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Frequency Response of Stochastic Dynamic Systems: A Modal Approach

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Abstract

A method to calculate the statistics of transfer functions from the probability density functions of the eigenvalues is proposed. Firstly, single-degree-of-freedom-systems are considered, and an approximation to the mean and variance is obtained using a hybrid Laplace's method and numerical integration method. Then, the method is extended to calculate the mean and variance of frequency response function of multiple-degrees-of-freedom dynamic systems. Proportional damping is assumed and the eigenvalues are considered to be independent. Results are derived for several probability density functions, including gamma, normal and lognormal distributions. The accuracy of the approach, for both single and multiple-degrees-of-freedom systems, is examined using the direct Monte Carlo simulation.

Keywords: stochastic dynamical systems; uncertainty propagation; asymptotic expansion.

1 Introduction

A structure is modelled as a dynamic system when vibrations are expected to occur during its lifespan, as for mechanical equipments, vehicles or aircraft structures. This modelling is generally done through the finite element method (FEM) (see, for example, Reference [1]), where the structure is divided into elements and a matrix of shape functions within each element is assumed. This discretization procedure is used to approximate the partial differential equation governing the motion of the structure with a matrix equation characterized by the system mass, stiffness and damping matrices and the forcing vector, respectively $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{f}$. Proportional viscous damping is assumed, that is, $\mathbf{C}$ is diagonalized by the matrix of eigenvectors ($\mathbf{\Phi}$) of $\mathbf{M}$ and $\mathbf{K}$ and matrices appearing in the system are positive definite. A necessary condition for the method to be accurate is that the dimension of each element is short compared to the wavelength (at most $1/7$-th of the wavelength). This condition is not a problem for low frequencies, but it is at higher frequencies. As frequency augments, the number of elements needed to obtain an accurate approximation to the response grows too large, and the response sensitivity to small variations of the structure parameters rises.

Response sensitivity to variations of the structure, like variation of parameters of the model, e.g. Young's modulus, Poisson's ratio, density, or other kind of error sources, e.g. errors in model of damping, can be evaluated through uncertainty quantification. Then, the system matrices can be considered as random matrices and, if the forcing term is considered deterministic, uncertainties in the system can be characterized by the joint probability density function (pdf) of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, or, equivalently, by the joint pdf of their eigenvalues ω_j, eigenvectors $\boldsymbol{\phi}_j$ and damping factors ζ_j. It is generally considered that, at low frequencies, the study of the response is best addressed by parametric approach. Then, stochastic finite element methods (SFEM) are applied to obtain response statistics or eigenvalues and eigenvectors statistics.

SFEM are principally divided into simulation-based methods (e.g. Monte Carlo Simulation (MCS), see Reference [2]) and expansion-based methods (perturbation method, spectral approach and stochastic reduced basis method (SRBM) [3]). A review on SFEM is given, for example, in Reference [4]. Application of perturbation methods to calculate response can be found, for example, in Reference [5]. Spectral approach methods are reviewed in Reference [6], where the most widely used spectral approach method is polynomial chaos (PC) [7]. Non-probabilistic approaches, like fuzzy finite element procedures [8] or interval finite element [9], have also been researched to obtain frequency response function characteristics. A different approach followed by Reference [10, 11] proposed exact analytical expressions of response statistics for a single-degree-of-freedom system. They were obtained from the pdf of eigenvalues, related to pdf of random parameters. Non-parametric uncertainty can also be considered by modeling system matrices as different types of random matrices, like gamma distribution matrices [12] or Wishart matrices [13], this approach works well at high frequencies.

A review of methods to calculate mean and variance of random eigenvalue problems is given in Reference [14]. The most widely used methods are based on the perturbation method, and work well when the uncertainties are small and the parameter distribution is Gaussian, see for example, References [15–19]. Other methods include simulation methods [20, 21], crossing theory [22], asymptotic methods [23–25], iterative methods [26] and expansion methods like stochastic reduced basis [27], Ritz method [28, 29], polynomial chaos [30] or dimensional decomposition method [31]. At high frequencies, several pdf for eigenvalues are assumed (e.g. Poisson's distribution [32], gaussian orthogonal ensemble [33]) to calculate space and frequency averaged energies in the context of SEA.

It is observed that, on the one hand, efforts have been made to obtain results on systems eigenvalues and eigenvectors statistics. On the other hand, few works investigate about response statistics based on those results. This work focuses on obtaining mean and variance of the response from eigenvalue pdf. The integrals appearing in the problem can be integrated exactly for uniform distribution. For other distributions, Laplace's method is applied when a good approximation is expected, and a numerical integration or a modified Laplace's method is applied elsewhere. The method is extended to multiple degrees of freedom (MDOF) systems, assuming uncorrelated eigenvalues and proportional damping. Numerical results allow to compare the proposed method and MCS.

2 Single-degree-of-freedom (SDOF)

Single-degree-of-freedom systems are often used to model structural dynamic systems as their solution can be easily obtained. They are also useful for general MDOF problems, as MDOF problems with proportional damping reduce to a linear combination of SDOF problems. For an SDOF system, the transfer function h is given by

$$h(\mathrm{i}\omega) = \frac{1}{(-\omega^2 + \mathrm{i}2\omega_n\zeta_n\omega + \omega_n^2)} \qquad \text{with} \qquad \mathrm{u} = h(\mathrm{i}\omega)\mathrm{f} \tag{1}$$

Here ω_n and ζ_n are respectively the natural frequency and the damping factor, ω is the frequency, u and f are the response and the forcing term. The real and imaginary parts of the transfer function and its absolute value are respectively given by

$$\Re\left(h(\omega,\omega_n)\right) = \frac{\omega_n^2 - \omega^2}{(\omega_n^2-\omega^2)^2 + 4\zeta_n^2\omega_n^2\omega^2} \tag{2}$$

$$\Im\left(h(\omega,\omega_n)\right) = \frac{-2\zeta_n\omega_n\omega}{(\omega_n^2-\omega^2)^2 + 4\zeta_n^2\omega_n^2\omega^2} \tag{3}$$

$$|h(\omega,\omega_n)| = \frac{1}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta_n^2\omega_n^2\omega^2}}. \tag{4}$$

In this set of equations, without any loss of generality, the squared natural frequency of the system ω_n^2 is assumed to be a random variable.

2.1 The general derivation of the response statistics

In this subsection, a method to calculate first and second moment of quantities from Eqs. (2) to (4) is proposed. A probability density function $f_x(x)$ is assumed for the random variable $x = \omega_n^2$, where $f_x(x \leq 0) = 0$. The first moment (mean) of real and imaginary part of the transfer function and the second moment of the transfer function, derived from Eqs. (2), (3) and (4), are given by (see, for example, Reference [34])

$$\mu_{\Re(h)} = \mathrm{E}\left[\Re(h)\right] = \int_{-\infty}^{+\infty} \frac{x f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx - \omega^2 \int_{-\infty}^{+\infty} \frac{f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx \tag{5}$$

$$\mu_{\Im(h)} = \mathrm{E}\left[\Im(h)\right] = -2\zeta_n\omega \int_{-\infty}^{+\infty} \frac{\sqrt{x} f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx \tag{6}$$

$$\mathrm{E}\left[|h|\right] = \sigma_h^2 + \mu_h^2 = \int_{-\infty}^{+\infty} \frac{f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx \tag{7}$$

where $\mu_h = \mu_{\Re(h)} + \mathrm{i}\mu_{\Im(h)}$ is the mean of the absolute value of the transfer function and σ_h is its standard deviation. The square of the mean is obtained by multiplying the mean by its complex conjugate, so that $\mu_h^2 = \mu_{\Re(h)}^2 + \mu_{\Im(h)}^2$. In the next section, integrals appearing in Eqs. (5), (6) and (7) are approximated with Laplace's method.

