



A second-moment approach for direct probabilistic model updating in structural dynamics

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ABSTRACT

Discrepancies between experimentally measured data and computational predictions are unavoidable for structural dynamic systems. Model updating methods have been developed over the past three decades to reduce this gap. Well established model updating methods exist when both the model and experimental measurements are deterministic in nature. However in reality, experimental results may contain uncertainty, for example arising due to unknown experimental errors, or variability in nominally identical structures. Over the past two decades probabilistic approaches have been developed to incorporate uncertainties in computational models. In this paper, the natural frequencies and the eigenvectors of the system are measured and assumed to be uncertain. A random matrix approach is proposed and closed-form expressions are derived for the mean matrix and the covariance matrix of the updated stiffness matrix. A perturbation technique is used to obtain a usable expression for the covariance matrix.

The new method is illustrated by three numerical examples highlighting the influence of the eigenfrequency uncertainties on the mean matrix and the influence of the eigenvector uncertainties on the covariance matrix.

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1. Introduction

In various areas of computational modeling, it has been established over the past three decades that uncertainties should be taken into account for credible predictions. In the context of structural dynamics, such uncertainties can be broadly divided into two categories, namely, parametric uncertainty and non-parametric uncertainty. Parametric uncertainty includes uncertainty in geometric parameters, friction coefficient, and the moduli of the materials involved. In contrast, non-parametric uncertainty can arise due to the lack of scientific knowledge about the model which is unknown a priori. Although this distinction is often made, the origin of uncertainty in real-life systems is not always obvious [1,2]. In this paper we propose an analytical approach to update random variables describing parametric uncertainty of a linear dynamical system using experimental data.

The deterministic finite element model updating problem [3–7] is well established, both in the development of methods and in application to industrial-scale structures [8,9]. The proposed methods can be broadly divided into two categories, namely, the non-parametric (or direct) and parametric approaches. In the non-parametric approach, developed

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Nomenclature			
$\text{cov}(\Omega^2)$	covariance matrix of Ω^2	$\text{tr}(X)$	trace of X
$\ X\ _S$	stochastic norm of X	$\text{vec}(\cdot)$	vectorization operator of a matrix
\hat{Y}	mean of Y estimated through a perturbation method	\hat{X}	maximum likelihood estimator of X
$C = \text{cov}(X)$	covariance matrix of X	$E\{\bullet\}$	expectation operator
$(\Sigma; \Psi)$	Kronecker product decomposition of a covariance matrix	$f(t)$	forcing vector
Ω	natural frequency matrix	K	initial estimate of the stiffness matrix
\otimes	Kronecker product of two matrices	M	mass matrix
\bar{X}	mean of X	n	number of degrees of freedom
$\ X\ _F$	Frobenius norm of X	$q(t)$	response vector
		X	mode shape matrix
		X'	transpose of X
		Y	corrected stiffness matrix

during early eighties, the system matrices (mass, stiffness and damping matrices) are updated directly so that the differences between the predicted data (natural frequencies, damping ratios, and mode shapes) and measured data are minimum according to a suitable norm. The non-parametric methods have some significant problems that have restricted their application in practical examples where the model has a large number of degrees of freedom. The methods do not give a clear physical insight into the modeling errors that are corrected, and the connectivity of the original model is not necessarily preserved. Furthermore the incompleteness of the measured data, in terms of the number of sensors used and the number of modes measured, leads to a highly underdetermined estimation problem. This is further complicated because the lower frequency modes are measured but the higher frequency modes have the greatest effect on the stiffness matrix. Despite these problems non-parametric methods can be applied to reduced order models for some limited objectives, such as controller performance or response prediction studies. The alternative, and increasingly popular, approach is parametric model updating where physical parameters (for example, joint stiffnesses, thicknesses) are selected and updated. The estimation is usually based on some kind of sensitivity analysis that minimizes the error between predicted results and test data from a single physical structure. The choice of updating parameters is an important aspect of the process and should always be justified physically. Model uncertainties should be located and parameterized sensitively to the predictions. Finally, the model should be validated by assessing the model quality within its range of operation and its robustness to modifications in the loading configuration, design changes, coupled structure analysis and different boundary conditions.

Collins et al. [10] developed a Bayesian approach to model updating using linearized sensitivities based on knowledge of the distributions of the unknown parameters and the vibration measurements. In these approaches, the randomness arises only from the measurement noise and the updating parameters take unique values, to be found by iterative correction to the estimated means, whilst the variances are minimized [11]. These statistical approaches have been extended to update parameter distributions using measured response distributions from multiple measurements. These include Bayesian methods [12–15], perturbation based methods [16] and the maximum likelihood method [17]. Hua et al. [18] considered an improved perturbation method where statistical correlations between the updating parameters were taken into account. Faverjon et al. [19] considered updating of uncertain systems using a polynomial chaos expansion. McFarland et al. [20] used Gaussian process emulators to update linear systems with parametric updating. Their approach is valid for both Gaussian and non-Gaussian random variables. Ren et al. [21] proposed a response surface-based finite-element-model updating technique using structural static responses. Adhikari and Friswell [22] proposed a sensitivity based model updating approach for distributed updating parameters expressed using the Karhunen–Loève expansion. Goller et al. [23] considered stochastic model updating for complex aerospace structures. More recently Khodaparast et al. [24] considered model updating with an interval description of uncertain variables.

The source of the uncertainty in the model must be considered when the objectives of the model updating problem are considered. The uncertainty may be epistemic, for example arising from the measurement process, that may be reduced by using more information. Alternatively the uncertainty may be aleatory, for example arising from manufacturing or material variability for multiple structures, that cannot be improved by increased information on a single structure. Determining the source of the uncertainty in a practical problem is difficult, and many updating methods do not make a clear distinction. Indeed the way a method is used can determine the type of uncertainty considered. For example, the Bayesian methods [10,11] specify the covariance matrix of the measurements, and the original papers assumed this represented the measurement errors with a view to estimating deterministic parameters for a single structure. However if the covariance matrix represented the variability in measurements taken on multiple structures then the methods could be used to estimate the covariance of the estimated parameters.

The majority of the research reported in the literature consider the parametric approach for stochastic model updating. The aim of this paper is to propose a simple and computationally efficient approach (i.e., avoiding Monte Carlo Simulation) to update stiffness matrices from experimental measurements of the natural frequencies and the corresponding eigenvectors. A direct second moment approach based on random matrix theory is developed in this paper. The outline of the paper is as follows. In Section 2 the direct deterministic updating methods are briefly presented. Then in Section 3, the context of the

probabilistic approach is given and the main mathematical tools used in this work are defined. The solutions obtained when only the eigenfrequencies are supposed to be random are given in Section 4, while in Section 5, the solutions are provided when both eigenfrequencies and eigenvectors are random. Section 6 gives three examples to illustrate this approach. Finally, based on the results and analytical formulations, a set of conclusions are drawn in Section 7.

2. A brief overview of direct modal updating

The equation of a discretized linear dynamical system can be expressed by a set of coupled second-order ordinary differential equations as

$$M\ddot{q}(t) + Kq(t) = f(t) \quad (1)$$

Here M is the mass matrix, K is the stiffness matrix, $q(t)$ is the vector of generalized coordinates, $f(t)$ is the applied forcing and t is the time. We consider the system has n degrees of freedom. The eigenvalue problem governing the natural frequencies and mode shapes associated with Eq. (1) is given by

$$Kx_j = \omega_j^2 Mx_j, \quad j = 1, 2, \dots, n \quad (2)$$

In the above equation ω_j are the natural frequencies and x_j are the mode shapes. We form the matrices of natural frequencies and mode shapes as

$$\Omega^2 = \text{diag } \omega_1^2, \omega_2^2, \dots, \omega_n^2 \quad \text{and} \quad X = [x_1, x_2, \dots, x_n] \quad (3)$$

The eigenvectors are mass normalized so that $x_j' M x_k = \delta_{jk}$ where δ_{jk} is the Kronecker delta function, and the prime denotes the matrix transpose. The relationships between M , K , X and Ω^2 can be expressed using the orthogonality properties as

$$X' M X = I, \quad X' K X = \Omega^2 \quad \text{and} \quad K X = M X \Omega^2 \quad (4)$$

We consider that experimentally measured natural frequencies and mode shapes are 'exact'. Updating the numerical model is required because the eigenvalues and eigenvectors computed from the original mass and stiffness matrices do not match with the corresponding measured quantities. The fundamental aim of model updating is to reduce this 'mismatch'. This can be done in one of the two ways, namely (a) by updating a set of physically realistic system parameters (geometric dimension, joint stiffness, thickness, mass density), or (b) directly updating the mass or stiffness matrix, or both.

The damping has been neglected so far. If the system is proportionally damped, then the undamped mode shapes and the damped mode shapes are identical. However, general non-proportionally damped systems usually have complex modes. For such general damped systems, model updating [25–31] can be done, for example by extending the classical modal sensitivity analysis [32] to complex modes [33,34]. Another alternative is the direct model updating approach proposed by Friswell et al. [35] and subsequently used by several researchers [36–48]. In this paper we consider updating the stiffness matrix directly when there are uncertainties in the measured undamped natural frequencies and mode shapes.

The mass matrix is positive definite and is considered to be error free. The equation of motion (1) can be expressed in terms of the weighted stiffness matrix using a coordinate transformation of the form

$$q = M^{-1/2} p \quad (5)$$

Substituting this in Eq. (1), premultiplying by $M^{-1/2}$ and assuming free vibration we have

$$\ddot{p}(t) + M^{-1/2} K M^{-1/2} p(t) = 0 \quad (6)$$

Suppose that the 'correct' stiffness matrix is Y . This implies that the eigenvalues and the eigenvectors generated by this matrix are equal to the measured values. The matrix Y should be symmetric and should satisfy the modal equation, that is

$$YX = MX\Omega^2 \quad \text{and} \quad Y = Y' \quad (7)$$

Additionally, we want to obtain the Y which is closest to our initial model. This implies that the following norm needs to be minimized:

$$d = \|M^{-1/2}(Y-K)M^{-1/2}\|_2 \quad (8)$$

Minimizing the above norm subject to the constraints in Eq. (7), Baruch and Bar-Itzhack [49] showed that the optimal value of the corrected stiffness matrix is obtained as

$$Y = K - KXX'M - MXX'K + MXX'KXX'M + MX\Omega^2X'M \quad (9)$$

In this paper we use this expression for the case when there are uncertainties in the measured data.

For the stochastic case, the deterministic matrices appearing in Eq. (9) become random matrices. In particular, the following points can be noted:

- M is the $n \times n$ deterministic mass matrix, where n is the number of degrees of freedom.
- K is the initial estimate of matrix Y . It is an $n \times n$ deterministic matrix obtained from an initial finite element model. In our case where n_{samp} stiffness matrices are derived from a Monte-Carlo simulation, K may be either a matrix drawn

from the n_{samp} stiffness matrix samples or the mean over all the Monte-Carlo simulated stiffness matrices. K^s denotes for a sample stiffness matrix from the Monte-Carlo simulation.

- X is the matrix of eigenvectors. Depending on the assumptions made in the next section, X may be either a deterministic matrix or a random matrix. In this latter case, X^s , a sample of X , is the eigenvector matrix related to the (K^s, M) eigenproblem. The size of X is $n \times p$, where p is the number of ‘measured’ modes ($p \leq n$).
- Ω^2 is a random diagonal matrix. Ω^{2s} , a sample of Ω^2 , is the diagonal matrix of the eigenvalues related to the (K^s, M) eigenproblem.

