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Dynamic stiffness of randomly parametered beams

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A finite element-based methodology is developed for the determination of the dynamic stiffness matrix of Euler–Bernoulli beams with randomly varying flexural and axial rigidity, mass density and foundation elastic modulus. The finite element approximation made employs frequency dependent shape functions and the analysis avoids eigenfunction expansion which, not only eliminates modal truncation errors, but also, restricts the number of random variables entering the formulations. Application of the proposed method is illustrated by considering two problems of wide interest in engineering mechanics, namely, vibration of beams on random elastic foundation and the problem of seismic wave amplification through randomly inhomogeneous soil layers. Satisfactory agreement between analytical solutions and a limited amount of digital simulation results is also demonstrated. © 1997 Elsevier Science Ltd.

1 INTRODUCTION

The direct dynamic stiffness method is a useful alternative to the more popular mode superposition method of vibration analysis. A vast amount of literature is available on the development of this method (see, for example, Refs [1–7]). Some of the notable features of the method, which can be discerned from the available literature[1–7] are:

1. the mass distribution of the element is treated in an exact manner in deriving the element dynamic stiffness matrix;
2. the dynamic stiffness matrix of one-dimensional structural elements taking into account the effects of flexure, torsion, axial motion, shear deformation effects and damping are exactly determinable, which, in turn, enables the exact vibration analysis of skeletal structures by an inversion of the global dynamic stiffness matrix;
3. the method does not employ eigenfunction expansions and, consequently, a major step of the traditional finite element analysis, namely, the determination of natural frequencies and mode shapes, is eliminated, which automatically avoids the errors due to series truncation. This makes the method attractive for situations in which a large number of modes participate in vibration;
4. since the modal expansion is not employed, *ad hoc* assumptions concerning the damping matrix being proportional to mass and/or stiffness is not necessary;
5. the method is essentially a frequency domain approach suitable for steady state harmonic or stationary random excitation problems; generalization to other type of problems through the use of Laplace transforms is also available;
6. the static stiffness matrix and consistent mass matrix appear as the first two terms in the Taylor expansion of the dynamic matrix in the frequency parameter.

The present paper considers the problem of determination of dynamic stiffness coefficients of a Euler–Bernoulli beam element that has randomly inhomogeneous mass, flexural and axial rigidities and which rests on an elastic foundation with random elastic foundation modulus. The system properties are modeled as a vector of homogeneous stochastic fields. The dynamic stiffness coefficients are, by definition, frequency dependent, and, when system properties are modeled as random fields, these coefficients become random variables for a fixed value of the driving frequency. As the driving frequency is varied, these stiffness coefficients become random processes evolving in the frequency parameter. Furthermore, the presence of damping in the system makes these random processes complex in nature. Thus, the problem on hand consists of the description of the stiffness coefficients as complex valued random processes. This problem is governed by a stochastic boundary value problem. The solution procedure employed in this study is based on the application of the finite element method, which uses frequency dependent shape functions. This offers a powerful means to discretize the system property random fields and relaxes the dependence of finite element mesh size with respect to the frequency of excitation. As a prelude to the development of the method, we first present a brief review of the literature on vibration analysis of structures with random parameter variations.

2 A BRIEF REVIEW OF THE LITERATURE

Problems of structural dynamics, in which the uncertainty in specifying mass and stiffness of the structure is modeled within the framework of probability and random processes, are currently receiving notable research attention. Mathematical problems arising in these studies consist of random differential and matrix eigenvalue problems, inversion of random matrix and differential operators and characterization of random matrix products. The study of these problems is rich with new challenges, not only from a mathematical point of view of developing acceptable solution procedures, but also, from the point of view of constructing appropriate probabilistic models for the uncertainties in terms of random fields, identifying parameters of these models, investigation of phenomenological features associated with structural system disorders and designing and performing laboratory experiments on an ensemble of disordered structural systems. Early studies in this field have been comprehensively reviewed by Ibrahim[8] and subsequent progress can be traced by studying references.[9–14] A notable development in recent years has been the generalization of the finite element methods to include random field models for the system elastic and mass properties, see the monographs by Ghanem and Spanos[12] and Kleiber and Hien.[14] Application of the stochastic finite element method to linear structural dynamics problems typically consists of the following key steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions; both Gaussian and non-Gaussian random fields have been proposed.
2. Replacement of the element property random fields by an equivalent set of a finite number of random variables. Several procedures are available for this purpose; these are the method of local averages,[15] mid-point method,[16] discretizing using deterministic shape functions[17] and stochastic shape functions,[18] method of weighted integrals,[19–22] discretization using Karhunen expansion and other orthogonal expansions for the element random fields[12,23–24] and optimal linear estimation-based methods.[25] The accuracy with which the random fields get represented primarily depends upon the size of the finite element used. The factors to be considered in the selection of mesh size which, in turn, influence the accuracy and efficiency of discretization are: stress and strain gradients; frequency range of interest, nature of the information available about the random field; correlation length of the random fields representing the element properties and the excitations; ability to model accurately the tails of the pdf especially when non-Gaussian models are being used; ability to model non-homogeneous random fields; stability of numerical inversion of the probability transformations; gradient of limit state function; and the desire to introduce a minimum number of random variables into the analysis.
3. Formulation of the system equations of motion of the form $M\ddot{X} + C\dot{X} + KX = F(t)$ where elements of M, C and K are random variables.
4. Several studies on the solution of these equations, based on mode superposition methods, Fourier domain solutions by inversion of the dynamic stiffness matrix $K - \omega^2 M + i\omega C$, direct integration either in configuration space or in the modal space, have been developed. A first step in the application of the mode superposition method is the solution of the matrix random eigenvalue problem $Kx = \lambda Mx$, which in itself has attracted the wide attention of researchers. The response analysis procedures include perturbation methods,[14,26,27] Monte Carlo simulation procedures and series expansion methods.[12,24]

Some of the previous studies by us on structural dynamics with random property variations include the study on analytical determination of probability density functions of natural frequencies and mode shapes of stochastic strings,[28] vibration energy flow in random ensemble of beam and axially vibrating rod assemblies[29] and harmonic response of axially vibrating rods and random soil layers by the use of the theorem of stochastic averaging.[30]

Application of the mode superposition method of vibration analysis to randomly parametered structures has two limitations: firstly, it requires the determination of the joint statistics of natural frequencies and mode shapes, which is, by no means, an easy task; and secondly, the series expansion introduces a large number of random variables, which is, at least, twice as many as the number of modes retained in the modal expansion which, in turn, increases the size of integration on joint probability distributions while evaluating the response statistics. It must also be noted that the size of the finite element mesh used in this type of analysis depends upon the highest mode to be included in the analysis with the size being smaller for higher frequencies of interest. Besides, when the stochastic mode shapes are not determined exactly, which, most often, will be the case, the question of orthogonality of mode shapes requires careful interpretation. The dynamic stiffness method is eminently suited to avoid these limitations and, motivated by this consideration, we have attempted in this paper to develop the dynamic stiffness matrix for randomly parametered beam elements.

