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Dynamical response of damped structural systems driven by jump processes

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ABSTRACT

The consideration of uncertainties in numerical models to obtain probabilistic descriptions of vibration response is becoming more desirable for practical problems. In this paper a new method is proposed to obtain statistical properties of the response of damped linear oscillators subjected to Lévy processes. Lévy processes can be used to model physical phenomena that feature jumps. These types of problems are relevant to many civil, mechanical and aerospace engineering problems such as aircrafts subjected to sudden turbulence, wind turbines subjected to hurricanes and automobiles running over pot-holes. The mathematical theory behind Lévy processes is briefly discussed with various examples. These processes are then used to formulate the damped oscillator equation driven by Lévy noise. A relevant existence and uniqueness result for the solution of stochastic differential equations driven by Lévy noise is presented and an explicit form of the solution is found. An Euler scheme is proposed to calculate sample paths of the solution. A numerical example involving an offshore 3 MW twin-blade wind turbine subjected to wind gust is considered to illustrate the application of the proposed method.

1. Introduction

A wide range of civil, mechanical and aerospace engineering systems are subjected to dynamic loads. Often the detailed description of such dynamic loads are not precisely available prior to the observation of such dynamic forces. Examples include buildings and bridges subjected to earthquake and wind loads, cars subjected to uneven road surface, wind turbines loaded by wind gust and aircrafts driven by turbulence. The subject of random vibration was developed since the mid-60s [1–4] to take account of random nature of dynamic *loading* acting on real-life systems. The theory of stationary linear random vibration is now quite a mature research field rich with many useful results which are now routinely used in practice. Some authors have considered stationary random vibration of linear uncertain systems (for example [5,6]). More recent research concerns nonlinear random vibration [7] and systems subjected to non-gaussian and non-stationary random processes [8–11]. In this paper we consider linear systems driven by Lévy processes.

Roberts and Spanos [12] and Redhorse and Spanos [13] used stochastic averaging for random vibration of nonlinear systems. We refer the readers to the recent review papers by Socha [14,15] for comprehensive details. The solution of linear non-stationary random vibration problem can also pose significant challenge. Grigoriu [16] considered Linear systems driven by martingale noise. Grigoriu and his co-authors [17–21] have considered oscillators driven by Lévy process. More recently Grigoriu [22] considered linear systems driven by Poisson white noise. The purpose of this paper is to develop practical computational methods to obtain response statistics of linear systems driven by Lévy process. Lévy processes can be used to model physical phenomena that feature jumps. These types of problems are relevant to many civil, mechanical and aerospace engineering problems such as aircrafts subjected to sudden turbulence, wind turbines subjected to hurricanes and automobiles running over pot-holes. The method developed in this paper is applied to an offshore wind turbine subjected to wind gust.

The outline of the paper is as follows. A brief overview Lévy processes is given in Section 2. The solution of stochastic differential equations with Lévy noise is discussed in Section 3. A numerical algorithm to the damped harmonic oscillator with Lévy noise is proposed in Section 4. A MATLAB ™ code to implement the proposed algorithm is given in Appendix. In Section 5 the proposed method is numerically applied to an off-shore 3 MW twin-blade wind turbine subjected to wind gust. Based on the study taken in the paper, a set of conclusions are drawn in Section 6.

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2. A brief review of Lévy processes

Processes with independent and stationary increments that are continuous in probability, called Lévy processes, can be used to model physical phenomena that feature jumps and have been widely investigated in the fields of economics and finance. The most famous example of a Lévy process is perhaps Brownian motion, the only process that has continuous sample paths. A list of other Lévy processes that have so far been mainly used in financial applications are given at the end of this section. Each of these processes has a different underlying probability distribution that governs the distribution and the size of the jumps of the process. We discuss some of these examples and their sample paths later on. For an introduction to Lévy processes, their simulation, and applications in finance readers are referred to Cont and Tankov [23] and Glasserman [24]. For a mathematically thorough exposition of Lévy processes we refer to the books by Bertoin [25], Jacob [26] and Sato [27]. In the following we aim to keep the necessary mathematical notation to a minimum and refer to the aforementioned textbooks for details.

A stochastic process $(X_t)_{t\geq 0}$ is a function $X_t(\omega)$ such that for any fixed parameter $t \geq 0$ the mappings $\omega \mapsto X(t,\omega)$ are random variables on some underlying probability space. On the other hand, if we fix an outcome ω then the function $t \mapsto X_t(\omega)$ is a sample path or realization of $(X_t)_{t\geq 0}$ corresponding to ω. Whenever all sample paths of the process $(X_t)_{t\geq 0}$ have a certain property we say that the process itself has this property. For example, if all sample paths of the process $(X_t)_{t\geq 0}$ are continuous then we say that $(X_t)_{t\geq 0}$ is continuous. We call a process $(X_t)_{t\geq 0}$ a Lévy process if it has stationary and independent increments and it satisfies the technical condition that X_t converges in distribution to X_s as $t \to s$. By saying that $(X_t)_{t\geq 0}$ has stationary increments we mean that $X_{t+s} - X_s$ has the same distribution as X_t. Let us denote the jump of $(X_t)_{t\geq 0}$ at time $t \geq 0$ by $\Delta X_t := X_t - X_{t-}$ where X_{t-} is the value of the process just before time t. We measure the jump at a fixed time t by choosing a reasonable subset $A \subset \mathbb{R}$, and say $\Delta X_t \in A$ if the jump at time t has a size belonging to A. Then

$$\nu(A) := \mathbb{E}\left(\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}\right) \tag{1}$$

is the expected number of jumps, in the interval $[0, 1]$, whose size belongs to A. We call ν the *Lévy measure* of $(X_t)_{t\geq 0}$. By

$$N(t,A) := \sum_{0<s\leq t} \chi_A(\Delta X_s) \tag{2}$$

we denote the number of jumps of size $\Delta X_s \in A$ which occur before or at the time t. We call $N(t, A)$ the *Poisson random measure* or *jump measure* of $(X_t)_{t\geq 0}$. In (2) we need to make the additional assumption that the closure of A does not contain 0, i.e. we exclude the number of jumps of size $\varepsilon > 0$, where ε is very small. The Lévy–Itô decomposition now tells us that every Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^n$ can be represented as the sum of a deterministic drift, a Brownian motion, and a pure-jump process independent of the Brownian motion:

$$X_t = bt + B^Q_t + \int_{\mathbb{R}^n} z\overline{N}(t,\mathrm{d}z) \tag{3}$$

where $b \in \mathbb{R}^n$, $(B^Q_t)_{t\geq 0}$ is a Brownian motion with covariance matrix Q and

$$\overline{N}(\mathrm{d}t,\mathrm{d}z) = \begin{cases} N(\mathrm{d}s,\mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s, & \text{if } |z| < R \\ N(\mathrm{d}s,\mathrm{d}z), & \text{if } |z| \geq R, \end{cases} \tag{4}$$

