European Journal of Mechanics A/Solids 34 (2012) 52–62

Contents lists available at SciVerse ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

Nonlocal elasticity based vibration of initially pre-stressed coupled nanobeam systems

T. Murmu*, S. Adhikari

College of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, Wales, UK

ARTICLE INFO

Article history:
Received 7 October 2010
Accepted 28 November 2011
Available online 10 December 2011

Keywords:
Double-nanobeam-system
Scale effects
Pre-stressed
In-phase and out-of-phase

ABSTRACT

Vibration analyses of coupled nanobeam system under initial compressive pre-stressed condition are presented. An elastically connected double-nanobeam-system is considered. Expressions for bending-vibration of pre-stressed double-nanobeam-system are formulated using Eringen's nonlocal elasticity model. An analytical method is proposed to obtain natural frequencies of the nonlocal double-nanobeam-system (NDNBS). Nano-scale effects and coupling spring effects in (i) in-phase type, (ii) out-of-phase type vibration; and (ii) vibration with one nanobeam fixed are examined. Scale effects in higher natural frequencies of NDNBS are also highlighted in this manuscript. Results reveal the difference (quantitatively) by which the pre-load affects the nonlocal frequency in the in-phase type and out-of-phase type vibrations mode of NDNBS.

1. Introduction

A beam is a simple model of one-dimensional continuous system (Timoshenko, 1953). Its importance in various engineering fields is well appreciated (Jennings, 2004). Beam-type structures are widely used in many branches of modern civil, mechanical and aerospace engineering. Recently, it is being extensively utilized as nanostructure components (Harik and Salas, 2003) for nanoelectromechanical (NEMS) and microelectromechanical systems (MEMS). Being important from the theoretical and engineering points of view, the dynamic problems involving one-dimensional continuous beam have drawn great deal of attention over the past few decades.

An important technological extension of the concept of the single beam is that of the complex coupled-beam-systems. One such simple coupled beam system is the double-beam-system. The double-beam-system is a continuous system consisting of two one-dimensional beams joined by an elastic medium represented by distributed vertical springs. Employing beam theories, several important works on vibration and buckling of elastically connected double-beam systems are reported. Vu et al. (2000) studied the vibration of homogenous double-beam system subjected to harmonic excitation. Erol and Gurgoz (2004) extended the analysis of Vu et al. (2000) to axially vibrating double-rod system coupled by translational springs and dampers. Oniszczuk (2000a) studied the free vibrations of two parallel simply supported beams continuously joined by a Winkler elastic layer. Undamped forced transverse vibrations of an elastically connected simply supported double-beam system were analysed. Free and forced vibration of double-string complex system was also investigated by Oniszczuk (2000b, 2000c). Hilal (2006) investigated the dynamic response of a double Euler–Bernoulli beam due to moving constant load. The effects of the speed of the moving load, the damping and the elasticity of the coupling viscoelastic layer on the dynamic responses of the beam system were presented. Vibration analysis of double-beam systems interconnected only at discrete points was reported by Hamada et al. (1983) and Gurgoz and Erol (2004). Buckling and the effect of a compressive load on the free and forced vibration on double-beam systems were reported by Zhang et al. (2008a, 2008b). Kelly and Srinivas (2009) carried out vibrations of elastically connected stretched beam systems. Analyses of double-beam systems by numerical techniques were also reported. Rosa and Lippiello (2007) presented non-classical boundary conditions and differential quadrature method for vibrating double-beams. Li and Hua (2007) presented spectral finite element analysis of elastically connected double-beam systems.

From the above discussion, it can be observed that the vibration theory of double-beam systems is well developed and studied in details. However, there are only few contributions dealing with the vibrations of beam-systems which are scale-dependent. Scale-dependent beams structures are those fabricated from nanomaterials. The nanomaterials are future generation engineering materials and have stimulated the interest of the scientific researcher's communities in physics, chemistry, biomedical and

* Corresponding author. Tel.: +44 7742214778; fax: +44 1792 295676.
E-mail addresses: murmutony@gmail.com, T.Murmu@swansea.ac.uk (T. Murmu).

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doi:10.1016/j.euromechsol.2011.11.010

engineering. These nanomaterials have special properties resulting from their nanoscale dimensions. Common examples of materials that exhibit interesting properties on the nanoscale include nanoparticles, nanowires and nanotubes (viz. carbon nanotubes, ZnO nanotubes). These nanomaterials have promising mechanical (tensile strength), chemical, electrical, optical and electronic properties (Dai et al., 1996; Bachtold et al., 2001; Kim and Lieber, 1999). Because of many desirable properties, nanomaterials are perceived to be the components for various nanoelectromechanical systems (NEMS) and nanocomposites. Structural beams fabricated from nanomaterials and of nanometre dimension are referred as nanobeams.

The understanding of dynamics of single-nanobeam (carbon nanotubes, nanowires) is important. The vibration characteristics of nanobeams can be employed for NEMS/MEMS applications (Pugno et al., 2005; Ke et al., 2005a, 2005b). Parallel to vibration of single-nanobeam, the study of vibrating multiple-nanobeam-system is also relevant for nanosensors and nanoresonators applications. The recent development of nano-optomechanical systems (NOMS) necessitates the use of vibrating double-nanobeam-systems.

The employment of double-nanobeam-systems in NOMS has been reported by various researchers. Frank et al. (2010) presented a dynamically reconfigurable photonic crystal nanobeam cavity. There work involved two closely situated parallel vibrating clamped double-nanobeam-systems. Eichenfield et al. (2009) described the design, fabrication, and measurement of a cavity nano-optomechanical system (NOMS). The NOMS consisting of two closely separated coupled nanobeams. The researchers fabricated the low dimension double-beam-system by depositing stoichiometric silicon nitride using low-pressure-chemical-vapour-deposition on a silicon wafer. Deotare et al. (2009) studied the coupled photonic crystal nanobeam cavities consisting of two parallel suspended nanobeams separated by a small gap. The use of vibration properties in double-nanobeam-system has also been reported by Lin et al. (2010). The authors studied the coherent mixing of mechanical excitations in nano-optomechanical structures. Most of the works reported here are experimental works.