3 GOVERNING EQUATIONS

The beam element considered in this study is shown in Fig. 1. It is assumed here that the axial forces are small to the extent that they do not affect the flexural deformations. It is also assumed that the behavior of the beam follows the Euler–Bernoulli hypotheses and the beam rests on Winkler's elastic foundation. The governing field equations of motion under these assumptions are given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 Y}{\partial x^2}+c_1\frac{\partial^3 Y}{\partial x^2\partial t}\right]+m(x)\frac{\partial^2 Y}{\partial t^2}$$
$$+c_2\frac{\partial Y}{\partial t}+k(x)Y=0 \quad (1)$$

$$\frac{\partial}{\partial x}\left[AE(x)\frac{\partial U}{\partial x}+c_3\frac{\partial^2 U}{\partial x\partial t}\right]=m(x)\frac{\partial^2 U}{\partial t^2}+c_4\frac{\partial U}{\partial t} \quad (2)$$

Here $Y(x,t)$ = transverse flexural displacement, $U(x,t)$ = axial displacement, $EI(x)$ = flexural rigidity, $AE(x)$ = axial rigidity, $m(x)$ = mass per unit length, $k(x)$ = elastic foundation modulus, c_1 and c_3 = strain rate dependent viscous damping coefficients and c_2 and c_4 = velocity dependent viscous damping coefficients. The quantities $k(x)$, $EI(x)$, $AE(x)$ and $m(x)$ in this study are modeled as mean-square bounded, homogeneous random fields and are taken to have the following form:

$$\begin{aligned} k(x) &= k_0[1+\epsilon_1 f_1(x)] \\ m(x) &= m_0[1+\epsilon_2 f_2(x)] \\ EI(x) &= EI_0[1+\epsilon_3 f_3(x)] \\ AE(x) &= AE_0[1+\epsilon_4 f_4(x)] \end{aligned} \quad (3)$$

Here '0' subscript indicates the mean values, $0<\epsilon_i\ll 1$ $(i=1,\ldots,4)$ are deterministic constants and the random fields $f_i(x)$ are taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$. Of the four random fields $f_i(x)$, $(i=1,\ldots,4)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ can be expected to be mutually correlated while $f_1(x)$ can be taken to be uncorrelated to these processes. This assumption of uncorrelatedness, however, is not crucial to the development of the approach described in this paper. Furthermore, since, $k(x)$, $m(x)$, $EI(x)$ and $AE(x)$ are strictly positive, $f_i(x)$

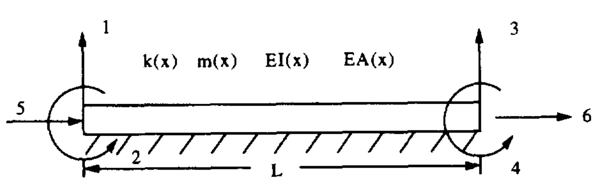

Fig. 1. Random beam element on Winkler's foundation.

$(i-1,\ldots,4)$ are required to satisfy the conditions $P[1+\epsilon f_i(x)\le 0]=0$. This requirement, strictly speaking, rules out the use of Gaussian models for these random fields. But, however, for small ϵ_i, it is expected that Gaussian models still can be used if the primary interest in the analysis is on finding the first few response moments and not on the response behavior near the tails of the probability distributions. To determine the dynamic stiffness matrix, we begin by applying harmonic forces, $P_i\exp[i\omega t]$ at the nodes $i=1,\ldots,6$ and seek to relate these forces to the nodal harmonic displacements $\delta_i\exp[i\omega t]$. Since linear system behavior is assumed, the solution to the field equations can be taken to be of the form

$$Y(x,t)=y(x)\exp[i\omega t] \quad (4)$$

$$U(x,t)=u(x)\exp[i\omega t] \quad (5)$$

Consequently, the equations governing $y(x)$ and $u(x)$ have the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+i\omega c_1\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]+[k(x)-m(x)\omega^2$$
$$+c_2 i\omega]y=0 \quad (6)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[AE(x)\frac{\mathrm{d}u}{\mathrm{d}x}+i\omega c_3\frac{\mathrm{d}u}{\mathrm{d}x}\right]+[\omega^2 m(x)-i\omega c_4]u=0 \quad (7)$$

To determine the element dynamic stiffness matrix, it is necessary to solve the above equations under two sets of boundary conditions given by

$$\begin{aligned} &y(0)=\delta_1,\quad \frac{\mathrm{d}y}{\mathrm{d}x}(0)=\delta_2,\quad y(L)=\delta_3, \\ &\frac{\mathrm{d}y}{\mathrm{d}x}(L)=\delta_4,\quad u(0)=\delta_5,\quad u(L)=\delta_6 \end{aligned} \quad (8)$$

and

$$\begin{aligned} &\left[EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+i\omega c_1\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right](0)=-P_2, \\ &\frac{\mathrm{d}}{\mathrm{d}x}\left[EI(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+i\omega c_1\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right](0)=P_1, \\ &\left[EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+i\omega c_1\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right](L)=P_4, \\ &\frac{\mathrm{d}}{\mathrm{d}x}\left[EI(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+i\omega c_1\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right](L)=-P_3, \\ &\left[AE\frac{\mathrm{d}u}{\mathrm{d}x}+i\omega c_3\frac{\mathrm{d}u}{\mathrm{d}x}\right](0)=-P_5, \\ &\left[AE\frac{\mathrm{d}u}{\mathrm{d}x}+i\omega c_3\frac{\mathrm{d}u}{\mathrm{d}x}\right](L)=P_6 \end{aligned} \quad (9)$$

Equations (6) and (7), together with the above boundary conditions, constitute a set of stochastic boundary value problems. Exact solutions to this type of problem are currently not available and, in this study, we use the finite element method to obtain approximate solutions.

4 FINITE ELEMENT APPROXIMATION

4.1 Shape functions

The selection of shape functions is based on the solution of the deterministic undamped field equations, that is, eqns (6) and (7) with $c_i = 0$ and $\epsilon_i = 0$, $(i = 1, \dots, 4)$. With the notations

$$b^4 = \frac{m_0\omega^2 - k_0}{EI_0}; \quad a^2 = \frac{m_0\omega^2}{AE_0} \tag{10}$$

the governing equations can be written as

$$\frac{d^4y}{dx^4} - b^4 y = 0 \tag{11}$$

$$\frac{d^2u}{dx^2} + a^2 u = 0 \tag{12}$$

To generate the shape functions, we need the solutions of the above pair of equations under 'binary' boundary conditions. The flexural motions are considered first and, depending on the sign of b^4, three cases arise:

Case 1: $b^4 > 0$

$$y(x) = c_1 \sin bx + c_2 \cos bx + c_3 \sinh bx + c_4 \cosh bx \tag{13}$$

Case 2: $b^4 = 0$

$$y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4 \tag{14}$$

Case 3: $b^4 < 0$

$$y(x) = c_1 \sin bx \sinh bx + c_2 \sin bx \cosh bx + c_3 \cos bx \sinh bx + c_4 \cos bx \cosh bx \tag{15}$$

Accordingly, the shape functions for these cases can be shown to be given by

$$\begin{Bmatrix} N_1(x,\omega) \\ N_2(x,\omega) \\ N_3(x,\omega) \\ N_4(x,\omega) \end{Bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{cS+Cs}{cC-1} & -\frac{1}{2}\frac{1+sS-cC}{cC-1} & -\frac{1}{2}\frac{cS+Cs}{cC-1} & \frac{1}{2}\frac{cC+sS-1}{cC-1} \\ \frac{1}{2}\frac{cC+sS-1}{b(cC-1)} & \frac{1}{2}\frac{-Cs+cS}{b(cC-1)} & -\frac{1}{2}\frac{1+sS-cC}{b(cC-1)} & -\frac{1}{2}\frac{-Cs+cS}{b(cC-1)} \\ -\frac{1}{2}\frac{S+s}{cC-1} & \frac{1}{2}\frac{C-c}{cC-1} & \frac{1}{2}\frac{S+s}{cC-1} & -\frac{1}{2}\frac{C-c}{cC-1} \\ \frac{1}{2}\frac{C-c}{b(cC-1)} & -\frac{1}{2}\frac{S-s}{b(cC-1)} & -\frac{1}{2}\frac{C-c}{b(cC-1)} & -\frac{1}{2}\frac{S-s}{b(cC-1)} \end{bmatrix} \times \begin{Bmatrix} \sin bx \\ \cos bx \\ \sinh bx \\ \cosh bx \end{Bmatrix} \quad (\text{Case 1: } b^4 > 0) \tag{16}$$

$$\begin{Bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-3}{L^2} & \frac{2}{L^3} \\ 0 & 1 & \frac{-2}{L^2} & \frac{1}{L^2} \\ 0 & 0 & \frac{3}{L^2} & \frac{-2}{L^3} \\ 0 & 0 & \frac{-1}{L^2} & \frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \tag{17}$$

