High dimensional model representation method for fuzzy structural dynamics

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ABSTRACT

Uncertainty propagation in multi-parameter complex structures possess significant computational challenges. This paper investigates the possibility of using the High Dimensional Model Representation (HDMR) approach when uncertain system parameters are modeled using fuzzy variables. In particular, the application of HDMR is proposed for fuzzy finite element analysis of linear dynamical systems. The HDMR expansion is an efficient formulation for high-dimensional mapping in complex systems if the higher order variable correlations are weak, thereby permitting the input–output relationship behavior to be captured by the terms of low-order. The computational effort to determine the expansion functions using the $\alpha-$cut method scales polynomially with the number of variables rather than exponentially. This logic is based on the fundamental assumption underlying the HDMR representation that only low-order correlations among the input variables are likely to have significant impacts upon the outputs for most high-dimensional complex systems. The proposed method is first illustrated for multi-parameter nonlinear mathematical test functions with fuzzy variables. The method is then integrated with a commercial finite element software (ADINA). Modal analysis of a simplified aircraft wing with fuzzy parameters has been used to illustrate the generality of the proposed approach. In the numerical examples, triangular membership functions have been used and the results have been validated against direct Monte Carlo simulations. It is shown that using the proposed HDMR approach, the number of finite element function calls can be reduced without significantly compromising the accuracy.

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1. Introduction

Uncertainties are unavoidable in the description of real-life engineering systems. The quantification of uncertainties plays a crucial role in establishing the credibility of a numerical model. In the context of structural dynamics, there may be uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficients, and geometric parameters such as length and thickness of different components. Such uncertainties can be modeled either by probabilistic approaches or by possibilistic approaches. In the context of probabilistic approaches, over the past two decades the stochastic finite element method (see for example [1–12]) and random matrix theory (see for example [13–19]) have been developed to systematically quantify uncertainty in dynamical systems. The reliable application of probabilistic approaches requires information to construct the probability density functions of uncertain parameters. This information may not be easily available for many complex practical problems. In such

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situations, non-probabilistic approaches such as interval algebra [20], convex models [21] and fuzzy set [22] based methods can be used. In this paper the uncertain variables describing the system parameters are modeled using fuzzy variables.

Over the past decade significant developments have taken place in fuzzy finite element analysis of linear dynamical systems (see for example the review papers [23–25]). A fuzzy variable can be viewed as a generalization of an interval variable. When an uncertain variable is modeled using the interval approach, the values of the variable lie within a lower and an upper bound. The fuzzy approach generalizes this concept by introducing a membership function. In the context of computational mechanics, the aim of a fuzzy finite element analysis is to obtain the range of certain output quantities (such as displacement, acceleration and stress) given the membership of data in the set of input variables. This problem, known as the uncertainty propagation problem, has taken center stage in recent research activities in the field. In principle an uncertainty propagation problem can be always solved using the so called direct Monte Carlo simulation. Using this approach, a large number of members of the parameter set are individually simulated and bounds are obtained from the resulting outputs. For most practical problems, direct Monte Carlo simulation is prohibitively computationally expensive. Therefore, the aim of the majority of current research is to reduce the computational cost. Since fuzzy variables are a generalization of interval variables, methods applicable for interval variables such as classical interval arithmetic [20], affine analysis [26,27] or vertex theorems [28] can be used. The Neumann expansion [29], component mode synthesis [30] and response surface based methods [31,32] have been proposed for fuzzy uncertainty propagation. In the context of dynamical systems, several authors have extended classical modal analysis to fuzzy modal analysis [33–39]. In fuzzy sets theory, the reference solution is given by the Extension Principle proposed by Zadeh [22]. This numerical implementation, named as the transformation method [40–43], allows the membership functions of the output solutions to be built by knowing the membership functions of input solutions and the relation between the input and output variables. A crucial factor in the efficiency of all of the methods is the number of input variables, often known as the 'dimensionality' of the problem. Here we investigate the use of the High Dimensional Model Representation (HDMR) approach to address this issue.

The HDMR technique is based on optimisation and projector operator theory. The HDMR expansion is an efficient formulation for high-dimensional mapping in complex systems if the higher order variable correlations are weak, thereby permitting the input–output relationship behavior to be captured by the terms of low-order. The computational effort to determine the expansion functions using the $\alpha-$cut method will scale polynomically with the number of variables rather than exponentially. This logic is based on the fundamental assumption underlying the HDMR representation that only low-order correlations among the input variables are likely to have significant impacts upon the outputs for most high-dimensional complex systems. This approach is particularly suitable for multi-parameter systems arising in real structural dynamical systems.

The outline of the paper is as follows. Section 2 gives a brief background of the fuzzy finite element method. The proposed High Dimensional Model Representation approach is described in Section 3. A step by step summary of the proposed method is given in Section 4, and four numerical examples illustrate the proposed method in Section 5. The approach is first illustrated for multi-parameter nonlinear mathematical test functions with fuzzy variables. The method is then integrated with a commercial Finite Element software (ADINA). The modal analysis of a simplified aircraft wing with fuzzy parameters has been used to illustrate the generality of the proposed approach. In the numerical examples, triangular membership functions have been used and the results have been validated against direct Monte Carlo simulations. Based on the studies in the paper, some conclusions are drawn in Section 7.

2. Background of the fuzzy finite element method

2.1. A brief overview of fuzzy variables

The concept of the fuzzy set was proposed by Zadeh [22] to represent imprecise information in a mathematically rigorous manner. Since then numerous books and papers have discussed the detail of fuzzy sets (see for example the review papers [23–25]). Here a very brief discussion is given for completeness. The mathematical description of fuzzy variables can be given in various levels of abstraction. The easiest way to describe a fuzzy variable is via interval variables. A scalar interval variable x^I can be defined as

$$x^I = [x_l, x_h] = \{x \in \mathbb{R} | x_l \leq x \leq x_h\} \tag{1}$$

The scalar interval variable can be extended to vector and tensor valued variables in a natural way. In contrast to the probabilistic description, the entries of a vector or tensor are always considered to be uncorrelated. Therefore, generalizing (1), a n-dimensional interval vector simply lies within a n-dimensional hypercube in $\mathbb{R}^n$.

A fuzzy set can be defined by extending the deterministic (crisp) value and combining interval algebra. Suppose X is a (real) space whose elements are denoted by x. The key idea in fuzzy set theory is the membership function $\mu_{\tilde{A}} : X \rightarrow M \subseteq [0,1]$ which maps X into the membership space M. If $\mu_{\tilde{A}}(x) = 1$ then x is definitely a member of the fuzzy set $\tilde{A}$. On the other hand, if $\mu_{\tilde{A}}(x) = 0$ then x is definitely not a member of the fuzzy set $\tilde{A}$. For any intermediate case $0 < \mu_{\tilde{A}}(x) < 0$ the

membership is uncertain. Therefore, using the membership function, a fuzzy set can be defined as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} \tag{2}$$

If $\sup_{x \in X} \mu_{\tilde{A}}(x) = 1$, then the fuzzy set $\tilde{A}$ is called normal. In this paper only normalized fuzzy sets are considered.