It is understood that controlling every parameter in experiments at nanoscale is difficult. Further, since molecular dynamics simulations are computationally expensive, analysis of nanostructures had been carried out by classical continuum theory. Extensive research over the past decade has shown that classical continuum models (Timoshenko, 1974) are able to predict the performance of 'large' nanostructures reasonably well. Classical continuum models are scale-free theory and it lacks the accountability of the effects arising from the size-effects. Experimental (Ruud et al., 1994; Wong et al., 1997; Sorop and Jongh, 2007; Kasuya et al., 1997; Juhasz et al., 2004) and atomistic simulations (Chowdhury et al., 2010) have evidenced a significant 'size-effect' in the mechanical properties when the dimensions of the nanostructures become 'small'. Size-effects are related to atoms and molecules that constitute the materials. The application of classical continuum models thus may be questionable in the analysis of 'smaller' nanostructures. Therefore, recently there have been research efforts to bring in the scale effects within the formulation by amending the traditional classical continuum mechanics. One widely used size-dependant theory is the nonlocal elasticity theory pioneered by Eringen (Eringen, 1972, 1983, 2002). Nonlocal elasticity accounts for the small-scale effects arising at the nanoscale level. Recent literature shows that the theory of nonlocal elasticity is being increasingly used (Peddieson et al., 2003; Sudak, 2003; Wang, 2005; Wang et al., 2006; Reddy, 2007; Wang and Wang, 2007; Wang and Varadan, 2007; Lu, 2007; Hu et al., 2008; Heireche et al., 2008; Reddy and Pang, 2008; Artan and Tepe, 2008; Sun and Liu, 2008; Aydogdu, 2009; Murmu and Pradhan, 2009a, 2009b; Pradhan and Murmu, 2010; Murmu and Adhikari, 2010a, 2012, 2011a, 2011b; Hao et al., 2010; Shen, 2010; Xiang et al., 2010; Murmu et al., 2011) for reliable and quick analysis of nanostructures viz. nanobeams, nanoplates, nanorings, carbon nanotubes, graphenes, nanoswitches and microtubules. For double-nanobeam-system, Murmu and Adhikari (2010b) studied the nonlocal effects in the longitudinal vibration of double-nanorod systems. Further using nonlocal elasticity Murmu and Adhikari (2010c) have proposed nonlocal transverse vibration analysis of coupled double-nanobeam-systems. The nonlocal elasticity has also potential in application in wide areas such as nanomaterials with defects (Pugno and Ruoff, 2004; Pugno, 2006a, 2006b).

In the nonlocal elasticity theory, the small-scale effects are captured by assuming that the stress at a point is a function of the strains at all points in the domain (Eringen, 1983). This is unlike classical elasticity theory. Nonlocal theory considers long-range inter-atomic interaction and yields results dependent on the size of a body. Some drawbacks of the classical continuum theory could be efficiently avoided and the size-dependent phenomena can be reasonably explained by the nonlocal elasticity theory. The majority of the existing works on nonlocal elasticity are pertaining to the free transverse vibration of single nanobeams. Though the mechanical studies of nanobeams may include vibration of multiple-walled nanotubes, the study of discrete multiple-nanobeam-system is particularly limited in literature.

Further it is observed that during the fabrication of nanostructures, the residual stresses can be developed within the structures. This initial residual stresses could significantly modify the mechanical and electrical properties of MEMS or NEMS devices. Strains are usually developed during the material growth and temperature relaxation. For a suspended structure, this process-induced strain may cause the axial residual stress within the structure. This calls for a deep understanding of its influence on the performance of the devices for the optimum design.

Therefore, based on the above discussion there is a strong encouragement to gain an understanding of the entire subject of vibration of complex-nanobeam-system and the mathematical modelling of such phenomena. In this paper an investigation is carried out to understand the small-scale effects in the free bending-vibration of nonlocal double-nanobeam-system (NDNBS) subjected to initial compressive pre-stressed load. The two nanobeams are subjected to initially pre-stress compressive loads. Initial pre-stressed compressive load may arise due to fabrication on nanobeam (Carr and Wybourne, 2003) or due to external applied compressive loads. Further, this paper presents a unique yet simple method of obtaining the exact solution for free vibration of double-nanobeam system. Equations for free bending-vibration of a pre-stressed double-nanobeam-system (NDNBS) are formulated within the framework of Eringen's nonlocal elasticity. The two nanobeams are assumed to be attached by distributed vertical transverse springs. These springs may represent the stiffness of an enclosed elastic medium, forces due to nano-optomechanical effects (Eichenfield et al., 2009; Deotare et al., 2009; Lin et al., 2010) or Vander Waals forces. An exact analytical method is proposed for solving the nonlocal frequencies of transversely vibrating NDNBS. The simplification in the computation is achieved based on the change of variables to decouple the set of two fourth-order partial differential equations. It is assumed that the two nanobeams in the NDNBS are identical, and the boundary conditions on the same side of the system are the same. Simply-supported boundary conditions are employed in this study. Explicit expressions for the natural frequencies of NDNBS are derived. The results are obtained for various vibration-phases of the NDNBS. The vibration phases include in-phase (synchronous) and out-of-phase (asynchronous) modes of vibration. The effects of (i) axial pre-stressed load, (ii) nonlocal parameter or scale coefficient, (iii) stiffness of the springs and (iv) the higher modes, on the frequency of the NDNBS are discussed.

2. Nonlocal elasticity for nanostructure applications

For completeness, here we provide a brief review of the theory of nonlocal elasticity. According to nonlocal elasticity, the basic equations for an isotropic linear homogenous nonlocal elastic body neglecting the body force are given as (Eringen, 1983)

$$\begin{aligned} \sigma_{ij,j} &= 0, \\ \sigma_{ij}(\mathbf{x}) &= \int_{\mathbf{V}} \phi(|\mathbf{x}-\mathbf{x}'|,\alpha) t_{ij} d\mathbf{V}(\mathbf{x}'), \quad \forall \mathbf{x} \in \mathbf{V} \\ t_{ij} &= H_{ijkl}\varepsilon_{kl}, \\ \varepsilon_{ij} &= \frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \end{aligned} \tag{1}$$

The terms σ_{ij}, t_{ij}, ε_{kl}, H_{ijkl} are the nonlocal stress, classical stress, classical strain and fourth-order elasticity tensors respectively. The volume integral is over the region **V** occupied by the body. The above equation (Eq. (1)) couples the stress due to nonlocal elasticity and the stress due to classical elasticity. The kernel function $\phi(|\mathbf{x}-\mathbf{x}'|,\alpha)$ is the nonlocal modulus. The nonlocal modulus acts as an attenuation function incorporating into constitutive equations the nonlocal effects at the reference point **x** produced by local strain at the source $\mathbf{x}'$. The term $|\mathbf{x}-\mathbf{x}'|$ represents the distance in the Euclidean form and α is a material constant that depends on the internal (e.g. lattice parameter, granular size, distance between the C–C bonds) and external characteristics lengths (e.g. crack length, wave length). Material constant α is defined as $\alpha = e_0 a/\ell$. Here e_0 is a constant for calibrating the model with experimental results and other validated models. The parameter e_0 is estimated such that the relations of the nonlocal elasticity model could provide satisfactory approximation to the atomic dispersion curves of the plane waves with those obtained from the atomistic lattice dynamics. The terms a and ℓ are the internal (e.g. lattice parameter, granular size, distance between C–C bonds) and external characteristics lengths (e.g. crack length, wave length) of the nanostructure.

Equation (1) is in partial-integral form and generally difficult to solve analytically. Thus a differential form of nonlocal elasticity equation is often used. According to Eringen (1983), the expression of nonlocal modulus can be given as

$$\phi(|\mathbf{x}|,\alpha) = \left(2\pi\ell^2\alpha^2\right)^{-1} K_0\left(\sqrt{\mathbf{x}\cdot\mathbf{x}}/\ell\alpha\right) \tag{2}$$

where K_0 is the modified Bessel function.

The equation of motion in terms of nonlocal elasticity can be expressed as

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \tag{3}$$

where f_i, ρ and u_i are the components of the body forces, mass density, and the displacement vector, respectively. The terms i, j takes up the symbols x, y, and z.

