UNCERTAINTY QUANTIFICATION IN STRUCTURAL DYNAMICS

S. Adhikari[a]

[a]School of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, UK

ABSTRACT: The analysis of uncertainty of very large dynamical systems over a wide range of frequency is a significant challenge. In this paper a new reduced-order computational approach for very large damped stochastic linear dynamical systems is proposed. The approach is based on transformation and reduction of the stochastic system in the modal domain. A Wishart random matrix distribution is considered for the eigensolution of the reduced-order system. The identification of the parameters of the Wishart random model has been discussed. The newly proposed approach is compared with the existing random matrix models using numerical case studies. Results from the new approach have been validated using an experiment on a vibrating plate with randomly attached spring-mass oscillators. One hundred nominally identical samples have been physically created and individually tested within a laboratory framework. A simple step-by-step simulation method for implementing the new computational approach in conjunction with general purpose finite element software has been outlined. The method is applied to an aircraft wing problem with uncertainty to illustrate the generality, portability and the non-intrusive nature of the proposed approach.

Key words: structural dynamics, uncertainty, random matrix, modal analysis

NOMENCLATURE

$\mathbf{f}(t)$	forcing vector
$\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$	mass, damping and stiffness matrices respectively
$\boldsymbol{\phi}_j$	undamped eigenvectors
$\mathbf{q}(t)$	response vector
δ_G	dispersion parameter of $\mathbf{G}$
n	number of degrees of freedom
$(\bullet)^T$	matrix transposition
$\mathbb{R}$	space of real numbers
$\mathbb{R}_n^+$	space $n \times n$ real positive definite matrices
$\mathbb{R}^{n\times m}$	space $n \times m$ real matrices
$\lvert\bullet\rvert$	determinant of a matrix
$\mathrm{etr}\{\bullet\}$	$\exp\{\mathrm{Trace}(\bullet)\}$
$\lvert\lvert\bullet\rvert\rvert_F$	Frobenius norm of a matrix, $\lvert\lvert\bullet\rvert\rvert_F = \left(\mathrm{Trace}\left((\bullet)(\bullet)^T\right)\right)^{1/2}$
$\sim$	distributed as
$\mathrm{Trace}(\bullet)$	sum of the diagonal elements of a matrix
pdf	probably density function

Email address: `S.Adhikari@swansea.ac.uk` (S. Adhikari)

URL: `http://engweb.swan.ac.uk/~adhikaris/` (S. Adhikari)

1. INTRODUCTION

The analysis of uncertainty of very large dynamical systems over a wide range of frequency is a significant challenge [1, 2]. In this paper a new reduced-order computational approach for damped stochastic linear dynamical systems is proposed using Wishart random matrix distribution.

The use of random matrices to model uncertainty was proposed by Soize [3–5] and subsequently researched by others [6–9]. Although the validity of the Wishart random matrix is proven in various structural dynamical problems over the past decade, the computational methods till date relies on the direct Monte Carlo Simulation. It is therefore crucial to develop efficient simulation method to consider real-life problems with very large matrix sizes. The equation of motion of a damped n-degree-of-freedom linear dynamic system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{f}(t) \in \mathbb{R}^n$ is the forcing vector, $\mathbf{q}(t) \in \mathbb{R}^n$ is the response vector and $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times n}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass, damping and stiffness matrices respectively. In order to completely quantify the uncertainties associated with system (1) we need to obtain the probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. Using the parametric approach, such as the stochastic finite element method [10], one usually obtains a problem specific covariance structure for the elements of system matrices. The nonparametric approach [3–5] on the other hand results in a central Wishart distribution (or more generally, the matrix variate Gamma distribution [11]) for the system matrices. Wishart matrix with properly selected parameters can be used for systems with both parametric uncertainty and nonparametric uncertainty [see for example 12–15].

The aim of this paper is to investigate an efficient simulation method to obtain frequency response function (FRF) statistics with Wishart matrices. The approach is based on transformation and reduction of the stochastic system in the modal domain. Recently in the context of parametric uncertainty, Pellissetti et. al. [16] proposed a meta-model approach in the modal domain for the calculation of frequency response statistics. Here, a Wishart random matrix distribution is considered for the eigensolution of the reduced-order system. The identification of the parameters of the Wishart random model has been discussed. The newly proposed approach is compared with the existing random matrix models using numerical case studies. Results from the new approach have been validated using an experiment on a vibrating plate with randomly attached spring-mass oscillators. One hundred nominally identical samples have been physically created and individually tested within a laboratory framework. A simple step-by-step simulation method for implementing the new computational approach in conjunction with general purpose finite element software has been outlined. The method is applied to an aircraft wing problem with uncertainty to illustrate the generality, portability and the non-intrusive nature of the proposed approach.

2. THE GENERALIZED WISHART RANDOM MATRIX MODEL

The generalized Wishart random matrix model was recently introduced by Adhikari [17]. Here we briefly review the necessary details. Suppose the mass and stiffness matrices corresponding to the baseline model are known. In addition to this, it is assumed that the dispersion parameters associated with these matrices are known. The dispersion parameter, proposed by Soize [3, 4], is a measure of uncertainty in the system and it is similar to the normalized standard deviation of a matrix. For example, the dispersion parameter associated with the stiffness matrix is defined as

$$\delta_K^2 = \frac{\mathbb{E}\{\|\mathbf{K} - \mathbf{K}_0\|_{\mathrm{F}}^2\}}{\|\mathbf{K}_0\|_{\mathrm{F}}^2} \tag{2}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix, and the symbol $\mathbb{E}\{...\}$ denotes the operation of averaging with respect to the corresponding probability distribution. The dispersion parameter δ_M associated with the mass matrix can be defined in a similar way. The dispersion parameters δ_M and δ_K can be obtained using the stochastic finite element method or experimental measurements [6]. Given the dispersion parameters δ_M and δ_K and the baseline mass and stiffness matrices $\mathbf{M}_0$ and $\mathbf{K}_0$, the parameters for the random matrices $\mathbf{M}$ and $\mathbf{K}$ can be obtained in closed-form. Various parameter selection options have been investigated [9] and the optimal parameters can be obtained via closed-form expressions using optimisation approaches.