for some constant $R \in [0,\infty]$. The triplet (b, Q, ν) is called the *Lévy triplet* of $(X_t)_{t\geq 0}$. The two cases in (4) represent the sum of the small $(|\Delta X_s| < R)$ and the large jumps $(|\Delta X_s| \geq R)$ of $(X_t)_{t\geq 0}$. Given a Lévy process $(X_t)_{t\geq 0}$ with Lévy triplet (b, Q, ν) we also have that

$$\int_{\mathbb{R}^n} \min(|z|^2, 1)\nu(\mathrm{d}z) < \infty, \tag{5}$$

and

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\xi^T X_t}) = \mathrm{e}^{t\psi(\xi)} \tag{6}$$

for $\xi \in \mathbb{R}^n$ where

$$\psi(\xi) = \mathrm{i}b^T\xi - \frac{1}{2}\xi^T Q\xi + \int_{|z|<1}\left(\mathrm{e}^{\mathrm{i}\xi\cdot z} - 1 - \mathrm{i}\xi^T z\right)\nu(\mathrm{d}z) + \int_{|z|\geq 1}(\mathrm{e}^{\mathrm{i}\xi^T z} - 1)\nu(\mathrm{d}z). \tag{7}$$

On the other hand, given a Lévy triplet (b, Q, ν) such that (5)–(7) holds there exists a corresponding Lévy process $(X_t)_{t\geq 0}$.

A Lévy process that has nondecreasing sample paths is called a *subordinator*. Many Lévy processes can be be represented as $(B_{G_t})_{t\geq 0}$ where $(B_t)_{t\geq 0}$ is a Brownian motion and $(G_t)_{t\geq 0}$ is a subordinator independent of $(B_t)_{t\geq 0}$.

Before we consider some examples of Lévy processes, let us point out some notational issues. In line with standard notation, we denote by $(B^Q_t)_{t\geq 0}$ a Brownian motion on $\mathbb{R}^n$ with covariance matrix Q, i.e. B^Q_t is normal $(0, tQ)$ distributed. The process $(B^I_t)_{t\geq 0}$ where I is the $n \times n$ identity matrix is called Wiener process, and in this instance it is custom to write $(W_t)_{t\geq 0}$ instead of $(B^I_t)_{t\geq 0}$. Let us consider some examples.

(i) (*Brownian motion*) Let $(W_t)_{t\geq 0}$ be a Wiener process on $\mathbb{R}^n$ and let Q be a $n \times n$ positive definite matrix. Let σ be a $n \times n$ matrix for which $\sigma\sigma^T = Q$. Then

$$X_t = \sigma W_t$$

is a Lévy process with normal $(0, tQ)$ distribution and characteristic exponent

$$\psi(\xi) = -\frac{1}{2}\xi^T Q\xi,$$

i.e. its Lévy triple is $(0, Q, 0)$. The process $(X_t)_{t\geq 0}$ is also called Brownian motion with covariance matrix Q.

(ii) (*Brownian motion with drift*) Adding a deterministic drift to the example above we find

$$X_t = bt + \sigma W_t$$

is a Lévy process on $\mathbb{R}^n$ with Lévy triple $(b, Q, 0)$ and is called Brownian motion with drift and covariance matrix Q.

(iii) (*Poisson process*) Let $(X_t)_{t\geq 0}$ be a Poisson process on $\mathbb{R}$ with intensity $\lambda > 0$. This is a Lévy process with Lévy triple $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac measure with mass at 1. Its characteristic exponent is given by

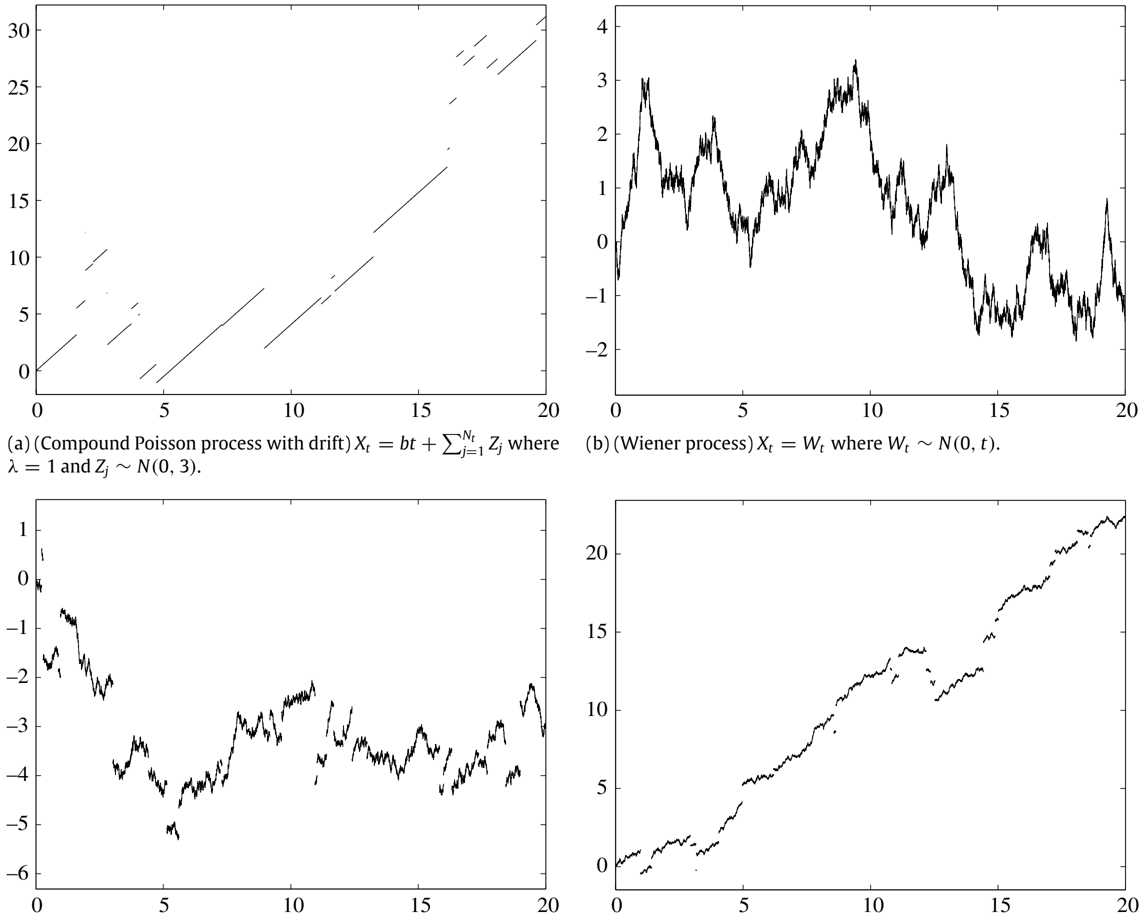
$$\psi(\xi) = \lambda\mathrm{e}^{\mathrm{i}\xi - 1}.$$

(iv) (*Compound Poisson process*) Let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of independent identically distributed random variables taking values in $\mathbb{R}^n$ with common distribution μ_Z and let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ that is independent of all Z_j. The compound Poisson process X_t defined by

$$X_t = \sum_{j=0}^{N_t} Z_j$$

is a Lévy process with Lévy triple $(0, 0, \lambda\mu_Z)$. The characteristic exponent is

$$\psi(\xi) = \int_{\mathbb{R}^n}\left(\mathrm{e}^{\mathrm{i}\xi^T z} - 1\right)\lambda\mu_Z(\mathrm{d}z).$$



(a) (Compound Poisson process with drift) $X_t = bt + \sum_{j=1}^{N_t} Z_j$ where $\lambda = 1$ and $Z_j \sim N(0, 3)$.