Assuming the kernel function ϕ as the Green's function, Eringen (1983) proposed a differential form of the nonlocal constitutive relation as

$$\sigma_{ij,j} + \mathscr{L}\left(f_i - \rho\ddot{u}_i\right) = 0 \tag{4}$$

where

$$\mathscr{L} = \left[1-(e_0 a)^2\nabla^2\right] \tag{5}$$

and ∇^2 is the Laplacian.

Using Eq. (5) the nonlocal constitutive stress–strain relation can be simplified as

$$\left(1-\alpha^2\ell^2\nabla^2\right)\sigma_{ij} = t_{ij} \tag{6}$$

Since we are considering nanobeams, in one dimensional form the constitutive relation is given by

$$\sigma(x) - (e_0 a)^2 \sigma''(x) = t_{ij} \tag{7}$$

For the stress resultant we use $\int_A \sigma dA$ and Eq. (7). We get

$$N - (e_0 a)^2 N''(x) = EAu'(x,t) \tag{8}$$

and the stress resultant in nonlocal moment's relation becomes

$$M - (e_0 a)^2 M''(x) = -EIw''(x,t) \tag{9}$$

Employing the nonlocal constitutive stress–strain relation, the one-dimensional equation of motion of a nonlocal Euler-Bernoulli beam can be written as (Reddy, 2007)

$$\begin{aligned} &EIw''''(x,t) - q(x) + (e_0 a)^2 q''(x) + Nw''(x,t) - (e_0 a)^2 Nw''''(x,t) \\ &\quad + m\ddot{w}(x,t) - (e_0 a)^2 m\ddot{w}''(x,t) = 0 \end{aligned} \tag{10}$$

where w denotes the deflection of the beam. The terms E, I and m are the Young's modulus, second moment of inertia and mass of the nonlocal beam, respectively. Term q is the distributed transverse load on the nonlocal beam. In the next Section we present the equation of motion of pre-stressed double-nanobeam-system.

3. Formulation of pre-stressed nonlocal double-nanobeam-systems

Consider a compressive pre-stressed nonlocal double-nanobeam-system (NDNBS) as shown in Fig. 1. The two nanobeams are denoted as Nanobeam-1 and Nanobeam-2. Vertically distributed springs attaches the two nanobeams. The stiffness of the springs is equivalent to the Winkler constant in a Winkler foundation model (Oniszczuk, 2000a). The springs can be used to substitute elastic medium, forces due to nano-optomechanical effects (Eichenfield et al., 2009; Deotare et al., 2009; Lin et al., 2010) or Vander Waals forces between the two nanobeams. These forces arise when the dimension of system approaches nanoscale. Generating a potential difference directly across the nanobeams an attractive electrostatic force can be induced between the two nanobeams (Frank et al., 2010). Thereby the spring stiffness can be varied between the nanobeams. The springs are considered to have stiffness, k. The two nanobeams are different where the length, mass per unit length and bending rigidity of the ith beam are L_i, m_i and $E_i I_i$ ($i = 1, 2$) respectively. These parameters are assumed to be constant along each nanobeam. The bending displacements over the two nanobeams are denoted by $w_1(x, t)$ and $w_2(x, t)$, respectively (Fig. 1).

The two nanobeams are subjected to initially pre-stress compressive loads. The initial axial stress effects can occur in micro/nanobeam-based devices. In the bottom-up or top-down

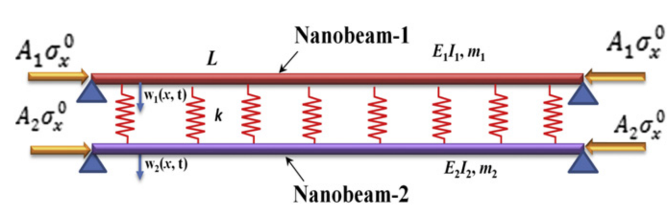

Fig. 1. Schematic diagram of elastically connected double-nanobeam-system subjected to pre-stressed compressive axial load.

fabrication of micro/nanobeam-like structures, the strains can develop during the material growth and temperature relaxation, resulting in axial compressive residual stress (Carr and Wybourne, 2003; Heireche et al., 2008; Murmu and Pradhan, 2009a).

Let the nanobeams subjected to initial pre-stress σ_x^0. Then the initial axial forces are given as

$$N_1^0 = A_1\sigma_x^0; \quad N_2^0 = A_2\sigma_x^0 \tag{11}$$

Employing the nonlocal elastic theory, the equations of motion of two pre-stressed nanobeams can be given by

nanobeam-1

$$E_1I_1w_1''''(x,t) + k[w_1(x,t) - w_2(x,t)] - (e_0a)^2k\left[w_1''(x,t) - w_2''(x,t)\right] + N_1^0w_1''(x,t) - (e_0a)^2N_1^0w_1''''(x,t) + m_1\ddot{w}_1(x,t) - (e_0a)^2m_1\ddot{w}_1''(x,t) = 0 \tag{12}$$

nanobeam-2

$$E_2I_2w_2''''(x,t) - k[w_1(x,t) - w_2(x,t)] + (e_0a)^2k\left[w_1''(x,t) - w_2''(x,t)\right] + N_2^0w_2''(x,t) - (e_0a)^2N_2^0w_2''''(x,t) + m_2\ddot{w}_2(x,t) - (e_0a)^2m_2\ddot{w}_2''(x,t) = 0 \tag{13}$$

Dots ($\cdot$) and primes ($'$) denote partial derivatives with respect to time t and position coordinate x, respectively. For the complete derivation of the equation of motion of a single nonlocal Euler-Bernoulli beam, one can see Ref. (Reddy, 2007)

4. Simplification of nonlocal double-nanobeam-systems

We assume that both the nanobeams and the forces within them are identical, that is

$$E_1I_1 = (E_2I_2) = EI \equiv \text{constant} \tag{14}$$

$$m_1 = m_2 = m \equiv \text{constant} \tag{15}$$

$$N_1^0 = N_2^0 = \hat{N} \equiv \text{constant (uniformly prestressed)} \tag{16}$$

Considering the Eqs. (12) and (13) and assumptions from Eqs. (14)–(16) we get the individual equations.

nanobeam-1

$$EIw_1''''(x,t) + k[w_1(x,t) - w_2(x,t)] - (e_0a)^2k\left[w_1''(x,t) - w_2''(x,t)\right] + \hat{N}w_1''(x,t) - (e_0a)^2\hat{N}w_1''''(x,t) + m\ddot{w}_1(x,t) - (e_0a)^2m\ddot{w}_1''(x,t) = 0 \tag{17}$$

nanobeam-2

$$EIw_2''''(x,t) - k[w_1(x,t) - w_2(x,t)] + (e_0a)^2k\left[w_1''(x,t) - w_2''(x,t)\right] + \hat{N}w_2''(x,t) - (e_0a)^2\hat{N}w_2''''(x,t) + m\ddot{w}_2(x,t) - (e_0a)^2m\ddot{w}_2''(x,t) = 0 \tag{18}$$

For the NDNBS we propose a change of variables (Vu et al., 2000)

$$w(x,t) = w_1(x,t) - w_2(x,t), \tag{19}$$

such that

$$w_1(x,t) = w(x,t) + w_2(x,t) \tag{20}$$

Here w (x, t) is the relative displacement of the main nano beam with respect to the auxiliary beam.

