^*, \lambda_{2N+1}, \cdots, \lambda_m. \tag{43}$$

It is further assumed that the remaining p eigenvalues, $\lambda_{2N+1}, \cdots, \lambda_m$, are purely real.

From Eq. (41) it is easy to observe that, when λ_k appear in complex conjugate pairs, $\mathbf{z}_k$ also appear in complex conjugate pairs, and when λ_k is real $\mathbf{z}_k$ can also be real. Corresponding to the $2N$ complex conjugate pairs of eigenvalues, the N eigenvectors together with their complex conjugates are called *elastic modes*. These modes are related to the N modes of vibration of the structural system. Physically, the assumption of "$2N$ complex conjugate pairs of eigenvalues" implies that all the elastic modes are oscillatory in nature, that is, they are subcritically damped. The modes corresponding to the "additional" p eigenvalues are called *nonviscous modes*. These modes are induced by the nonviscous effect of the damping mechanism. For stable passive systems the nonviscous modes are over-critically damped (i.e., negative real eigenvalues) and not oscillatory in nature.

Following [30] the normalization relationship satisfied by the modes of nonviscously damped systems may be expressed as

$$\mathbf{z}_k^T \left[2\lambda_k \mathbf{M} + \mathbf{G}(\lambda_k) + \lambda_k \frac{\partial[\mathbf{G}(s)]}{\partial s} \bigg|_{\lambda_k} \right] \mathbf{z}_k = \theta_k, \quad \forall k = 1, \cdots, m. \tag{44}$$

Note that Eq. (44) reduces to Eq. (9), the corresponding relationship for viscously damped systems, when $\mathbf{G}(s)$ is constant with respect to s.

5 Lancaster's Method for Nonviscously Damped Systems

5.1 Theory. In this section Lancaster's equations are extended to nonviscously damped systems. In reference [29], it was shown that the transfer function matrix of nonviscously damped systems can be expressed as

$$\mathbf{H}(s) = \sum_{k=1}^{2N+p} \frac{\mathbf{R}_k}{s - \lambda_k} \tag{45}$$

where the residue matrices $\mathbf{R}_k$ take the same form as given by Eq. (12). The difference between Eq. (45) and Eq. (10) comes from the fact that the sum in Eq. (45) is extended to nonviscous modes also. Due to the arrangement of the eigenvalues in Eq. (43), in addition to Eq. (11), which describes the relationships for the elastic modes, the following relationships also hold:

$$\lambda_{2N+k} = \lambda_{nv_k} \tag{46}$$

$$\mathbf{R}_{2N+k} = \mathbf{R}_{nv_k}, \quad 1 \leqslant k \leqslant p.$$

In the above $(\bullet)_{nv}$ denotes the nonviscous terms of $(\bullet)$. It has been mentioned earlier that for passive systems, the kind of systems we mostly encounter in practice, the nonviscous modes are usually over-critically damped. Thus, in contrast to the elastic modes, they do not produce any peaks in the transfer functions. As a consequence to this, the modal parameters corresponding to nonviscous modes cannot be obtained by usual techniques of experimental modal analysis. This is the fundamental difficulty in considering nonviscously damped systems. However, as shown by Adhikari [29], the nonviscous part of (45) may be quite small compared to that of the elastic part. Next, assuming the validity of Eq. (45), Lancaster's formulations are extended to nonviscously damped systems.

A major difficulty in relating the system matrices with the poles and residues is that, unlike viscously damped systems, the damping matrix $\mathbf{G}(s)$ is a function of s. To simplify the problem we consider only two limiting cases, (a) when $s \to \infty$, and (b) when $s \to 0$. Suppose

$$\lim_{s\to\infty} \mathbf{G}(s) = \mathbf{G}_\infty \in \mathbb{R}^{N\times N} \tag{47}$$

$$\text{and} \quad \lim_{s\to 0} \mathbf{G}(s) = \mathbf{G}_0 \in \mathbb{R}^{N\times N}, \tag{48}$$

where $\|\mathbf{G}_\infty\|, \|\mathbf{G}_0\| < \infty$.