(Case 2: $b^4 = 0$)

$$\begin{Bmatrix} N_1(x,\omega) \\ N_2(x,\omega) \\ N_3(x,\omega) \\ N_4(x,\omega) \end{Bmatrix} = \begin{bmatrix} -\frac{-c^2+C^2}{C^2-2+c^2} & \frac{CS+sc}{C^2-2+c^2} & -\frac{-CS+sc}{C^2-2+c^2} & 1 \\ -\frac{CS-sc}{b(C^2-2+c^2)} & \frac{C^2-1}{b(C^2-2+c^2)} & -\frac{-1+c^2}{b(C^2-2+c^2)} & 0 \\ \frac{2sS}{C^2-2+c^2} & -\frac{cS+sC}{C^2-2+c^2} & \frac{cS+sC}{C^2-2+c^2} & 0 \\ -\frac{sC-cS}{b(C^2-2+c^2)} & \frac{sS}{b(C^2-2+c^2)} & -\frac{sS}{b(C^2-2+c^2)} & 0 \end{bmatrix} \times \begin{Bmatrix} \sin bx \sinh bx \\ \sin bx \cosh bx \\ \cos bx \sinh bx \\ \cos bx \cosh bx \end{Bmatrix} \quad (\text{Case 3: } b^4 < 0) \tag{18}$$

Here, $C = \cosh bl$, $c = \cos bl$, $S = \sinh bl$ and $s = \sin bl$. Similarly, the shape functions for the axial deformations are obtained as

$$\begin{Bmatrix} N_5(x,\omega) \\ N_6(x,\omega) \end{Bmatrix} = \begin{bmatrix} -\cot aL & 1 \\ \operatorname{cosec} aL & 0 \end{bmatrix} \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} \tag{19}$$

Thus, the shape functions for all the three cases can be written in the form $N(x,\omega) = [\Gamma]\{s(x,\omega)\}$, where $s(x,\omega)$ is the 6×1 array of basis functions given by

Case 1: $b^4 > 0$

$$\{s(x,\omega)\} = [\sin bx, \cos bx, \sinh bx, \cosh bx, \sin ax, \cos ax]^{\mathrm{T}}$$

Case 2: $b^4 = 0$

$$\{s(x,\omega)\} = [1, x, x^2, x^3, \sin ax, \cos ax]^{\mathrm{T}}$$

Case 3: $b^4 < 0$

$$\{s(x,\omega)\} = [\sin bx \sinh bx, \sin bx \cosh bx, \cos bx \sinh bx, \cos bx \cosh bx, \sin ax, \cos ax]^{\mathrm{T}} \tag{20}$$

and Γ is a 6×6 frequency dependent matrix of constants. The superscript T appearing in the above equations represent matrix transpose. To construct finite element solutions to eqns (6) and (7) we seek the solutions in the form

$$y(x,t)=\sum_{j=1}^{4}a_j(t)N_j(x,\omega) \tag{21}$$

$$u(x,t)=\sum_{j=5}^{6}a_j(t)N_j(x,\omega) \tag{22}$$

Here $a_j(t)$ $(j=1,\dots,6)$ are the generalized coordinates representing the nodal displacements. The choice of $N_j(x,\omega)$ in eqn (21) depends on whether $b^4>0$, $b^4=0$ or $b^4<0$ which, in turn, depends upon the frequency ω. The formulation of the dynamic stiffness matrix is now carried out using the Lagrange equation. It may be added here that use of symbolic processing (using Maple V) has been made in this study in developing the shape functions described above and also in the following development of dynamic stiffness matrix.

4.2 Lagrange's equation

We first consider the beam element to be undamped. The expression for the kinetic and potential energies in terms of the assumed displacements (eqns (21) and (22)) can be obtained, respectively, as

$$T(t)=\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}a_i(t)a_j(t)I_{ij}(\omega)+\frac{1}{2}\sum_{i=5}^{6}\sum_{j=5}^{6}a_i(t)a_j(t)K_{ij}(\omega) \tag{23}$$

$$V(t)=\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}a_i(t)a_j(t)J_{ij}(\omega)+\frac{1}{2}\sum_{i=5}^{6}\sum_{j=5}^{6}a_i(t)a_j(t)L_{ij}(\omega) \tag{24}$$

where the frequency dependent quantities I_{ij}, K_{ij}, J_{ij} and L_{ij} are given by

$$I_{ij}(\omega)=\int_0^L m(x)N_i(x,\omega)N_j(x,\omega)\mathrm{d}x,\quad \text{for } i,j=1,\dots,4$$

$$K_{ij}(\omega)=\int_0^L m(x)N_i(x,\omega)N_j(x,\omega)\mathrm{d}x,\quad \text{for } i,j=5,6$$

$$J_{ij}(\omega)=\int_0^L\left[EI(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}N_i(x,\omega)\frac{\mathrm{d}^2}{\mathrm{d}x^2}N_j(x,\omega)+k(x)N_i(x,\omega)N_j(x,\omega)\right]\mathrm{d}x,\quad i,j=1,\dots,4$$

$$L_{ij}(\omega)=\int_0^L AE(x)\frac{\mathrm{d}}{\mathrm{d}x}N_i(x,\omega)\frac{\mathrm{d}}{\mathrm{d}x}N_j(x,\omega)\mathrm{d}x,\quad i,j=5,6 \tag{25}$$

The governing equations for the generalized coordinates $a_i(t)$ can now be obtained by using the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathscr{L}}{\partial\dot{a}_k}\right]-\frac{\partial\mathscr{L}}{\partial a_k}=0,\quad k=1,\dots,6 \tag{26}$$

where $\mathscr{L}(t)=T(t)-V(t)$ is the Lagrangian. This leads to the equations of motion

$$[I_{ij}]\{\ddot{a}_i\}+[J_{ij}]\{a_i\}=0\quad(i,j=1,\dots,4) \tag{27}$$

$$[K_{ij}]\{\ddot{a}_i\}+[L_{ij}]\{a_i\}=0\quad(i,j=5,6) \tag{28}$$

Since the motion is harmonic at frequency ω, it follows that $a_k(t)=A_k\exp[i\omega t]$. Consequently, the dynamic stiffness matrix D_u is given by

$$D_\mathrm{u}(\omega)=\begin{bmatrix}[-\omega^2I_{ij}(\omega)+J_{ij}(\omega)]_{(4\times4)} & 0_{(4\times2)}\\ 0_{(2\times4)} & [-\omega^2K_{ij}(\omega)+L_{ij}(\omega)]_{(2\times2)}\end{bmatrix}_{(6\times6)} \tag{29}$$

where the subscript u is used to denote undamped stiffness matrix and 0 denotes the null matrix. To make explicit the relationship between the stiffness coefficients $D_{\mathrm{u}ij}$ and the random fields $f_k(x)$ $(k=1,\dots,4)$, we write

$$D_{\mathrm{u}ij}(\omega)=\bar{D}_{\mathrm{u}ij}+\sum_{k=1}^{4}\sum_{r=1}^{4}\Gamma_{ik}\Gamma_{jr}W_{kr}\quad(i,j=1,\dots,4)$$

$$D_{\mathrm{u}ij}(\omega)=\bar{D}_{\mathrm{u}ij}+\sum_{k=5}^{6}\sum_{r=5}^{6}\Gamma_{ik}\Gamma_{jr}W_{kr}\quad(i,j=5,6) \tag{30}$$