2.1.1 Laplace's method

A Laplace integral (see, for example, Reference [35]) is an integral that can be approximated by

$$I(w) = \int_{x_1}^{x_2} g(x) e^{wy(x)}\, dx \backsim g(\theta) e^{wy(\theta)} \left[\frac{-2\pi}{wy''(\theta)}\right]^{\frac{1}{2}} \tag{8}$$

if g is continuous, y is twice continuously differentiable and $y'(\theta) = 0$, $y''(\theta) < 0$. We will assume $g(x) = 1$ and $w = 1$. Then

$$I(w) \backsim \sqrt{2\pi} e^{y(\theta)} \left[-y''(\theta)\right]^{-\frac{1}{2}}. \tag{9}$$

A general function $y(x,a)$, obtained from integrals appearing in Eqs.(5), (6) and (7), can be given by

$$y(x,a) = \ln(f_x(x)) + a\ln(x) - \ln(|h(\omega,x)|^2) \tag{10}$$

where

$$a = 0 \qquad \text{for} \qquad \int_{-\infty}^{+\infty} \frac{f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx \tag{11}$$

$$a = \frac{1}{2} \qquad \text{for} \qquad \int_{-\infty}^{+\infty} \frac{\sqrt{x} f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx \tag{12}$$

$$a = 1 \qquad \text{for} \qquad \int_{-\infty}^{+\infty} \frac{x f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\, dx. \tag{13}$$

Function $y(x,a)$ is maximum at $x = \theta_a$. Therefore, θ_a is the solution of

$$\frac{f_x'(\theta_a)}{f_x(\theta_a)} + \frac{a}{\theta_a} - \frac{2(\theta_a - \omega^2) + 4\omega^2\zeta_n^2}{(\theta_a-\omega^2)^2 + 4\zeta_n^2\omega^2\theta_a} = 0 \tag{14}$$

for which $y(x,a)$ is absolute maximum. The second derivative of $y(x,a)$ is

$$y''(x,a) = \bar{f}(x) - \frac{a}{x^2} - \bar{h}(x) \tag{15}$$

where

$$\bar{h}(x,\omega) \;=\; \frac{2}{(-\omega^2+x)^2+4\zeta_n^2\omega^2 x} - \left(\frac{2(-\omega^2+x)+4\zeta_n^2\omega^2}{(-\omega^2+x)^2+4\zeta_n^2\omega^2 x}\right)^2. \tag{16}$$

and $\bar{f}(x)$ depends on the pdf of the random variable x.

2.1.2 Hybrid Laplace-numerical integration and modified Laplace's methods

Laplace's method approximates Eq. (10) with a second degree polynomial given by the two first terms of Taylor expansion of $y(x)$ around its maximum θ_a. The method works well if the behavior of $y(x)$ in the vicinity of its absolute maximum point θ_a is well represented by the approximation used. From Eq.(10) it may be observed that $y(x)$ is obtained through the addition of three functions. The first one, $\ln(f_x(x))$ has its global maximum at μ, the global maximum and mean of $f_x(x)$, the pdf of the random variable x. The second function added to obtain $y(x)$ is $a\ln(x)$, an increasing function. The last function, $-\ln(|h(\omega,x)|^2)$, has its maximum at $\omega^2(1-2\zeta_n^2)$ and three points (x_{h1}, x_{h2} and x_{h3}) where its third derivative is zero

$$x_{h1} \;=\; \omega^2(1-2\zeta_n^2) - 2\sqrt{3}\omega^2\zeta_n\sqrt{1-\zeta_n^2} \tag{17}$$

$$x_{h2} \;=\; \omega^2(1-2\zeta_n^2) \tag{18}$$

$$x_{h3} \;=\; \omega^2(1-2\zeta_n^2) + 2\sqrt{3}\omega^2\zeta_n\sqrt{1-\zeta_n^2}. \tag{19}$$

Therefore, $y(x)$ can have up to two local maximums, the first one, $\theta_{a\,\mu}$, is close to μ and the second one ($\theta_{a\,\omega^2}$) is close to $\omega^2(1-2\zeta_n^2)$. A Newton iteration with these starting points should converge to the solution in few steps if the two maximums exist. Three roots of $y'''(x)=0$ can appear close to x_{h1}, x_{h2} and x_{h3}, respectively, $x_1 \approx x_{h1}$, $x_2 \approx x_{h2}$ and $x_3 \approx x_{h3}$. As formerly, Newton iteration is a procedure leading to the solution in few steps if the three points, x_{h1}, x_{h2} and x_{h3}, exist. Function $y(x)$ can have different shapes

- The function has two maximums and $y(\theta_{a\,\omega^2}) < y(\theta_{a\,\mu})$, then, Laplace's method is used.
- The function has two maximums and $y(\theta_{a\,\mu}) \leq y(\theta_{a\,\omega^2})$. Laplace approximation does not work well, in this case, a numerical integration (i.e. Gaussian quadrature, trapezoidal method) is a good alternative to approximate the integral.
- The function has only one maximum, $\theta_{a\,\mu}$, and x_1, x_2 and x_3 do not exist or are situated at the same side of $\theta_{a\,\mu}$, that is, $x_3 < \theta_{a\,\mu}$ or $\theta_{a\,\mu} < x_1$. Then, Laplace's method is used.

- The function has only one maximum, $\theta_{a\,\omega^2}$, and x_1, x_2 and x_3 exist and are such that $x_1 < \theta_{a\,\omega^2} < x_3$. In those situations, Laplace approximation can work well if no discontinuity has appeared both in $\theta(a\,\omega)$ and $y''(\theta(a\,\omega))$. Generally, Laplace approximation will provide a value smaller than the exact one. A different second order polynomial, having its maximum at θ_a and going through x_1 if $\theta_a > \mu$ and through θ_3 if $\theta_a \leq \mu$ can be used instead of the Taylor expansion used in Laplace method. Then

$$I(\omega) = e^{y(\theta_a)} \sqrt{\frac{-\pi(\theta_a - x_1)^2}{y(x_1) - y(\theta_a)}} \quad \text{if} \quad \theta_a > \mu \tag{20}$$

$$I(\omega) = e^{y(\theta_a)} \sqrt{\frac{-\pi(\theta_a - x_3)^2}{y(x_3) - y(\theta_a)}} \quad \text{if} \quad \theta_a \leq \mu. \tag{21}$$

  This second approximation to the integral provides, mostly, a larger value than the exact result. This approximation will be referred as modified Laplace's method.

The hybrid Laplace-numerical integration method applies Laplace's method when it leads to a good approximation and numerical approximation for other frequencies. The hybrid modified Laplace's method uses Laplace's method, modified Laplace's method or the numerical approximation depending on which method has been suggested.

3 FRF statistics for different probability density functions of the natural frequency

Any pdf of $x = \omega_n^2$ has to satisfy that $f_x(x \leq 0) = 0$, as it is not physically possible to have a negative or zero squared natural frequency. Numerical examples are provided, where $\mu_x = 9$, $\sigma_x = 1$ and the damping factor $\zeta_n = 0.1$ or $\zeta_n = 0.01$. The chosen pdfs, other than the uniform distribution, are assumed to be unimodal with maximum at $x = \mu$. The number of samples used in MCS is 5000.