For nonviscously damped systems, the transfer function matrix has the form

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s), \quad \text{where} \quad \mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K} \in \mathbb{C}^{N\times N}. \tag{49}$$

Rewrite the expression of the dynamic stiffness matrix as

$$\mathbf{D}(s) = s^2 \mathbf{M} \left[\mathbf{I}_N + \frac{\mathbf{M}^{-1}}{s} \left(\mathbf{G}(s) + \frac{\mathbf{K}}{s} \right) \right] \tag{50}$$

Taking the inverse of this equation and expanding the right-hand side one obtains

$$\mathbf{H}(s) = \frac{\mathbf{M}^{-1}}{s^2} + \frac{1}{s^3}(-\mathbf{M}^{-1}\mathbf{G}(s)\mathbf{M}^{-1}) + \frac{1}{s^4}(\mathbf{M}^{-1}[\mathbf{G}(s)\mathbf{M}^{-1}\mathbf{G}(s) - \mathbf{K}]\mathbf{M}^{-1}) + \cdots. \tag{51}$$

The expression of $\mathbf{H}(s)$ given by Eq. (19) holds for nonviscously damped systems provided the limit of the sums appearing in this equation is extended to $2N+p$, that is

$$\mathbf{H}(s) = \frac{1}{s}\left[\sum_{k=1}^{2N+p} \mathbf{R}_k\right] + \frac{1}{s^2}\left[\sum_{k=1}^{2N+p} \lambda_k \mathbf{R}_k\right] + \frac{1}{s^3}\left[\sum_{k=1}^{2N+p} \lambda_k^2 \mathbf{R}_k\right] + \frac{1}{s^4}\left[\sum_{k=1}^{2N+p} \lambda_k^3 \mathbf{R}_k\right] + \cdots. \tag{52}$$

Comparing Eqs. (51) and (52) it is clear that their right-hand sides are equal. Multiplying these equations by s and s^2 respectively and taking the limit as $s \to \infty$ one obtains

$$\sum_{k=1}^{2N+p} \mathbf{R}_k = O \tag{53}$$

$$\text{and} \quad \sum_{k=1}^{2N+p} \lambda_k \mathbf{R}_k = \mathbf{M}^{-1}. \tag{54}$$

Observe that the coefficients associated with the corresponding (negative) powers of s in the series expressions (51) and (52) cannot be equated because $\mathbf{G}(s)$ is also a function of s. However, in the limit when $s \to \infty$, the variation of $\mathbf{G}(s)$ becomes negligible as by Eq. (47) it approaches to $\mathbf{G}_\infty$. Considering the second term of the right-hand side of Eq. (51), equating it with the corresponding term of Eq. (52) and taking the limit as $s \to \infty$ one obtains

$$\sum_{k=1}^{2N+p} \lambda_k^2 \mathbf{R}_k = -\mathbf{M}^{-1} \mathbf{G}_\infty \mathbf{M}^{-1}. \tag{55}$$

It must be noted that this procedure cannot be extended to further lower order terms as all of them would be affected by the functional variation of $\mathbf{G}(s)$ from previous terms.

Equations (54) and (55) are equivalent to Lancaster's Eqs. (2) and (4). In view of Eqs. (11) and (46), from Eq. (54) one has

$$2\sum_{k=1}^{N} \Re(\lambda_{e_k} \mathbf{R}_{e_k}) + \sum_{k=1}^{p} \lambda_{nv_k} \mathbf{R}_{nv_k} = \mathbf{M}^{-1}$$

$$\text{that is,} \quad \mathbf{M}^{(r)} = \left[2\sum_{k=1}^{N} \Re(\lambda_{e_k} \mathbf{R}_{e_k}) + \sum_{k=1}^{p} \lambda_{nv_k} \mathbf{R}_{nv_k} \right]^{-1} \tag{56}$$

where $(\bullet)_e$ denotes the elastic parts of $(\bullet)$. Similarly, from Eq. (55), the damping matrix of damping functions evaluated at $s \to \infty$ can be expressed as

$$\mathbf{G}_\infty^{(r)} = -\mathbf{M}^{(r)} \left[2\sum_{k=1}^{N} \Re(\lambda_{e_k}^2 \mathbf{R}_{e_k}) + \sum_{k=1}^{p} \lambda_{nv_k}^2 \mathbf{R}_{nv_k} \right] \mathbf{M}^{(r)}. \tag{57}$$

The damping matrix can also be expressed in terms of the stiffness matrix. Following the approach outlined for viscously damped systems, and taking the limit as $s \to 0$ it may be shown that the reconstructed values of the stiffness and damping matrices are

$$\mathbf{K}^{(r)} = -\left[2\sum_{k=1}^{N} \Re(\mathbf{R}_{e_k}/\lambda_{e_k}) + \sum_{k=1}^{p} \mathbf{R}_{nv_k}/\lambda_{nv_k} \right]^{-1} \tag{58}$$

$$\mathbf{G}_0^{(r)} = \mathbf{K}^{(r)} \left[2\sum_{k=1}^{N} \Re(\mathbf{R}_{e_k}/\lambda_{e_k}^2) + \sum_{k=1}^{p} \mathbf{R}_{nv_k}/\lambda_{nv_k}^2 \right] \mathbf{K}^{(r)}. \tag{59}$$