where

$$\bar{D}_{\mathrm{u}ij}(\omega)=\sum_{k=1}^{4}\sum_{r=1}^{4}\Gamma_{ik}\Gamma_{jr}\int_0^L\left[\{k_0-m_0\omega^2\}s_k(x)s_r(x)+EI_0\frac{\mathrm{d}^2s_r(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2s_k(x)}{\mathrm{d}x^2}\right]\mathrm{d}x\quad(i,j=1,\dots,4)$$

$$\bar{D}_{\mathrm{u}ij}(\omega)=\sum_{k=5}^{6}\sum_{r=5}^{6}\Gamma_{ik}\Gamma_{jr}\int_0^L\left[-m_0\omega^2s_k(x)s_r(x)+AE_0\frac{\mathrm{d}s_r(x)}{\mathrm{d}x}\frac{\mathrm{d}s_k(x)}{\mathrm{d}x}\right]\mathrm{d}x\quad(i,j=5,6)$$

$$W_{kr} = \int_0^L \left[\{k_0\epsilon_1 f_1(x) - m_0\omega^2\epsilon_2 f_2(x)\} s_k(x) s_r(x) + EI_0\epsilon_3 f_3(x) \frac{d^2 s_r(x)}{dx} \frac{d^2 s_k(x)}{dx} \right] dx$$
$$(k, r = 1, \ldots, 4)$$
$$W_{kr} = \int_0^L \left[\{-m_0\omega^2\epsilon_2 f_2(x)\} s_k(x) s_r(x) + AE_0\epsilon_4 f_4(x) \frac{ds_r(x)}{dx} \frac{ds_k(x)}{dx} \right] dx$$
$$(k, r = 5, 6) \quad (31)$$

Since $W_{kr} = W_{rk}$, it follows that the summations in eqn (30) occur over only ten independent terms for flexure and three terms for axial motion. Accordingly, for flexural and axial motions, respectively, we can write

$$D_{uij} = \bar{D}_{uij} + \sum_{l=1}^{10} \alpha_{ij}^l X_l \quad (i,j = 1, \ldots, 4)$$
$$D_{uij} = \bar{D}_{uij} + \sum_{l=11}^{13} \alpha_{ij}^l X_l \quad (i,j = 5, 6) \quad (32)$$

Here

$$X_1 = W_{11}, \quad X_2 = W_{12}, \quad X_3 = W_{13}, \quad X_4 = W_{14},$$
$$X_5 = W_{22}, \quad X_6 = W_{23}, \quad X_7 = W_{24}, \quad X_8 = W_{33},$$
$$X_9 = W_{34}, \quad X_{10} = W_{44}, \quad X_{11} = W_{55}, \quad X_{12} = W_{56},$$
$$X_{13} = W_{66}$$
$$\alpha_{ij}^l = \Gamma_{ik}\Gamma_{jr}, \quad \text{for } l = 1, 5, 8, 10, 11, 13 \text{ and } k = r = 1, \ldots, 6$$
$$\alpha_{ij}^l = \Gamma_{ik}\Gamma_{jr} + \Gamma_{ir}\Gamma_{jk}, \quad \text{for } l = 2, 3, 4, 6, 7, 9 \text{ and } k \neq r, \quad k, r = 1, \ldots, 4 \text{ and for } l = 12, k, r = 5, 6 \quad (33)$$

Remarks:

1. For a fixed value of ω, X_l $(l = 1, \ldots, 13)$ are random variables since they arise as 'weighted integrals' of the random fields $f_i(x)$ $(i - 1, \ldots, 4)$; the random variability in the dynamics stiffness matrix of the beam element, thus, gets characterized in terms of these random variables;
2. since the shape functions used in this study are frequency dependent, it follows that the weighted integrals are also functions of frequency; this would mean that these integrals are random processes evolving in the frequency parameter ω;
3. these weighted integrals, thus, offer a means for discretizing the random fields for stochastic finite element dynamical analysis; the essence of this idea is not new: see, for example, the papers by Shinozuka,[19] Deodatis[18] and Takada[21,22] for application of this concept in *static* stochastic finite element applications; the paper by Bucher and Brenner[27] extends these formulations to dynamic applications through the use of consistent mass matrix formulation; the weighted integrals X_l developed in our study can be viewed as the generalization of a concept proposed in these earlier studies: this being achieved in terms of treatment of mass field and avoidance of free vibration analysis in the overall forced dynamical analysis;
4. X_l are linear functions of the random fields $f_i(x)$ and, therefore, if $f_i(x)$ are modeled as jointly Gaussian random fields, it follows that X_l are also jointly Gaussian; furthermore, since the dynamic stiffness coefficients are linear function of X_l, it follows that these coefficients are also Gaussian distributed;
5. since each of the random variables X_l is obtained as a sum of two (for axial deformation) or three (for flexure) random variables (see eqn (31)), and, the stiffness coefficients themselves are expressed as sum of ten (or three) random variables (see eqn (32)), one can hope to model stiffness coefficients as Gaussian distributed even when $f_i(x)$ are not Gaussian;
6. for $\epsilon_i = 0$ $(i = 1, \ldots, 4)$, that is, when the beam element is homogeneous, the dynamic stiffness matrix is given by $D_u(\omega) = \bar{D}_u(\omega)$. For this case, it can be shown that, upon simplification of eqn (31) one gets, the exact dynamic stiffness matrix reported in the literature[7] for a beam element on elastic foundation.

4.3 Damped beam element

The dynamic stiffness matrix derived in the last section is based on the assumption that the beam element is undamped. To allow for the effect of damping terms present in the field eqns (1) and (2) the following steps are adopted:

1. determine the damped dynamic stiffness matrix $D_1(\omega)$ for the deterministic homogeneous beam element, that is, $D_u(\omega)$ given by eqn (30) with $\epsilon_i = 0$ and
$$b^4 = \frac{m_0\omega^2 - k_0 + i\omega c_2}{EI_0 + i\omega c_1}; \quad a^2 = \frac{m_0\omega^2 - i\omega c_4}{AE_0 + i\omega c_3} \quad (34)$$
2. determine undamped dynamic stiffness matrix $D_2(\omega)$ for the deterministic homogeneous beam element given again by eqn (30) with $\epsilon_i = 0$ and the parameters a and b as per equation (10);
3. compute the contribution to the dynamic stiffness $D_d(\omega)$ from damping terms using $D_d(\omega) = D_1(\omega) - D_2(\omega)$; and
4. finally, the damped stochastic dynamic stiffness matrix is obtained as
$$D(\omega) = D_u(\omega) + D_d(\omega) \quad (35)$$

This formulation for damping terms is exactly valid if the system under study consists of an assembly of an undamped beam with dynamic stiffness $D_u(\omega)$ and a discrete damper connected in parallel with the damper dynamic stiffness matrix described by $D_d(\omega)$. However, when applied to the problem described by eqns (1) and (2), this way of including damping terms into the stochastic dynamic stiffness matrix is approximate in nature since the contribution to the dynamic stiffness coming from damping terms are computed here assuming that the beam element is homogeneous. This limitation can be overcome if the damping terms are also included in the definition of shape functions, in which case, these functions would become complex in nature and, consequently, the resulting weighted integrals would be complex valued. This would significantly increase the computational work, and, this is why the present approximation is being made. It will, however, be demonstrated through numerical examples in Section 5 that this approximation results in acceptable accuracy.