It is well known that the dynamic response of a proportionally damped stochastic system is characterized by the eigensolutions of the dynamic matrix

$$\mathbf{H} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}. \tag{3}$$

In general, when $\mathbf{K}$ and $\mathbf{M}$ are Wishart matrices, the matrix $\mathbf{H}$ is not a Wishart matrix (see e.g. [11] for its distribution). However, Adhikari [17] investigated the possibility of $\boldsymbol{\Xi}$ itself being a Wishart matrix. Since $\boldsymbol{\Xi}$ is a positive definite matrix, a Wishart matrix can be fitted using the maximum entropy principle [3, 4] just like the mass and stiffness matrices, provided the dispersion parameter and the baseline values are known. It was observed [17] that such a single Wishart matrix also provides a reasonable model. Thus generalized Wishart random matrix model does not strictly follow from the original Wishart matrix model of the system matrices and should be considered as one used for mathematical simplicity and computational efficiency, providing the qualitative (and in some cases quantitative) description of stochastic dynamics. The detailed derivation of such generalized Wishart matrix and its numerical and experimental validation can be found in [17]. It was shown that

$$\mathbf{H} \sim W_n\left(p, \boldsymbol{\Sigma}\right) \tag{4}$$

where $W_n(\bullet)$ denotes a n dimensional Wishart matrix. The parameters p and $\boldsymbol{\Sigma}$ can be obtained from the available data regarding the system, namely $\mathbf{M}_0$, $\mathbf{K}_0$, δ_M, and δ_K. Dynamical responses obtained using this generalized Wishart matrix have been validated [17] against the stochastic finite element method, full Wishart matrices and experiential results. Here we take this model to obtain the reduced random matrix.

3. THE REDUCED WISHART MATRIX APPROACH

Assuming all the initial conditions are zero and taking the Laplace transform of the equation of motion (1) we have

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{5}$$

where $(\bar{\bullet})$ denotes the Laplace transform of the respective quantities. The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices. The undamped eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2\mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{6}$$

where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system.

A high resolution model of a dynamical system can easily have several million degrees-of-freedom (that is n). On the other hand, it may be only few hundreds or thousands modes are necessary for the calculation of the dynamic response within the frequency range considered. Suppose the number of modes to be retained is m. In general $m \ll n$. We form the truncated undamped modal matrices

$$\begin{aligned} \boldsymbol{\Omega} &= \operatorname{diag}\left[\omega_1, \omega_2, \ldots, \omega_m\right] \in \mathbb{R}^{m\times m} \\ \text{and} \quad \boldsymbol{\Phi} &= \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_m\right] \in \mathbb{R}^{n\times m} \end{aligned} \tag{7}$$

so that

$$\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 \quad \text{and} \quad \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_m \tag{8}$$

where $\mathbf{I}_m$ is a m-dimensional identity matrix. Using these, Eq. (5) can be transformed into the modal coordinates as

$$\left[s^2\mathbf{I}_n + s\mathbf{C}' + \boldsymbol{\Omega}^2\right]\bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{9}$$

where and $(\bullet)'$ denotes the quantities in the reduced modal coordinates:

$$\mathbf{C}' = \boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\Phi} \in \mathbb{R}^{m\times m}, \quad \bar{\mathbf{q}} = \boldsymbol{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \boldsymbol{\Phi}^T\bar{\mathbf{f}} \tag{10}$$

For simplicity let us assume that the system is proportionally damped with deterministic modal damping factors $\zeta_1, \zeta_2, \ldots, \zeta_m$. Therefore, when we consider random systems, the matrix of eigenvalues $\boldsymbol{\Omega}^2$ in equation (9) will be a random matrix of dimension m. Suppose this random matrix is denoted by $\boldsymbol{\Xi} \in \mathbb{R}^{n\times n}$:

$$\boldsymbol{\Omega}^2 \sim \boldsymbol{\Xi} \tag{11}$$

From the definition of $\mathbf{H}$ in Eq. (3) it is clear that the dispersion parameter of $\boldsymbol{\Xi}$ and $\mathbf{H}$ are be the same. Since $\boldsymbol{\Xi}$ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\boldsymbol{\Psi}_r$ such that

$$\boldsymbol{\Psi}_r^T\boldsymbol{\Xi}\boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \tag{12}$$

Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\boldsymbol{\Xi}$. Recalling that $\boldsymbol{\Psi}_r^T\boldsymbol{\Psi}_r = \mathbf{I}_n$, from equation (9) we obtain

$$\bar{\mathbf{q}}' = \left[s^2\mathbf{I}_n + s\mathbf{C}' + \boldsymbol{\Omega}^2\right]^{-1}\bar{\mathbf{f}}' \tag{13}$$

$$= \boldsymbol{\Psi}_r\left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1}\boldsymbol{\Psi}_r^T\bar{\mathbf{f}}' \tag{14}$$

where

$$\boldsymbol{\zeta} = \mathrm{diag}\left[\zeta_1, \zeta_2, \ldots, \zeta_m\right] \in \mathbb{R}^{m\times m} \tag{15}$$

The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \boldsymbol{\Phi}\bar{\mathbf{q}}'(s) = \boldsymbol{\Phi}\boldsymbol{\Psi}_r\left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1}(\boldsymbol{\Phi}\boldsymbol{\Psi}_r)^T\bar{\mathbf{f}}(s)$$
$$= \sum_{j=1}^{m}\frac{\mathbf{x}_{r_j}^T\bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2}\mathbf{x}_{r_j}. \tag{16}$$

Here

$$\boldsymbol{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_m}\right] \tag{17}$$

$$\text{and}\quad \mathbf{X}_r = \boldsymbol{\Phi}\boldsymbol{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_m}\right] \tag{18}$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system. The Frequency Response Function (FRF) of the system can be obtained by substituting $s = \mathrm{i}\omega$ in Eq. (16). In the next section we summarize the Monte Carlo Simulation (MCS) based computational approach arising from this analysis.

4. SUMMARY OF THE COMPUTATIONAL APPROACH

A step-by-step method for implementing the new computational approach in conjunction with any general purpose finite element software is given below:

1. Form the deterministic mass and stiffness matrices $\mathbf{M}_0$ and $\mathbf{K}_0$ using the standard finite element method and the modal damping factors ζ_j. Select the number of modes $m < n$. The number of modes to be retained, m should be selected based on the frequency of excitation.
2. Solve the deterministic undamped eigenvalue problem

$$\mathbf{K}_0\boldsymbol{\phi}_{0j} = \omega_{0_j}^2\mathbf{M}_0\boldsymbol{\phi}_{0j}, \quad j = 1, 2, \ldots, m \tag{19}$$

and create the matrix

$$\boldsymbol{\Phi}_0 = \left[\boldsymbol{\phi}_{0_1}, \boldsymbol{\phi}_{0_2}, \ldots, \boldsymbol{\phi}_{0_m}\right] \in \mathbb{R}^{n\times m} \tag{20}$$

Calculate the ratio

$$\beta_H = \left(\sum_{j=1}^{m}\omega_{0_j}^2\right)^2 \Big/ \sum_{j=1}^{m}\omega_{0_j}^4 \tag{21}$$