The notion of support and $\alpha-$cuts of a fuzzy set play a crucial role in the computational and analytical methods involving fuzzy sets. The support of a fuzzy set $\tilde{A}$ is the crisp set of all $x \in X$, such that $\mu_{\tilde{A}}(x) > 0$, or mathematically

$$\text{supp}(\tilde{A}) = \{x | \mu_{\tilde{A}}(x) > 0, x \in X\} \tag{3}$$

The $\alpha-$cut or $\alpha-$level set of $\tilde{A}$ is the crisp set A_α such that

$$A_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha, x \in X, 0 < \alpha \leq 1\} \tag{4}$$

In the case of bounded supports, we also define the 0-cut as the largest closed interval $A_0 = [\inf(\text{supp}(\tilde{A})), \sup(\text{supp}(\tilde{A}))]$. Often $\alpha-$cuts are considered as intervals of confidence, since in case of convex fuzzy sets, they are closed intervals associated with a gradation of confidence between [0,1]. In Fig. 1 the concept of a fuzzy set is pictorially described.

2.2. Uncertainty propagation

The equation of motion of a damped n-degree-of-freedom linear structural dynamical system with fuzzy parameters can be expressed as

$$\mathbf{M}(\mathbf{x})\ddot{\mathbf{q}}(t) + \mathbf{C}(\mathbf{x})\dot{\mathbf{q}}(t) + \mathbf{K}(\mathbf{x})\mathbf{q}(t) = \mathbf{f}(t) \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^M$ is a fuzzy vector, $\mathbf{f}(t) \in \mathbb{R}^n$ is the forcing vector, $\mathbf{q}(t) \in \mathbb{R}^n$ is the response vector and $\mathbf{M} \in \mathbb{R}^{n \times n}$, $\mathbf{C} \in \mathbb{R}^{n \times n}$ and $\mathbf{K} \in \mathbb{R}^{n \times n}$ are the mass, damping and stiffness matrices, respectively. When fuzzyness in the system parameters, boundary conditions and geometry are considered, the system matrices become matrices with fuzzy entries. Uncertainties in system (5) are completely characterized by the membership functions of the entries of the vector $\mathbf{x}$. The main interests in structural dynamic analysis are (a) quantification of uncertainty in the eigenvalues and eigenvectors, and (b) quantification of uncertainty in the response $\mathbf{q}$ either in the time domain or in the frequency domain. Few researchers [32–34] have considered uncertainty quantification in the frequency response function of a linear dynamical systems with fuzzy variables. This problem, in effect, has to solve the uncertainty propagation problem, which can be expressed in a general way as

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \in \mathbb{R}^m \tag{6}$$

where $\mathbf{f}(\bullet) : \mathbb{R}^M \to \mathbb{R}^m$ is a smooth nonlinear function of the input fuzzy vector $\mathbf{x}$. The function $\mathbf{f}(\bullet)$ can be the frequency response function of Eq. (5) or the eigensolutions.

The uncertainty propagation of fuzzy variables through smooth nonlinear functions can be performed in two main ways:

- *Interval algebra based approach:* In this approach a fuzzy variable is considered as an interval variable for each $\alpha-$cut and propagated using classical interval arithmetic [20]. Unless the functions are simple and the number of variables is small, this approach produces a very large bound for the output variables. Such overestimates may not be useful for practical design decisions.
- *Global optimisation based approach:* Here for each $\alpha-$cut two optimisation problems are solved for each of the output quantities. The first optimisation problem aims to find the minimum value while the second optimisation problem aims to find the maximum value of the output quantity. By combining results for all $\alpha-$cuts, one can obtain the fuzzy description of the output quantities. The advantage of this approach is that the bounds are 'tight' and the wealth
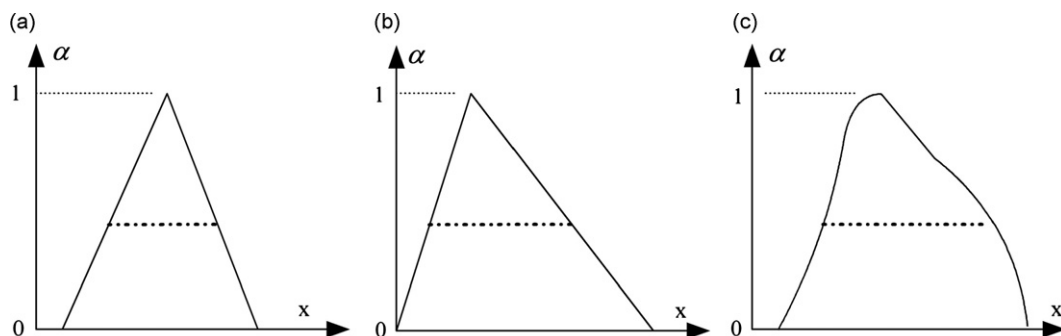

Fig. 1. Membership functions of a fuzzy variable; (a) symmetric triangular membership function; (b) asymmetric triangular membership function; (c) general membership function. The value corresponding to $\alpha = 1$ is the crisp (or deterministic) value. The range corresponding to $\alpha = 0$ is the widest interval. Any intermediate value of $0 < \alpha < 1$, yields an interval variable with a finite lower and upper bound.

of tools are available for mathematical optimisation can be employed in a reasonably straight-forward manner. The disadvantage is, of course, the computational cost as two optimisation problems need to be solved for every $\alpha-$cut and for every output quantity. For these reasons efficient numerical methods are necessary to use this approach.

In this paper the global optimisation based approach is adopted for uncertainty propagation. Suppose I is the set of integers and $r \in I$ is the number of $\alpha-$cuts used for each of the input fuzzy variables x_j, $j=1,2,\ldots,m$ in the numerical analysis. Therefore, $0 \le [\alpha_1, \alpha_2, \ldots, \alpha_r] < 1$ are the values of α for which we wish to obtain our outputs. Suppose $f_j(\mathbf{x}_{\alpha_k})$, $j = 1, 2, \ldots, M$ are the functions which relate the output y_{jk} to the interval variable $\mathbf{x}_{\alpha_k}$ associated with α_k for some $k \le r$ such that

$$y_{jk} = f_j(\mathbf{x}_{\alpha_k}), \quad j = 1, 2, \ldots, m; \quad k = 1, 2, \ldots, r \tag{7}$$

The fuzzy uncertainty propagation problem can be formally defined as the solution of the following set of optimisation problems

$$\left. \begin{array}{l} y_{jk_{\min}} = \min(f_j(\mathbf{x}_{\alpha_k})) \\ y_{jk_{\max}} = \max(f_j(\mathbf{x}_{\alpha_k})) \end{array} \right\} \forall \quad j = 1, 2, \ldots, m; \quad k = 1, 2, \ldots, r \tag{8}$$

In total there are $2mr$ optimisation problems. The computational cost of each of these optimisation problems depends on the dimensionality of the $\mathbf{x}_{\alpha_k}$ that is M. Suppose $c(M)$ is the computational cost associated with each optimisation problem. For example, if $f_j(\bullet)$ are monotonic functions, then the optimal values occur at the vertices of the M dimensional parallelepiped. For this case one has $c(M) \propto 2^M$. The total computational cost to solve the fuzzy uncertainty propagation problem (8) is therefore $2mrC(M)$. Since m, r and M are fixed by the problem, there are two ways to reduce the computational cost, namely (a) to reduce the number of function evaluations by using superior optimisation techniques, or (b) to reduce the cost of each function evaluation, for example, by 'replacing' the expensive function by a surrogate model. In this paper the second approach is adopted. Clearly if the cost of each function evaluation can be reduced then the overall computational cost to solve the uncertainty propagation problem will be reduced no matter what optimisation algorithm is used. De Munck et al. [44] used a Kriging based approach. Here we propose a High Dimensional Model Representation (HDMR) approach to efficiently generate the surrogate model. The aim is to 'build' the surrogate model with the minimum number of evaluations of the original function. Once the surrogate model is obtained, Monte Carlo simulation is used to obtain the minimum and maximum of the function.