4.4 Response variability

Given that the random fields $f_i(x)$ are taken to have zero mean, it follows from eqn (30) that

$$\langle D(\omega)\rangle = \bar{D}_u(\omega) + D_d(\omega) \tag{36}$$

where $\langle\cdot\rangle$ denotes the mathematical expectation operator. Thus, the stochastic component of the dynamic stiffness coefficient is given by

$$\begin{aligned} U_{ij}(\omega) &= D_{ij}(\omega) - [\bar{D}_u(\omega) + D_d(\omega)]_{ij} \\ &= \sum_{l=1}^{10} \alpha_{ij}^l X_l \quad (i,j = 1,\ldots,4) \\ &= \sum_{l=11}^{13} \alpha_{ij}^l X_l \quad (i,j = 5,6) \end{aligned} \tag{37}$$

It may be noted that, although the dynamic stiffness coefficients are complex valued, their random component $U_{ij}(\omega)$, as per the above equation, is real in nature. This feature arises because, in this study, we have utilized shape functions that are independent of damping terms and, hence, they are real. If interest is focused on the mean and mean-square values of the amplitude of the stiffness coefficients, it follows that:

$$\begin{aligned} \langle|D_{ij}(\omega)|\rangle &= \int_{-\infty}^{\infty} \{[\bar{D}_{ij}(\omega) + U_{ij}(\omega)] \\ &\quad \times [\bar{D}_{ij}(\omega) + U_{ij}(\omega)]^*\}^{1/2} p_{ij}(u;\omega)\mathrm{d}u \\ \langle|D_{ij}(\omega)|^2\rangle &= \int_{-\infty}^{\infty} \{[\bar{D}_{ij}(\omega) + U_{ij}(\omega)] \\ &\quad \times [\bar{D}_{ij}(\omega) + U_{ij}(\omega)]^*\} p_{ij}(u;\omega)\mathrm{d}u \end{aligned} \tag{38}$$

Here * denotes the complex conjugation and $p_{ij}(u;\omega)$ is the probability density function of $U_{ij}(\omega)$. It is clear that if $f_i(x)$ are taken to be Gaussian distributed, $U_{ij}(\omega)$ will be Gaussian distributed with $\langle U_{ij}(\omega)\rangle = 0$ and

$$\begin{aligned} \langle U_{ij}^2(\omega)\rangle &= \sum_{l=1}^{10}\sum_{m=1}^{10} \alpha_{ij}^l \alpha_{ij}^m \langle X_l X_m\rangle \quad (i,j = 1,\ldots,4) \\ &= \sum_{l=11}^{13}\sum_{m=11}^{13} \alpha_{ij}^l \alpha_{ij}^m \langle X_l X_m\rangle \quad (i,j = 5,6) \end{aligned} \tag{39}$$

Consequently, the determination of mean and mean-square values of the amplitude of the stiffness coefficients involves a one-dimensional integration over a Gaussian distribution. The determination of $p_{ij}(u;\omega)$, in turn, requires the determination of the covariance matrix of the weighted integrals X_l which can be obtained in terms of the covariance functions of the random fields $f_i(x)$ $(i = 1,\ldots,4)$.

5 NUMERICAL EXAMPLES AND DISCUSSION

We consider two numerical examples to illustrate the analytical formulations presented in the previous section. The first example is on a fixed–fixed beam resting on a elastic foundation and which is subjected to a harmonic support sinking at one end; the second example is on seismic wave amplification through a random soil layer modeled as a one-dimensional shear beam with randomly inhomogeneous mass density and shear modulus. Figure 2 shows the two examples considered together with the numerical values assumed for the system parameters: these values, for the two examples, being taken, respectively, from Refs 12 and 31. The particular choice of autocovariance functions in these examples is made only for the purpose of illustration and the theoretical results developed are expected to be valid for other forms of these functions also.

5.1 Example 1. Beam on elastic foundation

In this example, the flexural rigidity, mass density and foundation elastic modulus are taken to be independent, homogeneous random fields. It is assumed that $\epsilon_i = 0.05$, $(i = 1,3)$, and the autocovariance of the processes $f_i(x)$ are taken to be of the form

$$R_{ii}(\xi) = \cos\lambda_i\xi, \quad i = 1,3 \tag{40}$$

with $\lambda_i = 10\pi$ per unit length. It may be recalled that the dynamic stiffness coefficients for damped structural elements is complex in nature. When the structural element is random, the stiffness coefficients can be interpreted as being complex valued random processes evolving in the frequency parameter ω. In this study we focus our attention on computing the mean and standard deviation of the amplitude of the shear forces at the two ends of the beam as a function of the driving frequency ω. These two quantities, for the right and left

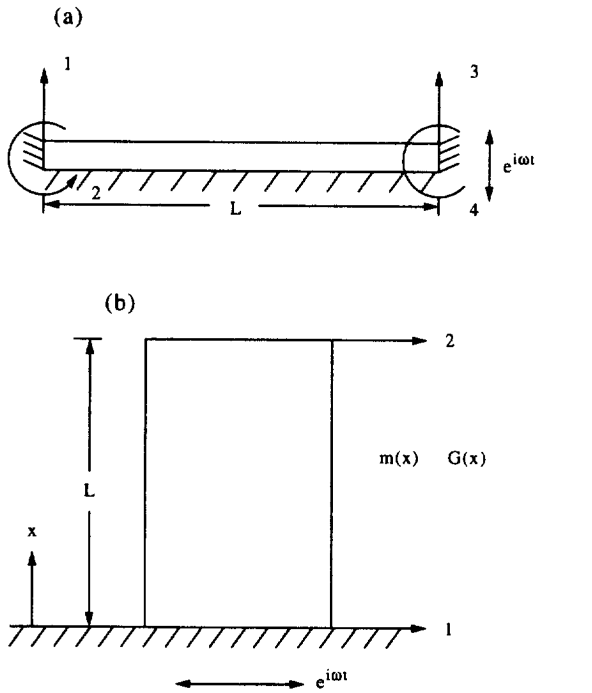

Fig. 2. (a) Random beam under harmonic support displacements; $EI_0 = 10{\cdot}0$, $L = 1{\cdot}0$, $m_0 = 0{\cdot}2$, $k_0 = 5{\cdot}0$, $\epsilon_i = 0{\cdot}05$ $(i = 1, 3)$, $c_1 = 0$, $c_2 = 1{\cdot}0$. (b) Harmonic wave amplification in random soil layer; $G_0 = 1{\cdot}03 \times 10^8\,\mathrm{N\,m^{-2}}$, $L = 30{\cdot}5\,\mathrm{m}$, $m_0 = 2005\,\mathrm{Kg\,m^{-3}}$, $c_3 = 10^{-3} G_0\,\mathrm{Ns}$, $c_4 = 0$, $\epsilon_i = 0{\cdot}05$ $(i = 3, 4)$.

ends of the beam are given, respectively, by $|D_{33}(\omega)|$ and $|D_{13}(\omega)|$. Figures 3 and 4 show a comparison of analytical results on the spectrum of the mean and standard deviation of these quantities with the results of 100 samples Monte Carlo simulations. In the numerical work, the computation of covariance matrix of X_l was carried out using a two-dimensional seventh-order Newton–Coates' integration scheme. The algorithm used in the simulation work to simulate samples of the dynamic stiffness matrix is outlined briefly in Appendix A. It must be emphasized that the simulation work treats the damping terms appearing in eqns (1) and (2) exactly and, therefore, the results obtained also serve to validate the approximation made in the analysis on handling damping terms. In producing these numerical results it is assumed that $f_i(x)$ are jointly Gaussian random fields. This, as has been already noted, affords a major simplification, since, in this approximation, the probability distribution of the sum $\sum_{l=1}^{10} \alpha_{ij}^{l} X_l$ also becomes Gaussian which, in turn, enables the calculation of the statistics of the stiffness coefficients by a one-dimensional integration over a Gaussian distribution. It may further be observed that the theoretical predictions compare well with the simulations results over the entire frequency range considered. This supports the approximations made in this study in the treatment of system randomness and damping in deriving the element dynamic stiffness matrix. The passage of the driving frequency through the system natural frequencies is observed to induce non-stationarity into the frequency evolution of the stiffness coefficients statistics with the variability being higher near resonance points and in some places standard deviations being as high as the mean itself. This is significant, since, the standard deviations of the beam property random fields are only 5% of the response means, which would mean that the system dynamics magnifies the structural