Thus, Eqs. (56), (57), (58) and (59) provide relationships equivalent to that of Lancaster's for nonviscously damped systems. One difficulty in employing this approach is that the damping matrix in the Laplace domain, $\mathbf{G}(s)$, can be obtained only at the two limiting values when $s \to \infty$, and $s \to 0$. That is, no clue regarding the functional variation of $\mathbf{G}(s)$ can be obtained between these two extreme values. Further note that, in contrast to viscously damped systems, the poles and the residues corresponding to the nonviscous modes appear in Eqs. (56)–(59). This is the biggest difficulty in applying these equations in practice. As mentioned earlier, it is still not possible to identify the nonviscous modes from the measured transfer functions. Due to this shortfall it appears that, although it is known that in general mechanical systems are nonviscously damped, one still has to use the expressions developed for viscously damped systems. This fact, in turn, also indicates that Lancaster's original equations are only approximate when applied to nonviscously damped systems and that the "amount of the approximation" depends on how "big" are the nonviscous terms. Next, these issues are discussed using a numerical example.

5.2 Numerical Examples. A three-degree-of-freedom system is used to illustrate the results derived in the last section. The mass and stiffness matrices are assumed to be

$$\mathbf{M} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{60}$$

$$\text{and} \quad \mathbf{K} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}. \tag{61}$$

The matrix of the damping functions is assumed to be of the form

$$\mathcal{G}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.5g(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{62}$$

where

$$g(t) = \delta(t) + (\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}); \quad \mu_1, \quad \mu_2 > 0. \tag{63}$$

The damping matrix in the Laplace domain, $\mathbf{G}(s)$, can be obtained by taking the Laplace transform of Eq. (62). The Laplace transform of $g(t)$ given by Eq. (63) can be obtained as

$$G(s) = 1 + \frac{(\mu_1 + \mu_2)s + 2\mu_1\mu_2}{s^2 + (\mu_1 + \mu_2)s + \mu_1\mu_2}. \tag{64}$$

This damping model is a linear combination of the viscous damping model and the GHM damping model [31,32]. Regarding the numerical values of the damping parameters, we assume $\mu_1 = 0.15$ and $\mu_2 = 0.1$.

Using Eq. (64), together with the expressions of the system matrices given by equations (60)–(62), it can be shown that the order of the characteristic polynomial, $m = 8$. It has been mentioned that (for lightly damped systems), among the m eigenvalues, $2N = 6$ appear in complex conjugate pairs (elastic modes) and the rest $p = m - 2N = 2$ eigenvalues become purely real (nonviscous modes). The eigenvalues (poles) and the residues of the system are shown in Table 2. Observe that the eigenvalues corresponding to the nonviscous modes are purely real and negative, implying that the nonviscous modes are stable and nonoscillatory in nature (i.e., over critically damped). For this simulation example, the residue matrices corresponding to these modes are calculated using the eigenvectors from Eq. (12). The eigenvectors, in turn are calculated using the method outlined in reference [29]. It

Table 2 Poles and residues of the three DOF system with non-viscous damping

Mode Number	Pole (λ_k)	Residue Matrix ($\mathbf{R}_k$)
Elastic mode 1	$-0.1296 + 0.6740i$	$\begin{bmatrix} 0.0107-0.0664i & -0.0030-0.0921i & 0.0107-0.0664i \\ -0.0030-0.0921i & -0.0282-0.1229i & -0.0030-0.0921i \\ 0.0107-0.0664i & -0.0030-0.0921i & 0.0107-0.0664i \end{bmatrix}$
Elastic mode 2	$1.1547i$	$\begin{bmatrix} -0.0722i & 0 & 0.0722i \\ 0 & 0 & 0i \\ 0.0722i & 0 & -0.0722i \end{bmatrix}$
Elastic mode 3	$-0.1312 + 1.4979i$	$\begin{bmatrix} -0.0118-0.0257i & 0.0007+0.0414i & -0.0118-0.0257i \\ 0.0007+0.0414i & 0.0235-0.0559i & 0.0007+0.0414i \\ -0.0118-0.0257i & 0.0007+0.0414i & -0.0118-0.0257i \end{bmatrix}$
Non-viscous Mode 1	-0.1373	$\begin{bmatrix} 0.0011 & 0.0022 & 0.0011 \\ 0.0022 & 0.0045 & 0.0022 \\ 0.0011 & 0.0022 & 0.0011 \end{bmatrix}$
Non-viscous Mode 2	-0.0912	$\begin{bmatrix} 0.0012 & 0.0024 & 0.0012 \\ 0.0024 & 0.0049 & 0.0024 \\ 0.0012 & 0.0024 & 0.0012 \end{bmatrix}$

must be noted that for experimental works the residue matrices are directly obtained from the measured transfer functions and determination of modes is not necessary.