3. Obtain the dispersion parameters δ_M and δ_K corresponding to the mass and stiffness matrices. This can be obtained from physical or computer experiments.
4. Obtain the dispersion parameter of the generalized Wishart matrix $\mathbf{H}$ in Eq (3) as [17]

$$\delta_H = \frac{\left({p_M}^2 + (p_K - 2 - 2n)\,p_M\right)\beta_H}{p_K\,(-p_M + n)\,(-p_M + n + 3)} + \frac{\left((-n-1)\,p_K + n^2 + 1 + 2n\right)\beta_H}{p_K\,(-p_M + n)\,(-p_M + n + 3)} + \frac{{p_M}^2 + (p_K - 2n)\,p_M + (1-n)\,p_K - 1 + n^2}{p_K\,(-p_M + n)\,(-p_M + n + 3)} \tag{22}$$

where

$$p_M = \frac{1}{\delta_M^2}\left\{1 + \{\mathrm{Trace}\,(\mathbf{M}_0)\}^2/\mathrm{Trace}\left({\mathbf{M}_0}^2\right)\right\} \tag{23}$$

$$p_K = \frac{1}{\delta_K^2}\left\{1 + \{\mathrm{Trace}\,(\mathbf{K}_0)\}^2/\mathrm{Trace}\left({\mathbf{K}_0}^2\right)\right\} \tag{24}$$

5. Calculate the parameters

$$\theta = \frac{(1+\beta_H)}{\delta_H^2} - (m+1) \quad\text{and}\quad p = [m+1+\theta] \tag{25}$$

where p is approximated to the nearest integer of $m + 1 + \theta$.

6. Create an $m \times p$ matrix $\mathbf{Y}$ such that

$$Y_{ij} = \omega_{0_i}\widehat{Y}_{ij}/\sqrt{\theta};\, i = 1, 2, \ldots, m;\, j = 1, 2, \ldots, p \tag{26}$$

where $\widehat{Y}_{ij}$ are independent and identically distributed (i.i.d.) Gaussian random numbers with zero mean and unit standard deviation.

7. Simulate the $m \times m$ Wishart random matrix

$$\boldsymbol{\Xi} = \mathbf{Y}\mathbf{Y}^T \quad\text{or}\quad \Xi_{ij} = \frac{\omega_{0_i}\omega_{0_j}}{\theta}\sum_{k=1}^{p}\widehat{Y}_{ik}\widehat{Y}_{jk}; \tag{27}$$

where $i = 1, 2, \ldots, m;\, j = 1, 2, \ldots, m$. Since $\boldsymbol{\Xi}$ is symmetric, only the upper or lower triangular part need to be simulated.

8. Solve the symmetric eigenvalue problem $(\mathbf{\Omega}_r, \mathbf{\Psi}_r \in \mathbb{R}^{m\times m})$ for every sample

$$\mathbf{\Xi}\mathbf{\Psi}_r = \mathbf{\Omega}_r^2\mathbf{\Psi}_r \tag{28}$$

and obtain the random eigenvector matrix

$$\mathbf{X}_r = \mathbf{\Phi}_0\mathbf{\Psi}_r = [\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_m}] \in \mathbb{R}^{n\times m} \tag{29}$$

9. Finally calculate the dynamic response in the frequency domain as

$$\bar{\mathbf{q}}_r(\mathrm{i}\omega) = \sum_{j=1}^{m} \frac{\mathbf{x}_{r_j}^T\bar{\mathbf{f}}(s)}{-\omega^2 + 2\mathrm{i}\omega\zeta_j\omega_{r_j} + \omega_{r_j}^2}\mathbf{x}_{r_j} \tag{30}$$

The samples of the response in the time domain can also be obtained from the random eigensolutions as

$$\mathbf{q}_r(t) = \sum_{j=1}^{m} a_{r_j}(t)\mathbf{x}_{r_j}, \tag{31}$$

where

$$a_{r_j}(t) = \frac{1}{\omega_{r_j}}\int_0^t \mathbf{x}_{r_j}^T\mathbf{f}(\tau)e^{-\zeta_j\omega_{r_j}(t-\tau)}\sin\left(\omega_{r_j}(t-\tau)\right)d\tau \tag{32}$$

The above procedure can be implemented very easily. When one implements this approach in conjunction with a general purpose commercial finite element software, the commercial software needs to be accessed only once to obtain the mean matrices $\mathbf{M}_0$ and $\mathbf{K}_0$ and solve the corresponding deterministic eigenvalue problem. This computational procedure proposed here is therefore 'non-intrusive'.

The main computationally intensive part of a random matrix based approach is the generation of the random matrices (by matrix multiplication in Eq. (27)) and solution of the eigenvalue problem. Both the matrix multiplication and the matrix eigenvalue problem scales approximately cubically with the dimension [18]. Therefore, the computational cost of the approach grows $\approx O(m^3)$ compared to $\approx O(n^3)$ for the full Wishart matrix based approach. Since $m \ll n$, the reduced approach is expected to be computationally efficient.

5. NUMERICAL AND EXPERIMENTAL VALIDATION

5.1. Plate with randomly inhomogeneous material properties: Numerical study

A rectangular cantilever steel plate is considered to illustrate the application of the proposed generalized Wishart random matrices in probabilistic structural dynamics. The deterministic properties are assumed to be $\bar{E} = 200 \times 10^9\text{N/m}^2$, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860\text{kg/m}^3$, $\bar{t} = 3.0\text{mm}$, $L_x = 0.998\text{m}$, $L_y = 0.59\text{m}$. These values correspond to the experimental study discussed in the next section. The following three methods are explicitly compared to gain further understanding of the proposed reduced-order method:

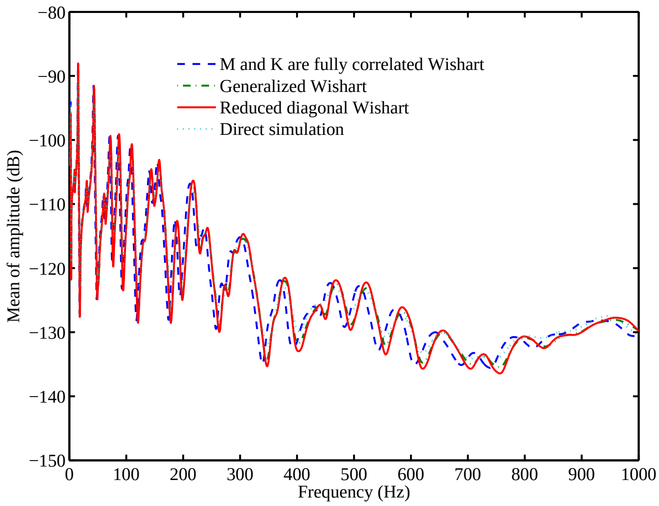
- *Method 1 - Mass and stiffness matrices are fully correlated Wishart matrices:* For this case $\mathbf{M} \sim W_n(p_M, \mathbf{\Sigma}_M)$, $\mathbf{K} \sim W_n(p_K, \mathbf{\Sigma}_K)$ with $\mathrm{E}[\mathbf{M}] = \mathbf{M}_0$ and $\mathrm{E}[\mathbf{M}] = \mathbf{M}_0$. This is similar to the approach proposed by [3, 4] (the original approach requires the simulation of Gamma matrices [11] which is computationally more expensive). This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n\times n$ generalized eigenvalue problem with two fully populated matrices. The computational cost of this approach is $\approx 2O(n^3)$.