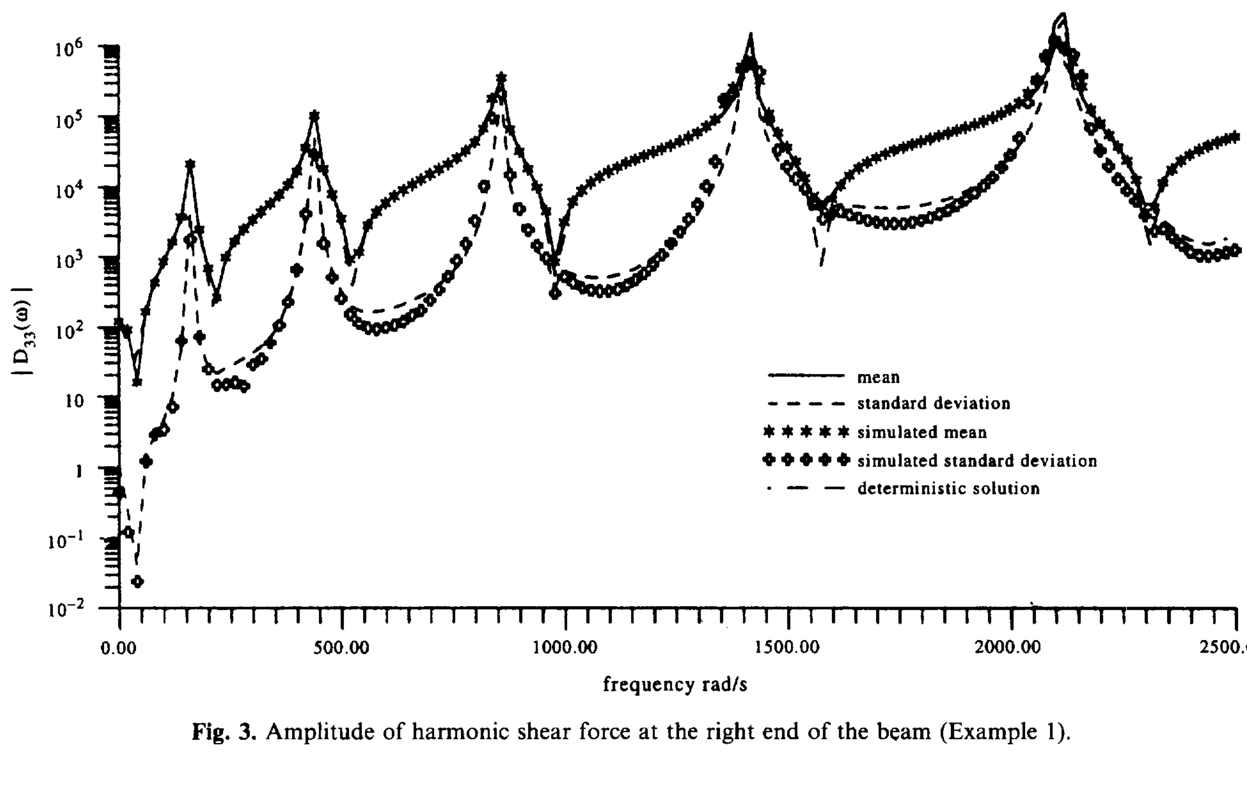

Fig. 3. Amplitude of harmonic shear force at the right end of the beam (Example 1).

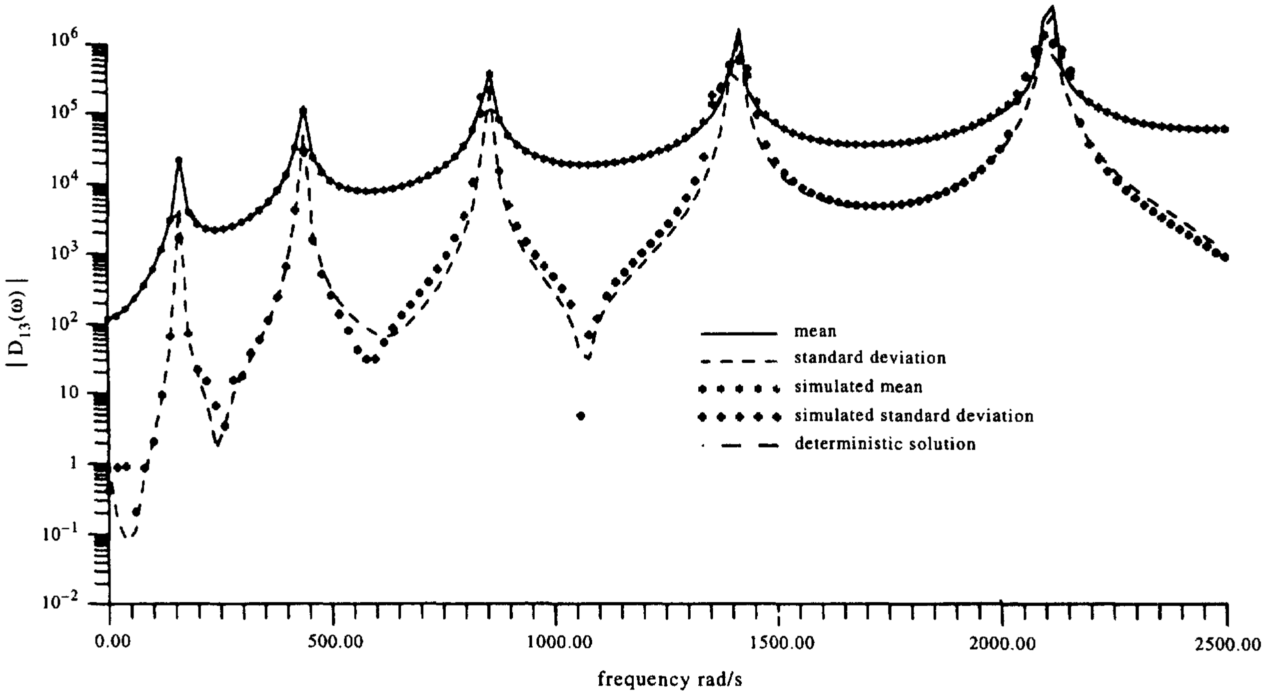

Fig. 4. Amplitude of harmonic shear force at the left end of the beam (Example 1).

uncertainties considerably. The mean results, however, are found to closely follow the deterministic results.

5.2 Example 2. Seismic wave amplification through a random soil layer

In this example, we consider the problem of seismic wave amplification through a stochastic soil layer modeled as a shear beam with randomly varying stiffness and mass properties. The soil layer is taken to be fixed at the base and free at the top (see Fig. 2(b)) and is assumed to have a depth of 30·5 m, shear modulus $G_0 = 1{\cdot}03 \times 10^8\,\mathrm{N\,m^{-2}}$, mass density $\rho_0 = 2005\,\mathrm{Kg\,m^{-3}}$ and damping coefficients $c_3 = 10^{-3}G_0$ and $c_4 = 0$. The mass and stiffness along the beam length are perturbed by independent, stationary random Gaussian fields with autocovariances given, respectively, by