Subtracting Eqn. (17) from Eqn. (18) gives

$$EI\left[w_1''''(x,t) - w_2''''(x,t)\right] + 2k[w_1(x,t) - w_2(x,t)] - 2(e_0a)^2k\left[w_1''(x,t) - w_2''(x,t)\right] + \hat{N}\left(w_1''(x,t) - w_2''(x,t)\right) - (e_0a)^2\hat{N}\left(w_1''''(x,t) - w_2''''(x,t)\right) + m\left[\ddot{w}_1(x,t) - \ddot{w}_2(x,t)\right] - m(e_0a)^2\left[\ddot{w}_1''(x,t) - \ddot{w}_2''(x,t)\right] = 0 \tag{21}$$

By introducing Eqs. (19) and (20) and using Eq. (21) we get two equations

$$EIw''''(x,t) + 2kw(x,t) - 2(e_0a)^2kw''(x,t) + \hat{N}w''(x,t) - (e_0a)^2\hat{N}w''''(x,t) + m\ddot{w}(x,t) - (e_0a)^2m\ddot{w}''(x,t) = 0 \tag{22}$$

$$EIw_2''''(x,t) + \hat{N}w_2''(x,t) - (e_0a)^2\hat{N}w_2''''(x,t) + m\ddot{w}_2(x,t) - (e_0a)^2m\ddot{w}_2''(x,t) = kw(x,t) - (e_0a)^2kw''(x,t) \tag{23}$$

It should be noted that when the nonlocal effects are ignored ($e_0a = 0$) and a single nanobeam is considered, the above equations revert to the equations of classical Euler-Bernoulli beam theory (Timoshenko, 1974). For the present analysis of coupled NDNBS, we see the simplicity in using Eq. (22). Here we will be dealing with Eq. (22) for coupled NDNBS.

5. Exact solutions of governing equations

Now we determine the solution of Eqs. (22) and (23). Let the solution of Eqn. (22) be

$$w(x,t) = W(x)e^{i\omega t} \tag{24}$$

where ω is the circular natural frequency and W (x) is the mode shape. Term i is the conventional imaginary number, $i = \sqrt{-1}$.

Substituting the Eq. (24) into Eq. (22) yields

$$A_1W''''(x) + A_2W''(x) - A_3W(x) = 0 \tag{25}$$

where

$$A_1 = EI - (e_0a)^2\hat{N}; A_2 = -m\omega^2(e_0a)^2 + \hat{N} - 2k(e_0a)^2; \quad A_3 = -m\omega^2 - 2k \tag{26}$$

The general solution of Eq. (25) is given as

$$W(x) = C_1\sin\ \alpha x + C_2\cos\ \alpha x + C_3\sinh\ \beta x + C_4\cosh\ \beta x \tag{27}$$

Where

$$\alpha^2 = \frac{1}{2A_1}\left(A_2 + \sqrt{A_2^2 + 4A_1A_3}\right) \tag{28}$$

$$\beta^2 = \frac{1}{2A_1}\left(A_2 - \sqrt{A_2^2 + 4A_1A_3}\right) \tag{29}$$

The terms C_1, C_2, C_3 and C_4 are the constants of integrations determined from the boundary conditions. Here α is not the nonlocal modulus.

6. Nonlocal boundary conditions of pre-stressed NDNBS

Now we present the mathematical expressions of the boundary conditions in NDNBS. The boundary conditions of simply supported conditions are described here. At each end of the nanobeams in NDNBS the displacement and the nonlocal moments are considered to be zero (Reddy, 2007). They can be mathematically expressed as

(nanobeam-1): at $x = 0$

$$w_1(0,t) = 0; \tag{30}$$

$$M_1(0,t) = -EIw_1''(0,t) + (e_0a)^2 m\ddot{w}_1(0,t) + (e_0a)^2 k[w_1(0,t) - w_2(0,t)] + (e_0a)^2 \hat{N} w_1''(0,t) = 0 \tag{31}$$

(nanobeam-1): at $x = L$

$$w_1(L,t) = 0; \tag{32}$$

$$M_1(L,t) = -EIw_1''(L,t) + (e_0a)^2 m\ddot{w}_1(L,t) + (e_0a)^2 k[w_1(L,t) - w_2(L,t)] + (e_0a)^2 \hat{N} w_1''(L,t) = 0 \tag{33}$$

(nanobeam-2): at $x = 0$

$$w_2(0,t) = 0; \tag{34}$$

$$M_2(0,t) = -EIw_2''(0,t) + (e_0a)^2 m\ddot{w}_1 - (e_0a)^2 k[w_1(0,t) - w_2(0,t)] + (e_0a)^2 \hat{N} w_2''(0,t) = 0 \tag{35}$$

(nanobeam-2): at $x = L$

$$w_2(L,t) = 0; \tag{36}$$

$$M_2(L,t) = -EIw_2''(L,t) + (e_0a)^2 m\ddot{w}_1 - (e_0a)^2 k[w_1(L,t) - w_2(L,t)] + (e_0a)^2 \hat{N} w_2''(0,t) = 0 \tag{37}$$

Now we utilise Eq. (19) and the above boundary condition simplifies to

NDNBS: at $x = 0$

$$w(0,t) = w_1(0,t) - w_2(0,t) = 0; \tag{38}$$

$$M_1(0,t) - M_2(0,t) = -EIw_1''(0,t) + (e_0a)^2 \left[m\ddot{w}_1(0,t) + k[w_1(0,t) - w_2(0,t)]\right] + EIw_2''(0,t) - (e_0a)^2 \left[m\ddot{w}_1 - k[w_1(0,t) - w_2(0,t)]\right] + (e_0a)^2 \hat{N}\left[w_1''(0,t) - w_2''(0,t)\right] = 0; \tag{39}$$

NDNBS: at $x = L$

$$w(L,t) = w_1(L,t) - w_2(L,t) = 0; \tag{40}$$

$$M_1(L,t) - M_2(L,t) = -EIw_1''(L,t) + (e_0a)^2 \left[m\ddot{w}_1(L,t) + k[w_1(L,t) - w_2(L,t)]\right] + EIw_2''(L,t) - (e_0a)^2 \left[m\ddot{w}_1 - k[w_1(L,t) - w_2(L,t)]\right] + (e_0a)^2 \hat{N}\left[w_1''(L,t) - w_2''(L,t)\right] = 0; \tag{41}$$

By the use of Eq. (19) and Eqs. (30)–(41); the boundary conditions effectively reduce to

$$W(0) = 0 \text{ and } W''(0) = 0, \quad W(L) = 0 \text{ and } W''(L) = 0, \tag{42}$$

Here it can be seen that the boundary conditions due to local elasticity and nonlocal elasticity are equivalent.

7. Vibration of NDNBS under pre-stressed condition

7.1. Out-of-phase vibration:($w_1 - w_2 \neq 0$)

Consider the case of the NDBNS when the both the nanobeams vibrate in out-of-phase sequence. The NDNBS is subjected to a pres-stress load. The configuration of the NDNBS with out-of-phase sequence of vibration ($w_1 - w_2 \neq 0$) is shown in Fig. 2a.