3. High Dimensional Model Representation (HDMR)

The High Dimensional Model Representation (HDMR) of an arbitrary M-dimensional function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^M$ can be derived by partitioning the identity operator $\mathcal{I}$, called $\mathcal{I}_M$ in the M-dimensional case and also in the 1D case hereafter, with respect to the projectors $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_M$. This can be expressed as follows [45–54]:

$$\mathcal{I}_M = \prod_{m=1}^{M} (\mathcal{P}_m + (\mathcal{I}_1 - \mathcal{P}_m)) = \underbrace{\prod_{m=1}^{M} \mathcal{P}_m}_{1 \text{ term}} + \underbrace{\sum_{m=1}^{M} (\mathcal{I}_1 - \mathcal{P}_m) \prod_{s \neq m} \mathcal{P}_s}_{\binom{M}{1} \text{ terms}} + \underbrace{\sum_{m=1}^{M} \sum_{s=m+1}^{M} (\mathcal{I}_1 - \mathcal{P}_m)(\mathcal{I}_1 - \mathcal{P}_s) \prod_{p \neq m,s} \mathcal{P}_p}_{\binom{M}{2} \text{ terms}}$$

$$+ \cdots + \underbrace{\sum_{m=1}^{M} \mathcal{P}_m \prod_{s \neq m} (\mathcal{I}_1 - \mathcal{P}_s)}_{\binom{M}{M-1} \text{ terms}} + \underbrace{\prod_{m=1}^{M} (\mathcal{I}_1 - \mathcal{P}_m)}_{1 \text{ term}} \tag{9}$$

composed of 2^M mutually orthogonal terms. The orthogonal representation of Eq. (9) is a manifestation of the HDMR and can be rewritten as [55,56],

$$f(\mathbf{x}) = f_0 + \sum_{i=1}^{M} f_i(x_i) + \sum_{1 \le i < j \le M} f_{ij}(x_i, x_j) + \cdots + f_{123\ldots M}(x_i, x_2, \ldots, x_M) = \sum_{l=0}^{M} \eta_l(\mathbf{x}) \tag{10}$$

where f_0 is a constant term representing the zeroth-order component function or the mean response of any response function $f(\mathbf{x})$. f_i is the first-order term expressing the effect of variable x_i acting alone upon the output $f(\mathbf{x})$, and this function is generally nonlinear. The function $f_{ij}(x_i, x_j)$ is a second-order term which describes the cooperative effects of the variables x_i and x_j upon the response. The higher order terms gives the cooperative effects of increasing numbers of input variables acting together to influence the output. The last term $f_{123\ldots M}(x_i, x_2, \ldots, x_M)$ contains any residual dependence of all the input variables locked together in a cooperative way to influence the output. Once all of the relevant component functions in Eq. (10) are determined and suitably represented, then the component functions constitute the HDMR, thereby replacing the original computationally expensive method of calculating the response by the computationally efficient meta model. Usually the higher order terms in Eq. (10) are negligible [47,57] such that an HDMR with only a few low-order correlations amongst the input variables is adequate to describe the output behavior. This in turn results in rapid convergence of the HDMR expansion [48,49], which has a finite number of terms and is always exact [47,52] in the least-square senses. Other

popular expansions (e.g., polynomial chaos) have been suggested [1], but they commonly have an infinite number of terms with some specified functions, such as Hermite polynomials [45,46,52].

To generate the HDMR approximation of any function, more precisely the cut-center based HDMR, first a reference point $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_M)$ has to be defined in the variable space. In the convergence limit, where all correlated functions in Eq. (10) are considered, the cut-HDMR is invariant to the choice of reference point $\overline{\mathbf{x}}$. However, in practice the choice of reference point $\overline{\mathbf{x}}$ is important for the cut-HDMR, especially if only the first few terms, say up to first and second order, in Eq. (10) are considered. Sobol [48] showed that the reference point $\overline{\mathbf{x}}$ at the middle of the input domain appears to be the optimal choice. The expansion functions are determined by evaluating the input–output responses of the system relative to the defined reference point along the associated lines, surfaces or sub-volumes in the input variable space. This process reduces to the following relationship for the component functions in Eq. (10):

$$f_0 = \int \mathrm{d}x f(\mathbf{x})$$

$$f_i(x_i) = \int \mathrm{d}x^i f(\mathbf{x}) - f_0$$

$$f_{ij}(x_i, x_j) = \int \mathrm{d}x^{ij} f(\mathbf{x}) - f_i(x_i) - f_j(x_j) - f_0 \tag{11}$$

where, $\int \mathrm{d}x^i$ means to integrate over all M variables except x_i and $\int \mathrm{d}x^{ij}$ means to integrate over all M variables except x_i and x_j, etc. These integrals are generally evaluated using numerical integration techniques. Substituting the component functions defined in Eq. (11) into Eq. (10), the general expression of the HDMR and can be expressed as

$$f(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_\beta \le M} f(x_{i_1}, \ldots, x_{i_\beta}; \overline{x}^{i_1, \ldots, i_\beta}) - (M - \beta) \sum_{1 \le i_1 < \cdots < i_{\beta-1} \le M} f(x_{i_1}, \ldots, x_{i_{\beta-1}}; \overline{x}^{i_1, \ldots, i_{\beta-1}})$$

$$+ \frac{(M - \beta + 1)!}{2!(M - \beta - 1)!} \sum_{1 \le i_1 < \cdots < i_{\beta-2} \le M} f(x_{i_1}, \ldots, x_{i_{\beta-2}}; \overline{x}^{i_1, \ldots, i_{\beta-2}}) - \cdots$$

$$\mp \frac{(M - 2)!}{(\beta - 1)!(M - \beta - 1)!} \sum_{1 \le i \le M} f(x_i; \overline{x}^i) \pm \frac{(M - 1)!}{\beta!(M - \beta - 1)!} f(\overline{\mathbf{x}}) \tag{12}$$

where β is the order of the HDMR approximation, $1 \le \beta \le (M - 1)$ and the + or − sign of the last term in Eq. (12) corresponds to β being even or odd, respectively. Considering the weak role of the higher order correlation effects, the approximation is likely to converge at a low HDMR order, say, $\beta \ll M$. The particular form of Eq. (12) for β = 1, 2, or 3 corresponds to first, second or third order HDMR and can be explicitly given as

$$\hat{f}(\mathbf{x}) = \sum_{1 \le i \le M} f(x_i; \overline{x}^i) - (M - 1) f(\overline{\mathbf{x}}), \quad \beta = 1 \tag{13}$$

$$\hat{f}(\mathbf{x}) = \sum_{1 \le i < j \le M} f(x_i, x_j; \overline{x}^{i,j}) - (M - 2) \sum_{1 \le i \le M} f(x_i; \overline{x}^i) + \frac{(M - 1)!}{2!(M - 3)!} f(\overline{\mathbf{x}}), \quad \beta = 2 \tag{14}$$

$$\hat{f}(\mathbf{x}) = \sum_{1 \le i < j < k \le M} f(x_i, x_j, x_k; \overline{x}^{i,j,k}) - (M - 3) \sum_{1 \le i < j \le M} f(x_i, x_j; \overline{x}^{i,j})$$

$$+ \frac{(M - 2)!}{2!(M - 4)!} \sum_{1 \le i \le M} f(x_i; \overline{x}^i) + \frac{(M - 1)!}{3!(M - 4)!} f(\overline{\mathbf{x}}), \quad \beta = 3 \tag{15}$$

The term $f(x_i; \overline{x}^i)$ is a function of the single x_i component (i.e., a cut along x_i through the reference point in the function space), while the other variables, $x_j \equiv \overline{x}_j, j \ne i$, are fixed at the reference point. In the same manner, $f(x_i, x_j; \overline{x}^{i,j})$ is the observed response for all the variables, $x_k \equiv \overline{x}_k, k \ne i, j$, fixed at the cut center except for x_i and x_j. A similar interpretation would apply to higher order HDMR terms.