3.1 Uniform distribution

The pdf of a uniform distribution is given by a constant α_c defined over the interval $x \in [c_1, c_2]$. Parameters α_c, c_1 and c_2 of the distribution can be expressed through its mean (μ_x) and variance (σ_x)

$$c_1 = \mu_x - \sqrt{3\sigma_x} \quad c_2 = \mu_x + \sqrt{3\sigma_x} \quad \alpha_c = \frac{1}{2\sqrt{3\sigma_x}}. \tag{22}$$

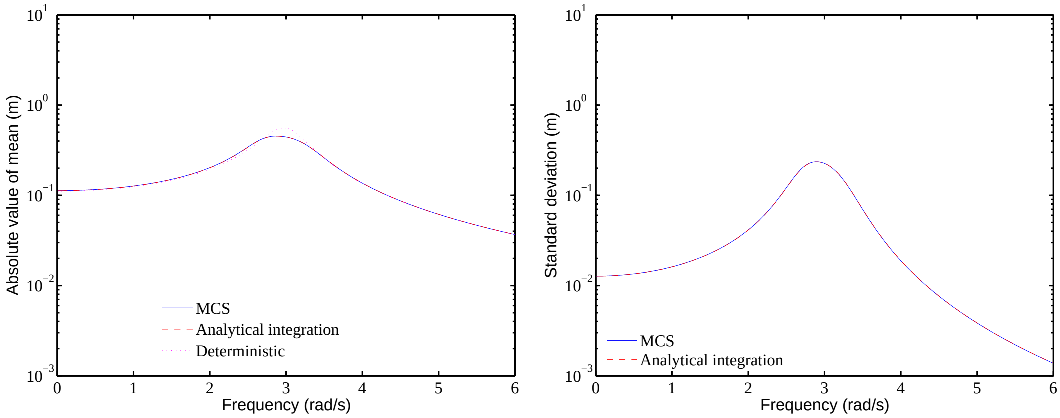
For this pdf, all integrals appearing in Eqs. (11), (12) and (13) can be calculated exactly. For example,

$$\text{if} \quad a = 0, \quad I(\omega) = \frac{1}{2\omega^2\zeta_n\sqrt{1-\zeta_n^2}} \arctan \frac{x-\omega^2(1-2\zeta_n^2)}{2\omega^2\zeta_n\sqrt{1-\zeta_n^2}} \tag{23}$$

$$\text{if} \quad a = 1, \quad I(\omega) = \frac{\log(\omega^2(\omega^2-2x(1-2\zeta_n^2))+x^2)}{2} - \tag{24}$$

$$\frac{\sqrt{1-4\zeta_n^2(1-\zeta_n^2)}}{2\zeta_n\sqrt{1-\zeta_n^2}} \arctan \frac{(\omega^2(2\zeta_n^2-1)+2x)\sqrt{1-4\zeta_n^2(1-\zeta_n^2)}}{\omega^2(2\zeta_n^2-1)2\zeta_n\sqrt{1-\zeta_n^2}} \tag{25}$$

An expression for $a = 0.5$ can also be calculated analytically. Figure 1 and Figure 2 compare the results of the analytical expressions and MCS for different ζ_n and same uniform distribution for x.



(a) Absolute value of mean. (b) Standard deviation.

Figure 1: Mean and standard deviation of the absolute value of the transfer function for uniform distribution, with $\zeta_n = 0.1$.

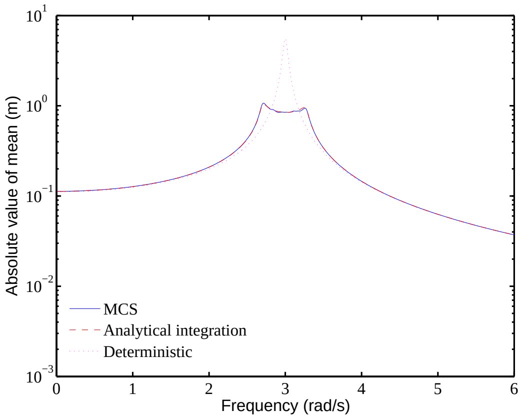
3.2 Normal distribution

The pdf $f_x(x)$ and $f_x'(x)/f_x(x)$ of the normal distribution conditional to $x > 0$ are given by

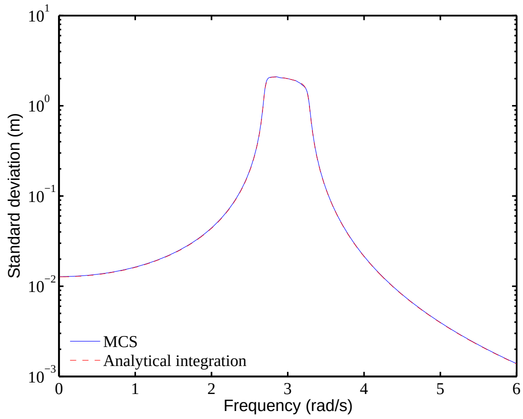
$$f_x(x) = \frac{e^{-(x-\mu_x)^2/2\sigma_x^2}}{\sqrt{2\pi\sigma_x^2}P(0)} \quad x \in (0, +\infty) \qquad \frac{f_x'(x)}{f_x(x)} = -\frac{x-\mu_x}{\sigma_x^2} \tag{26}$$

where μ_x and σ_x^2 are respectively the mean and the variance of the distribution and $P(0)$ is the probability of $x \leq 0$ for a normal distribution $N(\mu_x\,\sigma_x)$. From Eq. (14), the parameter θ_a can be identified as the solution to a fourth order polynomial

$$b_{1_a}x^4 + b_{2_a}x^3 + b_{3_a}x^2 + b_{4_a}x + b_{5_a} = 0 \tag{27}$$



(a) Absolute value of mean.



(b) Standard deviation.

Figure 2: Mean and standard deviation of the absolute value of the transfer function for uniform distribution, with $\zeta_n = 0.01$.

with coefficients

$$\begin{aligned} b_{1_b} &= -1 \\ b_{2_b} &= -\bar{\zeta}_\omega + \mu_x \end{aligned} \qquad \begin{aligned} b_{3_b} &= -\omega^4 + (a-2)\sigma_x^2 + \mu_x \bar{\zeta}_\omega \\ b_{4_b} &= (1+a)\sigma_x^2 \bar{\zeta}_\omega + \mu_x \omega^4 \\ b_{5_b} &= a\sigma_x^2 \omega^4 \end{aligned} \tag{28}$$

and $\bar{\zeta}_\omega = 2\omega^2 (2\zeta_n^2 - 1)$. A real solution close to the mean of the distribution, $\theta_{a\,\mu}$, is assumed and if a second solution $\theta_{a\,\omega^2}$ exists and is not a saddle point, then a relative minimum between the two relative maximums exists. The remaining real solution is a spurious value, and likely to be negative or close to zero. As indicated in subsubsection 2.1.2, Laplace method may be applied to approximate the integrals appearing at Eqs. (5), (6) and (7). The second derivatives of $y(x)$ is given by Eq. (15) where $\bar{h}(x, \omega)$ is given by Eq. (16) and $\bar{f}(x) = -\frac{1}{\sigma_x^2}$. Laplace's method approximation, assuming normal distribution, is given by