We consider the case when all the ends have simply-supported boundary conditions (Fig. 2). For simply-supported case, the use of the boundary conditions in (42) yields

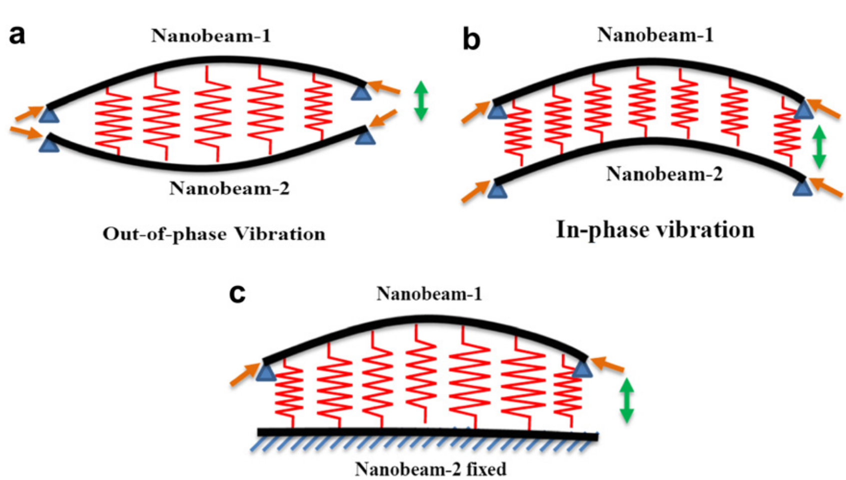

Fig. 2. Configuration of (a) out-of-phase vibration and (b) in-phase vibration of pre-stressed double-nanobeam-system, (c) vibration of double-nanobeam-system when one nanobeam is fixed.

$$C_2 = 0, \; C_4 = 0 \tag{43}$$

and

$$\begin{bmatrix} \sin \alpha L & \sinh \beta L \\ EI\alpha^2 \sin \alpha L & EI\beta^2 \sinh \beta L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{44}$$

For non- trivial solution, the determinant yields

$$\sin \alpha L \left(EI\beta^2 \sinh \beta L - EI\alpha^2 \sinh \beta L \right) = 0 \tag{45}$$

Thus the Eigen equation reduces to

$$\sin \alpha L = 0 \tag{46}$$

Therefore

$$\alpha L = n\pi, \quad n = 1, 2..... \tag{47}$$

From Eq. (28) we have

$$2A_1\alpha^2 = A_2 + \sqrt{A_2^2 + 4A_1A_3} \tag{48}$$

which yields,

$$A_1\alpha^4 - A_2\alpha^2 - A_3 = 0 \tag{49}$$

We introduce the following parameters for sake of simplicity and generality

$$\Omega = \sqrt{\frac{m\omega^2 L^4}{EI}}, \quad K = \frac{kL^4}{EI}, \quad \mu = \frac{e_0 a}{L}, \quad F = \frac{\hat{N}L^2}{EI} \tag{50}$$

Using Eq. (50) the expression of natural frequency of NDNBS is evaluated as

$$\Omega = \sqrt{\frac{(n\pi)^4 + 2K + 2K(\mu)^2(n\pi)^2 - F(n\pi)^2 - F(\mu)^2(n\pi)^4}{1 + (\mu)^2(n\pi)^2}}, \; n = 1, 2... \tag{51}$$

Note that when the nonlocal parameter μ is set to zero we get the expression of classical double-beam-system (Vu et al., 2000; Zhang et al., 2008a).

7.2. In-phase vibration: ($w_1 - w_2 = 0$)

Next, the in-phase sequence of vibration will be considered (Fig. 2b). For the present NDNBS, the relative displacements between the two nanobeams are absent ($w_1 - w_2 = 0$). Here we solve the Eq. (23) for the vibration of NDNBS. The vibration of nanobeam-2 would represent the vibration of the coupled vibrating system. We apply the same procedure for solving Eq. (23).

Using Eqs. (43)–(50) we can obtain the natural frequencies. The natural frequencies for the NDNBS in this case can be expressed as

$$\Omega = \sqrt{\frac{(n\pi)^4 - F(n\pi)^2 - F(\mu)^2(n\pi)^4}{1 + (\mu)^2(n\pi)^2}}, \quad n = 1, 2... \tag{52}$$

Note that in in-phase mode of vibration, the NDNBS is independent of stiffness of springs.

7.3. Vibration of nanobeam when one nanobeam is fixed: ($w_2 = 0$)

Consider the case of NDNBS when one of the two nanobeams (viz. nanobeam-2) is stationary ($w_2 = 0$). The schematic diagram is shown in Fig. 2c. Like the case of buckling, the NDNBS behaves like a vibrating beam embedded in an elastic medium .The elastic medium can be modelled as Winkler elastic foundation.

By following the same procedure as solution of Eq. (22), the explicit nonlocal frequency of NDNBS can be expressed as

$$\Omega = \sqrt{\frac{(r\pi)^4 + K + K(\mu)^2(r\pi)^2 - F(r\pi)^2 - F(\mu)^2(r\pi)^4}{1 + (\mu)^2(r\pi)^2}}, \quad r = 1, 2... \tag{53}$$

In fact when one of the nanobeam (viz. nanobeam-2) in NDNBS is fixed ($w_2 = 0$), the NDNBS vibrates as a beam on elastic medium.

8. Results and discussion

8.1. Coupled-carbon-nanotube-systems

The nonlocal theory for NDNBS illustrated here is a generalised theory and can be applied for the bending-vibration analysis of coupled carbon nanotubes, double ZnO nanobeam systems and double-nanobeam-systems for NOMS application; and in nanocomposites. The applicability of nonlocal elasticity theory in the analysis of single nanostructures (nanotubes and graphene sheet) has been established in various previous works. For the present study we assume two carbon nanotubes being elastically attached by an elastic medium. The properties of the nanobeams considered are that of a single-walled carbon nanotube (SWCNT) (Lu, 1997). An armchair SWCNT with chirality (5, 5) is considered. The radius of each individual SWCNT is assumed as 0.34 nm. Young's modulus, E, is taken as 0.971 TPa (Lu, 1997). Density ρ is taken as 2300 kg/m^3. Length of both nanotubes is considered as 5 nm. The frequency results of the NDNBS are presented in terms of the frequency parameters (Eq. (51)). The nonlocal parameter and the stiffness of the springs are computed as given in Eq. (50). Spring stiffness represents the stiffness of the enclosing elastic medium. Different values of spring parameters, K, are considered. This is because the elastic medium can be of low as well as of high stiffness (Liew et al., 2006). The values of K range from 5 to 500. Both the nanotubes (nanotube-1 and nanotube-2) are assumed to have the same geometrical and material properties. It should be noted that the coupled carbon nanotubes system is different from the conventional double-walled carbon nanotubes.

The nonlocal parameters are generally taken as $e_0 = 0.39$ (Eringen, 1983) and $a = 0.142$ nm (distance between carbon–carbon atoms). For carbon nanotubes and graphenes the range of $e_0a = 0$–2.0 nm has been widely used. In the present study we take the scale coefficient μ or nonlocal parameter in the similar range as $\mu = 0$–1.