- *Method 2 - Generalized Wishart Matrix [17]:* For this case $\mathbf{\Xi} \sim W_n\left(p, \mathbf{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\mathbf{\Xi}^{-1}\right] = \mathbf{\Omega}_0^{-2}$ and $\delta_{\mathbf{\Xi}} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem. The computational cost of this approach is $\approx O(n^3)$.

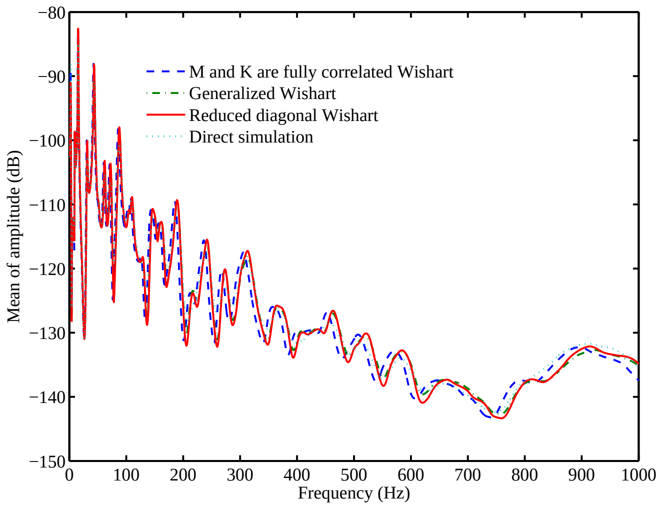
- *Method 3 - Reduced diagonal Wishart Matrix:* For this case $\widetilde{\mathbf{\Xi}} \sim W_m\left(\widetilde{p}, \widetilde{\mathbf{\Omega}}_0^2/\theta\right)$ with $\mathrm{E}\left[\widetilde{\mathbf{\Xi}}^{-1}\right] = \widetilde{\mathbf{\Omega}}_0^{-2}$ and $\delta_{\widetilde{\mathbf{\Xi}}} = \delta_H$. We used tilde to differentiate with the previous case. This requires the simulation of one $m \times m$ uncorrelated Wishart matrix and the solution of a $m \times m$ standard eigenvalue problem. For large complex systems m can be significantly smaller than n. The computational cost of this approach is $\approx O(m^3)$.

The methods are listed in the decreasing order of computational cost. Here we aim to verify their accuracy using numerical and experimental examples. It is assumed that the Young's modulus, Poissons ratio, mass density and thickness are correlated homogenous Gaussian random fields. The standard deviation of these random fields are considered to be 10, 10, 8 and 12 percent of the mean values respectively. An exponential correlation function with correlation length 0.2 times the lengths in each direction has been considered. Each of the random fields are simulated by expanding them using the Karhunen-Loève expansion [10] involving uncorrelated standard normal variables. A 10,000-sample Monte Carlo simulation is performed to obtain the frequency response functions (FRFs) of the system.

We want to verify how the results from the proposed reduced Wishart matrix approach correspond with the direct stochastic finite element Monte Carlo Simulation results. We are also interested to understand whether significant accuracy is lost when comparing the simplified method with the original approach involving two fully correlated Wishart matrices [3–5]. The discretized model has 4650 degrees-of-freedom so that $n = 4650$. In the reduced approach only 80 modes have been used, that is $m = 80$. The baseline model has 77 modes upto 1 kHz frequency considered in the numerical results. The number of modes used here are therefore sufficiently higher and expected to produce physically meaningful results. From the simulated random mass and stiffness matrices we obtain $\delta_M = 0.1133$ and $\delta_K = 0.2916$. Since 2% constant modal damping factor is assumed for all the modes, $\delta_C = 0$. The only uncertainty related information used in the random matrix approach are the values of δ_M and δ_K. The information regarding which element property functions are random fields, nature of these random fields (correlation structure, Gaussian or non-Gaussian) and the amount of randomness are not used in the proposed approach. The predicted mean of the amplitude using the direct stochastic finite element simulation and three Wishart matrix approaches are compared in Fig. 1 for the driving-point-FRF and a cross FRF. The predicted standard deviation of the amplitude using the direct stochastic finite element simulation and three Wishart matrix approaches are compared in Fig. 2 for the driving-point-FRF and the cross FRF. Among the three Wishart matrix approaches discussed here, the reduced diagonal Wishart matrix approach produces accurate results across the frequency range. A random eigenvalue problem involving a single 80×80 is solved for the reduced approach. For the generalized Wishart

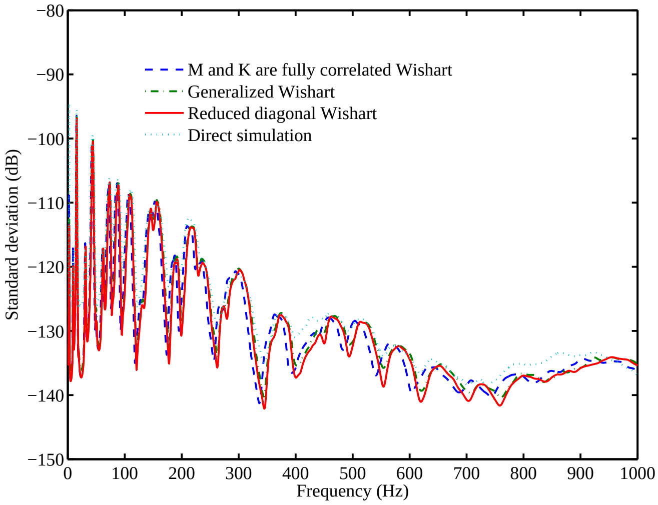


(a) The driving-point-FRF

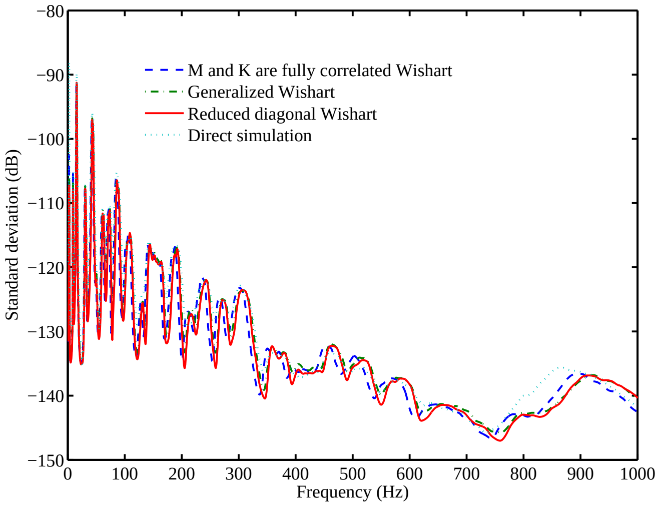


(b) A cross FRF

Fig. 1: Comparison of the mean of the amplitude obtained using the direct stochastic finite element simulation and three Wishart matrix approaches for the plate with randomly inhomogeneous material properties.