(b) (Wiener process) $X_t = W_t$ where $W_t \sim N(0, t)$.

(c) (Jump–diffusion) $X_t = \sigma W_t + \sum_{j=1}^{N_t} Z_j$ where $\sigma = 0.5$, $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda = 2$, and $Z_j \sim N(0, 1)$.

(d) (Jump–diffusion with drift) $X_t = bt + \sigma W_t + \sum_{j=1}^{N_t} Z_j$ where $b = 1, \sigma = 0.5$, $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda = 2$ and $Z_j \sim N(0, 3)$.

Fig. 1. Sample paths of selected Lévy processes.

(v) (*Jump–diffusion process*) Combining examples (i)–(iv) we arrive at the process

$$X_t = bt + \sigma W_t + \sum_{j=0}^{N_t} Z_j. \tag{8}$$

The Lévy triple of this process is $(b, Q, \lambda\mu_Z)$.

There are many other interesting Lévy processes that we will not consider in this paper. The following list is non-exhaustive: α-stable process, Gamma process, variance Gamma process, normal inverse, Gaussian process, generalized hyperbolic process, Meixner process, CGMY process, Pareto process, Weibull process, lognormal process.

Note however that any Lévy process can be approximated to arbitrary precision by a jump–diffusion process. The algorithm used to produce the sample path of the jump–diffusion in Fig. 1(d) is given in Appendix.

Whilst Lévy processes have nice features that make them suitable for modelling phenomena with jumps, the constraints of independence and stationarity of their increments might not be optimal for certain applications. A more general class of stochastic processes are Feller processes. Every Lévy process is also a Feller process, i.e. all the processes mentioned so far are Feller processes. Let us consider a simple extension of (7), i.e.

$$q(x, \xi) = \mathrm{i}b(x)^T\xi + \frac{1}{2}\xi^T Q(x)\xi + \int_{|z|<1} \left(\mathrm{e}^{\mathrm{i}\xi\cdot z} - 1 - \mathrm{i}\xi^T z\right)\nu(x, \mathrm{d}z) + \int_{|z|\geq 1} (\mathrm{e}^{\mathrm{i}\xi^T z} - 1)\nu(x, \mathrm{d}z).$$

Under certain conditions on the functions b, Q and ν a corresponding Feller process exists. These processes do not have stationary increments anymore and the distribution of their increments now depends on the position of the process in space. This topic is explored in detail in [28,26], and further constructions of Feller processes are given in Potrykus [29,30]. In a similar vein we can make the function b, Q and ν time-dependent, i.e. consider $b(t), Q(t)$ and $\nu(A; t)$. As an example, let $b : \mathbb{R}^+ \to \mathbb{R}$ and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions. Define

$$X_t = b(t) + \sigma(t)W_t + \sum_{j=1}^{N_t} Z_j \tag{9}$$

and note the more complex behavior in Fig. 2. These time-inhomogeneous stochastic processes are called additive processes. For an introduction to this topic see [23].

3. Solving stochastic differential equations with Lévy noise

There are several references which discuss ordinary differential equations with Lévy noise. We refer the reader to the papers

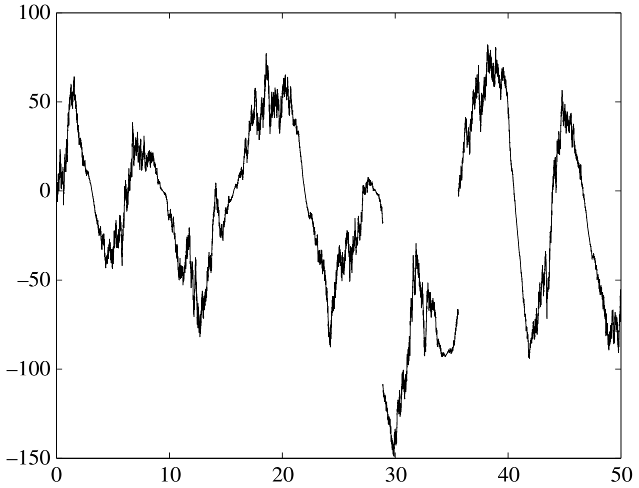


Fig. 2. Sample path of the process $X_t = b(t) + \sigma(t)W_t + \sum_{j=1}^{N_t} Z_j$ where $b(t) = 0.3\sin(t), \lambda = 0.1, Z_j \sim N(0, 1), \sigma(t) = 0.3\cos(\frac{t}{2})$.

by Grigoriu and his co-authors [17–21] and the book by Ideka and Watanabe [31] for an advanced mathematical treatise of this topic. Other more recent references are the ones by Applebaum [32], Protter [33], Oksendal and Sulem [34] as well as Peszat and Zabczyk [35]. This section features some more advanced mathematical concepts while Section 4 discusses the numerical implementation. The existence and uniqueness theorems for solutions of stochastic ordinary differential equations with additive Lévy noise can be found in all the above references in varying generality. We will only discuss results that are sufficiently general to cover the situation we consider later on and refer the reader to the literature for generalizations. Our main reference for the following is [34].

Using the same notation as in the previous section we are now interested in the existence and uniqueness of the solution of the differential equation

$$\mathrm{d}X_t = \alpha(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t + \int_{\mathbb{R}^n} \gamma(t, X_{t-}, z)\overline{N}(\mathrm{d}t, \mathrm{d}z) \tag{10}$$

where $X_0 = x_0 \in \mathbb{R}^n$ and $\alpha : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $(W_t)_{t\geq 0}$ is a one-dimensional Wiener process and $\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. In order to give (10) a rigorous meaning we write it as

$$X_t = X_0 + \int_0^t \alpha(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} \gamma(s, X_{s-}, z)\overline{N}(\mathrm{d}s, \mathrm{d}z).$$

Now, Theorem 1.19 in [34] gives us that if

$$|\sigma(t, x)|^2 + |\alpha(t, x)|^2 + \int_{\mathbb{R}} |\gamma(t, x, z)|^2 \nu(\mathrm{d}z) \leq C_1(1 + |x|^2) \tag{11}$$

for all $x \in \mathbb{R}^n$ and some constant $C_1 < \infty$, and if

$$|\sigma(t, x) - \sigma(t, y)|^2 + |\alpha(t, x) - \alpha(t, y)|^2 + \int_{\mathbb{R}} |\gamma(t, x, z) - \gamma(t, y, z)|^2 \nu(\mathrm{d}z) \leq C_2|x - y|^2 \tag{12}$$

for all $x, y \in \mathbb{R}$ and a constant $C_2 < \infty$, then there exists a unique well-behaved solution $(X_t)_{t\geq 0}$ such that $E(|X_t|^2) < \infty$ for all $t \geq 0$.