Next we obtain statistical moments of the corrected stiffness matrix Y .

3. Matrix variate distributions

In order to obtain the statistical moments of the matrix Y , we consider two cases based on increasingly physically realistic assumptions:

- *Case 1:* Ω^2 is random but X is deterministic. X may be either a matrix drawn from the n_{samp} matrices X^s samples or the mean of the Monte-Carlo simulated eigenvector matrices. From the point of view of experimental identification, this is the easiest case.
- *Case 2:* Both Ω^2 and X are random. This is the most general case.

We consider the measured natural frequencies and mode shapes to be normally distributed. In particular, the following assumptions are made:

- Each $\{\Omega_{ii}^2\}_{i=1..p}$ has a normal distribution. Then Ω^2 has a matrix variate normal distribution (see the definition below): $\Omega^2 \sim \mathcal{N}_{p,p}(\Omega^2, C_{\Omega^2})$.
- The diagonal elements of Ω^2 are in general assumed to be correlated with a (nondiagonal) covariance matrix C_{Ω^2} .
- When X is random, it is also assumed to have a matrix variate normal distribution. The elements of X are in general assumed to be correlated with a general nondiagonal covariance matrix $cov(X)$.

In this paper we are concerned with second-order statistics only. The maximum entropy principle suggests that given only the first two moments, the most appropriate distribution is Gaussian [50,51]. Therefore, since the only information given is the mean and covariance matrices of Ω^2 and X , we assume that they are Gaussian random matrices [52]. The matrix containing the square of the eigenvalues Ω^2 should have strictly positive diagonal entries. Clearly if Ω^2 is a Gaussian random matrix, this cannot be guaranteed with probability 1. In the following analytical derivations only first and second-order moments of Ω^2 are considered, which are finite. Because the tails of the probability density function of Ω^2 have not been used, the possible limitations arising from the Gaussian assumption for Ω^2 have not influenced our analytical second-moment results. For further developments, and some important notation and definitions, following Refs. [52–54] are given below:

- $vec(X)$ is the vector obtained by stacking the columns of X . The following property will be used [52]:

$$vec(AXB) = (B^T \otimes A) vec(X) \tag{10}$$

where \otimes denotes the Kronecker product.

- A random matrix X ($n \times p$) has a matrix normal distribution with a mean matrix \bar{X} ($n \times p$) if there exist matrices Σ ($p \times p$) and Ψ ($n \times n$) such that

$$vec(X) \sim \mathcal{N}_{np}(vec(\bar{X}), C = \Sigma \otimes \Psi) \tag{11}$$

This is denoted by $X \sim \mathcal{N}_{n,p}(\bar{X}, \Sigma, \Psi)$. The pdf of X is

$$p(X|\bar{X}, \Sigma, \Psi) = \frac{1}{\sqrt{2\pi} \det(\Sigma)^n \det(\Psi)^p} \exp\left\{ \text{tr}\left(-\frac{1}{2} \Psi^{-1}(X-\bar{X})\Sigma^{-1}(X-\bar{X})^T\right) \right\} \tag{12}$$

Some useful properties are given in [Appendices A and B](#).

- The covariance matrix of the random matrices X and Y is

$$cov(X, Y) = E\{vec(X-\bar{X})(vec(Y-\bar{Y}))^T\} \tag{13}$$

The variance matrix will be denoted $cov(X)$.

3.1. The Covariance Kronecker product decomposition

In statistics, it is natural to consider the covariance matrix of a normally distributed random $n \times p$ matrix Z as the Kronecker product [53]

$$cov(Z) = C = \Sigma \otimes \Psi \tag{14}$$

where Σ is a $p \times p$ matrix and Ψ is an $n \times n$ matrix. For example, the book by Gupta and Nagar [52] provides many relationships about the expectation of a product of matrices which are based on this decomposition. The Kronecker product decomposition allows the manipulation of smaller matrices. Indeed, covariance matrix C has $p^2 \times n^2$ elements whereas Σ and Ψ have only $p^2 + n^2$ elements. In this study, $\text{cov}(Z)$ may be derived from a Monte-Carlo simulation but we need to find Σ and Ψ .

3.1.1. Optimized solution

Van Loan and Pitsianis [55] considered the factorization of the covariance matrix, and more precisely minimized $F_Z(\Sigma, \Psi)$ defined by

$$F_Z(\Sigma, \Psi) = \|\text{cov}(Z) - \Sigma \otimes \Psi\|_F \tag{15}$$

The solution is only a rank-1 approximation. Moreover, in general, for a given covariance matrix (that is any symmetric positive semi-definite matrix), decomposition (14) is not always possible. However, for a given matrix C there exist two sets of matrices $\{\Sigma\}_{k=1..r}$ and $\{\Psi_k\}_{k=1..r}$ such that [56]

$$C = \sum_{k=1}^r \Sigma_k \otimes \Psi_k \tag{16}$$

with $r = \text{rank}(C)$. Σ_k (resp. Ψ_k) is related to the k -th left (resp. right) singular vector of C .

If the rank of C is too high, such a decomposition may be very costly and an approximation may be obtained by only summing up to $\bar{r} \ll r$.

3.1.2. Maximum likelihood estimation

The previous method to determine the Kronecker product decomposition required the covariance matrix C of a normal-distributed random matrix Z . However Dutilleul [54] estimated \bar{Z} , Σ , and Ψ from r measurements. This method consists of finding the matrices that lead to the maximum of the log-likelihood \mathcal{L} , given the r measurements. As the matrix variate distribution is normal, the log-likelihood function is

$$\mathcal{L} = -\frac{npnr}{2} \log(2\pi) - \frac{nr}{2} \log(\det(\Sigma)) - \frac{pr}{2} \log(\det(\Psi)) - \frac{1}{2} \sum_{k=1}^r \text{tr}(\Psi^{-1}(Z_k - \bar{Z})\Sigma^{-1}(Z_k - \bar{Z})') \tag{17}$$

Then, the maximum likelihood estimators of \bar{Z} , Σ , and Ψ are [54]

$$\begin{cases} \hat{\bar{Z}} = \frac{1}{2} \sum_{k=1}^r Z_k \\ \hat{\Psi} = \frac{1}{pr} \sum_{k=1}^r (Z_k - \hat{\bar{Z}})\hat{\Sigma}^{-1}(Z_k - \hat{\bar{Z}}) \\ \hat{\Sigma} = \frac{1}{nr} \sum_{k=1}^r (Z_k - \hat{\bar{Z}})\hat{\Psi}^{-1}(Z_k - \hat{\bar{Z}}). \end{cases} \tag{18}$$

An iterative procedure is necessary to evaluate $\hat{\Sigma}$ and $\hat{\Psi}$ [54]. This method will be used in the following to obtain the Kronecker decomposition product.

4. Updating with eigenvalue uncertainty

This is the case 1 introduced in the previous section. The statistical moments of the corrected stiffness matrix Y are obtained below.

4.1. Mean of Y

Considering that Ω^2 is the only random matrix, the mean of Y is easily determined as

$$\bar{Y} = K - KXX'M - MXX'K + MXX'KXX'M + MX\overline{\Omega^2}X'M \tag{19}$$

4.2. Covariance of Y

From (19) we have

$$Y - \bar{Y} = MX(\Omega^2 - \overline{\Omega^2})X'M \tag{20}$$

Following the definition given in the previous section, the covariance matrix of Y is

$$\text{cov}(Y) = E\{\text{vec}(Y - \bar{Y})\text{vec}(Y - \bar{Y})'\} \tag{21}$$

Considering (10) and (20), we have

$$\text{vec}(Y - \bar{Y}) = \text{vec}(AZA') \tag{22}$$

$$\text{vec}(Y - \bar{Y}) = (A \otimes A)\text{vec}(Z) \tag{23}$$

$$(\text{vec}(Y - \bar{Y}))' = \text{vec}(Z)'(A \otimes A)' \tag{24}$$

$$(\text{vec}(Y - \bar{Y}))' = \text{vec}(Z)'(A' \otimes A') \tag{25}$$

where

- $A = MX$
- $Z = \Omega^2 - \bar{\Omega}^2$

Hence the covariance matrix of Y is

$$\text{cov}(Y) = (A \otimes A)E\{\text{vec}(Z) \text{vec}(Z)'\}(A' \otimes A') \tag{26}$$

$$\text{cov}(Y) = ((MX) \otimes (MX))C_{\Omega^2} ((MX)' \otimes (MX)') \tag{27}$$

5. Updating with eigenvalue and eigenvector uncertainty

In this section Ω^2 and X are random matrices with matrix variate normal distributions:

- $\Omega^2 \sim \mathcal{N}_{p,p}(\bar{\Omega}^2, C_{\Omega^2})$;
- $X \sim \mathcal{N}_{n,p}(\bar{X}, C)$: \bar{X} is an $n \times p$ -matrix and C is an $np \times np$ -matrix.

In general, the matrices Ω^2 and X are themselves fully correlated random matrices. However, for analytical simplicity in the following Ω^2 and X are assumed to be uncorrelated. Under certain circumstances this assumption is valid even when mathematically the eigenvalues and eigenvectors are obtained from the same random eigenvalue problem. It is known that if a random matrix is rotationally invariant such as the Gaussian Orthogonal Ensemble (GOE) [51] or the Wishart random matrix with identity matrix as the parameter [57], the eigenvalues and eigenvectors become statistically uncorrelated. This condition is not met in general and therefore the assumption of uncorrelated Ω^2 and X will introduce some error when real-life measurements are used. It will be shown that only one term is influenced by the statistical correlation of Ω^2 and X . The calculations are carried out with the real covariance matrices first, and then the Kronecker product decomposition ($C = \Sigma \otimes \Psi$) will be used. Note that the relation between the elements of C , Ψ , and Σ is

$$E\{(X - \bar{X})_{ij}(X - \bar{X})_{kl}\} = C_{(j-1)n+i, (l-1)n+k} = \Psi_{ik} \Sigma_{jl} \tag{28}$$

5.1. The mean matrix: exact results

5.1.1. Real covariance matrix

The mean of the optimized stiffness matrix is

$$\bar{Y} = K - K E\{XX'\}M - M E\{XX'\}K + M E\{XX'KXX'\}M + M E\{X\Omega^2X'\}M \tag{29}$$

Note that among the various terms, only the term $E\{X\Omega^2X'\}$ is influenced by statistical correlation of Ω^2 and X . The terms $E\{XX'\}$, $E\{XX'KXX'\}$, $E\{X\Omega^2X'\}$ are evaluated in Appendix B. Then for a general covariance matrix, each element of \bar{Y} can be evaluated with relations (B.1), (B.4) and (B.7).