The notion of first, second order, etc. used in the HDMR does not imply the terminology commonly used either in the Taylor series or in the conventional polynomial based approximation formulae. In HDMR-based approximation, these terminologies are used to define the constant term, or for example, terms with one variable or two variables only. It is recognized that the lower order (e.g., first order or second order) function expansions in the HDMR, do not generally translate to linear or quadratic functions [49]. Each of the lower-order terms in the HDMR is sub-dimensional, but they are not necessarily low degree polynomials. The computational savings afforded by the HDMR are easily estimated. If the HDMR converges at β order with acceptable accuracy and considering s sample points for each variable, then the total number of numerical analyses needed to determine the HDMR is $\sum_{k=0}^{\beta} [M!/k!(M-k)!](s-1)^k$.

4. Proposed methodology

Based on the discussion in the previous section, the proposed methodology for the first-order HDMR approximation is easily implemented using the following steps:

1. Identify the range of input variables.
2. Select the reference point $\overline{\mathbf{x}}$, generally as the center of the input variable range.
3. Determine f_0, which is a constant term, representing the response at reference point $\overline{\mathbf{x}}$.
4. Generate regularly spaced sample points, as $x_i^1 = \overline{x}_i - (s-1)k_i/2$, $x_i^2 = \overline{x}_i - (s-3)k_i/2$, ..., $x_i^{(s+1)/2} = \overline{x}_i$, ..., $x_i^{s-1} = \overline{x}_i + (s-3)k_i/2$, $x_i^s = \overline{x}_i + (s-1)k_i/2$, along the variable axis x_i with reference $\overline{x}_i$ and uniform distance k_i. s denotes the number of HDMR sample points, and must be odd.
5. Estimate the responses at all these sample points.
6. Construct the HDMR approximation using following steps:
   - Interpolate each of the low dimensional (e.g., first-order) HDMR expansion terms $f(x_i; \overline{x}^i)$ as $f(x_i; \overline{x}^i) = \sum_{j=1}^{s} \varphi_j(x_i) f(x_i^j; \overline{x}^i)$. The response values are calculated in previous step and $\varphi_j(x_i)$ represents interpolation/shape functions. Presently we used moving least-squares interpolation functions, details of which can be found in [58].
   - Sum the interpolated values of HDMR expansion terms. This leads to the first-order HDMR approximation of the function $f(\mathbf{x})$ as follows:

$$\hat{f}(\mathbf{x}) = \sum_{1 \leq i \leq M} \sum_{j=1}^{s} \varphi_j(x_i) f(x_i^j; \overline{x}^i) - (M-1)f(\overline{\mathbf{x}}) \quad \beta = 1 \tag{16}$$

7. Perform Monte Carlo simulation on the approximated response function $\hat{f}(\mathbf{x})$ using uniform variables. Estimate the minimum–maximum range of the response for the particular α−cut.
8. Repeat the procedure for different α−cuts.

5. Application examples

The implementation of the HDMR approach in fuzzy analysis is illustrated with the help of two examples in this section. When comparing computational efforts in evaluating the response intervals, the number of actual FE analyses is chosen as the primary comparison tool because this indirectly indicates the CPU time usage. For full scale MCS, the number of original FE analyses is same as the sample size. When the responses are evaluated through full scale MCS, the CPU time is higher since a number of repeated FE analysis are required. However, in the proposed method MCS is conducted within the HDMR framework. Although the same sample size as for direct MCS is considered, the number of FE analyses is much lower. Hence, the computational effort expressed in terms of FE calculation alone should be carefully interpreted. Since the HDMR approximation leads to an explicit representation of the system responses, the MCS can be conducted for any sample size. Using a first-order HDMR approximation, the total cost of the original FE analysis entails a maximum of $(s-1) \times M + 1$ analyses for different α−cuts, where s is the number of sample points of the HDMR and M represents the number of fuzzy variables.

5.1. Example 1: cubic function with two variables

Consider a cubic function [59] of the following form:

$$f(\mathbf{x}) = 2.2257 - \frac{0.025\sqrt{2}}{27}(x_1 + x_2 - 20)^3 + \frac{33}{140}(x_1 - x_2) \tag{17}$$

with two fuzzy variables. The upper and lower limits of the variables are 16 and 4, respectively. To evaluate the response uncertainty associated with the fuzzy variables, the HDMR approximation is constructed by deploying five equally spaced sample points ($s = 5$) along each of the variable-axes. The reference point $\overline{\mathbf{x}}$ is chosen as 10. Fig. 2(a) and (b) shows the original function and the approximation error. It is noted that, the approximated function follows reasonable agreement with the original function. The bounds of the input variables are presented in Table 1 along with the reference point. Table 2 compares the results obtained by the present method using HDMR approximation with direct MCS using uniform variables with specified limits. A sampling size $N = 10^5$ is considered in direct MCS to compute the response bounds. A similar analysis is performed for different α−cuts and presented in Table 2. The total number of function calls for each α−cut is $9 (= (s-1) \times M + 1 = (5-1) \times 2 + 1)$. Table 2 shows that the maximum error in predicting the response bounds is 1.66 percent, compared to MCS.
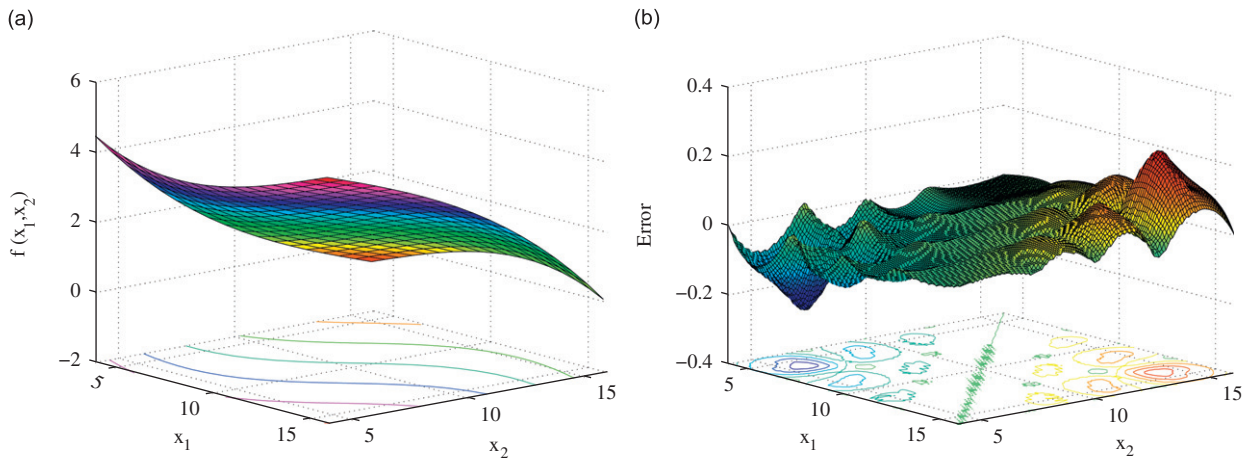
(a) (b)

Fig. 2. Plot of the original function and the approximation error using HDMR-based approach in Eq. (17): (a) original function and (b) approximation error.

