Nonlocal transverse vibration of double-nanobeam-systems

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Vibration analysis of double-nanobeam-systems is considered. Double-nanobeam-systems are important in nano-optomechanical systems and sensor applications. Expressions for free bending-vibration of double-nanobeam-system are established within the framework of Eringen's nonlocal elasticity theory. An analytical method is developed for determining the natural frequencies of the nonlocal double-nanobeam-system. Explicit closed-form expressions for natural frequencies are derived for the case when all four ends are simply-supported. The study highlights that the small-scale effects considerably influence the transverse vibration of double-nanobeam-systems. The nonlocal natural frequencies of double-nanobeam-system are smaller when compared to the corresponding local frequency values. The small-scale effects in the vibrating system are higher with increasing values of nonlocal parameter for the case of in-phase modes of vibration than in the out-of-phase modes of vibration. The increase in the stiffness of the coupling springs in double-nanobeam-system reduces the nonlocal effects during the out-of-phase modes of vibration.
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I. INTRODUCTION

Recently nanomaterials have intensely stimulated the interest of the scientific researcher's communities in physics, chemistry, and engineering. These nanomaterials have special properties resulting from their nanoscale dimensions. Common examples of materials that exhibit interesting properties on the nanoscale are nanoparticles, nanowires, and nanotubes (viz., carbon nanotubes, ZnO nanotubes), etc. These nanomaterials have promising mechanical (tensile strength), chemical, electrical, optical, and electronic properties.[1–3] Because of many desirable properties, these nanomaterials are perceived to be the components for various nanoelectromechanical systems and nanocomposites.

Nanomaterials are the basis material of many nanoscale objects. Nanoscale objects are referred to as nanostructures. For realistic designing of the nanostructures [viz., nanoresonantors,[4] nanoactuators,[5] nanomachines,[6] and nano-optomechanical systems (NOMSs) (Refs. 7 and 8] one must incorporate the small-scale effects and the atomic forces (e.g., van der Waals forces) to achieve solutions with acceptable accuracy. At the small-scale, the sizes of nanostructures often become prominent.[9,10] Both experimental and atomistic simulation results have shown a significant "size-effect" in the mechanical properties when the dimensions of these structures become small. Ignoring the small-scale effects and the atomic forces in sensitive nanodesigning fields may cause completely incorrect solutions and hence erroneous designs. Atomistic methods such as molecular mechanics simulation[11–15] are able to capture the small-scale effects and atomic forces. However these approaches are computationally prohibitive for nanostructures with large numbers of atoms. Thus analyses have been generally carried out by using the classical mechanics.

Extensive research over the past decade has shown that classical continuum models[16] are able to predict the performance of "large" nanostructures reasonably well. Classical continuum models are scale-free theory and it lacks the accountability of the effects arising from the small-scale. The application of classical continuum models may be questionable in the analysis of "smaller" nanostructures. Therefore, recently there have been research efforts to bring in the scale effects within the formulation by amending the traditional classical continuum mechanics. One widely used size-dependant theory is the nonlocal elasticity theory pioneered by Eringen.[17]

Nonlocal elasticity accounts for the small-scale effects arising at the nanoscale level. Recent literature[18–32] shows that the theory of nonlocal elasticity is being increasingly used for reliable and quick analysis of nanostructures, viz., nanobeams, nanoplates, nanorings, carbon nanotubes, graphenes, nanoswitches, and microtubules.

Extensive studies of nonlocal beams using nonlocal elasticity theory can be found in literature. Reddy[33] proposed elaborate expressions of nonlocal beam theories for bending, vibration, and buckling phenomenon. Wang *et al.*[34] presented a beam bending solutions based on nonlocal Timoshenko beam theory. A formulation of third-order nonlocal beam theories based on Eringen's nonlocal continuum theory and the Reddy and Leung higher-order beam models for bending, buckling, and vibration of nanobeams was reported by Niu *et al.*[35] Lim and Wang[36] introduced exact variational nonlocal stress modeling with asymptotic higher-order strain gradients for nanobeams. These nonlocal beam theories are generally utilized for analysis of carbon nanotubes.[20,21,37–41]

In the nonlocal elasticity theory the small-scale effects are captured by assuming that the stress at a point is a function of the strains at all points in the domain.[17] Nonlocal theory considers long-range interatomic interaction and yields results dependent on the size of a body. Some drawbacks of the classical continuum theory could be efficiently

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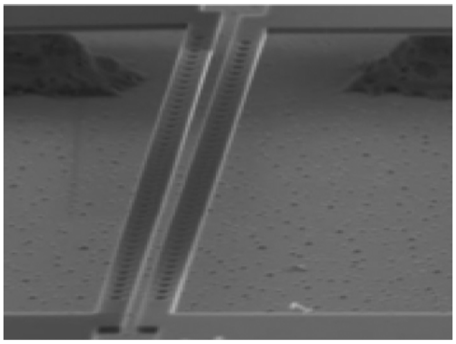

FIG. 1. Scanning electron microscope image of a double-nanobeam-system (Ref. 7).

avoided and the size-dependent phenomena can be reasonably explained by the nonlocal elasticity theory. The majority of the existing works on nonlocal elasticity are pertaining to the free transverse vibration of single nanobeams. Though the mechanical studies of nanobeams may include vibration of multiple-walled nanotubes, the study of discrete multiple-nanobeam-system is particularly limited in literature. However, for the scale-free (classical) double-beam-systems, some numbers of work on the vibration analysis have been reported.[42–45]

The study of double-nanobeam-system is extremely important in the area of NOMSs. The employment of double-nanobeam-systems has been reported by various researchers. Frank *et al.*[7] presented a dynamically reconfigurable photonic crystal nanobeam cavity. There work involved two closely situated parallel vibrating clamped double-nanobeam-systems (Fig. 1).