5.1.2. Kronecker product decomposition

When the covariance matrix is given as a Kronecker product, \bar{Y} can be calculated with relations (B.2), (B.6) and (B.9) as

$$\begin{aligned} \bar{Y} = & K - K(\bar{X}\bar{X}' + \text{tr}(\Sigma)\Psi)M - M(\bar{X}\bar{X}' + \text{tr}(\Sigma)\Psi)K + M(\bar{X}\bar{X}'K\bar{X}\bar{X}' + \text{tr}(\Sigma)\Psi K\bar{X}\bar{X}' + \Psi K\bar{X}\Sigma\bar{X}') \\ & + \text{tr}(\Psi K)\bar{X}\bar{X}\bar{X}' + \text{tr}(\Sigma\bar{X}'K\bar{X})\Psi + \bar{X}\Sigma\bar{X}'K\Psi + \text{tr}(\Sigma\Sigma)\Psi K\Psi + \text{tr}(\Sigma)^2\Psi K\Psi \\ & + \text{tr}(\Sigma)\bar{X}\bar{X}'K\Psi + \text{tr}(\Sigma\Sigma)\text{tr}(\Psi K)\Psi M + M(\bar{X}\bar{\Omega}^2\bar{X}' + \text{tr}(\Sigma\bar{\Omega}^2)\Psi)M \end{aligned} \tag{30}$$

5.2. The covariance matrix: exact results

The covariance matrix of Y is defined as

$$\text{cov}(Y) = E\{\text{vec}(Y - \bar{Y})\text{vec}(Y - \bar{Y})'\} \tag{31}$$

and we have

$$Y - \bar{Y} = K - \bar{Y} - KXX'M - MXX'K + MXX'KXX'M + MX\Omega^2X'M \tag{32}$$

with \bar{Y} being defined in (29) or in (30). Contrary to case 1 (see relation (21)), the covariance matrix expression is not simple and involves the evaluation of moments up to the 8th order. This expression is too complicated and such a calculation is much too time consuming: therefore, the covariance matrix of Y cannot be evaluated directly.

5.3. The mean and covariance matrices: perturbation results

A perturbation method is proposed to determine the covariance matrix when X is random. This method is presented first and then applied to the estimation of \bar{Y} : it will be possible to assess the perturbation method by comparing the results with the expressions obtained in Section 4.

The main assumption is that the random eigenfrequencies and eigenvectors are close to their mean. That is, these data may be written as follows:

$$X = \bar{X} + \Delta X \tag{33}$$

$$\Omega^2 = \bar{\Omega}^2 + \Delta\Omega^2 \tag{34}$$

where ΔX and $\Delta\Omega^2$ are supposed to be small, and are random matrices having the following matrix variate normal distributions:

$$\Delta X = \mathcal{N}_{n,p}([0], C) \quad \text{or} \quad \Delta X = \mathcal{N}_{n,p}([0], \Sigma, \Psi) \tag{35}$$

$$\Delta\Omega^2 = \mathcal{N}_{p,p}([0], C_{\Omega^2}) \quad \text{or} \quad \Delta\Omega^2 = \mathcal{N}_{p,p}([0], \Sigma_{\Omega^2}, \Psi_{\Omega^2}) \tag{36}$$

To estimate whether or not a stochastic variable ΔX is small compared to another, a stochastic norm $\|\cdot\|_S$ must be defined. This has been done by Stewart [58], and is defined as

$$\|\Delta X\|_S = E\{\|\Delta X\|_F\} \tag{37}$$

5.3.1. Estimation of \bar{Y}

By substituting (33) and (34) in (9), Y becomes

$$Y = K - K(\bar{X} + \Delta X)(\bar{X} + \Delta X)'M - M(\bar{X} + \Delta X)(\bar{X} + \Delta X)'K + M(\bar{X} + \Delta X)(\bar{X} + \Delta X)'K(\bar{X} + \Delta X)(\bar{X} + \Delta X)'M + M(\bar{X} + \Delta X)(\bar{\Omega}^2 + \Delta\Omega^2)(\bar{X} + \Delta X)'M \tag{38}$$

Then, as the mean of ΔX and $\Delta\Omega^2$ is null matrices, and keeping the terms only up to second order, the mean of Y is denoted by \tilde{Y} and estimated as

$$\tilde{Y} = K - K\bar{X}\bar{X}'M - K \text{Acov}(X)M - M\bar{X}\bar{X}'K - M \text{Acov}(X)K + M\bar{X}\bar{X}'K\bar{X}\bar{X}'M + M\bar{X}\bar{X}'K \text{Acov}(X)M + M\alpha\bar{X}'M + M\beta M + M\bar{X}\gamma\bar{X}'M + M\bar{X}\alpha'M + M \text{Acov}(X)K\bar{X}\bar{X}'M + M\bar{X}\bar{\Omega}^2\bar{X}'M + M\delta M \tag{39}$$

where

- the matrices $\alpha, \beta, \gamma, \delta$ are defined as

$$\alpha = E\{\Delta X \bar{X}'K \Delta X\} \tag{40}$$

$$\beta = E\{\Delta X \bar{X}'K\bar{X}' \Delta X'\} \tag{41}$$

$$\gamma = E\{\Delta X' K \Delta X\} \tag{42}$$

$$\delta = E\{\Delta X \bar{\Omega}^2 \Delta X'\} \tag{43}$$

The elements of these matrices may be evaluated by using (B.10), (B.12) and (B.14) or, by using (B.11), (B.13) and (B.15).

- The matrix $\text{Acov}(X)$ is defined as

$$\text{Acov}(X) = E\{\Delta X \Delta X'\} = E\{(X - \bar{X})(X - \bar{X})'\} \tag{44}$$

5.3.2. Expression of \tilde{Y} with the real covariance matrix

In this case, the jk element of matrices $\alpha, \beta, \gamma, \delta$, and $\text{Acov}(X)$ must be calculated element by element from (B.10), (B.12) and (B.14) as

$$\alpha_{jk} = \sum_{q=1}^p \sum_{r=1}^n (\bar{X}'K)_{qr} C_{(q-1)n+j, (k-1)n+r} \tag{45}$$

$$\beta_{jk} = \sum_{q=1}^p \sum_{r=1}^p (\bar{X}' K \bar{X})_{qr} C_{(q-1)n+j, (r-1)n+k} \tag{46}$$

$$\gamma_{jk} = \sum_{q=1}^n \sum_{r=1}^n K_{qr} C_{(j-1)n+q, (k-1)n+r} \tag{47}$$

$$\delta_{jk} = \sum_{q=1}^p \overline{\Omega^2}_{qq} C_{(q-1)n+j, (q-1)n+k} \tag{48}$$

$$\text{Acov}(X)_{jk} = \sum_{q=1}^p E\{\Delta X_{jq} \Delta X_{kq}\} = \sum_{q=1}^p C_{(q-1)n+j, (q-1)n+k} \tag{49}$$

5.3.3. Expression of \tilde{Y} with the Kronecker product decomposition

When the covariance matrix may be decomposed as a Kronecker product, relations (B.11), (B.13) and (B.15) may be used to calculate \tilde{Y} as

$$\begin{aligned} \tilde{Y} = & K - K \bar{X} \bar{X}' M - K \text{Acov}(X) M - \bar{M} \bar{X} \bar{X}' K - M \text{Acov}(X) K + \bar{M} \bar{X} \bar{X}' K \bar{X} \bar{X}' M + \bar{M} \bar{X} \bar{X}' K \text{Acov}(X) M + M \text{Acov}(X) K \bar{X} \bar{X}' M \\ & + M \Psi K \bar{X} \bar{X}' M + \text{tr}(\bar{X}' K \bar{X} \Sigma) M \Psi M + \text{tr}(K \Psi) \bar{M} \bar{X} \bar{X}' M + \bar{M} \bar{X} \bar{X}' K \Psi M + \bar{M} \bar{X} \bar{X}' \bar{X}' M + \text{tr}(\Sigma \bar{\Omega}^2) M \Psi M \end{aligned} \tag{50}$$

5.3.4. Estimation of cov(Y)

In Appendix C, it is proved that the covariance matrix of Y may be written as

$$\text{cov}(Y) = \sum_{i=1}^5 \sum_{j=1}^5 D_i \text{cov}(X) D_j' + D \text{cov}(\Omega^2) D' \tag{51}$$

where

$$\forall i = 1 \dots 5, D_i = (B_i' \otimes A_i) + (A_i \otimes B_i) \mathcal{P}_{np} \tag{52}$$

$$D = (M \bar{X}) \otimes (M \bar{X}) \tag{53}$$

where A_i , B_i , and \mathcal{P}_{np} are defined in Appendix C.

6. Numerical examples

A clamped–free square beam is studied in this section (length L , width b , thickness t). The characteristics of the beam are listed in Table 1. The beam is modeled as an Euler–Bernoulli beam and is discretized into n_{ele} elements. Each node has 2 degrees of freedom (dof) and the interpolation functions are cubic: in this study, $n_{ele} = 10$ and hence then the number of dofs, n , is equal to 20.

The rigidity $EI = E \times I$ (I : second moment of inertia) of each beam element is assumed to be a random variable. Thus

$$EI = EI_m + \Delta EI \tag{54}$$

where EI_m is deterministic and is evaluated from the section dimensions and Young’s modulus given in Table 1.

A Monte–Carlo simulation (MCS) was carried out: $r_{tot} = 4000$ random stiffness matrices K^s were calculated from r_{tot} samples of the rigidity EI^s . The mass matrix M is assumed to be deterministic. K is the ‘deterministic’ stiffness matrix evaluated for $EI = EI_m$: the deterministic eigenvectors X in case 1 were obtained by solving the deterministic eigenproblem associated with the matrices K and M .

Regarding case 2, by solving the eigenproblem related to the matrices K^s and M , r_{tot} samples of the first p modes $\{X, \Omega^2\}^s$ were obtained. The mean matrix \bar{X} (respectively $\bar{\Omega}^2$) and the covariance matrix C (respectively C_{Ω^2}) of X^s (respectively Ω^{2s}) were evaluated. A Kronecker product decomposition was then calculated following the procedure given in Section 3.1.2.

Relations (19), (27), (29), (30), (39), (50) and (51) were applied to calculate the first two moments of Y . In order to assess the accuracy of the results, the mean and the covariance matrix of Y were directly determined from the Monte–Carlo simulation: from each sample K^s a sample Y^s of Y was calculated with relation (9). Then the mean matrix and the covariance matrix of Y were determined from the r_{tot} samples Y^s . In the following, a 3D plot of the covariance matrices will be given. The standard deviation of the elements of the stiffness matrix Y are given in 2D plots: this corresponds to the

Table 1
Characteristics of the beam.

L (m)	b (mm)	t (mm)	E_m (GPa)	ρ (kg m ⁻³)	EI_m (N m ²)
1	20	20	210	7800	2.8×10^3

square root of the diagonal elements of the covariance matrix. Similarly, to show the results of the stiffness matrices, a vec operation had been performed on the mean matrix which is then plotted as a 2D curve. In order to assess the main assumption of the perturbation method is valid, the stochastic norm of the uncertainties on the eigenfrequencies $\|\Delta\Omega^2\|_S$ and the eigenvectors $\|\Delta X\|_S$ were also calculated.

In the following several cases are studied and compared to the results obtained from the direct MCS:

- the rigidity is almost uniform on the whole beam for each simulation,
- the rigidity of each element varies randomly along the beam for each simulation,
- the rigidity of only one element is random for each simulation.

6.1. Almost uniform rigidity

In this case, the rigidity of element e is

$$EI^e = EI_m + \Delta EI(1 + \sigma^e) \quad (55)$$

where:

- a realization of the random variable ΔEI is drawn for each simulation from a normal distribution: $\Delta EI \sim \mathcal{N}(0, \sigma_{EI})$. The influence of σ_{EI} will be addressed.
- σ^e is also a random variable; a realization is drawn for each element and each simulation according to a normal distribution $\Delta EI \sim \mathcal{N}(0, 5\%)$: the standard deviation of 5% is sufficiently small to consider that the rigidity is almost uniform along the beam.

It is important to mention that if the rigidity distribution was uniform for each simulation (i.e. $\sigma^e = 0$), the eigenvectors would be deterministic: this justifies that case 1 has been considered in this study.