(a) The driving-point-FRF



(b) A cross FRF

Fig. 2: Comparison of the standard deviation of the amplitude obtained using the direct stochastic finite element simulation and three Wishart matrix approaches for the plate with randomly inhomogeneous material properties.

approach, a random eigenvalue problem involving a single 4650×4650 is solved for every sample. When the mass and stiffness matrices are fully correlated Wishart matrices, a generalized eigenvalue problem involving two 4650×4650 matrices are solved for every sample. This example numerically validated the reduced random matrix approach proposed in the paper.

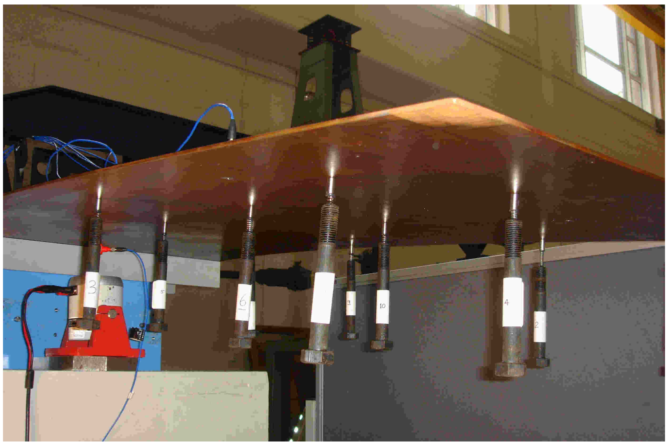
5.2. Plate with randomly attached spring-mass oscillators: Experimental study

We consider the dynamics of a steel cantilever plate with homogeneous geometric (i.e. uniform thickness) and constitutive properties (i.e. uniform Young's modulus and Poisson's ratio) described in the previous section. This uniform plate defines (as considered in the numerical studies in the previous section) the baseline system. The baseline model is perturbed by a set of spring-mass oscillators with different natural frequencies and attached randomly along the plate. The details of this experiment has been described in [19]. Here we give a very brief overview. The overall arrangement of the test-rig is shown in Fig. 3.

The plate is clamped along one edge using a clamping device. In total ten oscillators are used to simulate uncertainty in the system (see Fig. 3(b)). The spring is glue-welded with a magnet at the top and a mass at the bottom. The magnet at the top of the assembly helps to attach the oscillators at the bottom of the plate repeatedly without much difficulty. The stiffness of the ten springs used in the experiment are 16.800, 09.100, 17.030, 24.000, 15.670, 22.880, 17.030, 22.880, 21.360 and 19.800 kN/m. The oscillating mass of each of the 10 oscillators is 121.4g. Therefore the total oscillating mass is 1.214 Kg, which is 9.8% of the mass of the plate. The natural frequencies of the ten oscillators are obtained as 59.2060 43.5744, 59.6099, 70.7647, 57.1801, 69.0938, 59.6099, 69.0938, 66.7592 and 64.2752 Hz. The springs are attached to the plate at the pre-generated nodal locations using the small magnets located at the top the assembly. One hundred realizations of the oscillators are created (by hanging the oscillators at random locations) and tested individually in this experiment. We used a shaker to act as an impulse hammer. It generated impulses at a pulse rate of 20s and a pulse width of 0.01s. As seen in Fig. 3(a), six accelerometers are used as the response sensors.

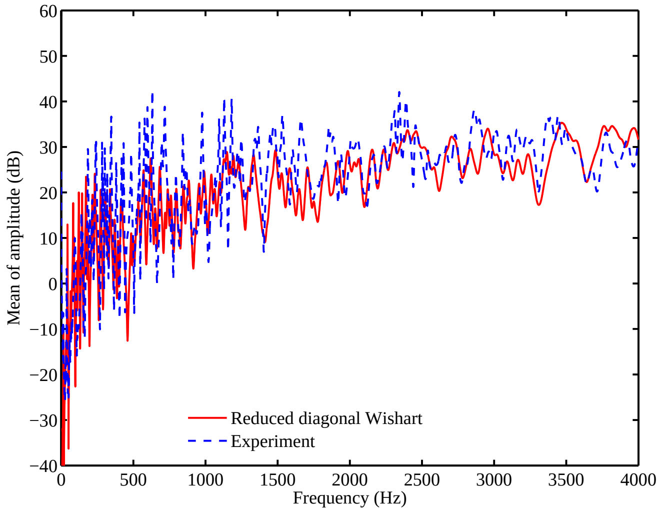
The mean of the amplitude from experiment and the proposed reduced diagonal Wishart matrix approach are compared in Fig. 4 for the driving-point-FRF and a cross FRF. The corresponding relative standard deviations are shown in Fig. 5. We obtained the relative standard deviation by divid-

(a) Arrangement of the test-rig showing the shaker and the accelerometers.

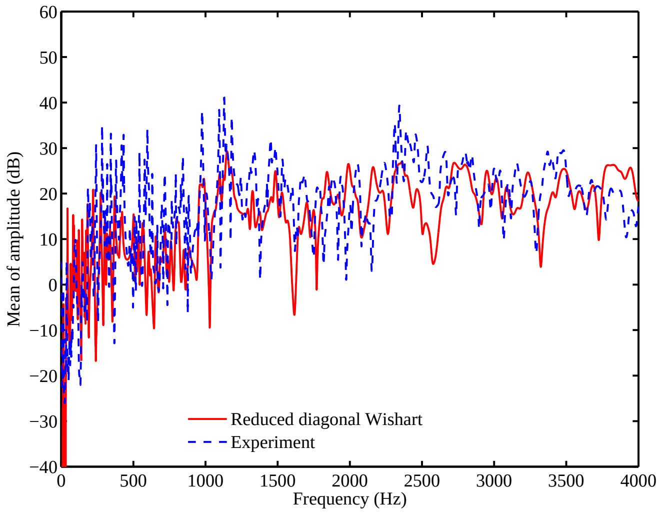


(b) Attached oscillators at random locations. The spring stiffness varies so that the oscillator frequencies are between 43 and 70 Hz.

Fig. 3: The test rig and one of the 100 realization of randomly attached oscillators. The cantilever plate is driven by an impulse. The position of the shaker (used as a impact hammer) and the accelerometers are shown in the figures.