We are now interested in the oscillator equation

$$\begin{cases} u''(t) + 2\zeta\omega_0 u'(t) + \omega_0^2 u(t) = F(t) + bt + \sigma W_t + J_t, \\ u(0) = u_0, \\ u'(0) = u_1, \end{cases} \tag{13}$$

where $\zeta \geq 0, \omega_0 \geq 0, F : [0, \infty) \to \mathbb{R}, b \in \mathbb{R}, \sigma \in \mathbb{R}, u_0, u_1 \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a one-dimensional Wiener process and

$$J_t := \sum_{j=1}^{N_t} Z_j \tag{14}$$

is a compound Poisson process where the Z_j's are independent identically distributed random variables and $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ. By

$$Y_t := bt + \sigma W_t + J_t \tag{15}$$

we define a jump–diffusion, i.e. $(Y_t)_{t\geq 0}$ is a Lévy process, see (8). Since the jump component of our Lévy process is a compound Poisson process we have only finitely many jumps per unit time. Indeed, every Lévy process with this kind of jump behavior has as jump component a compound Poisson process and can be represented by (15). There are many other interesting Lévy processes where the jump component J_t is not a compound Poisson process—see for examples the list we provided at the end of Section 2. As mentioned before, even these infinite activity Lévy processes can be approximated by processes of type (15), compare [23], and therefore we restrict ourselves in this paper to the case where the jump component of the Lévy noise is a compound Poisson process.

In order to solve (13) we introduce the vector

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$$

and obtain the following system of first order equations

$$\begin{cases} \mathrm{d}X_t^1 = X_t^2\,\mathrm{d}t \\ \mathrm{d}X_t^2 = \left(-2\zeta\omega_0 X_t^2 - \omega_0^2 X_t^1\right)\,\mathrm{d}t + F(t)\,\mathrm{d}t + bt\,\mathrm{d}t \\ \quad + \sigma\,\mathrm{d}W_t + \,\mathrm{d}J_t. \end{cases} \tag{16}$$

We can also write this as

$$\mathrm{d}X_t = AX_t\mathrm{d}t + H(t)\mathrm{d}t + K\mathrm{d}W_t + L\mathrm{d}J_t \tag{17}$$

where

$$\mathrm{d}X_t = \begin{pmatrix} \mathrm{d}X_t^1 \\ \mathrm{d}X_t^2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix},$$

$$H(t) = \begin{pmatrix} 0 \\ F(t) + bt \end{pmatrix}, \qquad K = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}, \qquad L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that $(X_t)_{t\geq 0}$ is a process on $\mathbb{R}^2$. Since J_t is the jump component of the Lévy process $(X_t)_{t\geq 0}$ the Lévy–Itô decomposition tells us that we can also write

$$\mathrm{d}J_t = \int_{\mathbb{R}} z\overline{N}(\mathrm{d}t, \mathrm{d}z).$$

Identifying the coefficients we can now check the validity of the conditions (11) and (12). We have

$$\alpha(t, X_t) := AX_t + H(t), \tag{18}$$

$$\sigma(t, X_t) := K, \tag{19}$$

$$\gamma(t, X_{t-}, z) := Lz, \tag{20}$$

and it easy to see that (11) and (12) are satisfied. Hence we know that a unique solution to (13) exists. The question we want to answer next is whether it is possible to find an explicit expression for this solution. This requires us to use Itô's formula for Lévy processes which the reader can find in any of the references given at the beginning of this section.

We rewrite (17) as

$$\mathrm{e}^{-At}\mathrm{d}X_t - \mathrm{e}^{-At}AX_t\mathrm{d}t = \mathrm{e}^{-At}\left(H(t)\mathrm{d}t + K\mathrm{d}W_t + L\int_{\mathbb{R}} z \cdot \overline{N}(\mathrm{d}t, \mathrm{d}z)\right).$$

Using Itô's formula for Lévy processes we also have after some calculations

$$d\left(e^{-At}X_t\right) = e^{-At}\,dX_t - e^{-At}AX_t\,dt \tag{21}$$

and thus

$$e^{-At}X_t - X_0 = \int_0^t e^{-As}H(s)\,ds + K\int_0^t e^{-As}\,dW_s + L\int_0^t\int_{\mathbb{R}} z e^{-As}\,\overline{N}(ds, dz).$$

We conclude that

$$X_t = e^{At}\left(X_0 + \int_0^t e^{-As}H(s)\,ds + K\int_0^t e^{-As}K\,dW_s + L\int_0^t\int_{\mathbb{R}} z e^{-As}\,\overline{N}(ds, dz)\right). \tag{22}$$

Let us emphasize again that the solution $(X_t)_{t\geq 0}$ is a process in $\mathbb{R}^2$ where the first component gives us $u(t)$ and the second component gives us $u'(t)$. Finally, let us remark that in order to arrive at (22) we did not use the fact that $(J_t)_{t\geq 0}$ is a compound Poisson process. Indeed, (22) holds in general for Lévy processes with Poisson random measure $\overline{N}(dt, dz)$.

In the next section we discuss how to solve Eq. (13) numerically. Note that one approach would be to simulate the explicit solution (22). It turns out however that the numerical approach using an Euler scheme is much simpler and hence preferable for practical purposes.

4. Numerical solution to the damped harmonic oscillator with Lévy noise

In this section, an Euler scheme for Eq. (13) driven by a jump–diffusion is derived. Topics such as accuracy and convergence of the Euler scheme for stochastic differential equations of type (17) are studied in e.g. [36,37]; see also the survey [38]. Other references for this section are [23,24,39,40].