$$R_{ii}(\xi) = \cos\lambda_i\xi, \quad i = 3, 4 \tag{41}$$

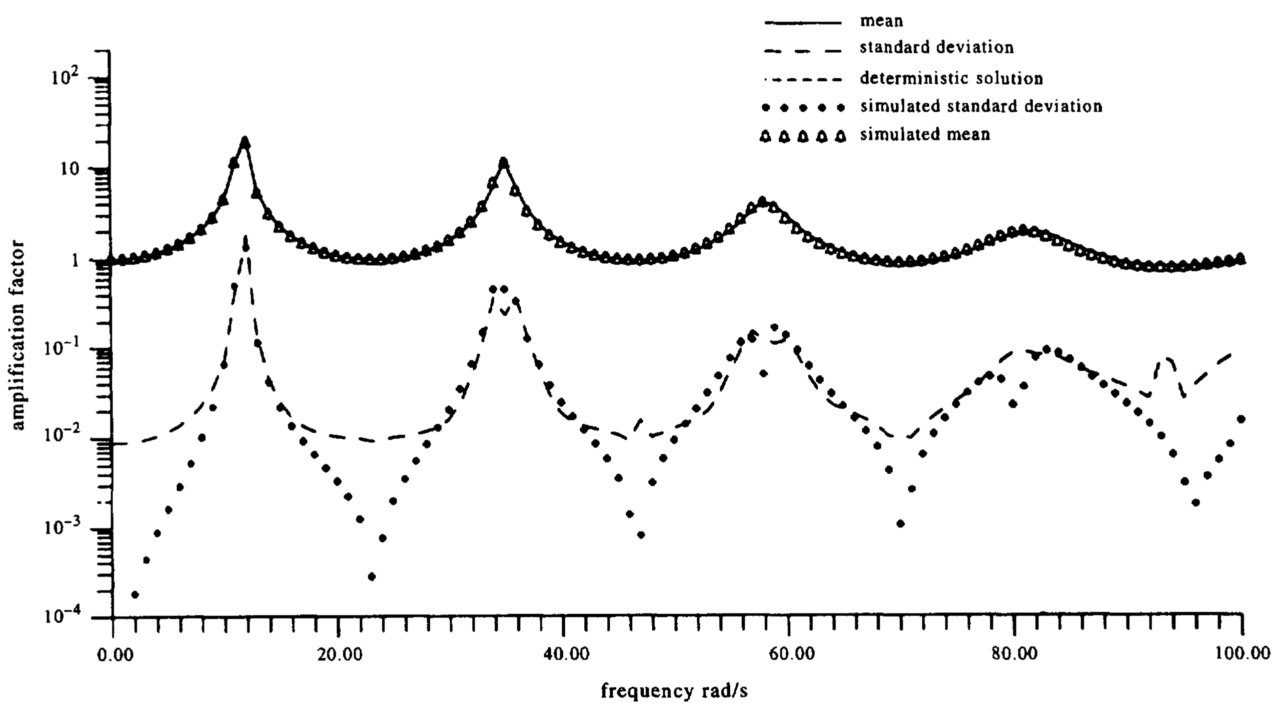

Fig. 5. Amplification factor for random soil layer (Example 2).

In the numerical work it is assumed that $\epsilon_3, \epsilon_4 = 0{\cdot}05$, $\lambda_3, \lambda_4 = 1{\cdot}03\,\mathrm{m}^{-1}$. The wave amplification factor can be defined as the ratio of the amplitude of a harmonic displacement at the top of the soil layer due to a unit harmonic displacement at the base (Fig. 2b).[31] This factor is given by

$$\Delta(\omega) = -\frac{[D_u(\omega) + D_d(\omega)]_{55}}{[D_u(\omega) + D_d(\omega)]_{56}} \tag{42}$$

The presence of damping in the system makes this factor complex valued. The analytically predicted mean and standard deviation of $|\Delta(\omega)|$ has been compared with the results of a 500 samples digital simulation (results in Fig. 5). As in the previous example, the theoretical results are again found to compare favorably with the simulation results with the comparisons being better near resonant frequencies. The trends in the variations of the mean and standard deviations with respect to ω are qualitatively similar to that observed in the previous example. The formulation developed can easily be extended to carry out stationary random vibration analysis. This is illustrated by considering the displacement of the top of the soil layer due to a base motion modeled as a stationary random process. The power spectral density of the displacement at the top of the soil layer in this case is given by

$$S(\omega) = |\Delta(\omega)|^2 S_b(\omega) \tag{43}$$

where $S_b(\omega)$ is the power spectral density function of the applied displacement. The theoretical results on mean and standard deviation of $S(\omega)$ for the case of $S_b(\omega)$ being a band limited white noise over a frequency range of 0–100 rad s^{-1} is shown in Fig. 6. A favorable comparison with simulated results is again observed.

6 CONCLUSIONS

The harmonic response of a general beam element with stochastically inhomogeneous properties is studied using finite element approximation based on frequency dependent shape functions. Expressions for mean and standard deviations of the dynamic stiffness coefficients as functions of the driving frequency are presented. The approach developed here offers an alternative to the more traditional mode superposition method of vibration response analysis. The need to find the random natural frequencies and random mode shapes as solutions of the associated random eigenvalue problem is eliminated. In the forced response analysis, since the method does not employ a series expansion in terms of random free vibration solutions, there would be no modal truncation errors and, also, the number of random variables arising in the discretization of the system property random fields remains constant for all frequencies. Thus, the proposed approach offers a powerful means for discretizing random fields for steady state harmonic and stationary random response analysis of random structures. The method is particularly advantageous if the operating frequency range lies in higher system natural frequency ranges and also when several structural modes contribute to the response. Further improvements to the method are possible with respect to the following aspects:

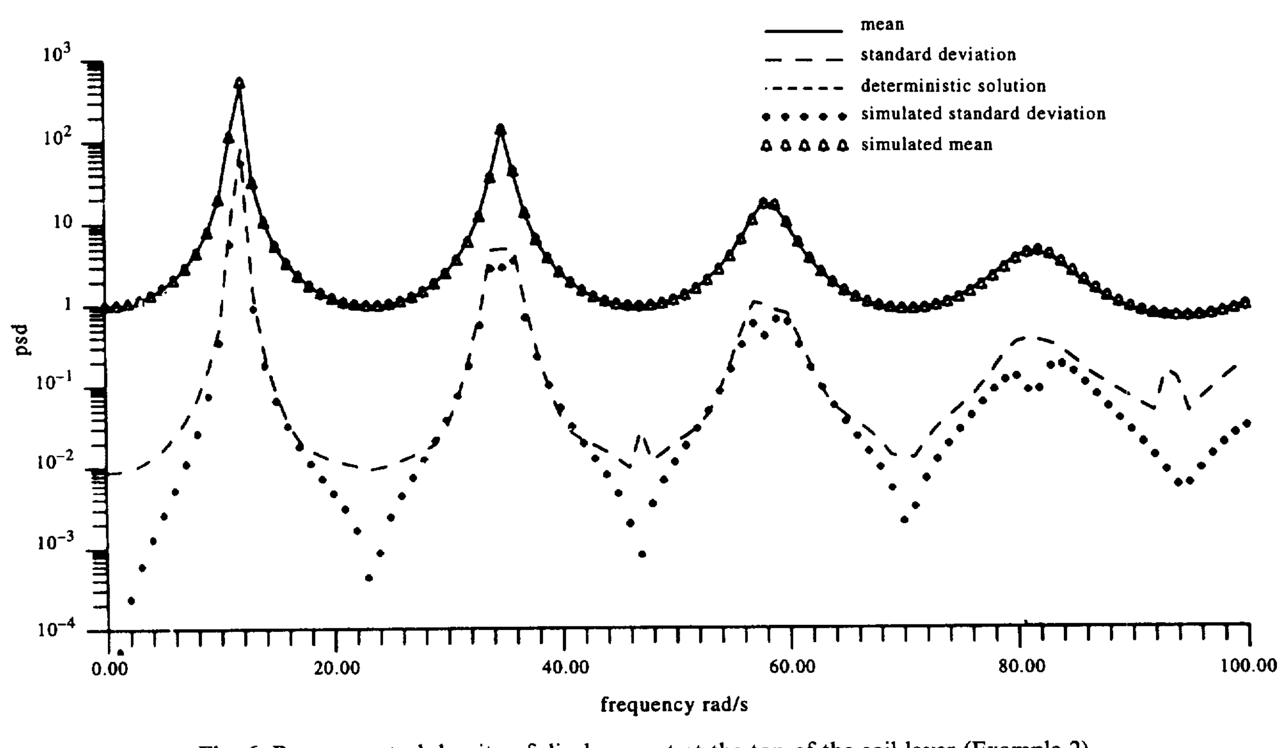

Fig. 6. Power spectral density of displacement at the top of the soil layer (Example 2).