(a) The driving-point-FRF

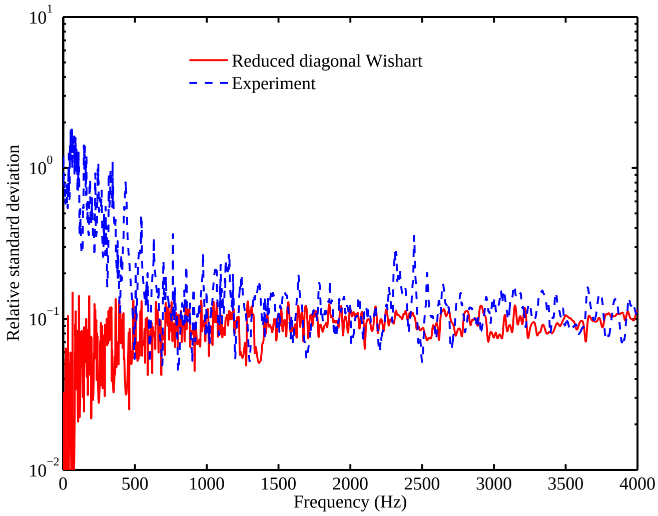


(b) A cross FRF

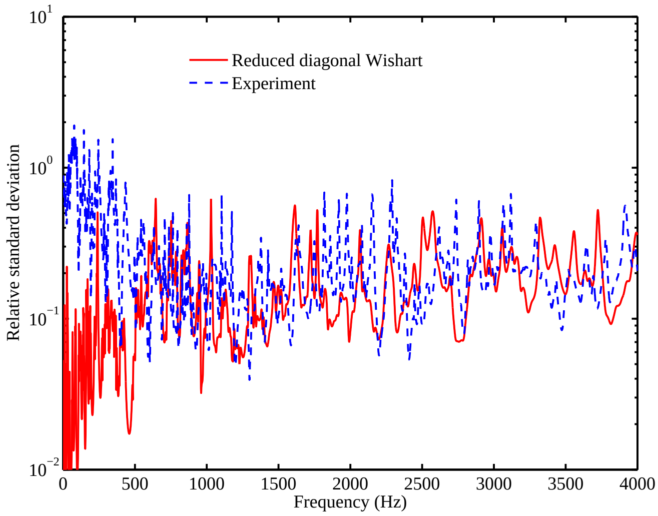
Fig. 4: Comparison of the mean of the amplitude obtained using the experiment and the proposed reduced diagonal Wishart matrix approach for the plate with randomly attached oscillators.

ing the standard deviation with the respective mean values. We have used 2000 samples in the Monte Carlo simulations. The first 600 modes are used in calculating the frequency response functions. This requires the simulation of a 600×600 Wishart random matrix only. Note that the dimension of full random matrix with reliable first 600 modes would be very large. A modal damping factor of 0.7% is assumed for all of the modes. A constant damping factor for all the modes is an assumption. Ideally one should identify damping using advanced methods. It is however a widely established practice to treat damping in a simple manner such as a constant modal damping factor as considered here.

Recall that there are four key parameters needed to implement the random matrix approach. They are respectively the mean and dispersion parameter of the mass and stiffness matrices. The mean system is considered to be the cantilever plate shown in Fig. 3. The mean mass and stiffness matrices are obtained using the standard finite element approach. The mean damping matrix for the experimental system is not obtained explicitly as constant modal damping factors are used. All 100 realiza-



(a) The driving-point-FRF



(b) A cross FRF

Fig. 5: Comparison of relative standard deviation of the amplitude obtained using the experiment and the proposed reduced diagonal Wishart matrix approach for the plate with randomly attached oscillators.

tions of the plate and oscillators were individually simulated. The dispersion parameters of the mass and stiffness matrices are obtained as $\delta_M^2 = 0.0286$ and $\delta_K^2 = 0.0017$. The only uncertainty related information used in the random matrix approach are the values of δ_M and δ_K. The explicit information regarding the spatial locations of the attached oscillators including their stiffness and mass properties are not used in fitting the Wishart matrices. This is aimed to depict a realistic situation when the detailed information regarding uncertainties in a complex engineering system is not available to the analyst.

In Fig. 4 and Fig. 5 observe that both means and standard deviations obtained from the proposed reduced diagonal Wishart matrix approach agree qualitatively with the experimental results. For the two FRFs shown here, the agreement is particularly good above 1 kHz frequency range. The discrepancies, especially in the low frequency regions, are perhaps due to incorrect values of the damping factors. The agreement with this limited experimental results shows that the proposed reduced computational approach might be applicable for uncertainty quantification of real-life dynamical systems. Next we integrate this approach with a general purpose finite element software.

6. APPLICATION AND INTEGRATION WITH GENERAL PURPOSE FINITE ELEMENT SOFTWARE

To illustrate the application of the method developed in the paper, we consider a simplified model of an aircraft wing. The model is shown in Fig. 6. The modeling is done using the ANSYS$^{\text{TM}}$ [20] finite element software. The wing is of uniform con-

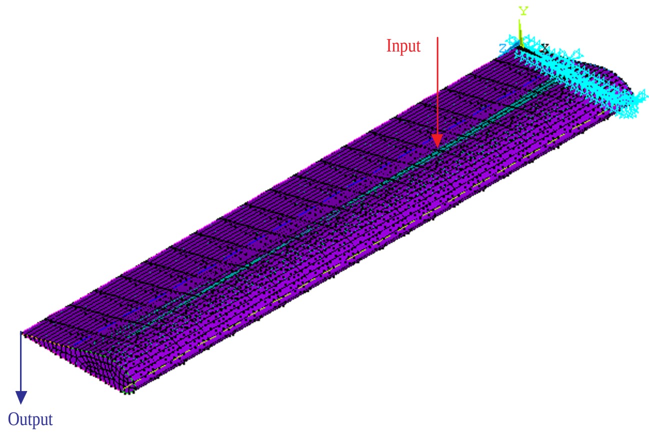
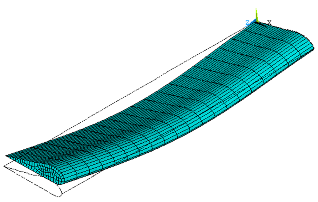


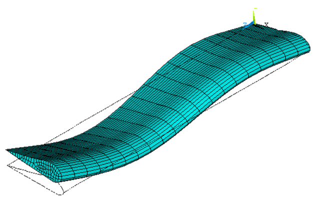
Fig. 6: The Finite Element (FE) model of an aircraft wing (5907 degrees-of-freedom). The width is 1.5m, length is 20.0m and the height of the aerofoil section is 0.3m. The material properties are: Young's modulus 262Mpa, Poisson's ratio 0.3 and mass density 888.10kg/m^3. Input node number: 407 and the output node number 96. A 2% modal damping factor is assumed for all modes.

figuration along its length and its cross-sectional area is defined by a straight-line and a spline. It is firmly attached to the body of the aircraft at one end and hangs freely at the other. The width and length of the wing are 1.5m and 20.0m respectively. The height of the aerofoil section is 0.3m. The wing is made of low density polyethylene with a Young's modulus of 262 MPa, Poisson's ratio of 0.3 and a density of 888.10kg/m^3. In this analysis, 8-noded brick element (SOLID45) is used with global element size as 0.10m. The total degrees of freedom of the system turned out to be 5907. It is assumed that the wing is excited by an unit harmonic excitation at the point shown in Fig. 6 (aimed to simulate excitation coming from the engine) and we are interested in the dynamic response of the tip of the leading edge of the wing. For the finite element model we obtain the input node number 407 and the output node number 96 corresponding to these two points. A constant 2% modal damping factor for all the modes for the calculation of dynamic response of the wing.