Eichenfield *et al.*[8] described the design, fabrication, and measurement of a cavity NOMS. The NOMS is consisting of two closely separated coupled nanobeams. The researchers[8] fabricated the low dimension double-nanobeam-system by depositing stoichiometric silicon nitride using low-pressure-chemical-vapor-deposition on a silicon wafer. Deotare *et al.*[46] studied the coupled photonic crystal nanobeam cavities consisting of two parallel suspended nanobeams separated by a small gap. The use of vibration properties in double-nanobeam-system has also been reported by Lin *et al.*[47] The authors studied the coherent mixing of mechanical excitations in nano-optomechanical structures. Chan *et al.*[48] used double-nanobeam-system to perform optical and mechanical design of a zipper photonic crystal optomechanical cavity. Most of the works reported here are experimental and computational works.

Therefore, based on the above discussion there is a strong encouragement to gain an understanding of the entire subject of vibration of complex-nanobeam-system and the mathematical modeling of such phenomena. In this paper an investigation is carried out to understand the small-scale effects in the free bending-vibration of a nonlocal double-nanobeam-system (NDNBS). Equations for free bending-vibration of a double-nanobeam-system (NDNBS) are formulated within the framework of Eringen's nonlocal elasticity.[17] The two nanobeams are assumed to be attached by distributed vertical transverse springs. These springs may represent the stiffness of an enclosed elastic medium[19] or van der Waals forces or forces due to optomechanical coupling[47,48] between the two nanobeams. An exact analytical method is shown for solving the nonlocal frequencies of transversely vibrating NDNBS. The simplification in the computation is achieved based on the change in variables to decouple the set of two fourth-order partial differential equations. It is assumed that the two nanobeams in the NDNBS are identical, and the boundary conditions on the same side of the system are the same. Simply-supported boundary conditions are employed in this study. Explicit expressions for the natural frequencies of NDNBS are derived. Results are obtained for various vibration-phase of the NDNBS. The vibration phases include in-phase and out-of-phase modes of vibration. The effects of (i) nonlocal parameter, (ii) stiffness of the springs, and (iii) the higher modes, on the resonance frequency of the NDNBS are discussed.

II. REVIEW OF NONLOCAL ELASTICITY

According to the nonlocal elasticity, the basic equations for an isotropic linear homogenous nonlocal elastic body neglecting the body force are given as[17]

$$\sigma_{ij,j}=0,$$

$$\sigma_{ij}(\boldsymbol{x})=\int_V \phi(|\boldsymbol{x}-\boldsymbol{x}'|,\alpha)t_{ij}dV(\boldsymbol{x}'),\qquad \forall\, \boldsymbol{x}\in \boldsymbol{V},$$

$$t_{ij}=H_{ijkl}\varepsilon_{kl},$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \tag{1}$$

The terms σ_{ij}, t_{ij}, ε_{kl}, H_{ijkl} are the nonlocal stress, classical stress, classical strain, and fourth order elasticity tensors, respectively. The volume integral is over the region $\boldsymbol{V}$ occupied by the body. The above equation [Eq. (1)] couples the stress due to nonlocal elasticity and the stress due to classical elasticity. The kernel function $\phi(|\boldsymbol{x}-\boldsymbol{x}'|,\alpha)$ is the nonlocal modulus. The nonlocal modulus acts as an attenuation function incorporating into constitutive equations the nonlocal effects at the reference point $\boldsymbol{x}$ produced by local strain at the source $\boldsymbol{x}'$. The term $|\boldsymbol{x}-\boldsymbol{x}'|$ represents the distance in the Euclidean form and α is a material constant that depends on the internal (e.g., lattice parameter, granular size, distance between the C–C bonds) and external characteristics lengths (e.g., crack length, wavelength). Material constant α is defined as $\alpha=e_0a/\ell$. Here e_0 is a constant for calibrating the model with experimental results and other validated models.[17] The parameter e_0 is estimated such that the relations of the nonlocal elasticity model could provide satisfactory approximation to the atomic dispersion curves of the plane waves with those obtained from the atomistic lattice dynamics. The terms α and ℓ are the internal and external characteristics of the nanostructure.

Equation (1) is in partial-integral form and generally difficult to solve analytically. Thus a differential form of nonlocal elasticity equation is often used.[18–32] According to Eringen,[17] the expression of nonlocal modulus can be given as

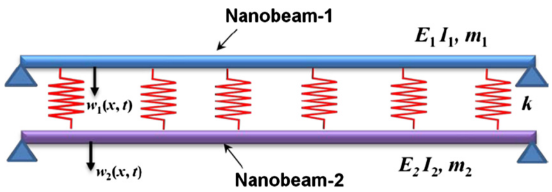

FIG. 2. (Color online) The configuration of a simply-supported double-nanobeam-system (NDNBS).

$$\phi(|\boldsymbol{x}|,\alpha) = (2\pi\ell^2\alpha^2)^{-1}K_0(\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}/\ell\alpha), \tag{2}$$

where K_0 is the modified Bessel function.

The equation of motion in terms of nonlocal elasticity can be expressed as[17]

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i, \tag{3}$$

where f_i, ρ, and u_i are the components of the body forces, mass density, and the displacement vector, respectively. The terms i, j take up the symbols x, y, and z.

Assuming the kernel function ϕ as the Green function, Eringen[17] proposed a differential form of the nonlocal constitutive relation as

$$\sigma_{ij,j} + \mathcal{L}(f_i - \rho\ddot{u}_i) = 0, \tag{4}$$

where

$$\mathcal{L} = [1 - (e_0 a)^2 \nabla^2], \tag{5}$$

and ∇^2 is the Laplace operator.