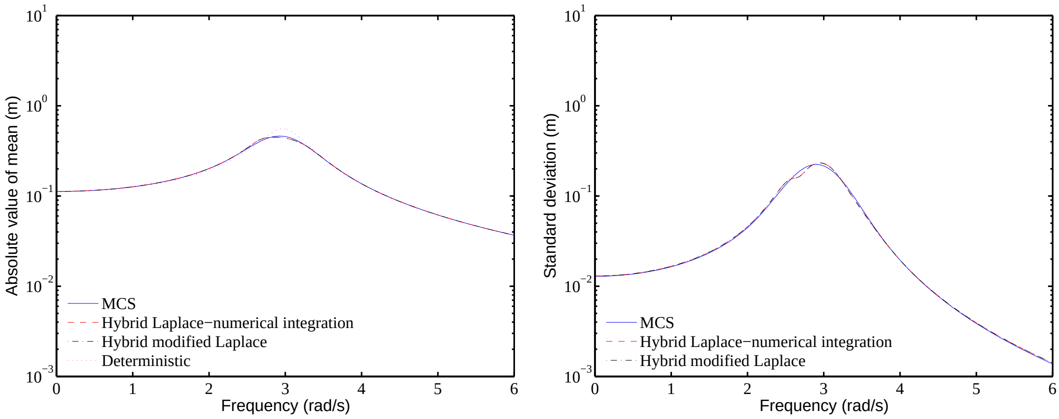
$$I(\omega, a) \backsim \frac{\theta_a^a \sqrt{2\pi} e^{-(\theta_a - \mu_x)^2/2\sigma_x^2}}{P(0)\sqrt{2\pi\sigma_x^2}\left((\theta_a - \omega^2)^2 + 4\zeta_n^2 \omega^2 \theta_a\right)^{a/2}} \left[\frac{1}{\sigma_x^2} - \frac{a}{\theta_a^2} + \bar{h}(\theta_a)\right]^{-\frac{1}{2}} \tag{29}$$

where a is given in Eqs. (11), (12) and (13). Another approximation of the integral can be obtained from Eqs. (20) and (21). Plots of approximations for an SDOF system with $\mu_x = 9$ and $\sigma_x = 1$ are shown, in Figure 3 for $\zeta_n = 0.1$ and in Figure 4 for $\zeta_n = 0.01$. It is observed that the method works better for higher damping.

3.3 Gamma distribution

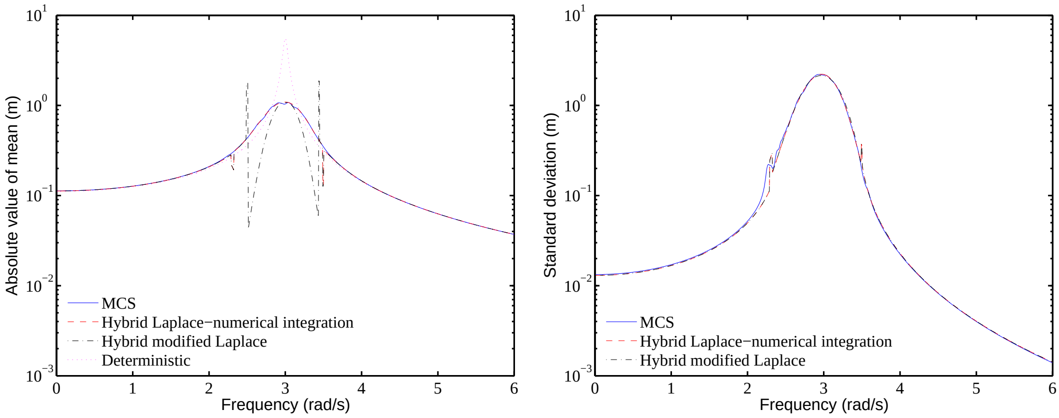
The probability density function $f_x(x)$ and $f_x'(x)/f_x(x)$ of the gamma distribution, defined in the interval $[0, \infty]$, are given by

$$f_x(x) = \frac{x^{\alpha_g - 1}}{\Gamma(\alpha_g)\beta_g^{\alpha_g}} e^{-x/\beta_g} \qquad \frac{f_x'(x)}{f_x(x)} = \frac{\alpha_g - 1}{x} - \frac{1}{\beta_g}. \tag{30}$$



(a) Absolute value of mean. (b) Standard deviation.

Figure 3: Mean and standard deviation of the absolute value of the transfer function for normal distribution, with $\zeta_n = 0.1$.



(a) Absolute value of mean. (b) Standard deviation.

Figure 4: Mean and standard deviation of the absolute value of the transfer function for normal distribution, with $\zeta_n = 0.01$.

Relationships between mean μ_x, variance σ_x^2 and parameters α_g and β_g are given by

$$\mu_x = \alpha_g \beta_g \qquad \sigma_x^2 = \alpha_g \beta_g^2 \qquad \alpha_g = \frac{\mu_x^2}{\sigma_x} \qquad \beta_g = \frac{\sigma_x}{\mu_x} \tag{31}$$

As formerly, θ_a is the real solution to the third order polynomial

$$b_{1_a} x^3 + b_{2_a} x^2 + b_{3_a} x + b_{4_a} = 0 \tag{32}$$

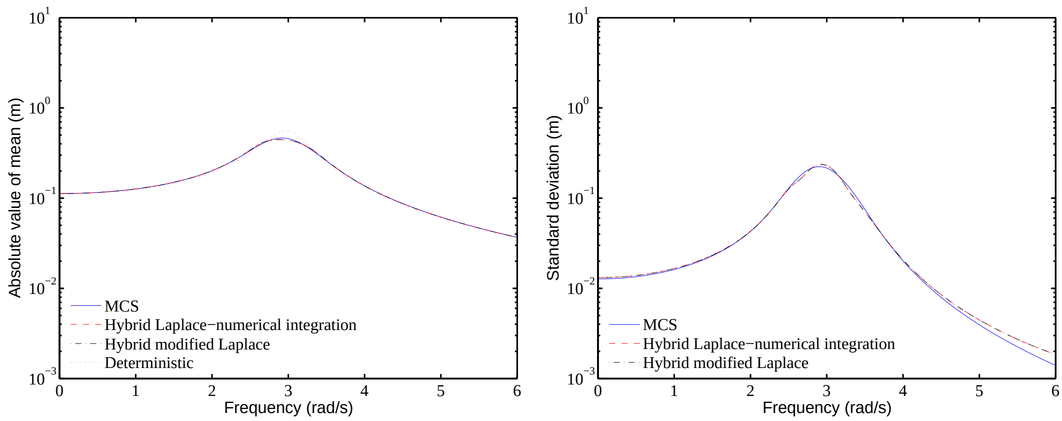
with coefficients

$$\begin{aligned} b_{1_a} &= -\frac{1}{\beta_g} & b_{2_a} &= -\frac{\bar{\zeta}_\omega}{\beta_g} + \alpha_g + a - 3 \\ b_{4_a} &= \omega^4(\alpha_g + a - 1) & b_{3_a} &= \bar{\zeta}_\omega \left(\alpha_g + a - 2\right) - \frac{\omega^4}{\beta_g} \end{aligned} \tag{33}$$