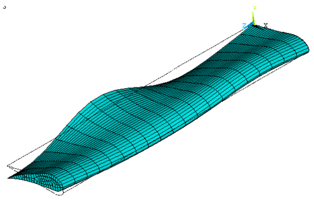
We are interested in frequency response of this system upto 1.0 kHz. The system has about 47 modes within this frequency range. Consequently we have used $m = 50$ in our calculations. Four selected modes are shown in Fig. 7 for illustration. For the system uncertainty, four values of the dispersion parameters, namely $\delta_M, \delta_K = 0.1, 0.2, 0.3$ and 0.4, are considered for the mass and the stiffness matrices. The method outlined in the previous section is used with 10,000 samples in the Monte Carlo simulation.



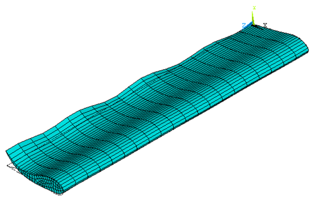
(a) Mode 3, frequency 19.047Hz



(b) Mode 5, frequency 53.628Hz



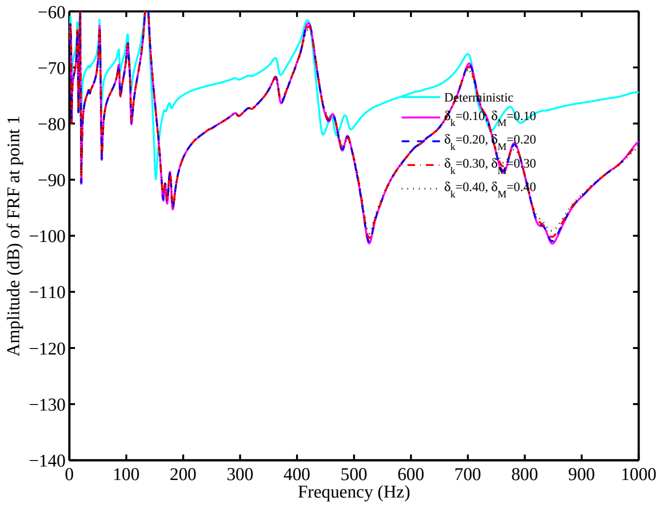
(c) Mode 10, frequency 168.249Hz



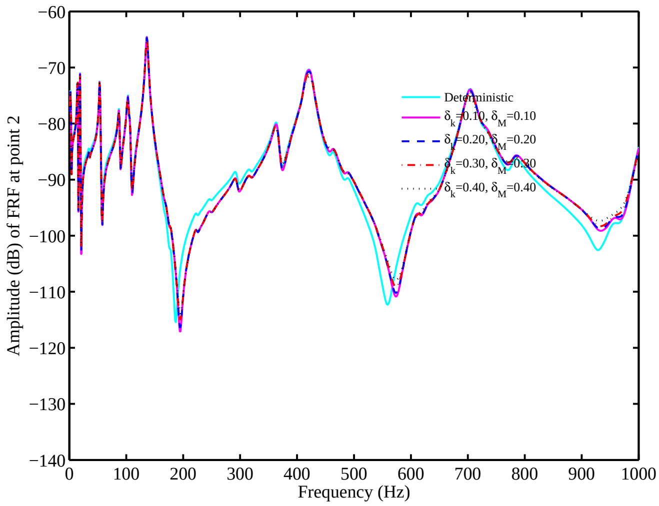
(d) Mode 20, frequency 403.711Hz

Fig. 7: Four selected mode shapes of the baseline model of the wing.

The baseline and mean of the amplitude of the driving-point-FRF obtained using the proposed reduced approach for the four sets of dispersion parameters are shown in Fig. 8(a). Except in the lower frequency range, the ensemble means of the response amplitude obtained for the four sets of dispersion parameters do not follows the deterministic result closely. However, the ensemble means for different sets of dispersion parameters are very close to each other. Unlike the driving-point-FRF,



(a) The driving-point-FRF

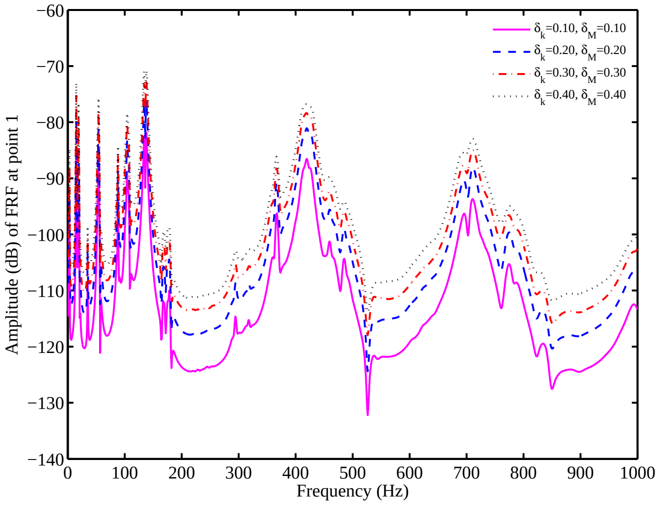


(b) A cross-FRF

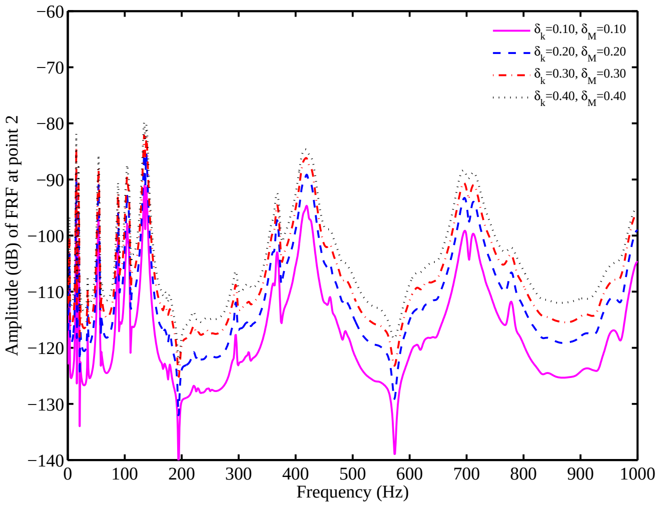
Fig. 8: Baseline and mean of the amplitude of the driving-point and a cross FRF obtained using the proposed reduced approach for the four sets of dispersion parameters.

the ensemble mean of the response amplitude of the cross-FRF shown in Fig. 8(b) follows the deterministic result relatively closely and also the means obtained for different sets of dispersion parameters are very close to each other.