8.2. Vibration response of coupled-vibrating-systems

This Section presents the vibration response of initially prestressed coupled-carbon-nanotubes-systems using the framework of nonlocal elasticity theory. Here we see the influence of small-scale (or nonlocality) on the natural frequency of the coupled-carbon-nanotube-systems. Curves have been plotted for natural frequencies against scale coefficient (nonlocal parameter) and depicted in Fig. 3. To signify the small-scale effect we introduce the parameter Frequency Reduction Percent (FRP). FRP is defined as

$$\mathrm{FRP} = \left(\frac{\Omega_{\text{Local Theory}} - \Omega_{\text{Nonocal Theory}}}{\Omega_{\text{Local Theory}}} \right) \times 100 \tag{54}$$

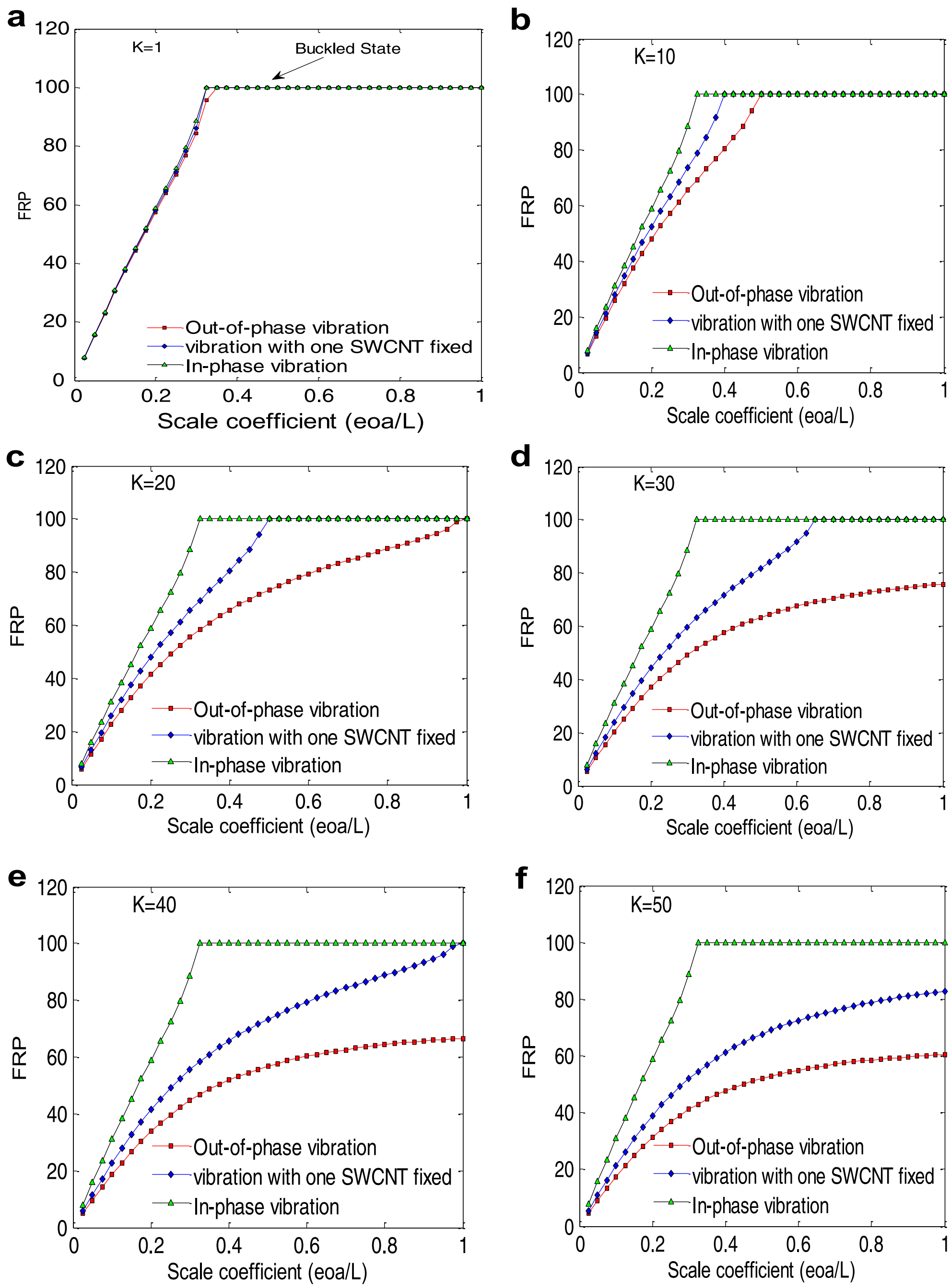

Fig. 3. Effect of scale coefficient ($\mu = e_0a/L$) on frequency reduction percent (FRP) for higher different values of stiffness of springs in coupled-SWCNT-systems, (a) $K = 1$; (b) $K = 10$; (c) $K = 20$; (d) $K = 30$; (e) $K = 40$; (f) $K = 50$.

8.2.1. Small-scale effects

Fig. 4a shows the variation of FRP against scale coefficient (e_0a/L). Both the coupled SWCNT is considered to be subjected to a constant initial compressive axial pre-load. A constant pre-load parameter of $F = 5$ is arbitrarily assumed. The stiffness parameter of the coupling springs is assumed as $K = 1$. From the figure it is observed that as the scale coefficient μ increases the FRP increases. This implies that for increasing scale coefficient the value of natural frequencies decreases. The reduction in natural frequency is due to the incorporation of nonlocal effects in the material properties of the carbon nanotubes.

The nonlocal effect reduces the stiffness of the material and hence the comparative lower natural frequencies. At a certain value of scale coefficient μ the FRP reaches 100%; and associated natural frequency has reached its maximum limit. Further increase of scale coefficient has no effect on the coupled SWCNT system and the curves become flat (Fig. 3a). This implies that the vibration system has reached the buckling state of mode. And the vibrating system cannot be considered to be vibrator any more.

Three cases of vibration response are considered here; case 1: out-of-phase vibration; case 2: vibration with one SWCNT fixed and case 3: in-phase type of vibration. On comparison of the three cases of coupled-carbon nanobeam system, the FRP for case 3 (in-phase vibration) is higher than the FRP for case 1 (out of phase vibration) and case 2 (one-SWCNT fixed). Though in all three cases the difference is slight (Fig. 3a). In other words, the scale coefficient

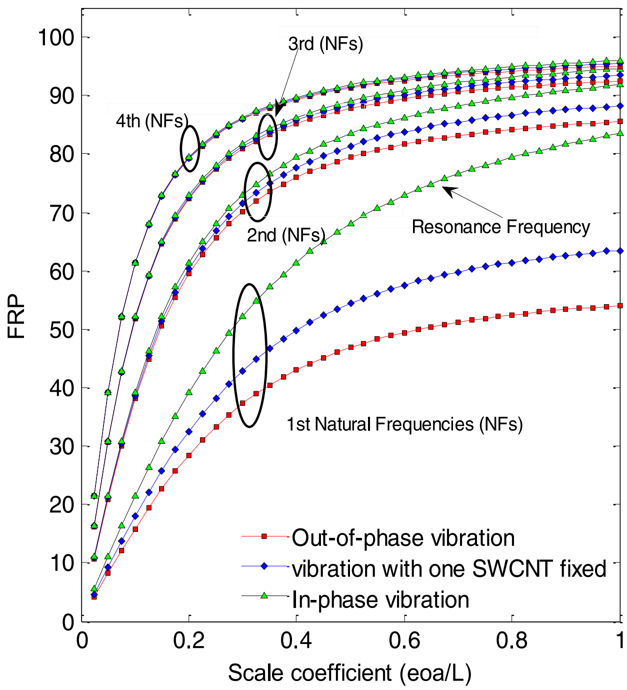

Fig. 4. Effect of scale coefficient ($\mu = e_0a/L$) on frequency reduction percent (FRP) for higher natural frequencies of coupled-SWCNT-systems.

reduces the in-phase natural frequencies (case 3) (thus higher FRP) compared to other cases considered. The relative higher FRP in case-3 (Eq. (51)) is due to the absence of coupling effect of the springs and the two SWCNT. Further we see that the values of the FRP for the case-2 are larger than the values of the FRP for the case-1. For Case-2 the coupled carbon nanotubes system becomes similar to the vibration characteristics of the single SWCNT with the effect of elastic medium.