and $\bar{\zeta}_\omega = 2\omega^2 (2\zeta_n^2 - 1)$. As indicated in subsubsection 2.1.2, one of the solutions is close to μ, and, if it exists, a second solution is close to $x_{h2} = \omega^2(1 - 2\zeta_n^2)$. The third real solution is the relative minimum situated between the two relative maximums. Otherwise, two of the solutions are complex and there is only one real solution. The second derivative of $y(x, a)$ is given at Eq. (15) where $\bar{h}(x, \omega)$ is given by Eq. (16) and $\bar{f}(x) = (1 - \alpha_g)/(x^2)$. From Eq. (9), an analytical approximation to the integrals appearing in Eqs. (11), (12) and (13) is given by

$$I(\omega, a) \backsim \frac{\sqrt{2\pi}\theta_a^{\alpha_g - 1 + a} e^{-\theta_a/\beta_g}}{\Gamma(\alpha_g)\beta_g^{\alpha_g}((\theta_a - \omega^2)^2 + 4\zeta_n^2\omega^2\theta_a)} \left[\frac{\alpha_g - 1 - a}{\theta_a^2} + \bar{h}(\theta_a)\right]^{-\frac{1}{2}} \tag{34}$$

where parameters α_g and β_g can be derived from Eq. (31) if the mean μ_x and the standard deviation σ_x of ω_n^2 are known. Plots of approximations for an SDOF system with $\mu_x = 9$ and $\sigma_x = 1$ are displayed, in Figure 5 for $\zeta_n = 0.1$ and in Figure 6 for $\zeta_n = 0.01$. It is observed that the method works better for higher damping.



(a) Absolute value of mean.

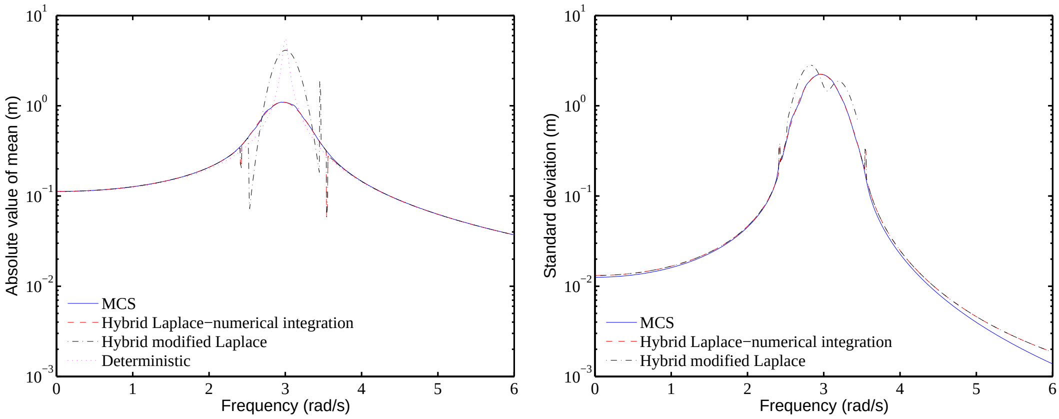
(b) Standard deviation.

Figure 5: Mean and standard deviation of the absolute value of the transfer function for gamma distribution, with $\zeta_n = 0.1$.

3.4 Lognormal distribution

A lognormal distribution is a probability distribution obtained by taking the exponential of a normal distribution of mean μ and standard deviation σ, $N(\mu, \sigma)$. The probability density function $f_x(x)$, and $f_x'(x)/f_x(x)$ of lognormal distribution, defined in the interval $(0, \infty)$, is given by

$$f_x(x) = \frac{e^{-(\ln x - \mu)^2/2\sigma^2}}{x\sqrt{2\pi\sigma^2}} \qquad \frac{f_x'(x)}{f_x(x)} = -\frac{\sigma^2 \ln x - \mu}{x\sigma^2}. \tag{35}$$



(a) Absolute value of mean.

(b) Standard deviation.

Figure 6: Mean and standard deviation of the absolute value of the transfer function for gamma distribution, with $\zeta_n = 0.01$.

Mean μ_x and variance σ_x of lognormal distribution are related to μ and σ by

$$\mu_x = e^{\mu+\sigma^2/2} \qquad \sigma_x^2 = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2} \tag{36}$$

$$\mu = \ln \mu_x - \frac{1}{2}\ln\left(1 + \frac{\sigma_x}{\mu_x^2}\right) \qquad \sigma = \ln\left(\frac{\sigma_x}{\mu_x^2} + 1\right). \tag{37}$$

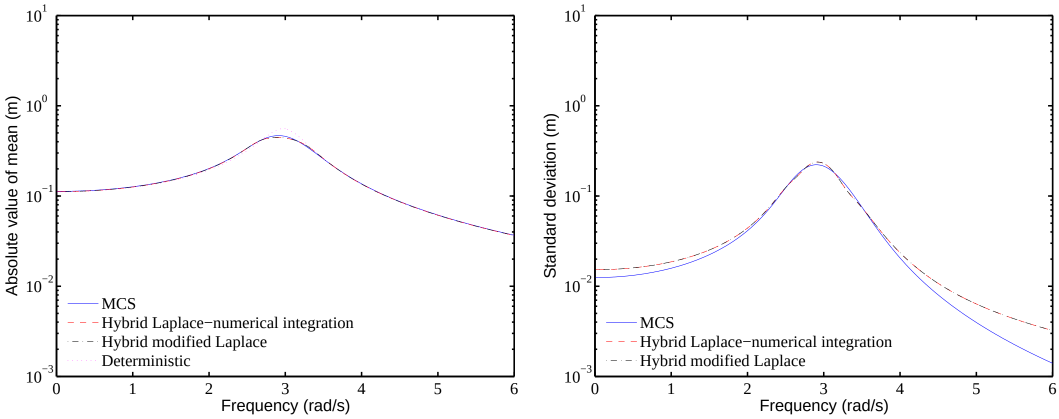
From Eq. (14) can be identified parameter θ_a as the solution to the equation

$$y'(\theta_a, a) = -\frac{\sigma^2 + \ln\theta_a - \mu}{\theta_a\sigma^2} + \frac{a}{\theta_a} - \frac{(2(\theta_a - \omega^2) + 4\zeta_n^2)}{(\theta_a - \omega^2)^2 + 4\zeta_n^2\theta_a} = 0 \tag{38}$$

As indicated in subsubsection 2.1.2 we will assume here that one solution, $\theta_{a\,\mu}$, is close to the mean of the distribution and that another relative maximum of $y(x)$, $\theta_{a\,\omega^2}$, can arise together with a relative minimum situated between those two maximums. We can then find an approximation to these solutions using Newton method. The following analytical approximation to the integrals appearing in Eqs. (11), (12) and (13) can be obtained using Eq. (9)