Let us remind ourselves that we can write our initial value problem (13) as the two-dimensional system of equations (16):

$$\begin{cases} dX_t^1 = X_t^2\,dt \\ dX_t^2 = \left(-2\zeta\omega_0 X_t^2 - \omega_0^2 X_t^1\right)\,dt + F(t)\,dt \\ \qquad + bt\,dt + \sigma\,dW_t + dJ_t. \end{cases} \tag{23}$$

Let us introduce some notation: In order to solve (23) numerically on the interval $[0, T]$ for some $T > 0$, we introduce an equidistant time discretization of this interval by choosing

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_N = T$$

with step size

$$\varDelta = \tau_{n+1} - \tau_n = \frac{T}{N}$$

for some fixed $N \in \mathbb{N}$. Similarly we adopt the standard notation

$$\Delta W = W_{\tau_{n+1}} - W_{\tau_n},$$

i.e. ΔW is the normal distributed $N(0; \varDelta)$ increment of the Wiener process $(W_t)_{t\geq 0}$ on $[\tau_n, \tau_{n+1}]$. In the same way we write for the increment of the jump component

$$\Delta J_n = J_{\tau_{n+1}} - J_{\tau_n}.$$

The Euler scheme for (23) then takes the form

$$\begin{aligned} Y_{n+1}^1 &= Y_n^1 + Y_n^2\varDelta \\ Y_{n+1}^2 &= Y_n^2 - 2\zeta\omega_0 Y_n^2\varDelta - \omega_0^2 Y_n^1\varDelta + F(\tau_n)\varDelta \\ &\quad + b\tau_n\varDelta + \sigma\Delta W + \Delta J_n. \end{aligned} \tag{24}$$

We can solve the first equation in (24) by writing

$$Y_n^2 = \frac{1}{\varDelta}(Y_{n+1}^1 - Y_n^1).$$

Inserting this into the second equation of (24) we arrive at

$$\begin{aligned} Y_{n+2}^1 &= (2 - 2\zeta\omega_0\varDelta)Y_{n+1}^1 + (2\zeta\omega_0\varDelta - \omega_0^2\varDelta^2 - 1)Y_n^1 \\ &\quad + F(\tau_n)\varDelta^2 + b\tau_n\varDelta^2 + \sigma\Delta W\varDelta + \Delta J_n\varDelta \end{aligned} \tag{25}$$

where we initialize

$$\begin{aligned} Y_1^1 &= u_0, \\ Y_1^2 &= u_1, \\ Y_2^1 &= Y_1^1 + Y_1^2\varDelta = u_0 + u_1\varDelta. \end{aligned}$$

A time discrete approximation $Y^{\varDelta}$ is said to converge strongly with order $\gamma > 0$ at time T to the solution Y of a given stochastic differential equation if there exists a positive constant C and a $\varDelta_0 > 0$ such that

$$\sqrt{E(|Y_T - Y_T^{\varDelta}|^2)} \leq C\varDelta^{\gamma}$$

for each maximum time step size $\varDelta \in (0, \varDelta_0)$. It is shown numerically in e.g. [41], Fig 5.1, that the above Euler scheme has strong order 0.5. Note that we could have chosen more complicated higher order schemes to improve various aspects of this iterative scheme. Here we want to focus on simulating the Lévy noise term. For a comparison of numerical schemes for solving stochastic differential equations we refer the interested reader to [38,40].

Let us now make some remarks concerning the iterative scheme (25). The jump ΔJ_n is multiplied by $\varDelta$, the time step. In other words when simulating the solution and implementing the jumps graphically, one should expect the jumps to get smaller as the mesh gets finer. Eventually, the jumps will no longer be visible. This suggests that the solution should be drawn as a continuous function, compare also the explicit solution (22). A more thorough discussion with examples is given in Section 5.

It is straightforward to simulate ΔB since Brownian motion has stationary increments and thus $B_{\tau_{n+1}} - B_{\tau_n}$ has distribution $N(0; \varDelta)$. Simulating normal distributed random variables is trivial with most software packages. Therefore it remains to take care of the term ΔJ_n. Since

$$\Delta J_n = J_{\tau_{n+1}} - J_{\tau_n} = \sum_{j=N_{\tau_n}}^{N_{\tau_{n+1}}} Z_j, \tag{26}$$

where $(N_t)_{t\geq 0}$ is Poisson process with intensity $\lambda > 0$, the only difficulty is to simulate the independent identically distributed random variables $Z_j, j = 1, 2, \ldots$ and the Poisson process $(N_t)_{t\geq 0}$. Assuming that can be done, we can interpret (26) as meaning that ΔJ_n is the sum of all jumps in the interval $[\tau_n, \tau_{n+1}]$ where the size of each jump is given by one of the Z_j's. Simulating random variables can be done using the acceptance–rejection technique—a description of this technique together with more specialised algorithms for simulating many of the more common probability distributions is given in [42]. The MATLAB ™ code that implements (25) is given in the Appendix. Note also, that

In line with our remarks at the end of Section 2 it is quite straightforward to make the coefficients in the noise that drives our physical system time-dependent, i.e. instead of (13) we now consider

$$\begin{cases} u''(t) + 2\zeta\omega_0 u'(t) + \omega_0^2 u(t) = F(t) + b(t) + \sigma(t)W_t + J_t, \\ u(0) = u_0, \\ u'(0) = u_1. \end{cases} \tag{27}$$

In this case the driving noise is given by

$$Y_t := b(t) + \sigma(t)W_t + J_t. \tag{28}$$

Note that $(Y_t)_{t\geq 0}$ as defined by (28) does no longer have stationary increments and hence is not a Lévy process—it is an additive process. The algorithm (25) then changes accordingly to

$$Y^1_{n+2} = (2 - 2\zeta\omega_0\Delta)Y^1_{n+1} + (2\zeta\omega_0\Delta - \omega_0^2\Delta^2 - 1)Y^1_n + F(\tau_n)\Delta^2 + b(\tau_n)\Delta^2 + \sigma(\tau_n)\Delta W\Delta + \Delta J_n\Delta. \tag{29}$$

Convergence results concerning Euler schemes for time-inhomogeneous equations of type (17) can be found in the references mentioned at the beginning of this section. In the same manner we can also introduce space-dependence in (28). Note however that introducing space-dependent coefficients would require us to make major modifications, for details concerning the diffusion case without jumps, see [40].

5. Numerical case study: An offshore wind turbine driven by wind gusts

We consider a typical wind turbine structure as shown in Fig. 3. The response of wind turbines driven by wind gust is of crucial importance to the power generation. In the high wind, wind turbines are often shutdown to prevent damage to the structure. Turbines are turned on again when the wind speed comes down below the dangerous level. Here we use Lévy process to model wind gust. Using a positive value of b in Eq. (8) one can model the increasing wind speed, for example as observed during the onset of a hurricane. A negative value b may also be used to model the retreat of a hurricane. The parameter b may be estimated from the time history of the wind velocity data. The energy operators aim to minimize any disruptions in the energy production arising due to the shutdown of the turbines due to the possibility of structural damage. The structural damage in turn depends of the dynamic response of the structure driven by the wind load. Here we use the proposed algorithm and a simplified structural model to understand the dynamic response.

This system is idealized by an Euler–Bernoulli beam fixed at the bottom ($x = 0$) and the idealization process is explained in the diagram. The bending stiffness of the beam is EI and the beam is attached to the base. Here x is the spatial coordinate, starting at the bottom and moving along the height of the structure. The beam has a top mass M. This top mass is used to idealize the rotor and blade system. The mass per unit length of the beam is m.