The standard deviations of the amplitude of the FRFs obtained using the proposed reduced approach for the four sets of dispersion parameters are shown in Fig. 9. As expected, the lower values



(a) The driving-point-FRF



(b) A cross-FRF

Fig. 9: Standard deviation of the amplitude of the driving-point and a cross FRF obtained using the proposed reduced approach for the four sets of dispersion parameters.

of δ_K and δ_M correspond to lower values of standard deviations and vice versa. The fluctuations in the standard deviations are larger in the high frequency rage compared to low frequency range. This study shows that, due to the non-intrusive nature, the proposed reduced method can be easily integrated with a commercially available general purpose finite element software for uncertainty quantification.

7. SUMMARY AND CONCLUSIONS

The discretized equation of motion of linear stochastic dynamical systems is characterized by random mass, stiffness and damping matrices. The possibility of using a single reduced Wishart random matrix model for the system is investigated in the paper. Closed-form analytical expressions of the parameters of the reduced Wishart random matrix have been given. The new approach requires the baseline mass and stiffness matrices and dispersion parameters associated with these matrices. The main novelty of the proposed approach compared to existing random matrix approaches are (a) it only requires the simulation of one random matrix, and (b) the size of the random matrix to be simulated can be significantly smaller compared to the original system matrices. The proposed approach is however limited to systems with proportional damping only. The core computational cost of the proposed Monte Carlo simulation based method consists of (a) the generation of a single Wishart random matrix of dimension equaling to the number of modes retained in the study, and (b) the solution of a standard random eigenvalue problem of reduced dimension. This computational efficiency is arising from the fact that the number of vibration modes is generally much smaller than the dimension of the system random matrices.

The proposed simulation approach is applied to the forced vibration problem of a plate with stochastically inhomogeneous properties. Numerical results shown that it is possible to predict the variation of the dynamic response using the new approach with an acceptable accuracy. The feasibility of adopting a single reduced Wishart random matrix model to quantify uncertainty in structural dynamical systems has also been investigated using experimental data. In particular, uncertainty in a vibrating plate due to disorderly attached spring-mass oscillators with random natural frequencies is considered. One hundred nominally identical dynamical systems were physically generated and individually tested in a laboratory setup. The uncertainty in the response of the main structure primarily emerges from the random attachment configurations of the subsystems having random natural frequencies. Two of the measured frequency response functions were used to validate the applicability of the proposed approach. To demonstrate the generality of the method, it has been integrated with the ANSYS finite element software. As an illustration, a model of the wing of an aircraft with uncertainty is considered. The new reduced approach opens up the possibility of performing uncertainty quantification on real-life large structural dynamic systems in a computationally efficient manner.

ACKNOWLEDGMENTS

SA gratefully acknowledges the support of The Leverhulme Trust for the award of the Philip Leverhulme Prize. Help by Dr R. Chowdhury on the ANSYS example is greatly acknowledged.

REFERENCES

[1] Pradlwarter H, Schuller G. Uncertain linear structural systems in dynamics: Efficient stochastic reliability assessment. Computers and Structures 2010;88(1-2):74–86.

[2] Schueller G, Pradlwarter H. Uncertain linear systems in dynamics: Retrospective and recent developments by stochastic approaches. Engineering Structures 2009; 31(11):2507 – 2517.

[3] Soize C. A nonparametric model of random uncertainties for reduced matrix models in structural dynamics. Probabilistic Engineering Mechanics 2000;15(3):277–294.

[4] Soize C. Maximum entropy approach for modeling random uncertainties in transient elastodynamics. Journal of the Acoustical Society of America 2001; 109(5):1979–1996. Part 1.

[5] Soize C. Random matrix theory for modeling uncertainties in computational mechanics. Computer Methods in Applied Mechanics and Engineering 2005; 194(12-16):1333–1366.

[6] Arnst M, Clouteau D, Chebli H, Othman R, Degrande G. A non-parametric probabilistic model for ground-borne vibrations in buildings. Probabilistic Engineering Mechanics 2006;21(1):18–34.

[7] Mignolet MP, Soize C. Nonparametric stochastic modeling of linear systems with prescribed variance of several natural frequencies. Probabilistic Engineering Mechanics 2008;23(2-3):267–278.

[8] Mignolet MP, Soize C. Stochastic reduced order models for uncertain geometrically nonlinear dynamical systems. Computer Methods in Applied Mechanics and Engineering 2008;197(45-48):3951–3963.

[9] Adhikari S. Wishart random matrices in probabilistic structural mechanics. ASCE Journal of Engineering Mechanics 2008;134(12):1029–1044.

[10] Ghanem R, Spanos P. *Stochastic Finite Elements: A Spectral Approach*. New York, USA: Springer-Verlag 1991.

[11] Gupta A, Nagar D. *Matrix Variate Distributions*. Monographs & Surveys in Pure & Applied Mathematics. London: Chapman & Hall/CRC 2000.

[12] Capiez-Lernout E, Pellissetti M, Pradlwarter H, Schuëller GI, Soize C. Data and model uncertainties in complex aerospace engineering systems. Journal of Sound and Vibration 2006;Published online.

[13] Pellissetti M, Capiez-Lernout E, Pradlwarter H, Soize C, Schueller GI. Reliability analysis of a satellite structure with a parametric and a non-parametric probabilistic model. Computer Methods in Applied Mechanics and Engineering 2008;198(2):344–357.

[14] Soize C, Capiez-Lernout E, Durand JF, Fernandez C, Gagliardini L. Probabilistic model identification of uncertainties in computational models for dynamical systems and experimental validation. Computer Methods in Applied Mechanics and Engineering 2008;198(1, Sp. Iss. SI):150–163.

[15] Adhikari S, Sarkar A. Uncertainty in structural dynamics: experimental validation of wishart random matrix model. Journal of Sound and Vibration 2009;323(3-5):802–825.

[16] Pichler L, Pradlwarter H, Schueller G. A mode-based meta-model for the frequency response functions of uncertain structural systems. Computers and Structures 2009;87(5-6):332 – 341.

[17] Adhikari S. Generalized Wishart distribution for probabilistic structural dynamics. Computational Mechanics 2010;45(5):495–511.

[18] Wilkinson JH. *The Algebraic Eigenvalue Problem*. Oxford, UK: Oxford University Press 1988.

[19] Adhikari S, Friswell MI, Lonkar K, Sarkar A. Experimental case studies for uncertainty quantification in structural dynamics. Probabilistic Engineering Mechanics 2009;24(4):473–492.

[20] *Release 10.0 Documentation for ANSYS*. Southpointe, 275 Technology Drive, Canonsburg, PA 15317, USA.