Using Eq. (5) the nonlocal constitutive stress-strain relation can be simplified as

$$(1 - \alpha^2\ell^2\nabla^2)\sigma_{ij} = t_{ij}. \tag{6}$$

Employing the nonlocal constitutive stress-strain relation, the one-dimensional equation of motion of a nonlocal Euler–Bernoulli beam can be written as[23,33]

$$EIw''''(x,t) - q(x) + (e_0a)^2 q''(x) + m\ddot{w}(x,t) - (e_0a)^2 m\ddot{w}''(x,t) = 0, \tag{7}$$

where w denotes the deflection of the beam. The terms E, I, and m are the Young's modulus, second moment of inertia, and mass of the nonlocal beam, respectively. Term q is the distributed transverse load on the nonlocal beam.

III. NDNBS

Consider the NDNBS as shown in Fig. 2. The two nanobeams of the NDNBS are referred to as nanobeam-1 and nanobeam-2. The nanobeams are considered to be of length L. Vertically distributed springs couples the two nanobeams. The springs may be used to substitute the electrostatic force,[7] elastic medium,[19] van der Waals forces or forces due to optomechanical coupling[47,48] between the two nanobeams. The springs have a stiffness k. In Ref. 7, it is reported that by applying a potential difference directly across the nanobeams, an attractive electrostatic force can be induced between the two nanobeams. Thereby the equivalent spring stiffness can be varied between the nanobeams.

Generally, the two nanobeams are different where the length, mass per unit length and bending rigidity of the ith beam are L_i, m_i, and E_iI_i $(i=1,2)$, respectively. These parameters are assumed to be constant along each nanobeam. The bending displacements over the two nanobeams are denoted by $w_1(x,t)$ and $w_2(x,t)$, respectively (Fig. 2).

Using Eq. (7) the governing nonlocal equations for NDNBS can be written as

Nanobeam-1:

$$E_1I_1w_1''''(x,t) + k[w_1(x,t) - w_2(x,t)] - (e_0a)^2 k[w_1''(x,t) - w_2''(x,t)] + m_1\ddot{w}_1(x,t) - (e_0a)^2 m_1\ddot{w}_1''(x,t) = 0. \tag{8}$$

Nanobeam-2:

$$E_2I_2w_2''''(x,t) - k[w_1(x,t) - w_2(x,t)] + (e_0a)^2 k[w_1''(x,t) - w_2''(x,t)] + m_2\ddot{w}_2(x,t) - (e_0a)^2 m_2\ddot{w}_2''(x,t) = 0, \tag{9}$$

where dots $(\cdot)$ and primes $(\,')$ denote partial derivatives with respect to time t and position coordinate x, respectively.

We assume that

$$E_1I_1 = (E_2I_2) = EI \equiv \text{constant}, \tag{10}$$

$$m_1 = m_2 = m \equiv \text{constant}. \tag{11}$$

Considering Eqs. (8) and (9) and using the assumptions from Eqs. (10) and (11) we get

Nanobeam-1:

$$EIw_1''''(x,t) + k[w_1(x,t) - w_2(x,t)] - (e_0a)^2 k[w_1''(x,t) - w_2''(x,t)] + m\ddot{w}_1(x,t) - (e_0a)^2 m\ddot{w}_1''(x,t) = 0, \tag{12}$$

Nanobeam-2:

$$EIw_2''''(x,t) - k[w_1(x,t) - w_2(x,t)] + (e_0a)^2 k[w_1''(x,t) - w_2''(x,t)] + m\ddot{w}_2(x,t) - (e_0a)^2 m\ddot{w}_2''(x,t) = 0. \tag{13}$$

For the NDNBS we employ a change in variables by considering $w(x,t)$ as the relative displacement of the nanobeam-1 with respect to the nanobeam-2:

$$w(x,t) = w_1(x,t) - w_2(x,t), \tag{14}$$

such that

$$w_1(x,t) = w(x,t) + w_2(x,t). \tag{15}$$

Subtracting Eq. (13) from Eq. (12) gives

$$EI[w_1''''(x,t) - w_2''''(x,t)] + 2k[w_1(x,t) - w_2(x,t)] - 2(e_0a)^2 k[w_1''(x,t) - w_2''(x,t)] + m[\ddot{w}_1(x,t) - \ddot{w}_2(x,t)] - (e_0a)^2 m[\ddot{w}_1''(x,t) - \ddot{w}_2''(x,t)] = 0. \tag{16}$$

By introducing Eqs. (14) and (15) and using Eq. (16) we get

$$EIw''''(x,t) + 2kw(x,t) - 2(e_0a)^2 kw''(x,t) + m\ddot{w}(x,t) - (e_0a)^2 m\ddot{w}''(x,t) = 0, \tag{17}$$

and

$$EIw_2''''(x,t) + m\ddot{w}_2(x,t) - (e_0a)^2 m\ddot{w}_2''(x,t) = kw(x,t) - (e_0a)^2 kw''(x,t), \tag{18}$$

It should be noted that when the nonlocal effects are ignored ($e_0a=0$) and a single nanobeam is considered, the above equations revert to the equations of classical Euler–Bernoulli beam theory. For the present analysis of coupled NDNBS, we see the simplicity in using Eq. (17).