Table 1
Bounds of the input fuzzy variables of Example 1.

$\alpha-$cut	x_{min}	x_{max}	$\overline{\mathbf{x}}$
0.0	4.0	16.0	10.0
0.2	5.2	14.8	10.0
0.4	6.4	13.6	10.0
0.6	7.6	12.4	10.0
0.8	8.8	11.2	10.0

Table 2
Comparison of response bounds obtained from HDMR approximation and MCS in Example 1.

$\alpha-$cut	Methods	f_{min}	f_{max}
0.0	HDMR	−0.5930	5.0438
	MCS	−0.5909	5.0456
	% error	−0.35	0.04
0.2	HDMR	−0.0280	4.4807
	MCS	−0.0276	4.4816
	% error	−1.66	0.02
0.4	HDMR	0.5352	3.9173
	MCS	0.5357	3.9176
	% error	0.09	0.01
0.6	HDMR	1.0988	3.3536
	MCS	1.0991	3.3537
	% error	0.02	0.00
0.8	HDMR	1.6623	2.7897
	MCS	1.6624	2.7897
	% error	0.00	0.00

5.2. Example 2: modal analysis of a simplified aircraft wing

This example involves the estimation of bounds of the natural frequencies of a vibrating aircraft wing with length L_y=1 m and width L_z=0.5 m, as shown in Fig. 3. The mass of the engine is attached to the plate, which represents the wing, through a spring at one third distance from the root of the wing. Poisson's ratio for the plate is taken as 0.33. The mass of the engine is taken as 0.80 times the mass of the wing. The stiffness of the spring is calculated from the second natural frequency of the wing. The thickness, mass density and elastic modulus of the wing are taken as fuzzy variables. The bounds of the variables are tabulated in Table 3. The response quantities, which are the natural frequencies of the wing structure, are represented by an HDMR expansion. Note that in this example, the responses (natural frequencies) are monotonic functions of the parameters. Furthermore, there is no mode crossing or veering in the parameter range considered. The proposed approach with HDMR approximation and full scale
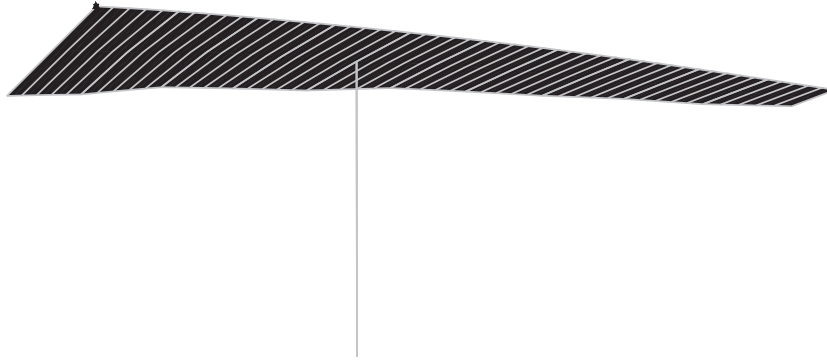
Fig. 3. The finite element model of a simplified aircraft wing with the engine modeled as a lumped mass attached to the wing via a spring at 1/3 span. The structure is modeled in ADINA [60] FE software. The 10×5 FE model of the plate consists of 50 2-noded solid elements and 303 nodes. A fixed boundary condition is applied at the left.

Table 3
Bounds of the input fuzzy variables of Example 2 at $\alpha-\mathrm{cut}=0$.

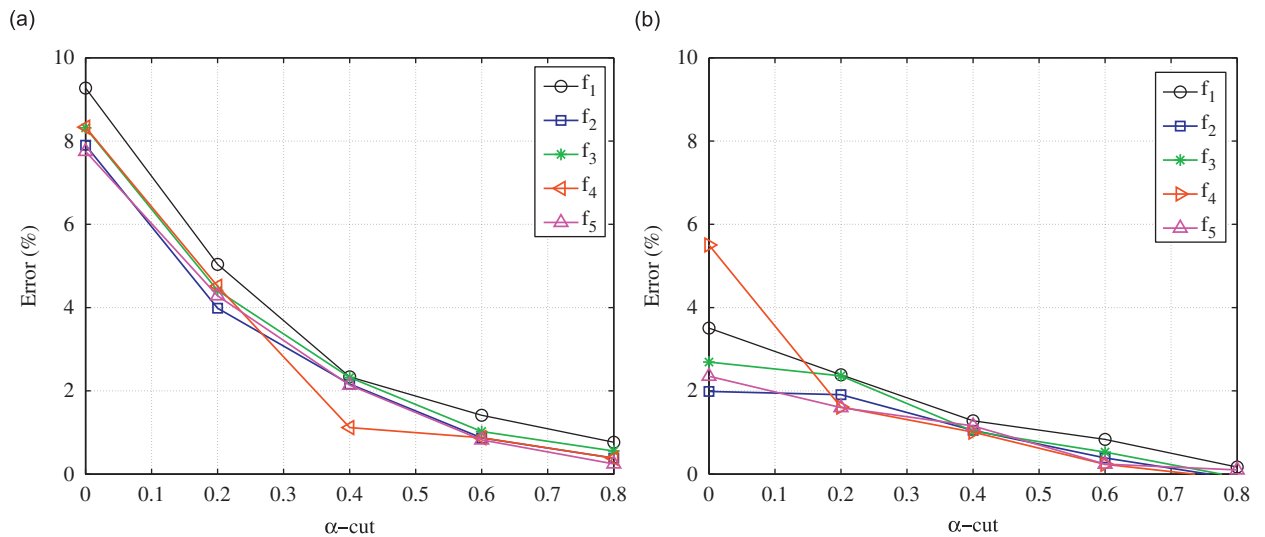| Variable	$x_{\min}$	$x_{\max}$	$\bar{x}$
Young's modulus (GPa)	29	109	69
Thickness (m)	0.1	0.5	0.3
Density (kg/m^3)	2000	4000	3000

Fig. 4. Errors in the response bounds of the natural frequencies using the HDMR method compared to MCS for Example 2: (a) errors in the lower bounds and (b) errors in the upper bounds.