The equation of motion of the beam is given by (see the book by Géradin and Rixen [43] for the derivation of this equation):

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m\ddot{w}(x,t) = f(t)\delta(x-L). \tag{30}$$

Blade passing frequency is typically kept below the first natural frequency, or between the first and second natural frequency of the system. For this reason we use an approximate single-degree-of-freedom (SDOF) model of the system. Following Blevins [44], around the first mode of vibration, the cantilever column with an end-mass (30) can be approximated as an SDOF oscillator with stiffness $\frac{3EI}{L^3}$ and mass $(0.24mL + M)$. The natural frequency of the equivalent oscillator can be given by

$$\omega_0 = \sqrt{\frac{3EI/L^3}{0.24mL + M}}. \tag{31}$$

The equation of motion of the damped oscillator is therefore given by Eq. (13) where ω_0 is defined in Eq. (31).

For the numerical calculations we consider a 3 MW offshore turbine considered by Tempel and Molenaar [45]. Table 1 shows the material and geometric properties of the constant speed two-bladed Opti-OWECS turbine.

Table 1
Material and geometric properties of the turbine structure (Tempel and Molenaar [45]).

Turbine structure properties	Numerical values
Length (L)	81 m
Average diameter (D)	3.5 m
Thickness (t_h)	0.075 m
Mass density (ρ)	7800 kg/m^3
Young's modulus (E)	2.1×10^{11} Pa
Rotational speed (ϖ)	22 r.p.m = 0.37 Hz
Top mass (M)	130,000 kg
Rated power	3 MW

The moment of inertia of the circular cross section can be obtained as

$$I = \frac{\pi}{64}D^4 - \frac{\pi}{64}(D - t_h)^4 \approx \frac{1}{16}\pi D^3 t_h = 0.6314\ \text{m}^4. \tag{32}$$

The mass density per unit length of the system can be obtained as

$$m = \rho A \approx \rho\pi D t_h/2 = 3.1817 \times 10^3\ \text{kg/m}. \tag{33}$$

Using these, one obtains the natural frequency of the structure

$$\omega_0 = \sqrt{\frac{3EI/L^3}{0.24mL + M}} = 1.9436\ \text{rad/s}. \tag{34}$$

We now plot the solutions to Eqs. (13) with $\omega_0 = 1.9436$. In this case the solution corresponds to the response of the rotor-head which is of primary practical importance. One of the interest in this study is to understand how the different parameters influence the behavior of the sample paths. The parameters one can choose are:

- ζ: damping ratio
- $F : [0, \infty) \to \mathbb{R}$: external force
- b: drift of Lévy noise
- σ: standard deviation of Brownian component
- λ: intensity of Poisson process in compound Poisson component
- Z_j's: size of the jumps given by distribution of the Z_j's
- u_0: initial displacement
- u_1: initial velocity.

As mentioned in Section 4, implementing Eq. (25) yields discontinuities whenever a jump happens. On the other hand, it is clear that the rotor head can only move continuously in space, i.e. the response should *not* have jumps. The explicit solution (22) confirms that the response should indeed be continuous as one integrates the discontinuities. It is important to note here the difference between implementing the solution via the Euler scheme and the exact physical solution. Although the Euler scheme dictates that jumps appear in the response, these jumps should not be implemented when programming the algorithm. Therefore, from now on the continuous response is presented graphically together with the discontinuous input noise to emphasize the influence of the jumps.

As we are most interested in the effects that the Lévy noise has on our system, we will assume that $F(t) \equiv 0$ as well as $u_0 = u_1 = 0$. We also fix $\zeta = 0.05$, i.e. we consider 5% damping. Since we are mainly interested in the jump-part of the noise, we furthermore consider a fixed value of $\sigma = 0.05$. In Fig. 4 a typical sample path of the input noise and the solution driven by this noise for $\lambda = 0.1$ is plotted. The jumps in the input noise coincide with sudden excitations in the dynamic response of the system. It is clear from the picture that we can indeed use Lévy noise to take into account sudden wind gusts. This type of behavior of the solution cannot be achieved by simply considering Wiener noise. In the same plot we show the standard deviation of the response calculated using

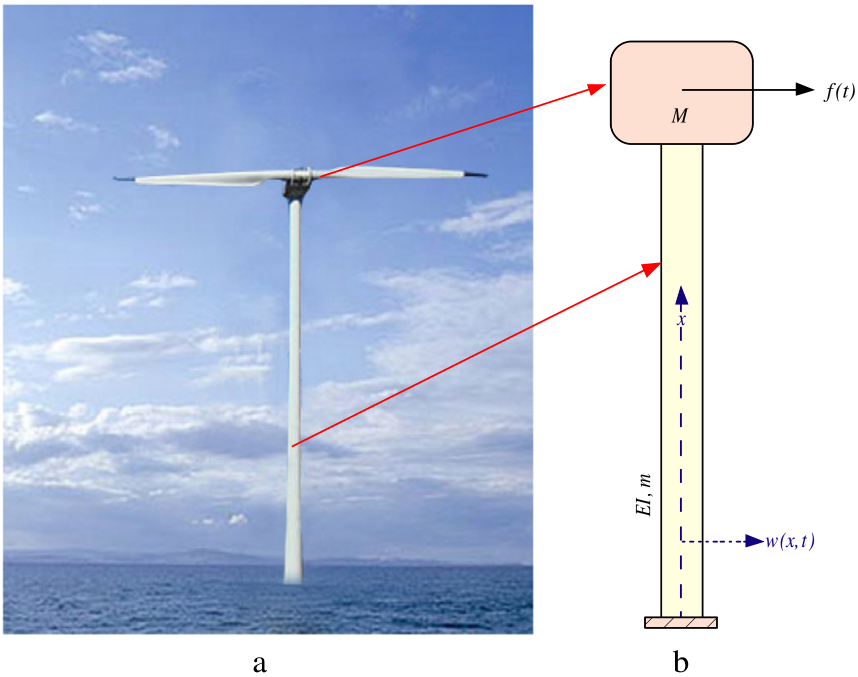
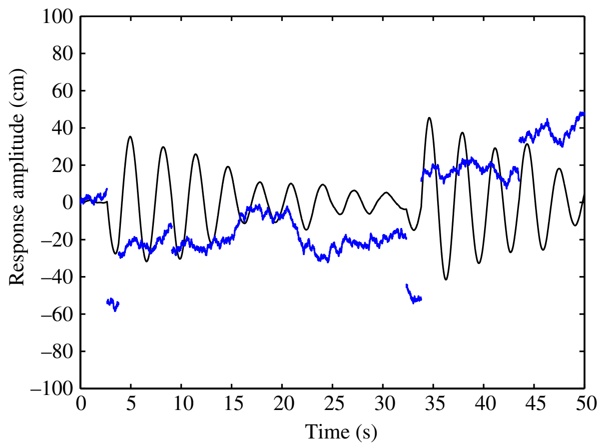
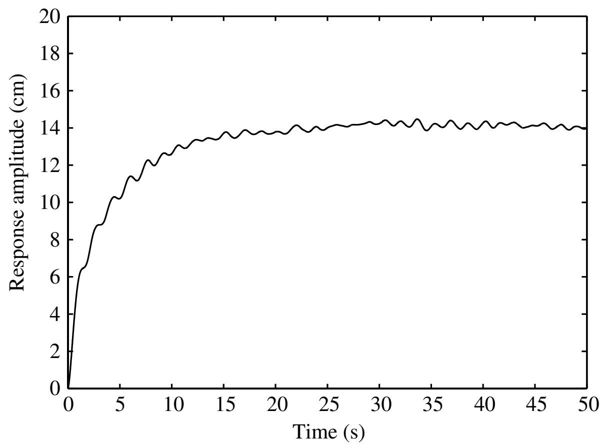