IV. VIBRATION OF NDNBS

Assuming that the relative motion $w(x,t)$ is one of its natural modes of vibration, the general solution of Eq. (17) is written as

$$w(x,t) = W(x)e^{i\omega t}, \tag{19}$$

Here ω is the frequency, $W(x)$ is the corresponding deformation shape of the NDNBS, and i is the conventional imaginary number $\sqrt{-1}$. Substituting the Eq. (19) into Eq. (17) yields

$$A_1 W''''(x) + A_2 W''(x) - A_3 W(x) = 0, \tag{20}$$

where the coefficients are

$$A_1 = EI; \quad A_2 = m\omega^2(e_0a)^2 - 2k(e_0a)^2; \quad A_3 = m\omega^2 - 2k. \tag{21}$$

The general solution of Eq. (20) can be given as

$$W(x) = C_1 \sin \alpha x + C_2 \cos \alpha x + C_3 \sinh \beta x + C_4 \cosh \beta x, \tag{22}$$

where

$$\alpha^2 = \frac{1}{2A_1}\left(A_2 + \sqrt{A_2^2 + 4A_1A_3}\right), \tag{23}$$

$$\beta^2 = \frac{1}{2A_1}\left(A_2 - \sqrt{A_2^2 + 4A_1A_3}\right). \tag{24}$$

The terms C_1, C_2, C_3, and C_4 are to be determined from the boundary conditions.

V. BOUNDARY CONDITIONS IN NDNBS

Now we present the mathematical expressions of the boundary conditions in NDNBS. The boundary conditions of simply-supported conditions are described here. At each ends of the nanobeams in NDNBS the displacement and the nonlocal moments are considered to be zero. They can be mathematically expressed as

Nanobeam-1:

$$w_1(0,t) = 0; \tag{25a}$$

$$M_1(0,t) = -EIw_1''(0,t) + (e_0a)^2 m\ddot{w}_1(0,t) + (e_0a)^2 k[w_1(0,t) - w_2(0,t)] = 0, \tag{25b}$$

and

$$w_1(L,t) = 0; \tag{26a}$$

$$M_1(L,t) = -EIw_1''(L,t) + (e_0a)^2 m\ddot{w}_1(L,t) + (e_0a)^2 k[w_1(L,t) - w_2(L,t)] = 0, \tag{26b}$$

Nanobeam-2:

$$w_2(0,t) = 0; \tag{27a}$$

$$M_2(0,t) = -EIw_2''(0,t) + (e_0a)^2 m\ddot{w}_1 - (e_0a)^2 k[w_1(0,t) - w_2(0,t)] = 0, \tag{27b}$$

and

$$w_2(L,t) = 0; \tag{28a}$$

$$M_2(L,t) = -EIw_2''(L,t) + (e_0a)^2 m\ddot{w}_1 - (e_0a)^2 k[w_1(L,t) - w_2(L,t)] = 0. \tag{28b}$$

Using Eq. (14) the boundary condition simplifies to

$$w(0,t) = w_1(0,t) - w_2(0,t) = 0; \tag{29}$$

$$M_1(0,t) - M_2(0,t) = -EIw_1''(0,t) + (e_0a)^2[m\ddot{w}_1(0,t) + k[w_1(0,t) - w_2(0,t)]] + EIw_2''(0,t) - (e_0a)^2[m\ddot{w}_1 - k[w_1(0,t) - w_2(0,t)]] = 0; \tag{30}$$

$$w(L,t) = w_1(L,t) - w_2(L,t) = 0; \tag{31}$$

$$M_1(L,t) - M_2(L,t) = = -EIw_1''(L,t) + (e_0a)^2\{m\ddot{w}_1(L,t) + k[w_1(L,t) - w_2(L,t)]\} + EIw_2''(L,t) - (e_0a)^2\{m\ddot{w}_1 - k[w_1(L,t) - w_2(L,t)]\} = 0. \tag{32}$$

By the use of Eq. (19) and the use of Eqs. (25a), (25b), (26a), (26b), (27a), (27b), (28a), (28b), and (29)–(32) the boundary conditions effectively reduces to

$$W(0) = 0 \text{ and } W''(0) = 0, \quad W(L) = 0 \text{ and } W''(L) = 0, \tag{33}$$

The above boundary conditions therefore simplifies to that of the classical single nanobeam. Here it should be noted that the local and nonlocal boundary conditions are effectively the same.

Similarly the boundary conditions of all ends clamped can be presented as

For nanobeam-1:

$$w_1(0,t) = 0; \quad w_1'(0,t) = 0, \tag{34}$$

$$w_1(L,t) = 0; \quad w_1'(L,t) = 0. \tag{35}$$

For nanobeam-2:

$$w_2(0,t) = 0; \quad w_2'(0,t) = 0, \tag{36}$$

$$w_2(L,t) = 0; \quad w_2'(L,t) = 0. \tag{37}$$

Using Eq. (14) the boundary condition of the clamped case becomes

$$w(0,t) = w_1(0,t) - w_2(0,t) = 0, \tag{38}$$

$$w'(0,t) = w_1'(0,t) - w_2'(0,t) = 0, \tag{39}$$

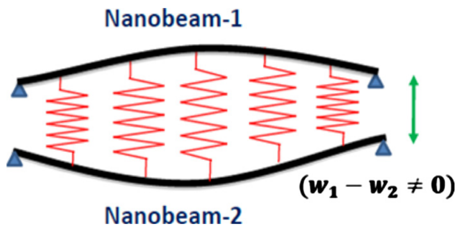

FIG. 3. (Color online) Out-of-phase vibration of the double-nanobeam-system $(w_1-w_2\neq 0)$.

$$w(L,t)=w_1(L,t)-w_2(L,t)=0, \tag{40}$$

$$w'(L,t)=w_1'(L,t)-w_2'(L,t)=0, \tag{41}$$

From Eq. (19) and the use of Eqs. (34)–(41) the effective boundary conditions reduces to

$$W(0)=0 \text{ and } W'(0)=0, \quad W(L)=0 \text{ and } W'(L)=0, \tag{42}$$

Similar procedures can be applied for other mixed boundary conditions such as the clamped free case.