MCS using the commercial FE code ADINA [60] are employed to evaluate the fuzzy characteristics of the natural frequencies of the structure. Therefore, computational efficiency, even for this simplified model, is a major practical requirement in solving the dynamic system. For the HDMR based approaches, a value of $s=5$ is selected. Using samples generated for the HDMR, approximation errors (compared with full scale MCS) in the response bounds are presented in Fig. 4. The errors in the response bounds at $\alpha-$cuts of 0 and 0.2 are quite high ($\approx 4-9$ percent), however, these errors can be reduced by taking more HDMR sample points (increase s), which demands more FE analyses. Thus the number of sample points (s) for the HDMR approximation will vary depending on the user's accuracy requirement. The full scale MCS results are obtained from 10^5 FE analyses. In contrast, only $13(=(s-1)\times M+1=(5-1)\times 3+1)$ FE analyses are required by HDMR for each $\alpha-$cut. In this problem, for each
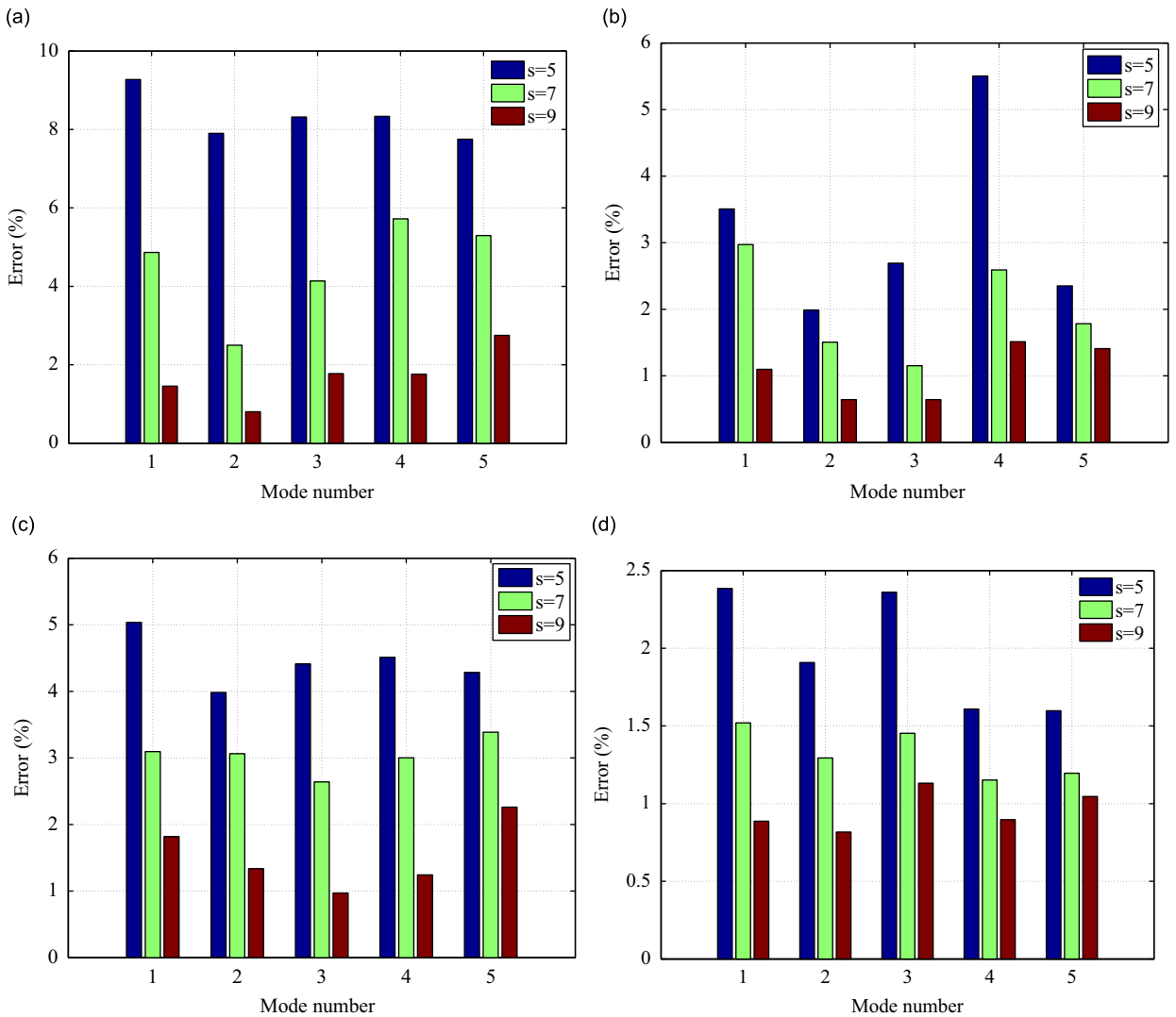
Fig. 5. Errors in the response bounds of the natural frequencies using the HDMR method compared to MCS for variation in the number of sample points (s) for Example 2: (a) errors in the lower bounds:α–cut=0, (b) errors in the upper bounds:α–cut=0, (c) errors in the lower bounds:α–cut=0.2 and (d) errors in the upper bounds:α–cut=0.2.

α–cut, the total CPU usage for full scale MCS is 10001 s, whereas for HDMR approximation it is only 4 s. Fig. 5 presents the variation of computed errors using HDMR approximation with increasing number of sample points. It is evident that, the percentage error reduces significantly with increasing number of sample points. However, this essentially increases the computational demand, e.g., the number of FE analyses required to construct HDMR approximation for s=5, 7 and 9 is 13, 19 and 25, respectively.

5.3. Example 3: time response of a single degree of freedom system

This example [61] examines a simple dynamic problem (ref. Fig. 6) of a hysteretic one degree of freedom system excited by a full sine-wave force impulse. The equation of motion is

$$m\ddot{r}(t) + c\dot{r}(t) + kr(t) + f_{nl}(t) = F_{ext}(t)$$

$$\dot{z} = \left(1 - \frac{1}{2}z^2(\text{sign}(\dot{r}z) + 1)\right)\frac{\dot{r}}{r_y}$$
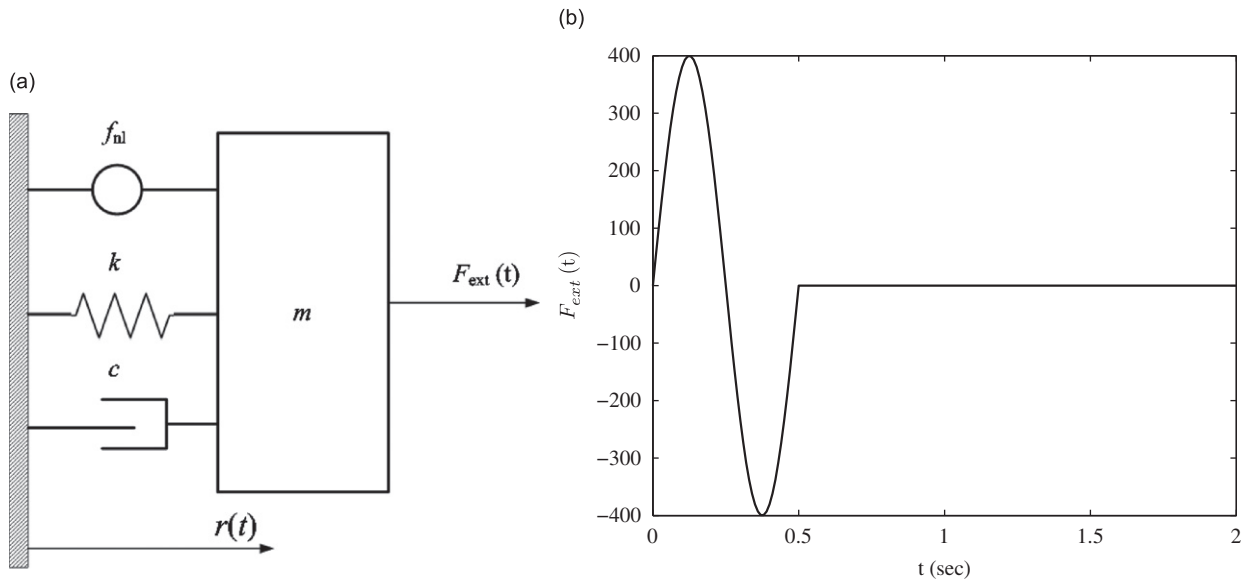
$$f_{nl} = f_y z \tag{18}$$

Fig. 6. A simple dynamic problem with an hysteretic force: (a) hysteretic system and (b) sine-wave force impulse.