Fig. 3. Idealization of a wind turbine using Euler Bernoulli beam with a top mass. The mass of the rotor-hub and blades is assumed to be M. The forcing due to wind $f(t)$ is assumed to be acting at the rotor hub; (a) actual system, (b) mathematical idealization.



(a) A typical sample path of the response and the input noise.



(b) Standard deviation of the response (calculated using 10,000 samples).

Fig. 4. Input noise and response of the rotor-head driven by Lévy noise without drift: $b = 0, \sigma = 0.05, \lambda = 0.1, Z_j \sim N(0, 0.5), \zeta = 0.05, \omega_0 = 1.9436$.

10,000 samples. The standard deviation is governed by the strength of the Wiener noise part of the input (that is the value of σ). It may be noted that after an initial transient period, the standard deviation of the response reaches a steady state. This behavior is expected for any dynamic system with damping.

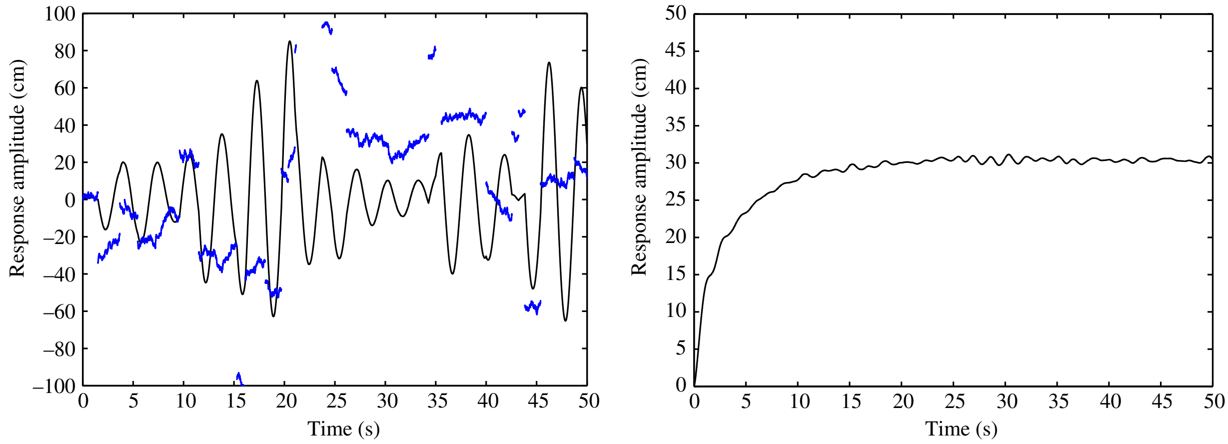
In Fig. 5 we consider $\lambda = 0.5$ and note that the number of jumps accordingly increases. By increasing the number of jumps we can model the behavior of the wind turbine under more erratic winds. Note that we chose to model the jump sizes (wind gusts) by $N(0, 0.1)$ distributed random variables. The amplitude of the standard deviation is also significantly different from the previous case. However, for both the cases the standard deviation of the response converges to a constant value depending on the value of σ.

Now we consider a non-zero value of the drift parameter b. In Fig. 6 we have plotted a sample path of the solution for $b = 0.03$ and $\lambda = 0.1$.

In Fig. 7 we have plotted a sample path of the solution for $\lambda = 0.5$. Note that the size of the jumps are more compared to the previous case since the value of λ is bigger. Due to the non-zero value of the drift term, the input and consequently the response have an increasing trend. This can be used to effectively model an incoming hurricane or other high wind-velocity weather events. Such event are crucial to the operation and safety of the wind turbines. As expected the the drift term has no effect on the standard deviation.

6. Conclusions

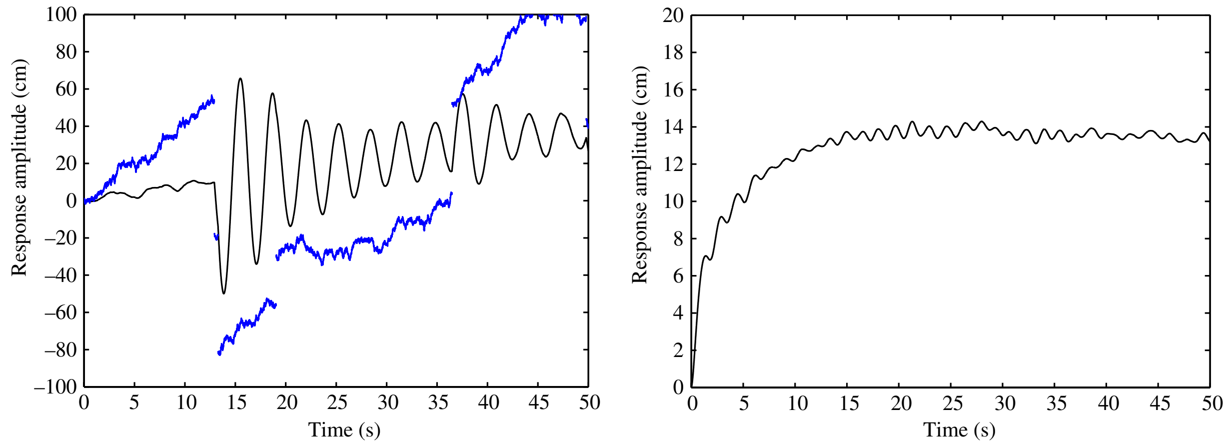
Dynamic response of damped linear oscillators subjected to Lévy noise is considered. Necessary mathematical backgrounds for Lévy processes have been discussed. Practical reasons behind the importance of Lévy processes in many civil, mechanical and aerospace engineering problems have been highlighted. One of the main contribution of the paper is the solution of stochastic differential equations driven by Lévy noise. An explicit simple simulation method has been proposed and as an example MATLAB ™



(a) A typical sample path of the response and the input noise.

(b) Standard deviation of the response (calculated using 10,000 samples).