VI. EXACT SOLUTIONS OF THE FREQUENCY EQUATIONS

A. Both nanobeams of NDNBS are vibrating out-of-phase; $(w_1-w_2\neq 0)$

Consider the case of the NDBNS when the both the nanobeams are vibrating with in-phase sequence and out-of-phase sequence. The configuration of the NDNBS with out-of-phase sequence of vibration $(w_1-w_2\neq 0)$ is shown in Fig. 3. In this section we solve the frequency for the out-of-phase vibration. Here we consider the case when all the ends have simply-supported boundary conditions (Fig. 2). For simply-supported case, the use of the boundary conditions Eq. (33) yields

$$C_2=0, \quad C_4=0, \tag{43}$$

and

$$\begin{bmatrix} \sin\alpha L & \sinh\beta L \\ EI\alpha^2\sin\alpha L & EI\beta^2\sinh\beta L \end{bmatrix}\begin{Bmatrix} C_1 \\ C_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{44}$$

For the nontrivial solution of Eq. (44), the determinant is zero and that yields

$$\sin\alpha L(EI\beta^2\sinh\beta L-EI\alpha^2\sinh\beta L)=0. \tag{45}$$

Therefore, the frequency equation is

$$\sin\alpha L=0, \tag{46}$$

which implies

$$\alpha L=n\pi, \quad n=1,2\ldots. \tag{47}$$

From Eq. (23)

$$2A_1\alpha^2=A_2+\sqrt{A_2^2+4A_1A_3}, \tag{48}$$

which yields,

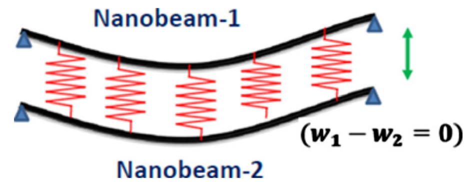

FIG. 4. (Color online) In-phase vibration of the double-nanobeam system $(w_1-w_2=0)$.

$$A_1\alpha^4-A_2\alpha^2-A_3=0. \tag{49}$$

Here we define the frequency, stiffness and nonlocal parameter as

$$\Omega=\omega L^2\sqrt{\frac{m}{EI}}, \quad K=\frac{kL^4}{EI}, \quad \mu=\frac{e_0a}{L}. \tag{50}$$

Using Eqs. (21) and (50), the natural frequency of NDNBS is

$$\Omega_n=\sqrt{\frac{(n\pi)^4+2K+2K(\mu)^2(n\pi)^2}{1+(\mu)^2(n\pi)^2}}, \quad n=1,2\ldots. \tag{51}$$

Similarly for the case of all ends clamped boundary condition, the roots of the following transcendental equation along with Eq. (23) gives the natural frequencies of the NDNBS.

$$-2+2\cos\alpha L\cos\beta L+\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\sin\alpha L\sin\beta L=0. \tag{52}$$

B. Both nanobeams of NDNBS are vibrating in-phase; $(w_1-w_2=0)$

Next, the in-phase sequence of vibration will be considered (Fig. 4). For the present NDNBS, the relative displacements between the two nanobeams are absent $(w_1-w_2=0)$. Here we solve the Eq. (18) for the vibration of NDNBS. The vibration of nanobeam-2 would represent the vibration of the coupled vibrating system. We apply the same procedure for solving Eq. (18). Using Eqs. (43)–(50) we can obtain the natural frequencies. The natural frequencies for the NDNBS in this case can be expressed as

$$\Omega_n=\sqrt{\frac{(n\pi)^4}{1+(\mu)^2(n\pi)^2}}, \quad n=1,2\ldots \tag{53}$$

For this case the vibration of NDNBS is independent of the stiffness of the connecting springs and therefore the NDNBS can be effectively treated as a single nanobeam.

C. One nanobeam of NDNBS is stationary; $(w_2=0)$

Consider the case of NDNBS when one of the two nanobeams (viz., nanobeam-2) is stationary $(w_2=0)$. The schematic diagram is shown in Fig. 5.

Using the equations from nonlocal elasticity [Eqs. (1)–(6)], the governing equation for the NDNBS [Eq. (17)] in this case reduces to

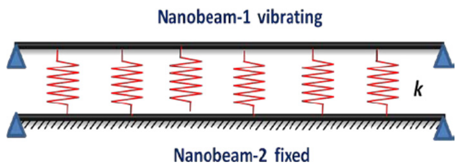

FIG. 5. (Color online) Vibration of NDNBS with one nanobeam stationary.

$$EIw''''(x,t) + kw - (e_0a)^2kw''(x,t) + m\ddot{w}(x,t) - (e_0a)^2m\ddot{w}''(x,t) = 0, \quad (54)$$

and the equivalent nonlocal boundary conditions are

$$w(0) = 0 \text{ and } w''(0) = 0, \quad w(L) = 0 \text{ and } w''(L) = 0, \quad (55)$$

In this case the NDNBS behaves as if the nanobeam is embedded in an elastic medium. The elastic medium can be modeled as Winkler elastic foundation.[19] The stiffness of the elastic medium is denoted by k. By following the same procedure as solution of Eq. (17), the nonlocal frequency of NDNBS can be explicitly expressed as

$$\Omega_r = \sqrt{\frac{(r\pi)^4 + K + K(\mu)^2(r\pi)^2}{1 + (\mu)^2(r\pi)^2}}, \quad r = 1,2\ldots, \quad (56)$$

where K is the stiffness parameter of the coupling springs and μ is nonlocal parameter as defined in Eq. (50). In fact when one of the nanobeam (viz., nanobeam-2) in NDNBS is fixed ($w_2=0$), the NDNBS behaves as a nanobeam on an elastic medium.