Table 4
Description and bounds of the input fuzzy variables of Example 3 at α−cut$= 0$.

Variable	Description	Units	x_{min}	x_{max}	$\bar{\mathbf{x}}$
m	Mass	kg	850	1150	1000
k	Stiffness	N/m	2800	5200	4000
c	Damping coefficient	N/m/s	100	400	250
f_y	Yield force	N	280	520	400
r_y	Yield displacement	m	0.0055	0.0145	0.01
F_p	Force amplitude	N	160	640	400
T_p	Force period	s	0.25	1.25	0.50

Table 5
Comparison of response bounds obtained from HDMR approximation and MCS in Example 3.

α−cut	Methods	$s=5$		$s=7$		$s=9$	
		f_{min}	f_{max}	f_{min}	f_{max}	f_{min}	f_{max}
0.0	HDMR	0.128322	0.782131	0.125322	0.752131	0.1219475	0.746679
	MCS	0.123172	0.745435	0.123172	0.745435	0.123172	0.745435
	% error	−4.18	−4.92	−1.75	−0.90	0.99	−0.17
0.2	HDMR	0.229965	0.617833	0.209965	0.607833	0.2097942	0.601483
	MCS	0.206222	0.601133	0.206222	0.601133	0.206222	0.601133
	% error	−11.51	−2.78	−1.82	−1.11	−1.73	−0.06
0.4	HDMR	0.349907	0.529117	0.339907	0.479117	0.338671	0.479036
	MCS	0.338883	0.473107	0.338883	0.473107	0.338883	0.473107
	% error	−3.25	−11.84	−0.30	−1.27	0.06	−1.25

for constants r_y and f_y. The response quantity is defined as follows:

$$f(\mathbf{x}) = \max_t |\ddot{r}(t)| \tag{19}$$

To evaluate the response uncertainty associated with the fuzzy variables, the HDMR approximation is constructed by deploying five equally spaced sample points ($s = 5$) along each of the variable-axes. The description of each fuzzy variable and their bounds are presented in Table 4 along with the reference points. Table 5 compares the results obtained by the present method using HDMR approximation with direct the MCS using uniform variables with specified limits. A sampling
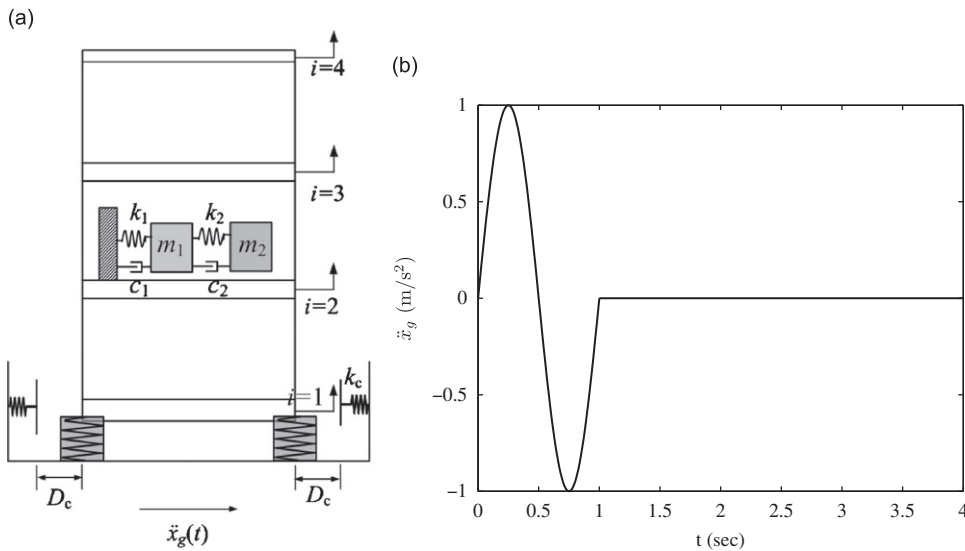
(a)

(b)

Fig. 7. Problem statement (Example 4); (a) Base isolated structure with an equipment isolation system on the second floor; (b) Acceleration history. (sine-wave force impulse).

Table 6
Description and bounds of the input fuzzy variables of Example 4 at $\alpha-cut = 0$.

Variable	Description	Units	x_{min}	x_{max}	$\overline{x}$
m_f	Floor mass	kg	4200	7800	6000
k_f	Floor stiffness	N/m	2.1×10^7	3.9×10^7	3.0×10^7
c_f	Floor damping coefficient	N/m/s	24 000	96 000	60 000
d_y	Isolation yield displacement	m	0.02	0.08	0.05
f_y	Isolation yield force	N	8000	32 000	20 000
d_c	Isolation contact displacement	m	0.35	0.65	0.50
k_c	Isolation contact stiffness	N	2.1×10^7	5.7×10^7	3.0×10^7
m_1	Mass of Block 1	kg	400	800	500
m_2	Mass of Block 2	kg	80	160	100
k_1	Stiffness of Spring 1	N/m	2000	4000	2500
k_2	Stiffness of Spring 2	N/m	8.0×10^4	1.60×10^5	1.0×10^5
c_1	Damping coefficient of damper 1	N/m/s	343	371	350
c_2	Damping coefficient of damper 2	N/m/s	196	212	200
T	Pulse excitation period	s	0.8	1.6	1.0
A	Pulse amplitude	m/s/s	0.50	2.50	1.00

size $N=10^5$ is considered in direct MCS to compute the response bounds. A similar analysis is performed for different $\alpha-$cuts and presented in Table 5. The total number of function calls for each $\alpha-$cut is 29 ($= (s-1) \times M+1 = (5-1) \times 7+1$). Table 5 shows that the maximum error in predicting the response bounds is 11.84 percent, compared to MCS for $s=5$. However, as expected, the error reduces to about 1.73 percent for increase in number of sample points to $s=9$. Note there are seven variables, hence $2^7=128$ function calls are required using the vertex method, while only 57 ($= (s-1) \times M+1 = (9-1) \times 7+1$) function evaluations are needed for $s=9$ in HDMR approximation. This clearly shows the efficiency of the HDMR-based approach over the popular vertex approach.

5.4. Example 4: time response of a six degrees of freedom system

This example considers a four-story building excited by a single period sinusoidal pulse of ground motion, studied by Gavin and Yau [61]. Fig. 7(a) shows the four-storey building with isolation systems and Fig. 7(b) presents the acceleration history. The building contains isolated equipment resting on the second floor. The motion of the ground floor is resisted mainly by base isolation bearings [61] and if their displacement exceeds $D_c(= 0.50$ m) then an additional stiffness force contributes to the resistance. The mass, stiffness and damping coefficients, m_f, k_f and c_f, respectively, at each floor are assumed to be identical.

There are two isolated masses, representing isolated, shock-sensitive equipment resting on the second floor. The larger mass $m_1(=500$ kg) is connected to the floor by a relatively flexible spring, $k_1(=2500$ N/m), and a damper, $c_1(=350$ N/m/s), representing the isolation system. The smaller mass $m_2(=100$ kg) is connected to the larger mass by a relatively stiff spring,

Table 7
Comparison of response bounds obtained from HDMR approximation and MCS in Example 4.