The results are given in Figs. 1–4. They were obtained for $p = n = 20$ with only $r = 100$ samples drawn from the r_{tot} MCS samples, and for $\sigma_{EI} = 10\%$. In that case the relative uncertainties on the eigenvectors and the eigenvalues are

$$\frac{\|\Delta X\|_S}{\|\bar{X}\|_F} \simeq 0.8\% \quad (56)$$

$$\frac{\|\Delta\Omega^2\|_S}{\|\Omega^2\|_F} \simeq 7.3\% \quad (57)$$

In the following the uncertainty on the eigenvectors (resp. eigenfrequencies) is quantified with relation (56) (resp. relation (57)).

In Fig. 1 all of the curves are superposed, showing a good agreement between the solutions proposed in this paper and the one obtained from the MCS solution:

$$\frac{\|\bar{X}_{case\ 1; case\ 2} - \bar{X}_{MCS}\|_F}{\|\bar{X}_{MCS}\|_F} = 2.3\% \quad (58)$$

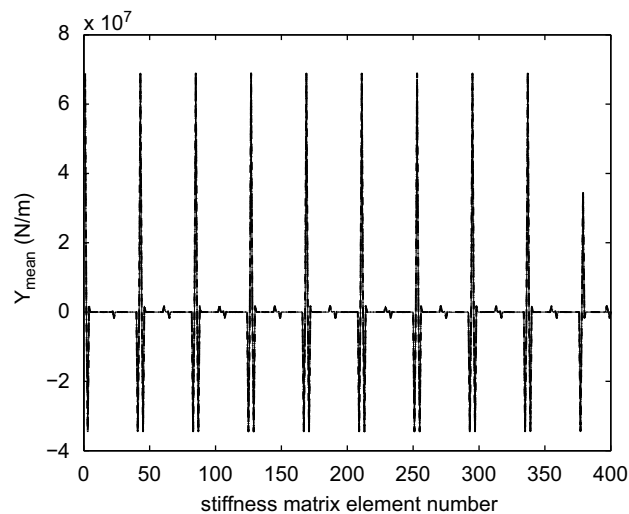


Fig. 1. $\text{vec}(\bar{Y})$ from MCS, case 1 solution, exact case 2 solution, perturbed case 2 solution, Kronecker product decomposition case 2 solution.

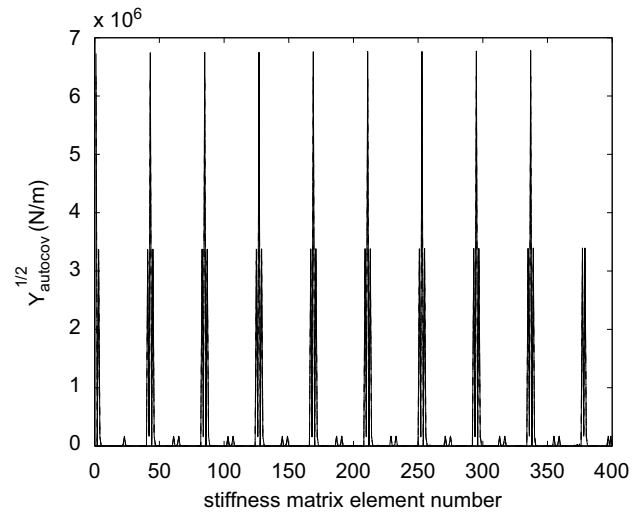


Fig. 2. Standard deviation of the elements of $\text{vec}(Y)$ from Monte-Carlo Simulation, case 1 solution, perturbed case 2 solution.

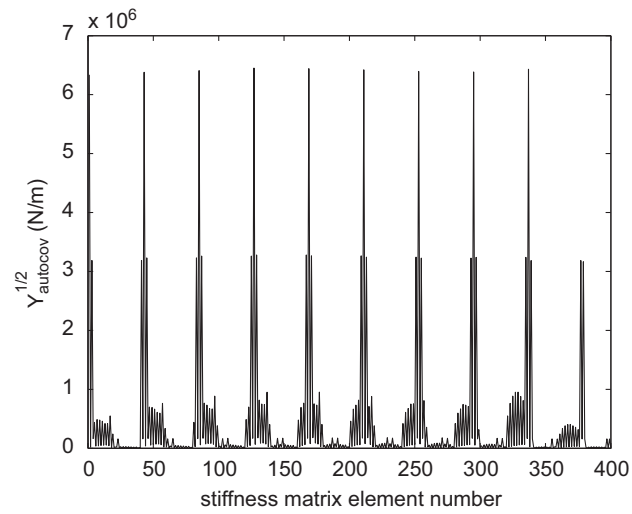


Fig. 3. Standard deviation of the elements of $\text{vec}(Y)$ from the Kronecker product decomposition case 2 solution.

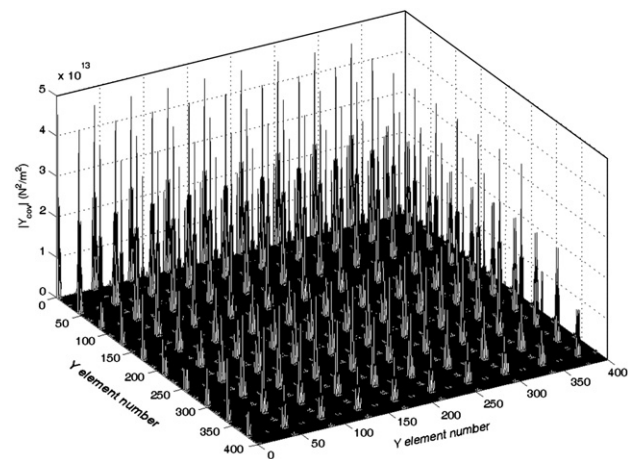


Fig. 4. Covariance matrix of Y from MCS.

From Fig. 2, the same conclusions may be drawn for the covariance matrices, except that the peaks obtained from the MCS are slightly higher (6%):

$$\frac{\|\text{COV}_{\text{case 1, case 2}} - \text{COV}_{\text{MCS}}\|_F}{\|\text{COV}_{\text{MCS}}\|_F} = 6.3\% \quad (59)$$

In the following the error on the mean matrix (resp. covariance matrix) is quantified with relation (58) (resp. relation (59)).

Fig. 3 shows some parasitic oscillations that increase the discrepancy with respect to the MCS solution when the Kronecker product decomposition solution is used:

$$\frac{\|\text{COV}_{\text{case 2, Kronecker}} - \text{COV}_{\text{MCS}}\|_F}{\|\text{COV}_{\text{MCS}}\|_F} = 22.3\% \quad (60)$$

A 3D representation of the covariance matrix calculated from the MCS is given in Fig. 4: the covariance matrices obtained from the solutions proposed in this paper are very similar. It shows that the uncertainties are spread over all of the beam showing correlation between the vertical displacement of distant nodes.

Relations (56) and (57) show that the uncertainty on the eigenvectors is much smaller than on the eigenvalues. Thus the eigenvectors may be considered as deterministic and case 1 is then applicable. This explains why the results obtained for case 2 are the same as for case 1: the contribution of the eigenvector uncertainties is negligible compared to the uncertainties of the eigenfrequencies. Relation (56) also shows that the main assumption of the perturbation method is fulfilled: the results given by this method are then very good.

The influence of the number of samples was studied by comparing the results obtained with 10, 30, 100, 300, 500, 1000 and 4000 samples. Naturally, the results show that the errors decrease with the number of samples, although the errors remain quite small even for $r=10$. Similarly the influence of the uncertainty on the rigidity, σ_{EI} , was studied. Fig. 5 shows the influence of this parameter on the input quantities (eigenvectors and eigenfrequencies): the uncertainty on the eigenfrequencies increases more quickly than that on the eigenvectors. Obviously the errors on the mean matrix and the covariance matrix decrease with σ_{EI} , but they are still good up to 20%. All these of results are shown in Figs. 6 and 7(b).

6.2. Non-uniform rigidity distribution

In this case, the rigidity of an element e is given by

$$EI^e = EI_m + \Delta EI^e \quad (61)$$

where ΔEI^e is drawn for each element and each simulation according to a normal distribution: $\Delta EI^e \sim \mathcal{N}(0, \sigma_{EI})$.

This rigidity distribution infers a stronger uncertainty on the eigenvectors than on the eigenfrequencies as shown in Fig. 8.

In this case the uncertainties do not spread over the beam and Fig. 9 shows that a vertical displacement is strongly correlated only with the vertical displacements of the adjacent nodes. This is a significant difference with the previous example. The covariance matrix given by the case 1 solution is very different from the one obtained from the MCS and is very similar to the one obtained in the previous example, as shown in Fig. 10(a): when the eigenvectors are supposed to be deterministic, the uncertainties spread over the beam. In contrast, the perturbation method case 2 solution is a good estimate of the covariance matrix obtained from the MCS (see Fig. 10(b)). Fig. 11(a) and (b) confirms these conclusions: even for a small value of σ_{EI} (1%) and a large r , the case 1 solution cannot provide a covariance matrix close to the MCS

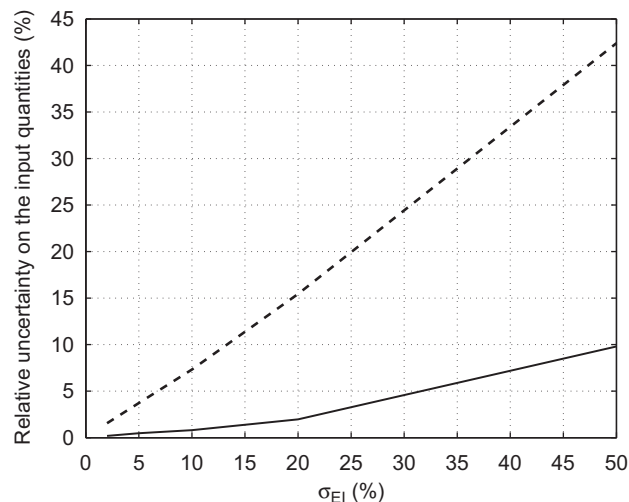


Fig. 5. Uncertainty on the eigenvectors (plain line) and the eigenfrequencies (dashed line).

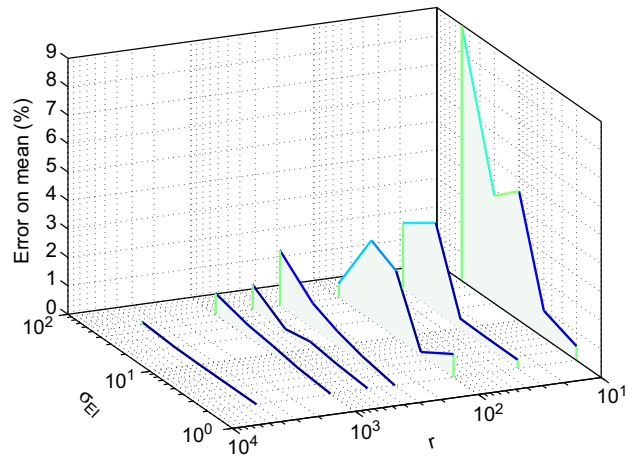


Fig. 6. Error on the mean matrix.