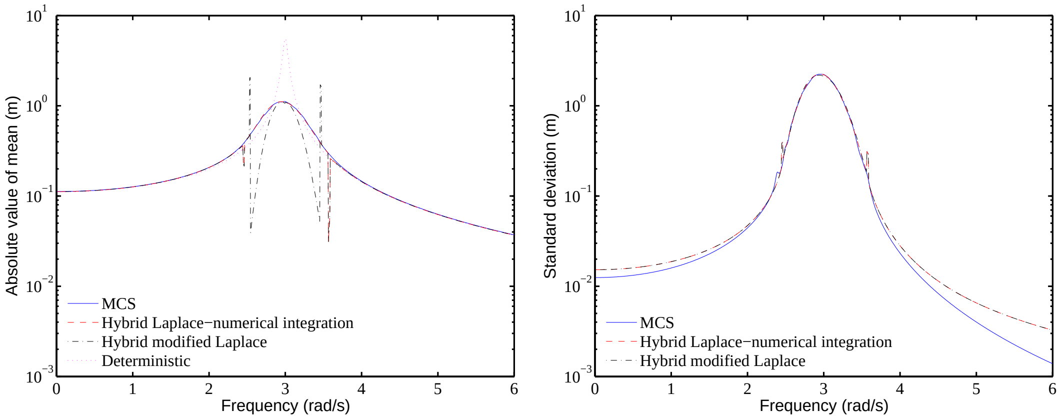
$$I(\omega, a) \backsim \frac{\theta_a^{a-1} e^{-(\ln\theta_a - \mu)^2/2\sigma^2}}{\sqrt{\sigma^2}((\theta_a - \omega^2)^2 + 4\zeta_n^2\omega^2\theta_a)}\left[\frac{\sigma^2(a-1) - \ln\theta_a + \mu}{\sigma^2\theta_a^2} + \bar{h}(\theta_a)\right]^{-\frac{1}{2}} \tag{39}$$

Parameters μ and σ can be found if μ_x and σ_x are known. Plots of the approximations for an SDOF system with $\mu_x = 9$ and $\sigma_x = 1$ are displayed, in Figure 7 for $\zeta_n = 0.1$ and in Figure 8 for $\zeta_n = 0.01$. As for normal and gamma distributions, the method works better for higher damping.



(a) Absolute value of mean. (b) Standard deviation.

Figure 7: Mean and standard deviation of the absolute value of the transfer function for lognormal distribution, with $\zeta_n = 0.1$.



(a) Absolute value of mean. (b) Standard deviation.

Figure 8: Mean and standard deviation of the absolute value of the transfer function for lognormal distribution, with $\zeta_n = 0.01$.

4 Multiple-degrees-of-freedom (MDOF)

4.1 Response calculation

Applying the finite element method to structural dynamic systems leads, generally, to an MDOF problem where a displacement vector $\mathbf{x}$ is the unknown. The frequency response vector of the MDOF system is given by (see, for example,Reference [1])

$$\begin{aligned}\mathbf{u} &= \mathbf{\Phi}[s^2\mathbf{I} + s2\boldsymbol{\zeta}\mathbf{\Omega} + \mathbf{\Omega}^2]^{-1}\mathbf{\Phi}^T\bar{\mathbf{f}} = \mathbf{\Phi}\mathbf{H}'\mathbf{\Phi}^T\mathbf{f} \\ &= \mathbf{\Phi}\left[\sum_{j=1}^{N}\frac{\mathbf{e}_j\mathbf{e}_j^T}{-\omega^2 - 2\mathrm{i}\omega\zeta_j\omega_j + \omega_j^2}\right]\mathbf{F}\end{aligned} \tag{40}$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\Omega}$ are respectively the matrices of damping parameters and eigenvalues, $\mathbf{e}_j$ is the j-th unit vector, or j-th column of an identity matrix, and matrix $\mathbf{H}^{'}$ is therefore diagonal. We denote by $\mathbf{u}^*$ and $\mathbf{H}^{'*}$ the complex conjugate of $\mathbf{u}$ and $\mathbf{H}^{'}$ respectively. The j-th diagonal element of matrix $\mathbf{H}^{'}$ is denoted by h_j', and $h_j^{'*}$ is its complex conjugate. Vector $\boldsymbol{\Phi}_i$ is the i-th row of matrix $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi}_{i_j}$ is its j-th element. The j-th element of vector $\mathbf{F} = \boldsymbol{\Phi}^T\mathbf{f}$ is denoted by F_j. Uncertainty is introduced by the diagonal terms of $\mathbf{H}'$, and therefore, all other vectors and matrices are deterministic. In the proposed method, a pdf is assumed for each random eigenvalue, the eigenvalues are assumed independent and eigenvectors are assumed deterministic. These assumptions are not valid for high frequencies, where overlap between eigenvalues appears and eigenvectors are not deterministic. Therefore, the proposed method is expected to work well at low frequencies. From Eq. (40), an expression of u_i, the i-th term of vector $\mathbf{u}$, and of $|\mathrm{u}_i|^2$ can be derived

$$\mathrm{u}_i = \sum_{j=1}^{N} \boldsymbol{\Phi}_{i_j} h_j' F_j \tag{41}$$

$$|\mathrm{u}_i|^2 = \mathrm{u}_i^T \mathrm{u}_i^* = \sum_{j=1}^{N}\sum_{k=1}^{N} F_j h_j' \boldsymbol{\Phi}_{i_j} \boldsymbol{\Phi}_{i_k} h_k^{'*} F_k \tag{42}$$

We denote $C_j = F_j \boldsymbol{\Phi}_{i_j}$. Expression of $|\mathrm{u}_i|^2$ can be simplified

$$\begin{aligned} |\mathrm{u}_i|^2 &= \sum_{j=1}^{N} C_j^2 h_j^{'} h_j^{'*} + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} C_j C_k \left(h_j^{'} h_k^{'*} + h_j^{'*} h_k^{'} \right) && (43) \\ &= \sum_{j=1}^{N} C_j^2 |h_j^{'}|^2 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} C_j C_k 2\Re(h_j^{'} h_k^{'*}) && (44) \end{aligned}$$

The mean of $|\mathrm{u}_i|^2$ is equal to the second moments of $|\mathrm{u}_i|$ and u_i.

4.2 Mean of the real and imaginary part of the response

Expressions of real and imaginary part of u_i can be derived from Eq. (41)

$$\Re(\mathrm{u}_i) = \sum_{j=1}^{N} C_j \Re\left(h_j'\right) \quad \text{with} \quad \Re\left(h_j'\right) = \frac{\omega_j^2 - \omega^2}{(\omega_j^2 - \omega^2)^2 + 4\zeta_j^2\omega^2\omega_j^2} \tag{45}$$

$$\Im(\mathrm{u}_i) = \sum_{j=1}^{N} C_j \Im\left(h_j'\right) \quad \text{with} \quad \Im\left(h_j'\right) = \frac{-2\zeta_j\omega\omega_j}{(\omega_j^2 - \omega^2)^2 + 4\zeta_j^2\omega^2\omega_j^2} \tag{46}$$

Mean is a linear operator, therefore, only means $< \Re\left(h_j'\right) >$ and $< \Im\left(h_j'\right) >$ are required to calculate $< \Re(\mathrm{u}_i) >$ and $< \Im(\mathrm{u}_i) >$. These means are given by

$$< \Re\left(h_j'\right) > = I_{2_j} - \omega^2 I_{1_j} \tag{47}$$

$$< \Im\left(h_j'\right) > = -2\zeta_j\omega I_{3_j} \tag{48}$$

If uncorrelated random variables are assumed, integrals I_{1_j}, I_{2_j} and I_{3_j} are given by respectively by Eqs. (11), (13) and (12) where $x = x_j$. Analytical approximation to these integrals have already been discussed for different distributions of x_j.