$\alpha-$cut	Methods	$s=5$		$s=7$		$s=9$	
		$f_{\min}$	$f_{\max}$	$f_{\min}$	$f_{\max}$	$f_{\min}$	$f_{\max}$
0.0	HDMR	−3.25070	1.38073	−3.58787	1.24123	−3.72870	1.20025
	MCS	−3.73387	1.17023	−3.73387	1.17023	−3.73387	1.17023
	% error	12.94	−17.99	3.91	−6.07	0.14	−2.57
0.2	HDMR	−2.28589	0.69951	−2.22898	0.71897	−2.18899	0.71999
	MCS	−2.17684	0.72059	−2.17684	0.72059	−2.17684	0.72059
	% error	−5.01	2.92	−2.39	0.22	−0.56	0.08
0.4	HDMR	−0.99515	−0.02426	−0.97515	0.71999	−0.96209	−0.02051
	MCS	−0.96237	−0.02012	−0.96237	−0.02068	−0.96237	−0.02012
	% error	−3.41	−20.55	−1.33	2.76	0.03	−1.94

$k_1(=10^5$ N/m), and a damper, $c_2(=200$ N/m/s), representing the equipment itself. Descriptions of the fuzzy variables are listed in Table 6. The limit state is defined by the combination of three failure modes leading to system failure and has the following form:

$$f(\mathbf{x}) = 12.50\left(0.04 - \max_t |x_{f_i}(t) - x_{f_{i-1}}(t)|\right)_{i=2,3,4} + \left(0.50 - \max_t |\ddot{x}_g(t) + \ddot{x}_{m_2}(t)|\right) + 2.0\left(0.25 - \max_t |x_{f_2}(t) - x_{m_1}(t)|\right) \quad (20)$$

where $x_{f_i}(t)$ refers to the displacement of ith floor and $(x_{f_i}(t) - x_{f_{i-1}}(t))$ is the inter storey drift. $\ddot{x}_g(t)$ is the ground acceleration and $\ddot{x}_{m_2}(t)$ is the acceleration of the smaller mass block. The displacement of the larger mass block is $x_{m_1}(t)$, and represents the displacement of the equipment isolation system. The limit state in Eq. (20) is the overall representation of three failure modes. The first term describes the damage to the structural system due to excessive deformation. The second term represents the damage to equipment caused by excessive acceleration. The last term represents the damage of the isolation system. The weighing factors, multiplied with each term in Eq. (20), are mainly to emphasize the equal contribution of the individual failure modes to the overall failure of system. It is desirable that (a) inter storey drift is limited to 0.04 m, (b) the peak acceleration of the equipment is less than 0.50 m/s², and (c) the displacement across the equipment isolation system is less than 0.25 m. Eq. (20) signifies overall system failure, which does not necessarily occur when one or two of the above mentioned failure criteria is satisfied.

Similar to the previous example, the approximate response function is constructed by deploying five, seven and nine equally spaced sample points (s=5, 7, 9) along each of the variable-axes. Table 7 compares the results obtained by the present method using HDMR approximation with direct MCS using uniform variables with specified limits. A sampling size N=10⁵ is considered in direct MCS to compute the response bounds. A similar analysis is performed for different $\alpha-$cuts and presented in Table 7. The total number of function calls for each $\alpha-$cut is 61 ($=(s-1) \times M + 1 = (5-1) \times 15 + 1$) for s=5. It is evident that the total number of original function evaluations is significantly less compared to the direct simulation.

6. Solution strategies

There are two important aspects related to the use of the HDMR approach in fuzzy systems, namely the accuracy of the approximation and the computational efficiency. Most of the discussion in this paper has concerned the accuracy, where it is vital that the 'exact' solution is obtained for the examples so that the errors in the HDMR approximation may be quantified. The exact solution has been obtained using a Monte Carlo simulation with a large number of sample points; this approach is guaranteed to obtain an excellent approximation to the solution if sufficient samples are taken, at the expensive of high computational cost. Often the extrema in the response are attained at boundaries in the set of parameters and these regions may be sparsely populated with samples, particularly when the number of parameters is large. Alternative exact methods, such as the vertex theorems, work well when the response is monotonic in the parameters, but will miss extrema at interior points of the parameter set. Direct optimisation methods can work well, but obtaining global optima often rely on good estimates of the optimum parameters, or alternatively require stochastic optimisation methods which are computationally intensive.

This paper has shown that the HDMR approach can give reasonably accurate estimates of the response bounds. The computed error tends to decrease significantly with increasing number of sample points (e.g., as the intervals become smaller). However, the computed error does not go to zero as the intervals become smaller. This is because the HDMR approximations (e.g., first order, second order, etc.) are essentially built on truncated series. Thus there will always be some truncation error, irrespective of number of sample points considered.

The computational efficiency of the approach is more difficult to quantify as this depends on the details of the method used to obtain the response bounds of the full system. However, the HMDR approximation may be used either on its own or in collaboration with a full system optimisation. In this discussion we will assume that the only significant

computational burden is calculating the response of the full model. It has been emphasized that a first-order HDMR approximation requires $(s-1) \times M + 1$ function evaluations, where s is the number of HDMR samples and M is the number of parameters. The vertex method for interval analysis requires 2^M function evaluations, which increases significantly for large M and is not guaranteed to give the correct extrema for non-monotonic functions. Exact methods to estimate the response bounds are likely to require even more functional evaluations than the vertex method and hence the HDMR approach can be very efficient for large numbers of parameters.

There is a second use of the HDMR approach. Although the results are only approximate, the HDMR optimal parameters will be close to the optimal parameters of the full system. Thus the HDMR approximation can be used to determine the parameter vertices that are likely to lead to the extreme responses, identify the boundaries in parameter space for further investigation, or provide initial estimates of interior points in parameter space that are likely to give extreme responses. In this way the overall computational cost of finding exact bounds in the response significantly reduced.

7. Conclusions

In the uncertainty assessment of real-life problems with fuzzy variables, the response functions are most often specified implicitly using a finite element code. This paper proposes a new computational method to predict the uncertainty bounds of dynamical systems with fuzziness in loads, material properties and geometric parameters in an efficient manner. It utilized the excellent properties of the HDMR for multivariate function approximation and moving least-squares as an interpolation scheme. The HDMR method is applied to different α−cuts to obtain the fuzzy description of the output variables.

Four numerical examples show the performance of the proposed method. Comparisons are made with the full scale simulation to evaluate the accuracy and computational efficiency of the proposed method. These numerical examples show that the proposed method not only yields accurate estimates of the response bounds compared to the direct simulation approach for highly nonlinear problems, but also reduces the computational effort significantly. Due to a small number of original function evaluations, the proposed HDMR approximations are very effective, particularly when a response evaluation entails costly finite element analysis, or other computationally expensive numerical methods. The proposed approach provides the desired accuracy to the predicted response with the least number of function evaluations. In order to reduce the approximation error further, a second-order HDMR approximation could be used, but the number of function evaluations would be increased quadratically. An optimum number of sample points, s, must be chosen in the function approximation by the HDMR, depending on the desired accuracy and computational resources. A very small number of sample points should be avoided as the HDMR approximation may not capture the nonlinearity within the domain of sample points and this could affect the estimated response significantly.

Acknowledgements

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