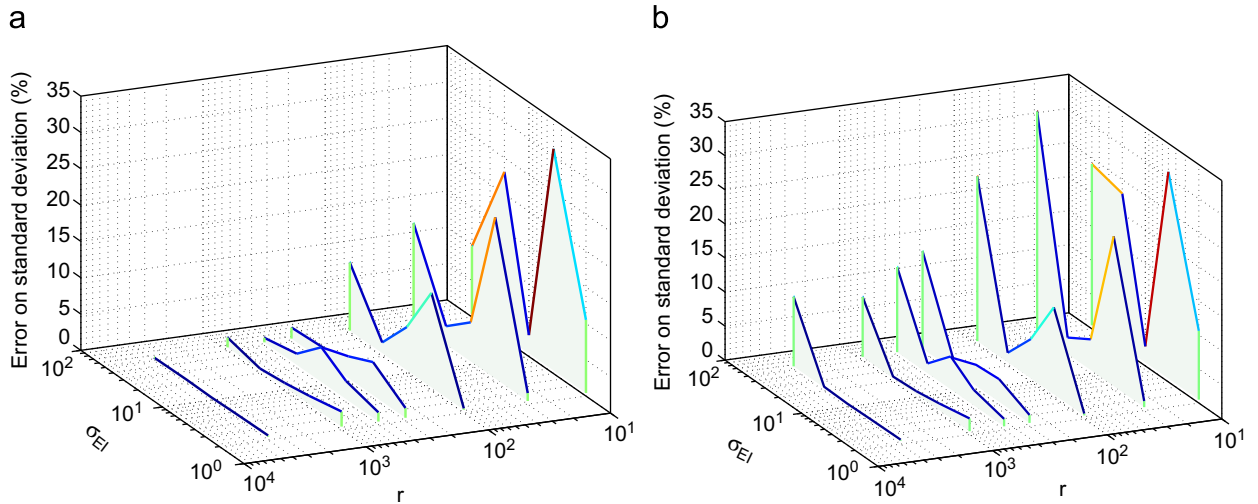


Fig. 7. Error on the covariance matrix with respect to direct Monte Carlo simulation—almost uniform rigidity distribution. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

solution (error always greater than 55%), whereas the perturbation method solution can give a good estimation of the covariance matrix, provided enough samples are considered, even for a quite high value of σ_{EI} (20%). This proves the strong influence on the eigenvectors on the covariance matrix estimation.

Regarding the mean matrix, Fig. 12(a) shows that the case 1 solution agrees well with the MCS solution up to $\sigma_{EI} = 20\%$ whereas Fig. 12(b) shows that the perturbation method case 2 solution agrees well with the MCS solution only up to $\sigma_{EI} = 10\%$: beyond this limit the uncertainty on the eigenvectors is not small enough. It is important to note that the results are very good even when few samples are used.

The good results obtained from the case 1 solution shows that the influence of the eigenvector uncertainty on the mean matrix is weak.

6.3. Random local rigidity

In this case, the rigidity of only one element is random. In this example the rigidity of the third element is considered, so that

$$EI^3 = EI_m + \Delta EI \tag{62}$$

where the ΔEI realizations are drawn for each element and each simulation according to a normal distribution: $\Delta EI \sim \mathcal{N}(0, \sigma_{EI})$. The rigidity of the other elements is assumed to be deterministic and equal to EI_m . Fig. 13 shows that the uncertainties on the eigenvectors are much stronger than on the eigenfrequencies, which remains quite small at approximately 4% even for $\sigma_{EI} = 50\%$.

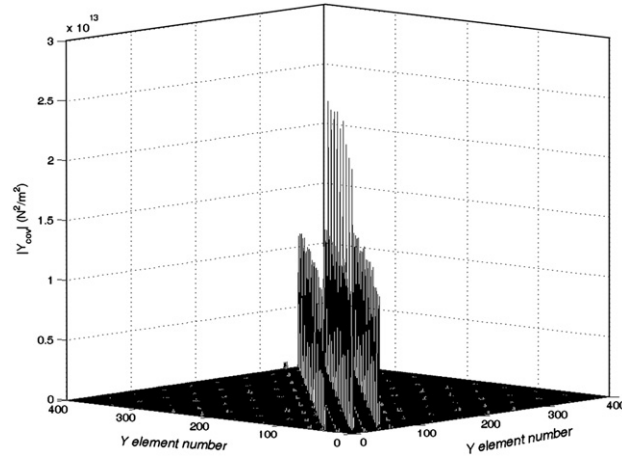


Fig. 8. Uncertainty on the eigenvectors (plain line) and the eigenfrequencies (dashed line).

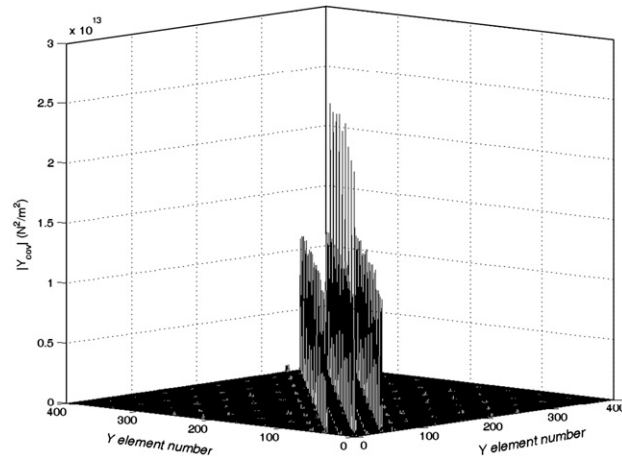


Fig. 9. Covariance matrix of Y from MCS, $\sigma_{EI} = 10\%$, $r = 100$.

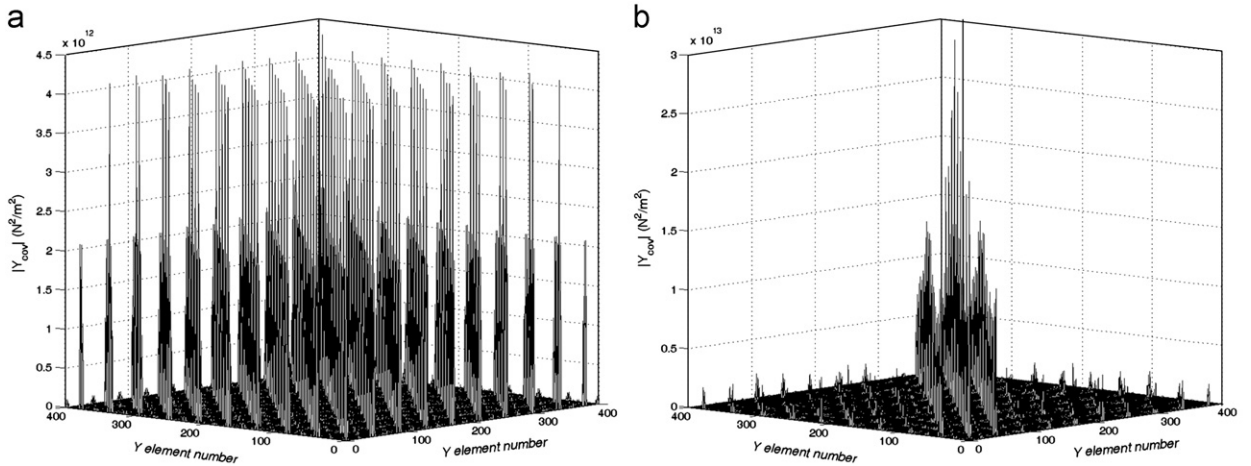


Fig. 10. Covariance matrix of Y , $\sigma_{EI} = 10\%$, $r = 100$, non-uniform rigidity distribution. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

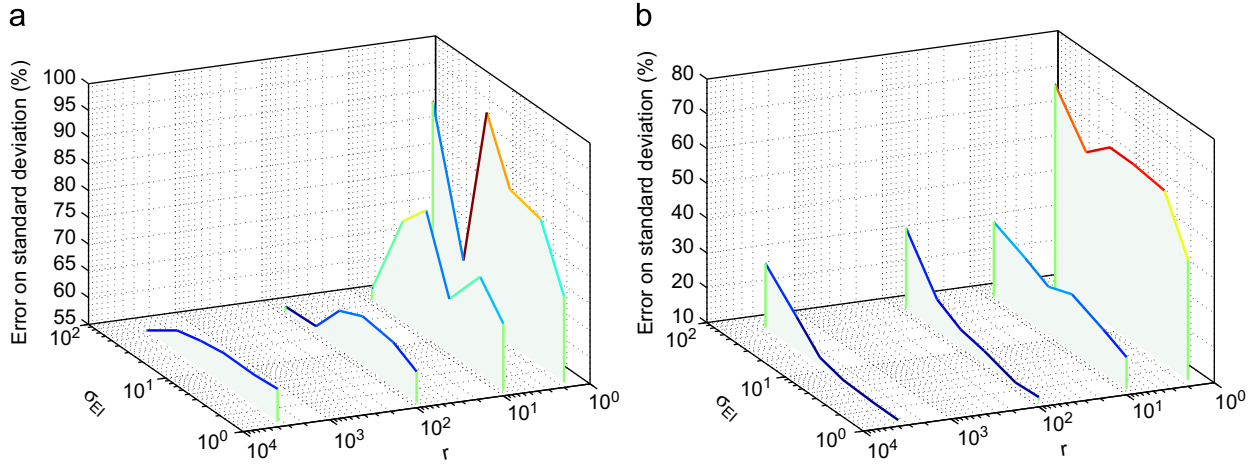


Fig. 11. Error on the covariance matrix with respect to direct Monte Carlo simulation—non-uniform rigidity distribution. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

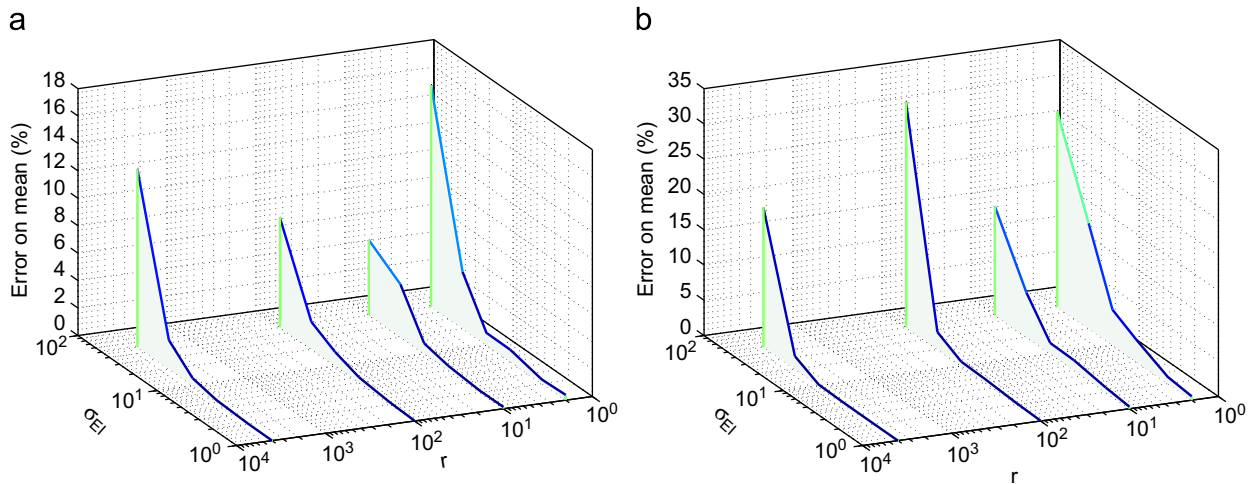


Fig. 12. Error on the mean matrix with respect to direct Monte Carlo simulation—non-uniform rigidity distribution. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

The uncertainty location appears very clearly in the covariance matrix, as shown in Fig. 14, where the covariance matrix obtained from the MCS is plotted: the only significant values are associated to the vertical displacements of the element with three nodes. Similar to the previous example, the perturbation method case 2 solution agrees well with the MCS simulation regarding the covariance matrix, whereas the case 1 solution does not give good results and, once again, spreads the uncertainties along the beam (see Fig. 15(a)). This is confirmed by Fig. 16(a) obtained from several values of σ_{EI} and several numbers of samples r for case 1: the error in all cases is greater than 90%. Fig. 16(b) shows that for the perturbation method case 2 solution, for $r \geq 10$ and $\sigma_{EI} \leq 20\%$, the error on the covariance matrix is almost constant and equal to 49%. This error may be considered as high but, for all these cases, the covariance matrix representation is similar to the one shown in Fig. 15(b), which is quite similar to the MCS result; however, the main peaks are underestimated by 25% and many small parasitic peaks appear where no such peaks are present in the MCS simulation solution.