Using the poles and the residues corresponding to the elastic and nonviscous modes it may be verified that Eqs. (56)–(59) are satisfied *exactly*. The interest here, however, is to understand the effect of neglecting the nonviscous modes. Thus, from Eq. (56) one obtains

$$\mathbf{M}^{(r)} \approx \left[2 \sum_{k=1}^{N} \Re(\lambda_{e_k} \mathbf{R}_{e_k}) \right]^{-1} = \begin{bmatrix} 2.9977 & -0.0047 & -0.0023 \\ -0.0047 & 2.9904 & -0.0047 \\ -0.0023 & -0.0047 & 2.9977 \end{bmatrix}. \tag{65}$$

The above value of the reconstructed mass matrix is sufficiently close to the actual value in Eq. (60). From Eq. (58), the reconstructed stiffness matrix using only the elastic modes becomes

$$\mathbf{K}^{(r)} \approx -\left[2 \sum_{k=1}^{N} \Re(\mathbf{R}_{e_k} / \lambda_{e_k}) \right]^{-1} = \begin{bmatrix} 4.0 & -2.0039 & 00 \\ -2.0039 & 4.4290 & -2.0039 \\ 0 & -2.0039 & 4.0 \end{bmatrix} \tag{66}$$

which is again close to the actual value given by Eq. (61).

Taking the Laplace transform of Eq. (62) and considering the limiting cases as $s \to \infty$, and $s \to 0$ one obtains

$$\mathbf{G}_{\infty} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{67}$$

$$\text{and} \quad \mathbf{G}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{68}$$

Now, using only the elastic mode, from Eq. (57) one obtains

$$\mathbf{G}_{\infty}^{(r)} \approx -\mathbf{M}^{(r)} \left[2 \sum_{k=1}^{N} \Re(\lambda_{e_k}^2 \mathbf{R}_{e_k}) \right] \mathbf{M}^{(r)} = \begin{bmatrix} 0.0003 & -0.0018 & 0.0003 \\ -0.0018 & 1.4915 & -0.0018 \\ 0.0003 & -0.0018 & 0.0003 \end{bmatrix}. \tag{69}$$

Similarly, after neglecting the nonviscous terms, Eq. (59) results

$$\mathbf{G}_{0}^{(r)} \approx \mathbf{K}^{(r)} \left[2 \sum_{k=1}^{N} \Re(\mathbf{R}_{e_k} / \lambda_{e_k}^2) \right] \mathbf{K}^{(r)} = \begin{bmatrix} -0.0004 & 0.0317 & -0.0004 \\ 0.0317 & 1.6540 & 0.0317 \\ -0.0004 & 0.0317 & -0.0004 \end{bmatrix}. \tag{70}$$

Observe that the reconstructed damping matrix shown in Eq. (69) is close to its exact value given by Eq. (67) while that given by Eq. (70) differs significantly from the true value given by Eq. (68). Although in practice the nonviscous modes cannot be measured, this study gives the confidence that reasonable estimates of the mass and stiffness matrices and also the damping matrix in the high frequency region may be obtained using modified Lancaster's equations.

6 Conclusions

A method for identification of damping in the context of multiple-degree-of-freedom linear systems has been developed. The approach adopted in this paper is based on the poles and the residues of the measured transfer functions. The newly developed approach extends the applicability of Lancaster's original contribution by avoiding the direct use of modes, thus bypassing the difficulty regarding the normalization, which *a priori* needs the mass and the damping matrices.

For viscously damped systems, the method developed here can identify the exact damping matrix provided the full set of poles and residues are known exactly. The effects of measurement noise have been investigated using a numerical example. It was shown that some of the relationships developed here can be applied to moderately noisy data. Finally, the approach is extended to a class of nonviscously damped systems where the damping forces depend on the past history of the velocities via convolution integrals over some kernel functions. For nonviscously damped systems, the application of Lancaster's original method (in terms of the poles and the residues) provides only approximate estimates of the system matrices because some poles and residues (those corresponding to the nonviscous modes) cannot be "measured" using the conventional experimental modal analysis techniques. The nature of this approximation has been investigated using a numerical example.

The initial numerical study suggests that it might be possible to put the method into practice. However, much remains to be done to consolidate, test and extend both the theory and the methods of practical application. Most immediate of these is the experimental testing of the proposed procedure. Beside these, there are a number of interesting issues. For nonviscously damped systems, a fundamental limitation of the method developed so far is that the matrix of damping functions, $\mathbf{G}(s)$, can only be obtained at two extreme values when $s \to \infty$, and $s \to 0$. This illustrates that, in principle, using the method as it stands now, it is *not* possible to obtain the functional variation of nonviscous damping functions. This demands new research regarding identification of nonviscous damping.

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