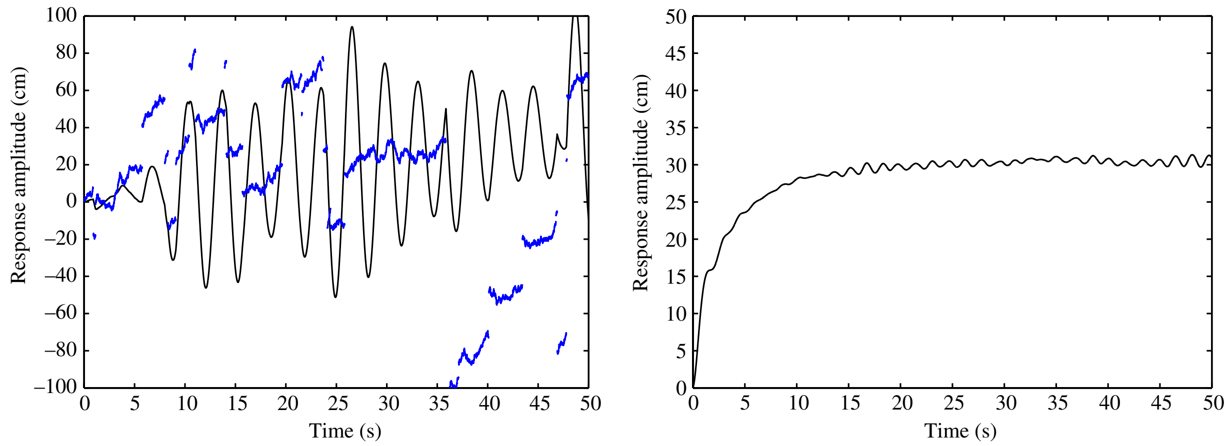
Fig. 5. Input noise and response of the rotor-head driven by Lévy noise without drift: $b = 0$, $\sigma = 0.05$, $\lambda = 0.5$, $Z_j \sim N(0, 0.5)$, $\zeta = 0.05$, $\omega_0 = 1.9436$.



(a) A typical sample path of the response and the input noise.

(b) Standard deviation of the response (calculated using 10,000 samples).

Fig. 6. Input noise and response of the rotor-head driven by Lévy noise with drift: $b = 0.03$, $\sigma = 0.05$, $\lambda = 0.1$, $Z_j \sim N(0, 0.5)$, $\zeta = 0.05$, $\omega_0 = 1.9436$.



(a) A typical sample path of the response and the input noise.

(b) Standard deviation of the response (calculated using 10,000 samples).

Fig. 7. Input noise and response of the rotor-head driven by Lévy noise with drift: $b = 0.03$, $\sigma = 0.05$, $\lambda = 0.5$, $Z_j \sim N(0, 0.5)$, $\zeta = 0.05$, $\omega_0 = 1.9436$.

scripts have been given to implement the new procedure in digital computers.

Our implementation of a damped oscillator driven by Lévy noise gives us a realistic approach to consider actual physical systems. The method can be readily extended to proportionally damped multiple degree of freedom systems. Via a numerical example, it was shown how the proposed method enables one to take into account sudden wind gust when studying the vibrations of a wind turbine. As we have remarked earlier, it is possible to further extend our model by making some of the noise parameters time- and

Listing 1: Code for generating sample paths of jump–diffusions

```
% Calculates X(t) = b*t+sigma*W(t)+J(t)
% where b = drift of the process, sigma = scaling of Wiener process,
% W(t) = Wiener process, J(t) = compound Poisson process with N(0,jumpPar1)
% distributed jump sizes and intensity lambda
%
% OUTPUT: yValues contains the values of a sample path of X(t) at discrete time
% steps

b=0; lambda=1; jumpPar1=0.1; sigma=0.05; %
T=10; xMesh=2^(-10);                      % problem parameters

N=poissrnd(lambda*T);
U=sort(unifrnd(0,T,1,N));
Z=normrnd(0,jumpPar1,1,N);                % jump size distribution
W=normrnd(0,xMesh^(1/2),1,T/xMesh+1);     % Wiener process


for x=1:1:T/xMesh+1
    J=0;
    for j=1:N
        if U(j)<(x-1)*xMesh
            J=J+Z(j);
        end
    end
 yValues(x)=b*(x-1)*xMesh+sigma*sum(W(1:x))+J;
end
```

Listing 2: Code for generating sample paths of the solution of (13)

```
% Solves u''(t)+2*zeta*w0*u'(t)+w0*w0*u(t)=F(t)+b*t+sigma*W(t)+J(t)
% where u(t) = displacement of the mass relative to its equilibrium position,
% zeta = damping ratio, w0 = natural frequency, F(t) = externally applied
% forcing function, b = drift, sigma = scaling of Wiener process, W(t) = Wiener
% process, J(t) = compound Poisson with N(0,jumpPar1) distributed jump sizes
% and intensity lambda, u0 = initial displacement of the mass, u1 = initial
% velocity of the mass
%
% OUTPUT: yValues contains the values of a sample path of the solution at
% discrete time steps

zeta=0.05; w0=1.9436; F=inline('0','x'); b=0; %
sigma=0.05; lambda=0.1; jumpPar1=0.5; u0=0;   %
u1=0; T=60; xMesh=2^(-10);                    % problem parameters

yValues=zeros(1,T/xMesh+1);
yValues(1)=u0;
yValues(2)=yValues(1)+u1*xMesh;

N=poissrnd(lambda*T);
U=sort(unifrnd(0,T,1,N));
Z=normrnd(0,jumpPar1,1,N);
W=normrnd(0,xMesh^(1/2),1,T/xMesh+1);
J=zeros(1,T/xMesh+1);

x=1;
for j=1:N
    while (U(j)>=x*xMesh)
        x=x+1;
    end
    if (x*xMesh<=T)
        J(x)=J(x)+Z(j);
    end
end

for x=1:1:T/xMesh-1
       yValues(x+2)=(2-2*zeta*w0*xMesh)*yValues(x+1)+...
         +(2*zeta*w0*xMesh-w{0}^{2}*xMesh^{2}-1)*yValues(x)+...
         +F(x*xMesh)*xMesh^{2}+b*x*xMesh*xMesh^{2}+sigma*W(x)*xMesh+...
         +(J(x+1)-J(x))*xMesh;
end
```

space-dependent to cover even more general situations. As an example, a wind turbine in a hurricane that lasts several hours can be considered. For the duration of the hurricane it might be sensible to include a non-zero drift term in the input noise. Further work should not only include studying these generalizations and their numerical implementations, but should also focus on finding the Lévy noise that best fits real observations. The authors hope to undertake some of this work in the future.

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Appendix. Algorithm for numerical solution of Lévy-driven stochastic differential equations

In this section we give the algorithm for creating sample paths of jump–diffusion processes (Listing 1) as well as the algorithm that implements the solution of our Lévy-driven stochastic differential equation (Listing 2). The algorithms will be given as MATLAB ™ code but can quite easily be implemented in any other programming language.

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