VII. RESULTS AND DISCUSSIONS

The nonlocal theory for NDNBS illustrated here is a generalized theory and can be applied for the bending-vibration analysis of coupled carbon nanotubes, double ZnO nanobeam systems, and double-nanobeam-systems for NOMS application.[7,8,46–48] The reliability of nonlocal elasticity theory in the analysis of nanostructures (nanotubes and graphene sheet) can be observed in various earlier works.[18–21,32,37–41]

For the present study, the properties of the nanobeams are considered that of a single-walled carbon nanotube (SWCNT).[49] An armchair SWCNT with chirality (5, 5) is considered (Fig. 6). The radius of each individual SWCNT is assumed as 0.34 nm. Young's modulus, E, is taken as 0.971 TPa.[49] The density is considered as 2300 kg/m^3. The length is taken as 20 nm. The frequency results of the NDNBS are presented in terms of the frequency parameters [Eq. (51)]. The nonlocal parameter and the stiffness of the springs are computed as given in Eq. (50). Different values of spring parameters, K, are considered. Spring stiffness represents the stiffness of the enclosing elastic medium. Both low and high stiffness of springs are assumed. Values of K range from 5 to 500. Both the nanotubes (nanotube-1 and nanotube-2) are assumed to have the same geometrical and material properties.

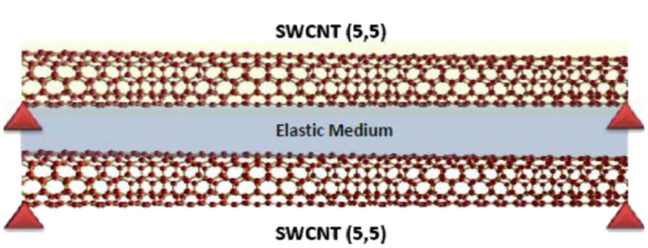

FIG. 6. (Color online) Coupled carbon nanotube system.

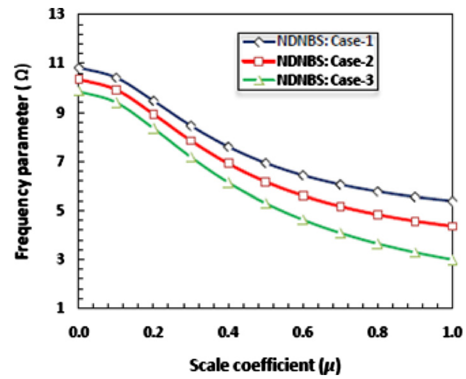

FIG. 7. (Color online) Variation in the frequency parameter (Ω) with the scale coefficient (μ) for different cases of NDNBS. Case-1: when both the coupled nanotubes vibrates in out-of-phase sequence ($w_1-w_2 \neq 0$); case-2: when one of the nanotubes in NDNBS is stationary ($w_2=0$); case-3; when both nanotubes vibrates with in-phase sequence ($w_1-w_2=0$), respectively.

In general the nonlocal parameters may be taken as $e_0=0.39$ (Ref. 17) and $a=0.142$ nm (distance between carbon-carbon atoms). For carbon nanotubes and graphenes, the range of $e_0a=0-2.0$ nm has been widely used.[20–22,27] In the present study we take the scale coefficient μ or nonlocal parameter in the similar range as $\mu=0-1$.[50]

A. Effect of small-scale on vibrating NDNBS

Figure 7 shows the variation in the frequency parameter with the scale coefficient for different cases of NDNBS. Three different cases of NDNBS are considered. Case-1, case-2 and case-3 depicts the conditions (i) when both the nanotubes vibrates in the out-of-phase sequence ($w_1-w_2 \neq 0$), (ii) when one of the nanotubes in NDNBS is stationary ($w_2=0$), (iii) when both the nanotubes vibrates with in-phase sequence ($w_1-w_2=0$), respectively. The results for the frequency parameter Ω are in the dimensionless form as in Eq. (50). The stiffness parameter of the springs is assumed to be constant ($K=10$). Unless stated the frequency parameter would denote the parameter associated with the first natural frequencies (in-phase and out-of-phase type vibration). The SWCNTs are referred here as nanobeams.

Figure 7 shows that as the values of the nonlocal parameter μ increases, the frequency parameter Ω decreases. The decreasing trend of the frequency parameter against increasing nonlocal parameter happens for all the three cases considered (viz., case-1, case-2 and case-3). This decrease in value of the frequency parameters is due to the assimilation of small-scale effects in the NDNBS. Therefore by the nonlocal elastic model the size effects are reflected in the NDNBS.

On comparison of the three cases of NDNBS (viz., case-1, case-2, case-3), the frequency parameter for the case-1 (NDNBS vibrating with out-of-phase sequence) is larger than the frequency parameter for case-2 and case-3. The relative higher frequency in case-1 [Eq. (51)] is due to the coupling effect of the spring and the two nanobeams (nanotubes). The presence of springs for case-1 makes the NDBNS stiffer and increases the stiffness of the system. For case-2 the stiffness effect due to the auxiliary nanobeam (nanobeam 2) is absent. Thereby there is effective lower stiffness parameter in case-2.

In addition it is seen that the values of the frequency parameter for the case-2 is larger than the values of the frequency parameter for the case-3. In case-3 the frequency parameter is relatively less because the NDBNS becomes independent of the effect of the spring stiffness. The NDNBS vibrates in in-phase sequence. For case-3 the NDNBS becomes similar to the frequency of the single nanobeam without the effect of elastic medium. In other words the whole NDNBS can be treated as a vibrating single nanobeam and the coupling internal structure is effectless (case-3). Further, it is important to note that the above mentioned frequency behavior for NDBNS (Fig. 7) is amplified for larger values of the scale coefficient values.