8.2.2. Effect of stiffness of coupling springs

To illustrate the influence of stiffness of the springs on the natural frequency of the coupled-carbon-nanotube-systems curves have been plotted for FRP against the scale coefficient. Fig. 3(a–f) depicts the effect of stiffness of the springs on the natural frequencies of coupled systems. Different values of stiffness parameter of the coupling springs are considered. Ratio K is considered in range 1–50. The initial pre-stress load, F is constant and assumed as 5. As the spring stiffness K of the coupling springs increases the FRP decreases. Considering all values of K and comparing the three cases of coupled-carbon nanobeam system, it is noticed that the FRP for case 3 (in-phase vibration) is larger than the FRP for case 1 (out of phase vibration) and case 2 (one-SWCNT fixed). These different changes of FRP with increasing scale coefficient for the three different cases are more amplified as the stiffness parameter (K/F) of the spring's increases. For case 1(out-of-phase vibration) and case 2 (one-SWCNT fixed) the FRP reduces with increasing values of K. This observation implies that case 1(out-of-phase vibration) and case 2 (one-SWCNT fixed) are less affected by scale effects. Comparing between case 1 and case 2, it is seen the FRP is lesser for out-of-phase vibration than for vibration in case 2. Thus the out-of-phase vibration phenomenon is less affected by small-scale or nonlocal effects. This phenomenon in out-of-phase vibration can be attributed to the fact that the coupling springs in the vibrating system dampens the nonlocal effects.

In-phase vibration of coupled system is unchangeable with increasing stiffness of springs. This is accounted due to the in-phase behaviour of vibration. For in-phase type of vibration the coupled system behaves as if a single SWCNT without the effect of internal elastic medium. In other words the whole coupled system can be treated as a single nano element and the coupling internal structure is effectless.

As discussed in the previous sub Section that at certain value of scale coefficient μ the FRP reaches 100%; and the vibration system has reached the buckling state of mode (Lu, 2007). By increasing stiffness parameter of the springs K, the 'saturation' of FRP (100%) happens at higher scale coefficient. This can be observed in the Fig. 3(a–f); that with increasing K the curves shifts towards the right (Fig. 3). This is significantly prominent in out-of phase vibration phenomenon; and in vibration with one SWCNT fixed (the former is highly affected). Thus, coupling medium (springs, elastic medium, and forces due to nano-optomechanical effects) in the initially pre-stressed coupled vibration system plays an important role from being buckled easily. For designing of coupled-nanobeam-systems the interrelation of coupling medium, nonlocal effects and the initial pre-stress load thus becomes important. The in-phase type of vibration here is unaffected by increasing K because of its independence of coupling medium as discussed earlier.

8.2.3. Analysis of higher natural frequencies

To see the effect of higher natural frequencies of coupled carbon nanotubes systems, curves have been plotted for higher natural frequencies. Fig. 4 shows the effect of scale coefficient (μ) on frequency reduction percent (FRP) for higher natural frequencies of coupled-SWCNT-systems. We have considered exclusive natural frequencies. The sequence of higher modes is based accordingly when nonlocal elasticity is negligible. Three cases of vibration response are considered (a) out-of-phase vibrations (b) one SWCNT fixed and (c) in-phase vibration. The coupling springs is chosen such that the stiffness parameter, $K = 30$. The initial pre-stress condition is neglected ($F = 0$). From Fig. 4 we see that with increase of higher natural frequencies the FRP for all case of vibration response increases. This implies that the higher natural frequencies of the coupled system are significantly reduced due to the nonlocal effects. Further we see that the small-scale effects are more pronounced for higher natural frequencies as steeper curves are found for higher natural frequencies. Further, it is also noticed that the difference between the in-phase type vibration, out-of-phase type vibration and vibration with one SWCNT fixed become less for higher modes of buckling loads. Thus it can be concluded that although the small-scale effects are more in prominent higher natural frequencies, the effect of stiffness of coupling springs reduces the nonlocal effects. Here it should be noted that the resonance frequency is the natural frequency in first in-phase type vibration.

8.2.4. Effect of initial compressive pre-stress load on coupled system

To illustrate the influence of initial compressive pre-stress load on the vibration response of the coupled-carbon-nanotube-systems, curves have been plotted for FRP against the scale coefficient. Fig. 5(a–f) depicts the effect of compressive pre-stress load F on the natural frequencies of coupled systems. Constant stiffness of the springs is considered, i.e. as $K = 5$. Different values of compressive pre-stress load F are considered. Preload F is assumed to be in range 1–6. As the load F increases, the frequency decreases and the vibrating system is 'saturated' at lesser values of scale coefficient.

At the saturation state, the vibrating system reaches the buckling state of mode (FRP is 100%). By increasing F, the curves shift to the left (lower scale coefficient). This is unlike the effect of coupling springs, where by increasing the stiffness parameter the curves