Fig. 17(a) and (b) shows that the mean matrix is very well estimated by the solutions given either by the case 1 approach or the case 2 approach, even when very few samples are used and for large uncertainties on the rigidity. This is in good agreement with the previous conclusion: the mean matrix is mainly controlled by the eigenfrequencies which have, in this example, a small uncertainty.

7. Conclusions

A probabilistic model updating method is developed to deal with uncertainties of measured eigenfrequencies and eigenvectors. The mean optimized stiffness matrix and the covariance matrix are calculated. A first case (case 1) assuming

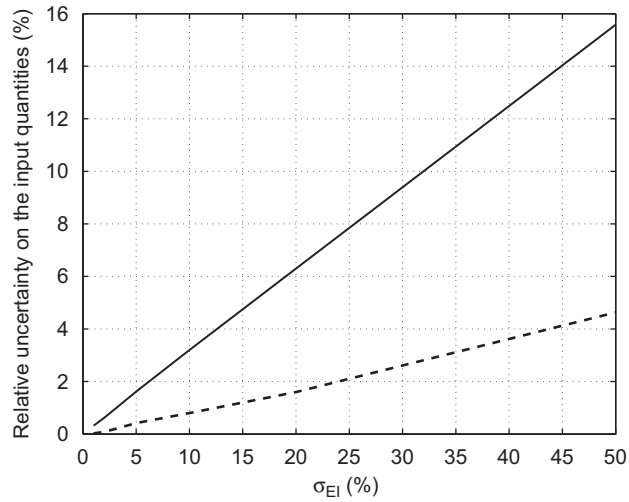


Fig. 13. Uncertainty on the eigenvectors (plain line) and the eigenfrequencies (dashed line)—random local rigidity.

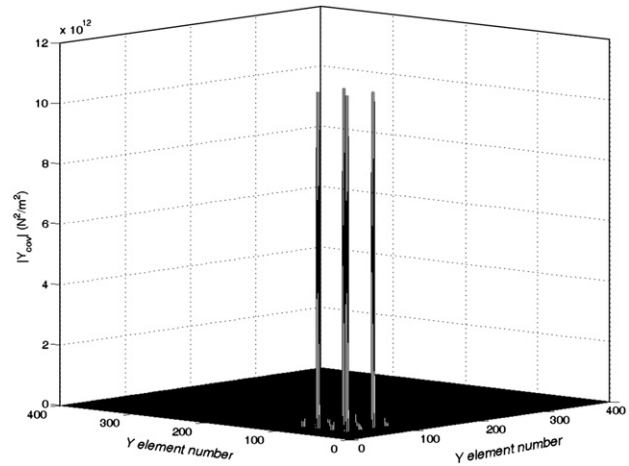


Fig. 14. Covariance matrix of Y from MCS, $\sigma_{EI} = 10\%$, $r = 100$, random local rigidity.

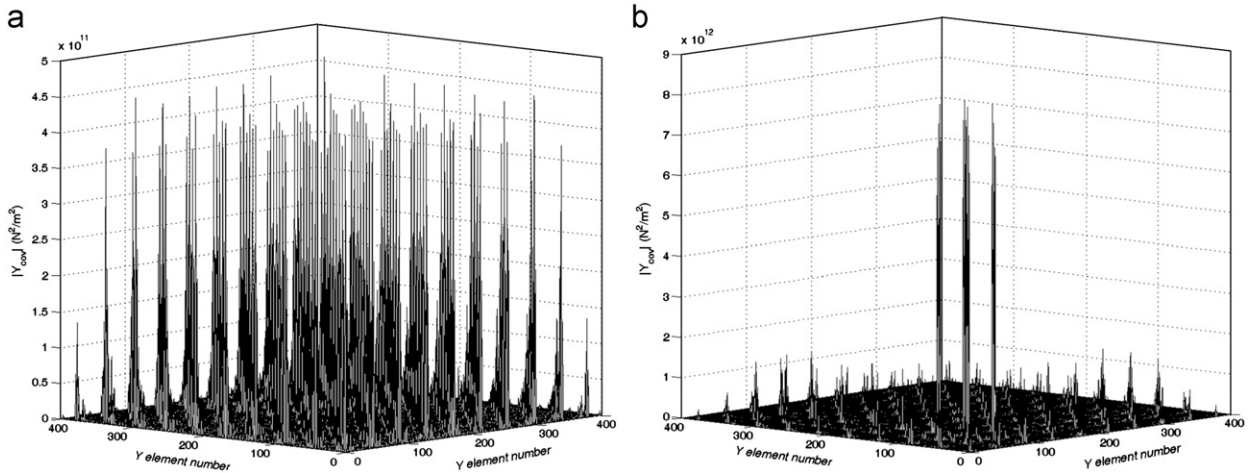


Fig. 15. Covariance matrix of Y, $\sigma_{EI} = 10\%$, $r = 100$, random local rigidity. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

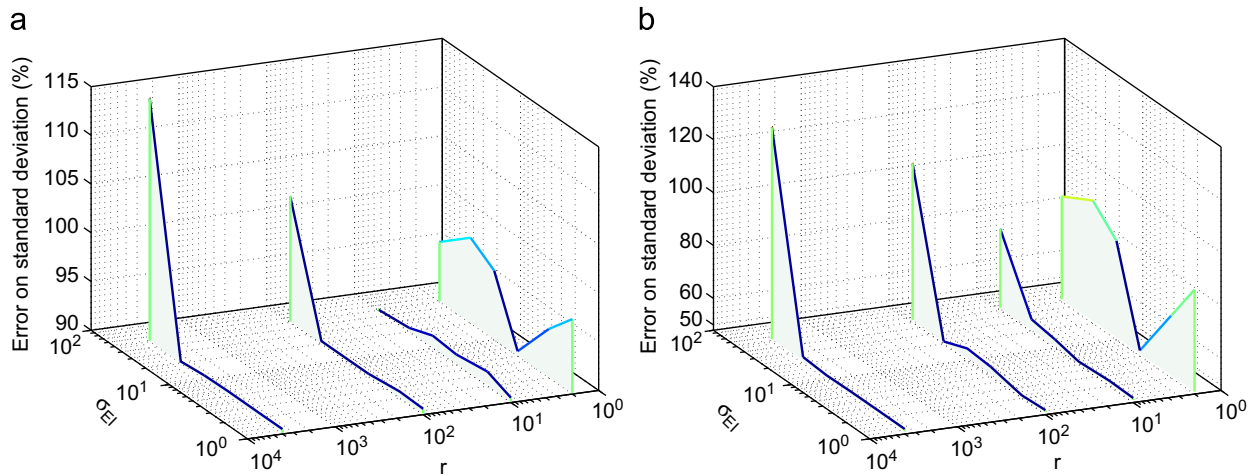


Fig. 16. Error on the covariance matrix with respect to direct Monte Carlo simulation—random local rigidity. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

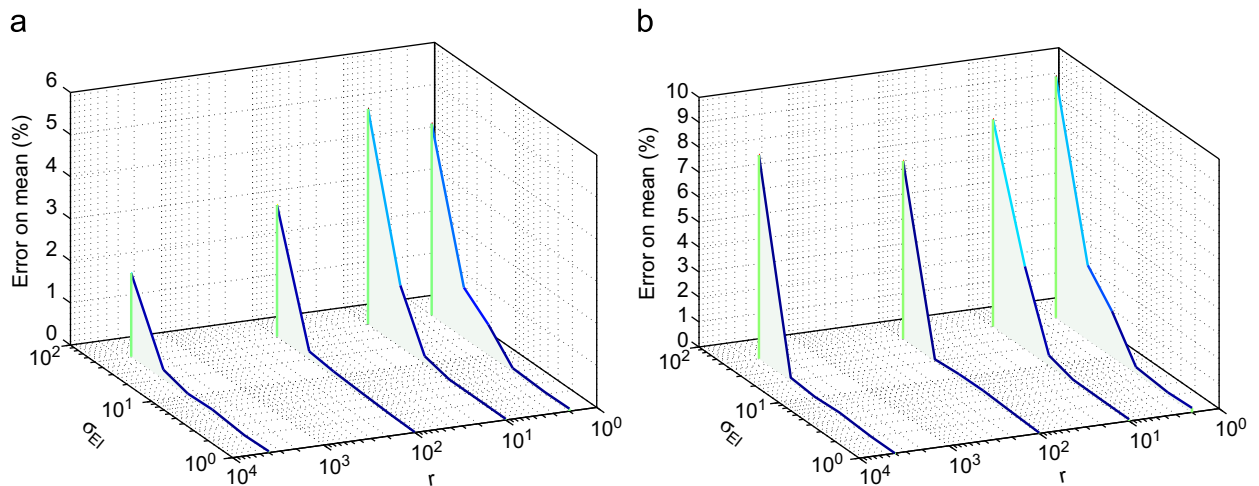


Fig. 17. Error on the mean matrix with respect to direct Monte Carlo simulation—random local rigidity. (a) Solution for case 1. (b) Perturbation method based solution for case 2.

that the uncertainty on the eigenvectors is negligible has been considered whereas a second case (case 2) has taken into account the uncertainties on both the eigenfrequencies and the eigenvectors. In case 2, a perturbation method is used to derive an estimation of the covariance matrix.

The results are tested on three examples involving random uncertainty on the rigidity along a beam: depending on the example, the uncertainty on the eigenvectors and the eigenfrequencies may be weak or strong. The solutions provided by the probabilistic analysis are compared to Monte-Carlo Simulation solutions.

The mean optimized stiffness matrix seems to be sensitive to the eigenfrequency uncertainties only. Thus the solution provided by the case 1 analysis gives results in a good agreement with the MCS results: the accuracy depends mainly on the uncertainty on the eigenfrequencies whereas it is only weakly sensitive to the number of samples used. The results for the mean matrix are not as good when the perturbation method case 2 solution is used, but they still remain acceptable.

The covariance matrix appears to be very sensitive to the eigenvector uncertainty. The covariance matrix obtained from the case 1 solution is in poor agreement with the MCS solution when the eigenvector uncertainty is not weak. When the rigidity uncertainty distribution is almost uniform along the beam, the uncertainty on the eigenvectors is weak and then the case 1 solution may give a rather good estimation of the covariance matrix. The perturbation method case 2 solution takes into account the eigenvector uncertainty: then a quite good agreement with the MCS solution is reached for all of the examples. Furthermore the perturbation method (case 2 solution) allows the characterization of the distribution of the uncertainties (almost uniform/non-uniform/localized).

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Appendix A. Matrix variate normal distribution properties

Following Gupta and Nagar [52], the characteristic function will be used to derive some expectations such as $E\{x_{i_1 j_1} \dots x_{i_r j_r}\}$.