4.3 Variance of the response

Expressions of the second moment of response can be derived from Eq. (44), remembering that mean is a linear operator and that uncorrelated random variables are assumed

$$
\begin{aligned}
m_2 =< |\mathrm{u}_i|^2 > &= \sum_{j=1}^{N} C_j^2 < |h_j'|^2 > + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} C_j C_k 2 < \Re(h_j' h_k'^*) > && (49) \\
< |\mathrm{u}_i|^2 > &= \sum_{j=1}^{N} C_j^2 I_{1_j} + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} C_j C_k 2 < \Re(h_j' h_k'^*) > && (50) \\
< \Re(h_j' h_k'^*) > &= \left(I_{2_j} - \omega^2 I_{1_j}\right)\left(I_{2_k} - \omega^2 I_{1_k}\right) + 4\zeta_j \zeta_k \omega^2 I_{3_j} I_{3_k} && (51)
\end{aligned}
$$

Second moment is given by

$$m_2 = \mu_{\mathrm{x}_i}^2 + \sigma_{\mathrm{x}_i}^2 \tag{52}$$

where mean of response and squared value of mean are given by

$$
\begin{aligned}
\mu_{\mathrm{u}_i} &= < \Re(\mathrm{u}_i) > +\mathrm{i} < \Im(\mathrm{u}_i) > && (53) \\
\mu_{\mathrm{u}_i}^2 &= < \Re(\mathrm{u}_i) >^2 + < \Im(\mathrm{u}_i) >^2 . && (54)
\end{aligned}
$$

And finally

$$\sigma_{\mathrm{u}_i}^2 =< |\mathrm{u}_i|^2 > -\mu_{\mathrm{u}_i}^2 . \tag{55}$$

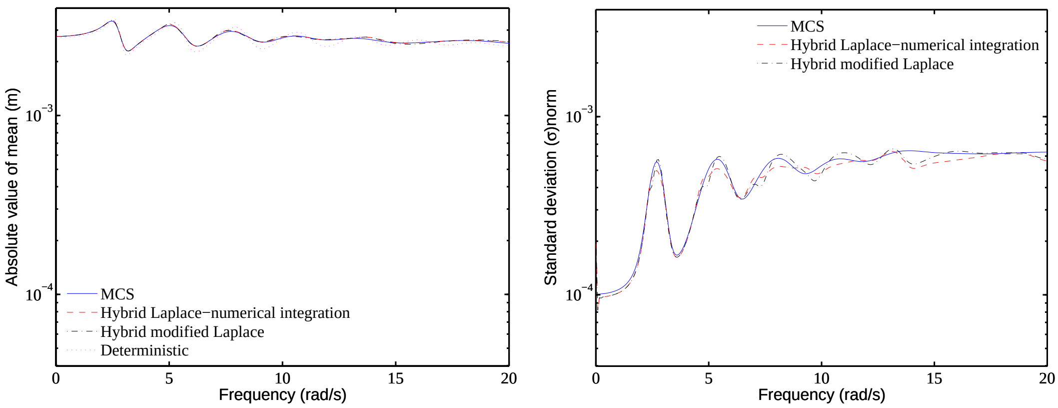
In the next section, numerical results are shown for an MDOF system with random eigenvalues with the different distributions already exposed.

4.4 Numerical example

The system considered has mass and stiffness matrices **M** and **K**, and forcing vector **f**

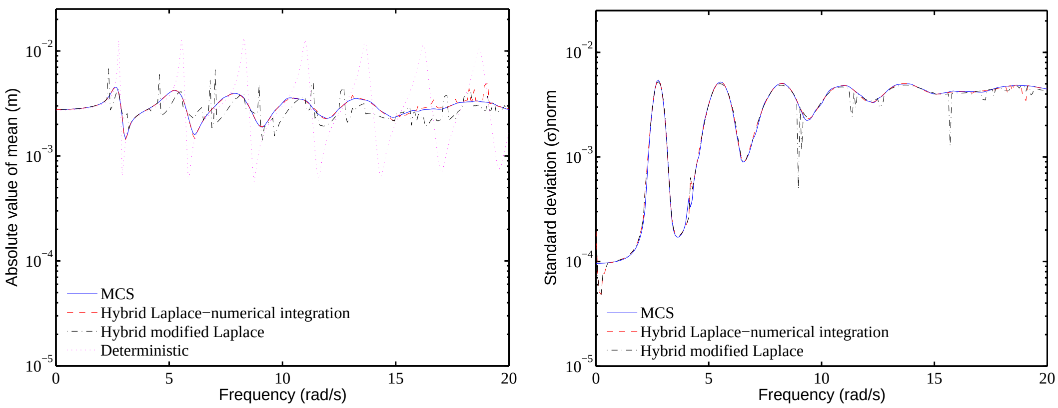
$$
\mathbf{M} = m\mathbf{I}, \quad \mathbf{K} = k \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{56}
$$

with **I** the identity matrix. The number of degrees of freedom of the system is 20, therefore, matrices **M** and **K** become 20×20 matrices. Mass and stiffness constants are given by $m = 1$ and $k = 350$. As already stated, eigenvectors used are the ones obtained from **M** and **K** matrices. Mean eigenvalues are the ones obtained from the deterministic **M** and **K** matrices. The standard deviation of each eigenvalue is expressed by a percentage of the mean. This percentage is considered to be uniform between $[10, 15]$ percent of the mean. Damping factors are assumed to be all equal to $\zeta_j = 0.01$ or to $\zeta_j = 0.1$. The number of samples for MCS is 5000. Results are given for normal distribution in Figure 9 and Figure 10, for gamma distribution in Figure 11 and Figure 12 and for lognormal distribution in Figure 13 and Figure 14.



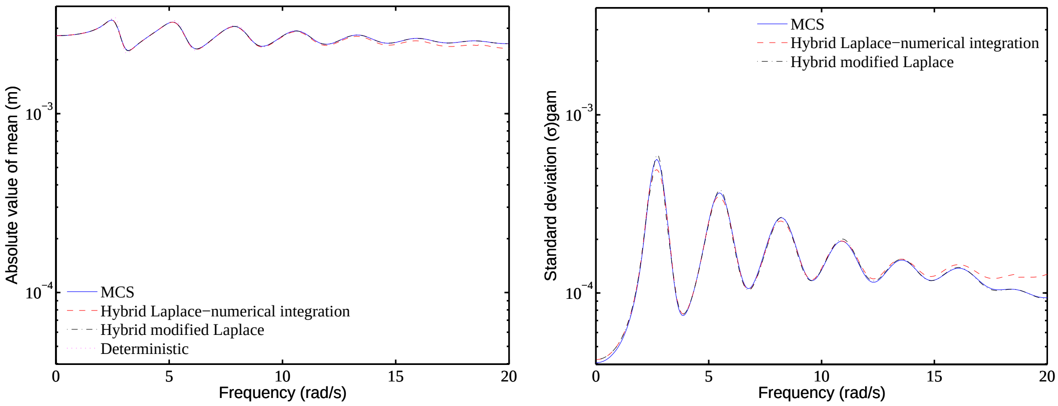
(a) Absolute value of mean. (b) Standard deviation.

Figure 9: Mean and standard deviation of the absolute value of the transfer function for normal distribution, with $\zeta_n = 0.1$.