In general, it is worth noticing that the small-scale effects in NDNBS are higher with increasing nonlocal parameter in the in-phase vibration than in the out-of-phase vibration. This is because the stiffness of the springs in out-of-phase vibration reduces the nonlocal effects. For different set of boundary conditions this behavior may be different. As small-scale effects reduce the natural frequencies of the vibrating nanosystem, nanoresonators can be designed so as to vibrate in the out-of-phase modes. Thus the nanoresonators would have higher resonant frequencies. This implies the significance of the use of small-scale effects in NDNBS. If some damping properties are present within the NDNBS, then the damping behavior could be effective in out-of-phase mode vibration (similar to stiffening NDNBS). Thus one of the nanobeams would act as a vibration absorber. However the damping behavior would be ineffective in the in-phase mode vibration. This vibration behavior will be important for study in NOMS.

B. Effect of the stiffness of the coupling springs on NDNBS

To illustrate the effect of the smaller and higher values of the stiffness of the coupling springs, curves have been plotted for the frequency parameter against the stiffness parameter of springs. Figure 8 depicts the variation in the frequency parameter (Ω) with the stiffness parameter of the springs (K) for the three cases of NDBNS. The stiffness parameter of the coupling springs are taken as $K=0$, 20, 40, 60, 80 and 100. The nonlocal parameter μ is considered as 0.5. As the stiffness parameter of the coupling springs increases the frequency parameter increases. Difference of frequencies between case-1 and case-2 is amplified for higher values of stiffness parameter of springs. Case-3 of NDNBS is unchangeable with increasing stiffness of springs due to the in-phase behavior of vibration.

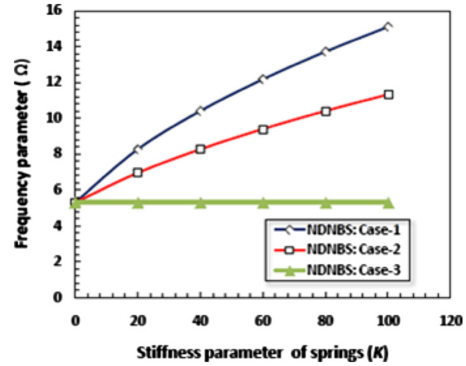

FIG. 8. (Color online) Variation in the frequency parameter (Ω) with the stiffness parameter (K) of the springs for different cases of NDNBS.

C. Analysis of higher modes of NDNBS

To see the effect of higher natural frequencies of NDNBS, curves have been plotted for frequency parameter against the first eight natural frequencies (Fig. 9). Curves have been illustrated for general NDNBS (both nanobeam vibrating) and single nanobeam embedded in an elastic medium. The nonlocal parameter μ is considered as 0.5. The first, second, third, and fourth pair of natural frequencies consists of in-phase and out-of-phase modes of NDNBS, respectively. In in-phase mode the two nanobeams vibrate in the same direction while in out-of-phase mode the beam vibrate in opposite direction. In in-phase modes of vibration the natural frequencies are independent of stiffness of springs. The corresponding pairs of in-phase and out-of-phase modes are also sometimes referred to as submodes. Here nonlocal parameter is assumed as $\mu=0.5$. From the Fig. 9 it is observed that the frequency parameter increases with increase in number of modes (wave numbers). The higher natural frequencies of NDNBS are lesser than the NDNBS with one nanobeam fixed. This is due to inclusion of both in-phase and antiphase modes in the system of NDNBS.

For the comprehension of the vibration of NDNBS with small-scale effects, the natural frequencies of the double-nanobeam system for different nonlocal parameters and stiffness parameters are plotted in a three dimensional graph. Figures 10–12 show the plots for the first, second and third

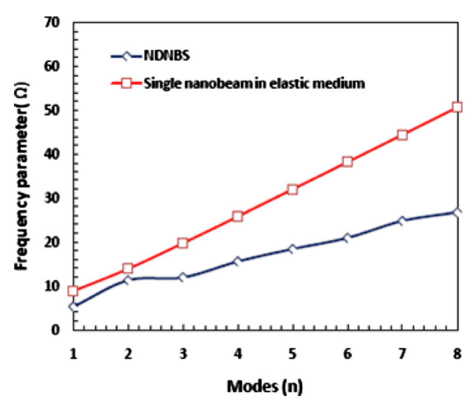

FIG. 9. (Color online) Variation in the frequency parameter (Ω) with the number of modes (n) for (i) NDNBS and (ii) single nanobeam in elastic medium.

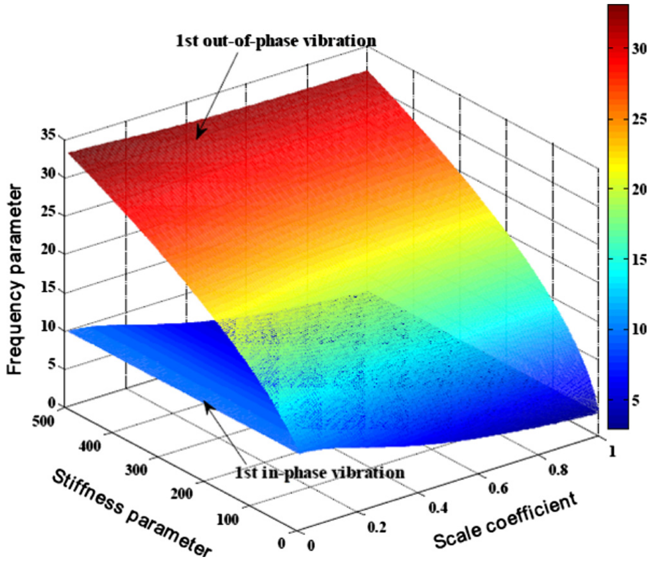

FIG. 10. (Color online) The variation in the frequency parameter (Ω) for the first in-phase and out-of-phase vibrations of NDNBS as a function of the stiffness parameter (K) and the scale coefficient (μ).

in-phase and out-of-phase vibrations respectively. The stiffness parameter K of the coupling springs are considered in the range of 0–500.