Consider:

- X , an $n \times p$ random matrix: $X \sim \mathcal{N}_{n,p}(\bar{X}, C)$, where \bar{X} is the mean of X .
- Z , an $n \times p$ matrix.
- ΔX , an $n \times p$ random matrix: $\Delta X \sim \mathcal{N}_{n,p}([0], C)$, where $[0]$ is the $n \times p$ null matrix.

The characteristic function is

$$\phi_X(Z) = E\{\exp(\text{tr}(\iota Z' X))\} \tag{A.1}$$

$$\phi_X(Z) = E\{\exp(\text{vec}(Z)' \text{vec}(X))\} \tag{A.2}$$

$$\phi_X(Z) = \exp(\text{vec}(Z)' \text{vec}(\bar{X}) - \frac{1}{2} \text{vec}(Z)' C \text{vec}(Z)) \tag{A.3}$$

$$\phi_X(Z) = \exp(h(Z)) \tag{A.4}$$

with

$$\iota = \sqrt{-1} \tag{A.5}$$

$$h(Z) = \iota \sum_{i=1}^n \sum_{j=1}^p \bar{x}_{ij} z_{ij} - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^p \sum_{k=1}^n \sum_{l=1}^p z_{ij} C_{(j-1)n+i, (l-1)n+k} z_{kl} \tag{A.6}$$

Indeed, according to (28), the covariance between the elements X_{ij} and X_{kl} is $C_{\alpha, \beta}$ with

- $\alpha = (j-1)n + i$ and $\beta = (l-1)n + k$
- or, equivalently:

$$\begin{cases} j = \lfloor \frac{\alpha-1}{n} \rfloor + 1 \\ i = \alpha - (j-1)n \end{cases} \quad \begin{cases} l = \lfloor \frac{\beta-1}{n} \rfloor + 1 \\ k = \beta - (l-1)n \end{cases}$$

The following characteristic function property will be used:

$$\frac{\partial^r \phi_X(Z)}{\partial z_{i_1 j_1} \dots \partial z_{i_r j_r}} \Big|_{Z=0} = \iota^r E \left\{ \prod_{k=1}^r x_{i_k j_k} \right\} \tag{A.7}$$

A.1. $E\{x_{ij}\}$

$$\frac{\partial \phi_X(Z)}{\partial z_{ij}} = \exp(h(Z)) \frac{\partial h(Z)}{\partial z_{ij}} \tag{A.8}$$

From (A.6), we have

$$\frac{\partial h(Z)}{\partial z_{ij}} = \iota \bar{x}_{ij} - \sum_{k=1}^n \sum_{l=1}^p C_{(j-1)n+i, (l-1)n+k} z_{kl} \tag{A.9}$$

Thus

$$E\{x_{ij}\} = \frac{1}{\iota} \frac{\partial h(Z)}{\partial z_{ij}} \Big|_{Z=0} = \bar{x}_{ij} \tag{A.10}$$

A.2. $E\{X_{ij}X_{kl}\}$

$$\frac{\partial^2 \phi_X(Z)}{\partial Z_{ij} \partial Z_{kl}} = \exp(h(Z)) \left[\frac{\partial h(Z)}{\partial Z_{ij}} \frac{\partial h(Z)}{\partial Z_{kl}} + \frac{\partial^2 h(Z)}{\partial Z_{ij} \partial Z_{kl}} \right] \tag{A.11}$$

From (A.9) we have

$$\frac{\partial^2 h(Z)}{\partial Z_{ij} \partial Z_{kl}} = -C_{(j-1)n+i, (l-1)n+k} \tag{A.12}$$

Thus

$$E\{X_{ij}X_{kl}\} = \frac{1}{i^2} \frac{\partial^2 \phi_X(Z)}{\partial Z_{ij} \partial Z_{kl}} \Big|_{Z=0} = \bar{X}_{ij} \bar{X}_{kl} + C_{(j-1)n+i, (l-1)n+k} \tag{A.13}$$

A.3. $E\{X_{ij}X_{kl}X_{mq}X_{rs}\}$

To simplify the equations, the following notations will be used:

$$h^{i_1 j_1 \dots i_r j_r} = \frac{\partial^r h(Z)}{\partial Z_{i_1 j_1} \dots \partial Z_{i_r j_r}} \tag{A.14}$$

$$\partial_{ij} = \partial Z_{ij} \tag{A.15}$$

The ϕ_X third partial derivative has to be calculated:

$$\frac{\partial^3 \phi_X(Z)}{\partial_{ij} \partial_{kl} \partial_{mq}} = e^{h(Z)} [h^{ij kl} h^{mq} + h^{ij kl} h^{mq} + h^{ij mq} h^{kl} + h^{kl mq} h^{ij} + h^{ij kl} h^{mq}] \tag{A.16}$$

However from (A.12) we have

$$\frac{\partial^3 h(Z)}{\partial_{ij} \partial_{kl} \partial_{mq}} = 0 \tag{A.17}$$

Then all the derivatives greater than the second order cancel. Therefore

$$\frac{\partial^3 \phi_X(Z)}{\partial_{ij} \partial_{kl} \partial_{mq}} = e^{h(Z)} [h^{ij kl} h^{mq} + h^{ij kl} h^{mq} + h^{ij mq} h^{kl} + h^{kl mq} h^{ij}] \tag{A.18}$$

Then

$$\begin{aligned} \frac{\partial^4 \phi_X(Z)}{\partial_{ij} \partial_{kl} \partial_{mq} \partial_{rs}} &= e^{h(Z)} [h^{ij kl} h^{mq} h^{rs} + h^{ij kl} h^{mq} h^{rs} + h^{ij mq} h^{kl} h^{rs} + h^{kl mq} h^{ij} h^{rs} + h^{ij rs} h^{kl} h^{mq} + h^{ij kl} h^{rs} h^{mq} + h^{ij kl} h^{mq} h^{rs} \\ &\quad + h^{ij kl} h^{mq} h^{rs} + h^{ij mq} h^{kl} h^{rs} + h^{ij rs} h^{kl} h^{mq}] \end{aligned} \tag{A.19}$$

From relations (A.9), (A.12) and (A.19) we have

$$\begin{aligned} E\{X_{ij}X_{kl}X_{mq}X_{rs}\} &= \frac{\partial^4 \phi_X(Z)}{\partial_{ij} \partial_{kl} \partial_{mq} \partial_{rs}} \Big|_{Z=0} \\ &= \bar{X}_{ij} \bar{X}_{kl} \bar{X}_{mq} \bar{X}_{rs} + C_{(j-1)n+i, (l-1)n+k} \bar{X}_{mq} \bar{X}_{rs} + C_{(j-1)n+i, (q-1)n+m} \bar{X}_{kl} \bar{X}_{rs} + C_{(l-1)n+k, (q-1)n+m} \bar{X}_{ij} \bar{X}_{rs} \\ &\quad + C_{(j-1)n+i, (s-1)n+r} \bar{X}_{kl} \bar{X}_{mq} + C_{(l-1)n+k, (s-1)n+r} \bar{X}_{ij} \bar{X}_{mq} + C_{(q-1)n+m, (s-1)n+r} \bar{X}_{ij} \bar{X}_{kl} \\ &\quad + C_{(j-1)n+i, (l-1)n+k} C_{(q-1)n+m, (s-1)n+r} + C_{(j-1)n+i, (q-1)n+m} C_{(l-1)n+k, (s-1)n+r} \\ &\quad + C_{(j-1)n+i, (s-1)n+r} C_{(l-1)n+k, (q-1)n+m} \end{aligned} \tag{A.20}$$

Appendix B. Some useful expectations of a product of matrices

B.1. $E\{XX'\}$

From relation (A.13), the ik element of $E\{XX'\}$ is

$$E\{XX'\}_{ik} = \sum_{j=1}^p E\{X_{ij}X_{kj}\} = \sum_{j=1}^p \bar{X}_{ij} \bar{X}_{kj} + \sum_{j=1}^p C_{(j-1)n+i, (j-1)n+k} \tag{B.1}$$

If $C = \Sigma \otimes \Psi$, this relation may be rewritten as

$$E\{XX'\} = \bar{X}\bar{X}' + \text{tr}(\Sigma)\Psi \tag{B.2}$$

B.2. $E\{XX'KXX'\}$

The im element of $E\{XX'KXX'\}$ is

$$E\{XX'KXX'\}_{im} = \sum_{j=1}^p \sum_{k=1}^n \sum_{l=1}^n \sum_{q=1}^p K_{kl} E\{x_{ij}x_{kj}x_{lq}x_{mq}\} \quad (\text{B.3})$$

This expression may be evaluated from (A.20):

$$\begin{aligned} E\{XX'KXX'\}_{im} = & \sum_{j=1}^p \sum_{k=1}^n \sum_{l=1}^n \sum_{q=1}^p \bar{x}_{ij}\bar{x}_{kj}K_{kl}\bar{x}_{lq}\bar{x}_{mq} + C_{(j-1)n+i,(j-1)n+k}K_{kl}\bar{x}_{lq}\bar{x}_{mq} + C_{(j-1)n+i,(q-1)n+l}K_{kl}\bar{x}_{kj}\bar{x}_{mq} \\ & + C_{(j-1)n+k,(q-1)n+l}K_{kl}\bar{x}_{ij}\bar{x}_{mq} + C_{(j-1)n+i,(q-1)n+m}K_{kl}\bar{x}_{kj}\bar{x}_{lq} + C_{(j-1)n+k,(q-1)n+m}K_{kl}\bar{x}_{ij}\bar{x}_{lq} \\ & + C_{(q-1)n+l,(q-1)n+m}K_{kl}\bar{x}_{ij}\bar{x}_{kj} + C_{(j-1)n+i,(j-1)n+k}K_{kl}C_{(q-1)n+l,(q-1)n+m} + C_{(j-1)n+i,(q-1)n+l}K_{kl}C_{(j-1)n+k,(q-1)n+m} \\ & + C_{(j-1)n+i,(q-1)n+m}K_{kl}C_{(j-1)n+k,(q-1)n+l} \end{aligned} \quad (\text{B.4})$$

When $C = \Sigma \otimes \Psi$, $E\{XX'KXX'\}$ can easily be derived from relation (B.3) which may be rewritten as

$$\begin{aligned} E\{XX'KXX'\}_{im} = & \sum_{j=1}^p \sum_{k=1}^n \sum_{l=1}^n \sum_{q=1}^p \bar{x}_{ij}\bar{x}_{kj}K_{kl}\bar{x}_{lq}\bar{x}_{mq} + \Psi_{ik}\Sigma_{jj}K_{kl}\bar{x}_{lq}\bar{x}_{mq} + \Psi_{il}\Sigma_{jq}K_{kl}\bar{x}_{kj}\bar{x}_{mq} \\ & + \Psi_{kl}\Sigma_{jq}K_{kl}\bar{x}_{ij}\bar{x}_{mq} + \Psi_{im}\Sigma_{jq}K_{kl}\bar{x}_{kj}\bar{x}_{lq} + \Psi_{km}\Sigma_{jq}K_{kl}\bar{x}_{ij}\bar{x}_{lq} + \Psi_{lm}\Sigma_{qq}K_{kl}\bar{x}_{ij}\bar{x}_{kj} \\ & + \Psi_{ik}\Sigma_{jj}K_{kl}\Psi_{lm}\Sigma_{qq} + \Psi_{il}\Sigma_{jq}K_{kl}\Psi_{km}\Sigma_{jq} + \Psi_{im}\Sigma_{jq}K_{kl}\Psi_{kl}\Sigma_{jq} \end{aligned} \quad (\text{B.5})$$