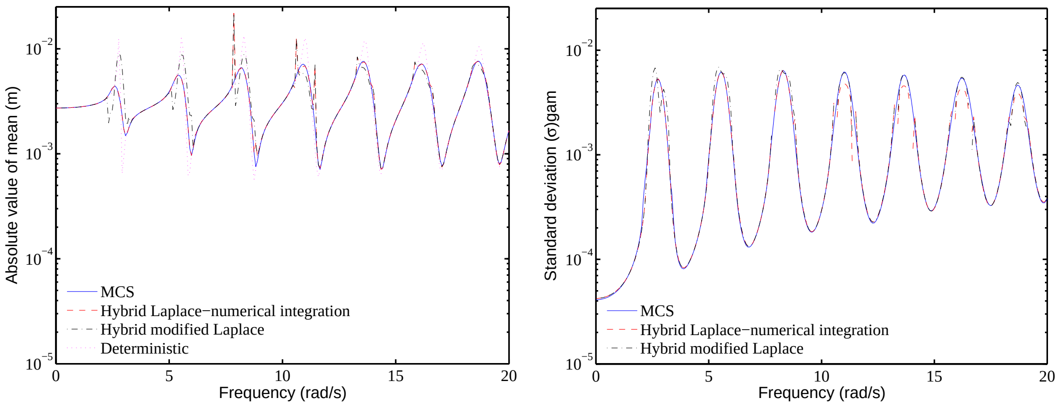
(a) Absolute value of mean. (b) Standard deviation.

Figure 10: Mean and standard deviation of the absolute value of the transfer function for normal distribution, with $\zeta_n = 0.01$.



(a) Absolute value of mean. (b) Standard deviation.

Figure 11: Mean and standard deviation of the absolute value of the transfer function for gamma distribution, with $\zeta_n = 0.1$.



(a) Absolute value of mean. (b) Standard deviation.

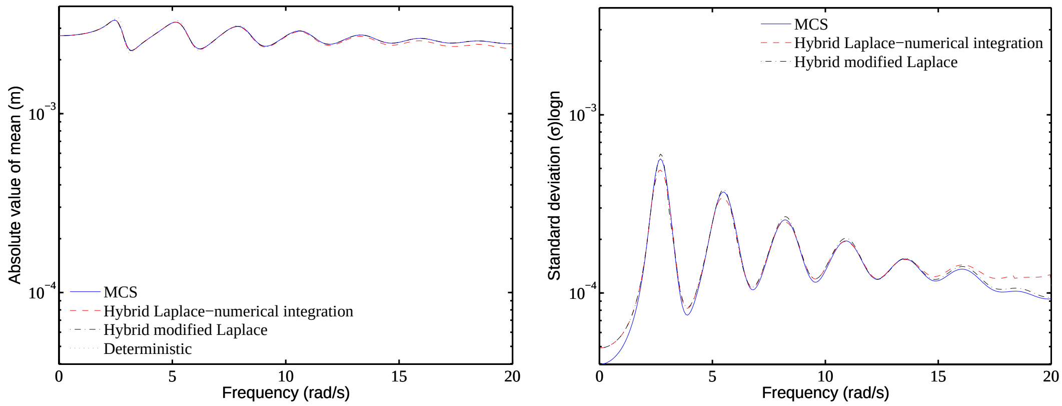
Figure 12: Mean and standard deviation of the absolute value of the transfer function for gamma distribution, with $\zeta_n = 0.01$.

5 Results and discussion

5.1 Discussion of the proposed methods

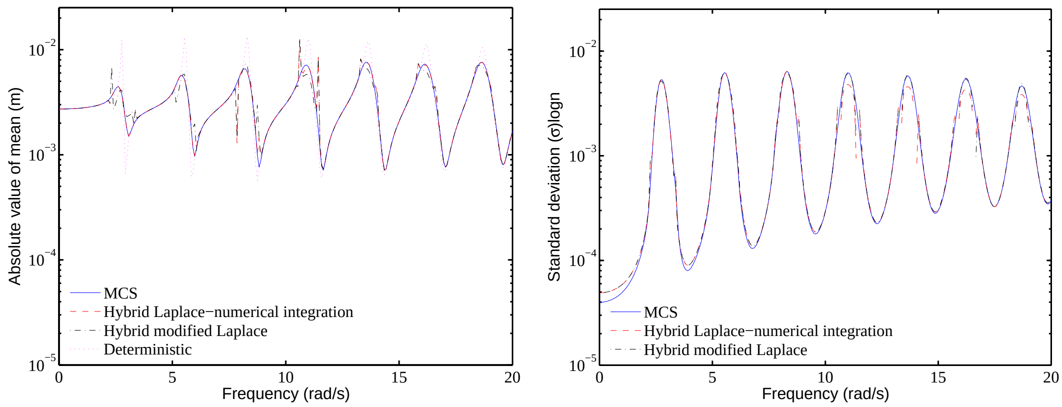
In this paper, the mean and variance of response are calculated from the pdf of eigenvalues. This method needs the calculation of three integrals per frequency and degree of freedom. Unfortunately, exact analytical integration is only available for uniform distribution. The main problem of the method is the calculation of the integrals. Numerical calculation of the integrals can become computationally expensive for large systems. This problem can be addressed if the integrals are approximated using one of the two proposed methods.

The first method is a hybrid method between Laplace's method and numerical in-



(a) Absolute value of mean. (b) Standard deviation.

Figure 13: Mean and standard deviation of the absolute value of the transfer function for lognormal distribution, with $\zeta_n = 0.1$.



(a) Absolute value of mean. (b) Standard deviation.

Figure 14: Mean and standard deviation of the absolute value of the transfer function for lognormal distribution, with $\zeta_n = 0.01$.

tegration. Laplace's method is used to approximate the integrals at those frequencies where the method is supposed to give a good approximation, and numerical integration is used for the remaining frequencies. The second method also approximates the integrals with Laplace's method when it is supposed to give a good approximation. The remaining integrals are approximated, when possible, with a modified Laplace's method, proposed in this paper, and with numerical integration at remaining frequencies. For those frequencies where the modified Laplace's method can be applied, it is normally verified that Laplace's method provides an approximation smaller than the exact value, while the modified Laplace's method provides mostly a larger approximation. The modified Laplace's approximation gives good approximation at resonance points both for lognormal and normal distribution, but results are too large when dealing with gamma distribution.

5.2 Summary of results

It can be observed that damping has an effect on standard deviation of the response. The higher is the damping, the smaller is the standard deviation compared to the mean. This is observed for both SDOF and MDOF systems, but is more evident for MDOF systems. This effect is independent of the distribution of the random variable. Comparing results of mean and standard deviation for the SDOF system, it can be observed that they are similar for different pdfs of the eigenvalue. This can be seen for the MDOF system only for frequencies near the first natural frequency. For higher frequencies, the mean of the FRF for normal distribution appears to be more damped than the ones obtained with other distributions, and standard deviation is generally larger than the one for other distributions. Values of the mean and standard deviation of the FRF for the MDOF system for lognormal, gamma and uniform distribution are almost coincident for every frequency.

6 Conclusions

In this paper, a method was proposed to calculate response statistics from the pdf of eigenvalues. Uncorrelated eigenvalues and proportional damping model is assumed. Mean of the real and imaginary parts of response vector and the second moment of its absolute value are calculated by making use of the Laplace's method and a modified Laplace's method. It is observed that the accuracy of the proposed method depends on the accuracy of the evaluation of the integrals appearing in the method. The accuracy of the integrals in turn depends on the pdf of the random variable and on the damping factor. The assumption of uncorrelated eigenvalues is valid for low frequencies, where little overlap between eigenvalues distribution is observed. At higher frequencies, joint pdf of eigenvalues and eigenvectors should be considered.

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