The nonlocal parameter μ is varied from 0 to 1. From Fig. 10 it is observed that the increase in the nonlocal parameter have a reducing effect on the natural frequencies of the NDNBS. However this reduction cannot be experienced at higher stiffness parameters values. The stiffness of the springs has a reducing effect on the small-scale effects of the NDNBS. It should be noted that for in-phase vibration the natural frequencies are independent of coupling springs. Figure 11 shows the plot for second in-phase and out-of-phase vibration respectively.

Compared to first natural frequencies (Fig. 10) the frequencies for second natural frequencies (Fig. 11) are more affected by scale effects. This is in line with the transverse vibration of uncoupled single nanobeam. In addition it is

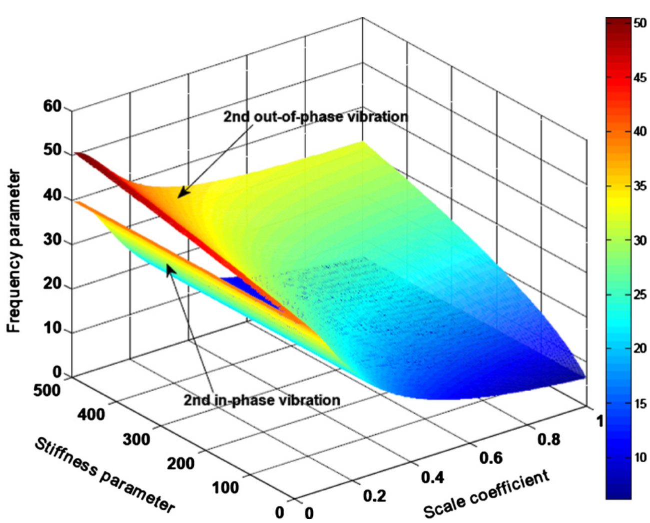

FIG. 11. (Color online) The variation in the frequency parameter (Ω) for the second in-phase and out-of-phase vibrations of NDNBS as a function of the stiffness parameter (K) and the scale coefficient (μ).

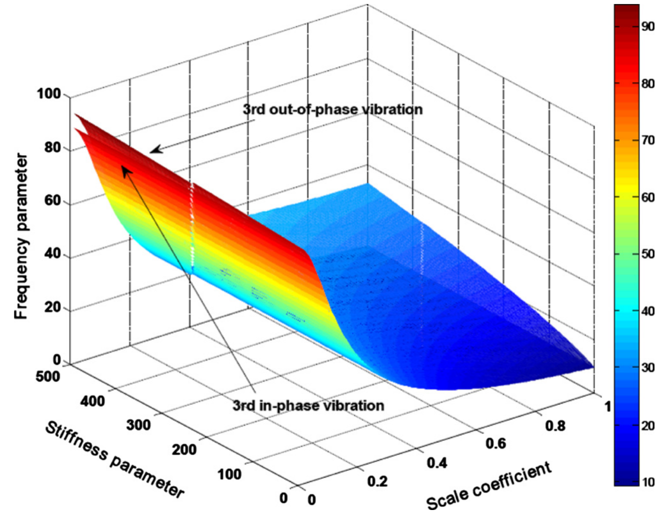

FIG. 12. (Color online) The variation in the frequency parameter (Ω) for the third in-phase and out-of-phase vibrations of NDNBS as a function of the stiffness parameter (K) and the scale coefficient (μ).

observed that the stiffness effect is less in both second in-phase and out-of-phase modes of vibration. The stiffness effect is further reduced between third in-phase and out-of-phase natural frequencies (Fig. 12). Thus it can be concluded that although the small-scale effects are more in higher frequencies (modes), the effect of stiffness of coupling springs reduces the nonlocal effects.

The present work provides an analytical solution that serves as a benchmark for further investigation of more complex n-nanobeam systems such as triple-nanobeam systems. Different boundary conditions at the ends will result in different vibration behavior. This would find application in the design of nanoresonators. The present work could also be useful in the study of double-nanoplate system for future NOMS studies.

VIII. CONCLUSIONS

In this paper, theoretical nonlocal elasticity is developed for the free bending-vibration of a double-nanobeam-system (NDNBS). An exact analytical method is developed for determining the nonlocal frequencies of transversely vibrating NDNBS. The in-phase and the out-of-phase vibrations are examined in details. The study shows that nonlocal effects are important in the transverse vibration of NDNBS. Nonlocal effects reduce the frequencies of the NDNBS. Increasing the stiffness of the springs in NDNBS reduces the nonlocal effects. The small-scale effects in NDNBS are more prominent with the increasing nonlocal parameter in the in-phase vibration than in the out-of-phase vibration. The frequencies in the in-phase vibration are independent of the stiffness of the springs. The NDNBS then can be treated as a single nanobeam. On the other hand, the frequencies of NDNBS in the out-of-phase vibration increase with the increasing stiffness parameter of the coupling springs. This study gives physical insights which may be useful for the design and vibration analysis of NOMSs, nanoresonators and sensors applications. Further this work provides an analytical solution which could serve as the starting point for further investigation of more complex n-nanobeam systems.

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