- selection of damping dependent shape functions would enable a more accurate treatment of damping terms in deriving the dynamics stiffness matrix. This would also permit random field models for damping terms;
- inclusion of non-Gaussian models for random fields $f_i(x)$ which satisfy the requirement that $P[1 + \epsilon_i f_i(x) < 0] = 0$;
- orthogonal decomposition of weighted integrals to further reduce the number of characteristic random variables entering formulations.

Work on achieving these improvements, and also in applying the approach to dynamics of built up skeletal structures, is currently being carried out.

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APPENDIX A

A.1 Algorithm for digital simulation of dynamic stiffness matrix

The sample solutions for the dynamic stiffness matrix are obtained by solving the eqns (6) and (7) under the boundary conditions given by eqns (8) and (9). The solution strategy is based on the conversion of the given boundary value problems into a set of equivalent initial value problems. These initial value problems, in turn, are solved numerically using a fourth-order Runge–Kutta algorithm. A similar procedure has been used by us in a recent study on dynamics of translating cables[32] and here we extend the formulations to the case of inhomogeneous beams. To illustrate briefly the procedure used, we consider only the flexural motions and introduce the variables

$$\bar{y}_1 + i\bar{y}_2 = y, \quad \bar{y}_3 + i\bar{y}_4 = \frac{dy}{dx},$$
$$\bar{y}_5 + i\bar{y}_6 = [EI(x) + i\omega c_1]\frac{d^2y}{dx^2}, \tag{A.1}$$
$$\bar{y}_7 + i\bar{y}_8 = \frac{d}{dx}[\bar{y}_5 + i\bar{y}_6]$$

Equations (6) and (7) can now be recast into the form

$$\frac{d\bar{y}}{dx} = Q^{-1}S\bar{y} \tag{A.2}$$

where Q and S are 8×8 matrices with

$Q_{ii} = 1, \quad \text{for } i = 1,2,5,6,7,8,$

$Q_{ii} = EI(x), \quad \text{for } i = 3,4, \quad Q_{34} = -Q_{43} = c_1\omega$

$S_{ii} = 0 \text{ for } i = 1,\ldots,8$

$$S_{13} = 1{\cdot}0, \quad S_{24} = 1{\cdot}0, \quad S_{35} = 1{\cdot}0, \quad S_{46} = 1{\cdot}0,$$
$$S_{57} = 1{\cdot}0 \quad S_{68} = 1{\cdot}0$$
$$S_{71} = S_{82} = m(x)\omega^2 - k(x), \quad S_{81} = -S_{71} = -c_2\omega \tag{A.3}$$

All the other elements of Q and S which are not specified above are zero. Consider the 8×8 matrix of W of fundamental solutions of eqn (A.2) which are obtained by solving (A.2) under the *initial conditions*

$$W_{ij}(x = 0) = \delta_{ij} \tag{A.4}$$

where δ_{ij} is the Kronecker delta function. That is, the jth column of the matrix W consists of solution of the vector differential equation with initial conditions $\bar{y}_j(0) = 1$ and $\bar{y}_k(0) = 0$ for $k \neq j$.

Any other solution $y(x)$ of equation (A.2) can be expressed as a linear combination of the elements of W as

$$\bar{y}(x) = W\gamma \tag{A.5}$$

where the vector γ needs to be selected to satisfy the prescribed boundary conditions. The boundary conditions described in eqns (8) and (9) are re-written as

$$y(0) = \Delta_{R1} + i\Delta_{I1}\,\frac{dy}{dx}(0) = \Delta_{R2} + i\Delta_{I2}$$
$$y(L) = \Delta_{R3} + i\Delta_{I3}\,\frac{dy}{dx}(L) = \Delta_{R4} + i\Delta_{I4} \tag{A.6}$$

and

$$\left[EI\frac{d^2y}{dx^2} + i\omega c_1\frac{d^2y}{dx^2}\right](0) = p_{R2} + ip_{I2},$$
$$\frac{d}{dx}\left[EI(x)\frac{d^2y}{dx^2} + i\omega c_1\frac{d^2y}{dx^2}\right](0) = p_{R1} + ip_{I1},$$
$$\left[EI\frac{d^2y}{dx^2} + i\omega c_1\frac{d^2y}{dx^2}\right](L) = p_{R4} + ip_{I4}, \tag{A.7}$$
$$\frac{d}{dx}\left[EI(x)\frac{d^2y}{dx^2} + i\omega c_1\frac{d^2y}{dx^2}\right](L) = p_{R3} + ip_{I3}$$

Using the boundary conditions on displacements, we get

$$\begin{Bmatrix} \Delta_{R1} \\ \Delta_{I1} \\ \Delta_{R2} \\ \Delta_{I2} \\ \Delta_{R3} \\ \Delta_{I3} \\ \Delta_{R4} \\ \Delta_{I4} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ W_{11}(L) & W_{12}(L) & W_{13}(L) & W_{14}(L) & W_{15}(L) & W_{16}(L) & W_{17}(L) & W_{18}(L) \\ W_{21}(L) & W_{22}(L) & W_{23}(L) & W_{24}(L) & W_{25}(L) & W_{26}(L) & W_{27}(L) & W_{28}(L) \\ W_{31}(L) & W_{32}(L) & W_{33}(L) & W_{34}(L) & W_{35}(L) & W_{36}(L) & W_{37}(L) & W_{38}(L) \\ W_{41}(L) & W_{42}(L) & W_{43}(L) & W_{44}(L) & W_{45}(L) & W_{46}(L) & W_{47}(L) & W_{48}(L) \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \end{Bmatrix} \tag{A.8}$$

which can be written concisely as

$$\Delta = \mathbf{S}_1 \gamma \tag{A.9}$$

Similarly, from the imposition of boundary forces one gets

$$\begin{Bmatrix} p_{R1} \\ p_{I1} \\ p_{R2} \\ p_{I2} \\ p_{R3} \\ p_{I3} \\ p_{R4} \\ p_{I4} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ W_{71}(L) & W_{72}(L) & W_{73}(L) & W_{74}(L) & W_{75}(L) & W_{76}(L) & W_{77}(L) & W_{78}(L) \\ W_{81}(L) & W_{82}(L) & W_{83}(L) & W_{84}(L) & W_{85}(L) & W_{86}(L) & W_{87}(L) & W_{88}(L) \\ W_{51}(L) & W_{52}(L) & W_{53}(L) & W_{54}(L) & W_{55}(L) & W_{56}(L) & W_{57}(L) & W_{58}(L) \\ W_{61}(L) & W_{62}(L) & W_{63}(L) & W_{64}(L) & W_{65}(L) & W_{66}(L) & W_{67}(L) & W_{68}(L) \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \end{Bmatrix} \tag{A.10}$$

which, again, can be written compactly as

$$P = \mathbf{S}_2 \gamma \tag{A.11}$$

It follows from eqns (A.9) and (A.10) that

$$P = \mathbf{S}_2 \mathbf{S}_1^{-1} \boldsymbol{\Delta} = D_s \boldsymbol{\Delta} \tag{A.12}$$

where $D_s = \mathbf{S}_2\mathbf{S}_1^{-1}$ is an alternative version of dynamic stiffness matrix which takes into account the complex nature of stiffness coefficients arising from presence of damping terms and which relates real and imaginary parts of nodal displacements to nodal forces. It can be shown that the elements of the $4 \times 4|D(\omega)|$ and $8 \times 8|D_s|$ matrices are related by

$$|D_{ij}(\omega)| = [D^2_{s(2i-1),(2j-1)} + D^2_{s2,(2j-1)}]^{1/2}, \quad i,j = 1,\ldots,4 \tag{A.13}$$