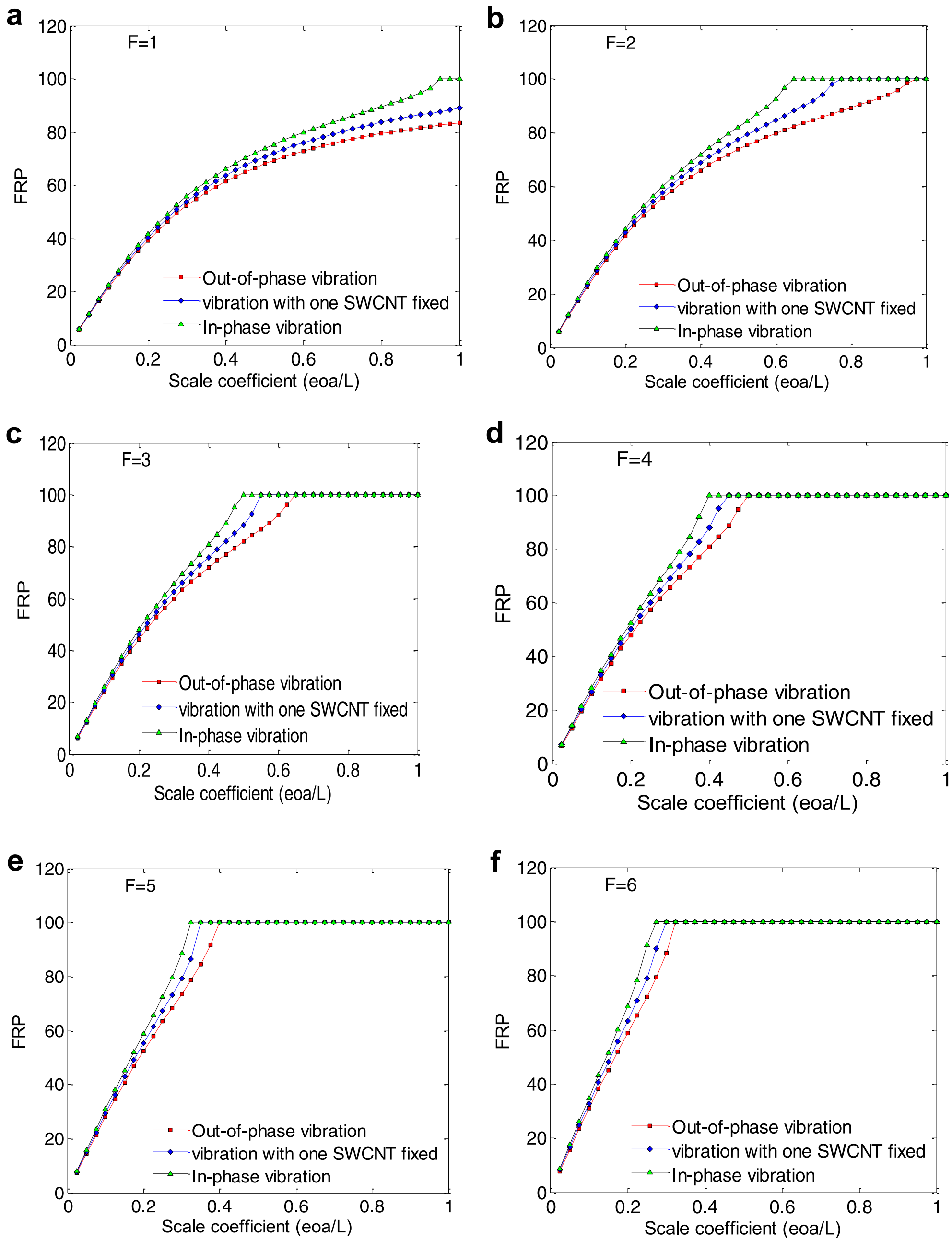

Fig. 5. Variation of frequency reduction percent (FRP) against scale coefficient ($\mu = e_0a/L$) for different initial pre-stress load, (a) $F = 1$; (b) $F = 2$; (c) $F = 3$; (d) $F = 4$; (e) $F = 5$; (f) $F = 6$.

shift to the right. All the three types of vibration phenomenon (out-of-phase vibration, vibration with one SWCNT fixed and in-phase vibration) are equally affected by the increasing pre-load.

8.2.5. Analysis of higher natural frequencies for initial pre-stress load

To see the effect of higher natural frequencies of initially pre-stressed coupled carbon nanotubes systems, curves have been plotted for higher natural frequencies. Fig. 6 completely illustrates the effect of scale coefficient (μ) on frequency reduction percent (FRP) for higher natural frequencies of coupled-SWCNT-systems. The coupling springs is chosen such that the stiffness parameter, $K = 30$. The initial pre-stress condition is assumed constant and taken as $F = 5$. From Fig. 6 we see that with the increase of higher buckling loads, the FRP for all cases of vibration response increases. We see that the small-scale effects are more pronounced for higher natural frequencies as steeper curves are found for higher natural frequencies. The difference between the in-phase type vibration, out-of-phase type vibration and vibration with one SWCNT fixed become less for higher modes of vibration. With the application of initial pre-stress, the buckled state is reached at certain scale coefficient values. With increasing natural frequencies the buckling state is reached at closer range of small-scale coefficient (Fig. 6). This is also true for in-phase, out-of-phase and vibration with one SWCNT fixed. Here it should be noted that the resonance frequency is the natural frequency in first in-phase type vibration. The buckling state reached at first in-phase type vibration is referred as critical buckling state (Fig. 6). The critical buckling state in vibration of NDNBS is reached at approximately $\mu = 0.33$.

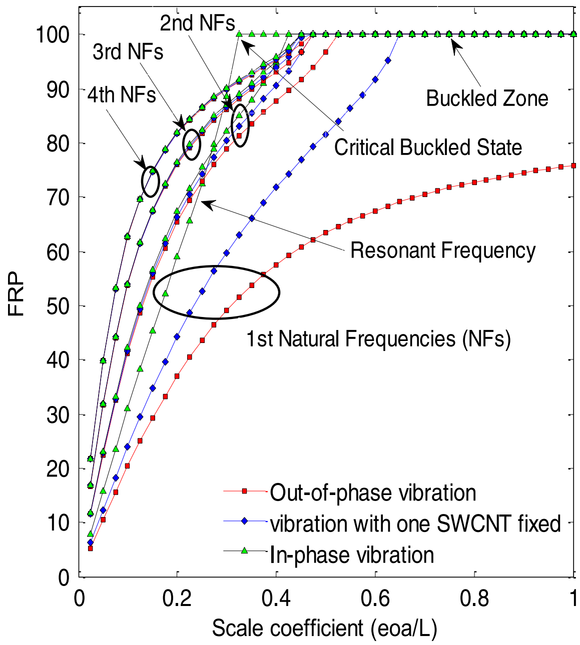

Fig. 6. Effect of scale coefficient ($\mu = e_0a/L$) on frequency reduction percent (FRP) for higher natural frequencies of initially pre-stressed coupled-SWCNT-systems.

In the end, we say that this present work can be extended to triple-nanobeam systems (TNBS). The work on vibration of TNBS is underway. Further different boundary conditions at the ends will result in different vibration and buckling behaviour which will be interesting to study. This would find practical application in the design of nanosensors, nanoresonators and also for devising better NOMS structures. The present work could also be useful in the study of double-nanoplate system for future NOMS studies. In summary, the present work illustrated here provides an analytical solution that serves as a benchmark for further investigation of more complex *n*-nanobeam systems using scale-based nonlocal elastic theory.

9. Conclusions

In this paper a theoretical scale-based nonlocal elasticity is considered for the stability and free bending-vibration of a pre-stressed double-nanobeam-system (NDNBS). An exact analytical method is developed for determining the nonlocal frequencies of transversely vibrating NDNBS. Study of NDNBS is applied to a twin single-walled carbon nanotube system coupled by elastic medium. The study shows that nonlocal effects are important in the transverse vibration. Nonlocal effects reduce the frequencies of the NDNBS(coupled-nanotube-system). Increasing the stiffness of the springs in coupled nanosystems reduces the nonlocal effects. The small-scale effects in coupled nano systems are more prominent with the increasing nonlocal parameter in the in-phase vibration than in the out-of-phase motion condition. For pre-stressed double nanobeam system (couple nanotubes) the natural frequencies is reduced with increasing pre-stressed load. At a certain value of the scale coefficient the frequency is reduced to zero and the double nanobeam system considered to be buckled. The increasing scale effects contribute to the phenomenon of buckled state for one carbon nanotubes. Finally, this present study gives physical insights which may be useful for the design and vibration analysis of nano-optomechanical systems (NOMS), nanocomposites and sensor applications.

Acknowledgements

The authors greatly acknowledge the funding from the Leverhulme Trust via the Philip Leverhulme Prize.

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