Therefore we have

$$\begin{aligned} E\{XX'KXX'\} = & \bar{X}\bar{X}'K\bar{X}\bar{X}' + \text{tr}(\Sigma)\Psi K\bar{X}\bar{X}' + \Psi K\bar{X}\Sigma\bar{X}' + \text{tr}(\Psi K)\bar{X}\Sigma\bar{X}' + \text{tr}(\Sigma\bar{X}'K\bar{X})\Psi + \bar{X}\Sigma\bar{X}'K\Psi \\ & + \text{tr}(\Sigma)\bar{X}\bar{X}'K\Psi + \text{tr}(\Sigma)^2\Psi K\Psi + \text{tr}(\Sigma\Sigma)\Psi K\Psi + \text{tr}(\Sigma\Sigma)\text{tr}(\Psi K)\Psi \end{aligned} \quad (\text{B.6})$$

B.3. $E\{X\Omega^2X'\}$

X and Ω^2 are uncorrelated, and Ω^2 is a diagonal matrix. Then the il element of $E\{X\Omega^2X'\}$ is

$$\begin{aligned} E\{X\Omega^2X'\}_{il} = & \sum_{k=1}^p E\{x_{ik}x_{lk}\} E\{\Omega_{kk}^2\} \\ = & \sum_{k=1}^p \bar{x}_{ik}\bar{x}_{lk}\overline{\Omega^2}_{kk} + C_{(k-1)n+i,(k-1)n+l}\overline{\Omega^2}_{kk} \end{aligned} \quad (\text{B.7})$$

If $C = \Sigma \otimes \Psi$, this relation may be rewritten as

$$E\{X\Omega^2X'\}_{il} = (\bar{X}\overline{\Omega^2}\bar{X}')_{il} + \sum_{k=1}^p \Psi_{il}\Sigma_{kk}\overline{\Omega^2}_{kk} \quad (\text{B.8})$$

Therefore

$$E\{X\Omega^2X'\} = \bar{X}\overline{\Omega^2}\bar{X}' + \text{tr}(\Sigma\overline{\Omega^2})\Psi \quad (\text{B.9})$$

B.4. $E\{\Delta X B \Delta X\}$

B is a $p \times n$ matrix. As the mean matrix of ΔX is the null matrix, the il element of $E\{\Delta X B \Delta X\}$ is

$$E\{\Delta X B \Delta X\}_{il} = \sum_{j=1}^p \sum_{k=1}^n E\{\Delta x_{ij}\Delta x_{lk}\} B_{jk} = \sum_{j=1}^p \sum_{k=1}^n B_{jk} C_{(j-1)n+i,(l-1)n+k} \quad (\text{B.10})$$

If $C = \Sigma \otimes \Psi$, this relation may be rewritten as

$$E\{\Delta X B \Delta X\} = \Psi B' \Sigma \quad (\text{B.11})$$

B.5. $E\{\Delta X B \Delta X'\}$

B is a $p \times p$ matrix. Then the il element of $E\{\Delta X B \Delta X'\}$ is

$$E\{\Delta X B \Delta X'\}_{il} = \sum_{j=1}^p \sum_{k=1}^p E\{\Delta x_{ij}\Delta x_{lk}\} B_{jk} = \sum_{j=1}^p \sum_{k=1}^p B_{jk} C_{(j-1)n+i,(k-1)n+l} \quad (\text{B.12})$$

If $C = \Sigma \otimes \Psi$, this relation may be rewritten as

$$E\{\Delta X' B \Delta X'\} = \text{tr}(B \Sigma) \Psi \tag{B.13}$$

B.6. $E\{\Delta X' B \Delta X\}$

B is an $n \times n$ matrix. Then the il element of $E\{\Delta X' B \Delta X\}$ is

$$E\{\Delta X' B \Delta X\}_{il} = \sum_{j=1}^n \sum_{k=1}^n E\{\Delta x_{ji} \Delta x_{kl}\} B_{jk} = \sum_{j=1}^n \sum_{k=1}^n B_{jk} C_{(i-1)n+j, (l-1)n+k} \tag{B.14}$$

If $C = \Sigma \otimes \Psi$, this relation may be rewritten as

$$E\{\Delta X' B \Delta X\} = \text{tr}(B \Psi) \Sigma \tag{B.15}$$

Appendix C. Estimation of cov(Y): perturbation method

By neglecting the second order terms; and using (9) and (39), $Y - \tilde{Y}$ may be calculated as

$$Y - \tilde{Y} = -K\bar{X} \Delta X' M - K \Delta X \bar{X}' M - M\bar{X} \Delta X' K - M \Delta X \bar{X}' K + M\bar{X}\bar{X}' K\bar{X} \Delta X' M + M\bar{X}\bar{X}' K \Delta X \bar{X}' M + M\bar{X} \Delta X' K\bar{X}\bar{X}' M + M \Delta X \bar{X}' K\bar{X}\bar{X}' M + M\bar{X}\bar{\Omega}^2 \Delta X' M + M \Delta X \bar{\Omega}^2 \bar{X}' M + M\bar{X} \Delta \Omega^2 \bar{X}' M \tag{C.1}$$

$$Y - \tilde{Y} = \sum_{i=1}^5 (A_i \Delta X B_i + B_i' \Delta X' A_i') + M\bar{X} \Delta \Omega^2 \bar{X}' M \tag{C.2}$$

with

$$\begin{aligned} A_1 &= -K, & B_1 &= \bar{X}' M \\ A_2 &= -M, & B_2 &= \bar{X}' K \\ A_3 &= M\bar{X}\bar{X}' K, & B_3 &= \bar{X}' M \\ A_4 &= M, & B_4 &= \bar{X}' K\bar{X}\bar{X}' M \\ A_5 &= M, & B_5 &= \bar{\Omega}^2 \bar{X}' M \end{aligned}$$

Then the covariance matrix is

$$\text{cov}(Y) = E\{\text{vec}(Y - \tilde{Y}) \text{vec}(Y - \tilde{Y})'\} \tag{C.3}$$

$$\begin{aligned} \text{cov}(Y) &= \sum_{i=1}^5 \sum_{j=1}^5 E\{\text{vec}(A_i \Delta X B_i) \text{vec}(A_j \Delta X B_j)'\} + \text{vec}(A_i \Delta X B_i) \text{vec}(B_j' \Delta X' A_j')' \\ &\quad + \text{vec}(B_i' \Delta X' A_i') \text{vec}(A_j \Delta X B_j)'\} + \text{vec}(B_i' \Delta X' A_i') \text{vec}(B_j' \Delta X' A_j')' \\ &\quad + E\{\text{vec}(M\bar{X} \Delta \Omega^2 \bar{X}' M) \text{vec}(M\bar{X} \Delta \Omega^2 \bar{X}' M)'\} \end{aligned} \tag{C.4}$$

where it has been taken into account the independence between X and Ω^2 , and the null mean for ΔX and $\Delta \Omega^2$. By using relation (10), this latter expression may be transformed:

$$\begin{aligned} \text{cov}(Y) &= \sum_{i=1}^5 \sum_{j=1}^5 E\{(B_i' \otimes A_i) \text{vec}(\Delta X) \text{vec}(\Delta X)' (B_j \otimes A_j) + (B_i' \otimes A_i) \text{vec}(\Delta X) \text{vec}(\Delta X)' (A_j' \otimes B_j) \\ &\quad + (A_i \otimes B_i') \text{vec}(\Delta X') \text{vec}(\Delta X)' (B_j \otimes A_j) + (A_i \otimes B_i') \text{vec}(\Delta X') \text{vec}(\Delta X)' (A_j' \otimes B_j)\} \\ &\quad + \text{cov}(Y)|_{\text{case 1}} \end{aligned} \tag{C.5}$$

where

$$\text{cov}(Y)|_{\text{case 1}} = ((M\bar{X}) \otimes (M\bar{X})) \text{cov}(\Omega^2) ((M\bar{X})' \otimes (M\bar{X})') \tag{C.6}$$

is the covariance matrix (27) obtained for case 1 with $X = \bar{X}$.

Using the permutation matrix \mathcal{P}_{np} , $\text{vec}(\Delta X')$ is expressed in function of $\text{vec}(\Delta X)$ [52]:

$$\text{vec}(\Delta X') = \mathcal{P}_{np} \text{vec}(\Delta X) \tag{C.7}$$

where \mathcal{P}_{np} is an $np \times np$ -matrix:

$$\mathcal{P}_{np} = \sum_{i=1}^n \sum_{j=1}^p H(i,j) \otimes H(i,j)' \quad (\text{C.8})$$

where element (k,l) of the $n \times p$ -matrix $H(i,j)$ is [52]

$$H(i,j)_{k,l} = \delta_{ik} \delta_{jl} \quad (\text{C.9})$$

Then (C.5) is rewritten as

$$\begin{aligned} \text{cov}(Y) = & \sum_{i=1}^5 \sum_{j=1}^5 E\{(B_i' \otimes A_i) \text{vec}(\Delta X) \text{vec}(\Delta X)' (B_j \otimes A_j) + (B_i' \otimes A_i) \text{vec}(\Delta X) \text{vec}(\Delta X)' \mathcal{P}'_{np}(A_j' \otimes B_j) \\ & + (A_i \otimes B_i') \mathcal{P}_{np} \text{vec}(\Delta X) \text{vec}(\Delta X)' (B_j \otimes A_j) + (A_i \otimes B_i') \mathcal{P}_{np} \text{vec}(\Delta X) \text{vec}(\Delta X)' \mathcal{P}'_{np}(A_j' \otimes B_j)\} \\ & + \text{cov}(Y)|_{\text{case 1}} \end{aligned} \quad (\text{C.10})$$

$$\begin{aligned} \text{cov}(Y) = & \sum_{i=1}^5 \sum_{j=1}^5 (B_i' \otimes A_i) \text{cov}(X) (B_j \otimes A_j) + (B_i' \otimes A_i) \text{cov}(X) \mathcal{P}'_{np}(A_j' \otimes B_j) \\ & + (A_i \otimes B_i') \mathcal{P}_{np} \text{cov}(X) (B_j \otimes A_j) + (A_i \otimes B_i') \mathcal{P}_{np} \text{cov}(X) \mathcal{P}'_{np}(A_j' \otimes B_j) + \text{cov}(Y)|_{\text{case 1}} \end{aligned} \quad (\text{C.11})$$

$$\text{cov}(Y) = \sum_{i=1}^5 \sum_{j=1}^5 [(B_i' \otimes A_i) + (A_i \otimes B_i') \mathcal{P}_{np}] \text{cov}(X) [(B_j \otimes A_j) + \mathcal{P}'_{np}(A_j' \otimes B_j)] + \text{cov}(Y)|_{\text{case 1}} \quad (\text{C.12})$$

Finally the covariance matrix of Y may be written as

$$\text{cov}(Y) = \sum_{i=1}^5 \sum_{j=1}^5 D_i \text{cov}(X) D_j' + D \text{cov}(\Omega^2) D' \quad (\text{C.13})$$

where

$$\forall i = 1 \dots 5, \quad D_i = (B_i' \otimes A_i) + (A_i \otimes B_i') \mathcal{P}_{np} \quad (\text{C.14})$$

$$D = (M\bar{X}) \otimes (M\bar{X}) \quad (\text{C.15})$$

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