Distribution of Eigenvalues of Linear Stochastic Systems

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ABSTRACT: Dynamic characteristics of linear structural systems are governed by the natural frequencies and the mode-shapes. In this paper the statistical properties of the eigenvalues of linear dynamic systems are considered. It is assumed that the mass and the stiffness matrices are smooth, continuous and at least twice differentiable functions of a random parameter vector. The random parameter vector is assumed to be standard Gaussian or can be transformed to standard Gaussian. Two approaches are proposed to obtain moments, cumulants and probability density functions of the eigenvalues. The first approach is based on a perturbation expansion of the eigenvalues about an optimal point which is best is some sense. This optimal point is obtained by using the concepts borrowed from structural reliability analysis. The second approach is based on asymptotic analysis. Moments of the eigenvalues are obtained by asymptotic expansion of a multidimensional integral involving the joint probability density function of the random variables. Based on theses methods, two simple expressions of the probability density functions of the eigenvalues are derived. A numerical example is given to compare the proposed methods with Monte Carlo simulations.

1 INTRODUCTION

Characterization of the natural frequencies and the mode-shapes play a fundamental role in the analysis and design of engineering dynamic systems. The determination natural frequency and mode shapes require the solution of an eigenvalue problem. Eigenvalue problem also arises in the context of stability analysis of structures. This problem could either be a differential eigenvalue problem or a matrix eigenvalue problem, depending on whether a continuous model or a discrete model is used to describe the given vibrating system. Description of real-life engineering structural systems is inevitably associated with some amount of uncertainties in specifying material properties, geometric parameters, boundary conditions and applied loads. For these reason it is necessary to consider *random eigenvalue problems*. Several studies have been conducted on this topic since the mid-sixties. The study of probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics. The paper by Boyce (1968) and the book by Scheidt & Purkert (1983) are useful source of information on early works in this area of research and also provide a systematic account of different approaches to random eigenvalue problems. Several review papers, for example by Ibrahim (1987), Benaroya & Rehak (1988), Benaroya (1992), Manohar & Ibrahim (1999) and Manohar & Gupta (2002), have appeared in this field which summarize the current as well as the earlier works.

In this paper discrete linear systems or discretized continuous systems are considered. The random eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j. \tag{1}$$

Here λ_j and $\boldsymbol{\phi}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$, the mass and the stiffness matrices, are assumed to be smooth, continuous and at least twice differentiable functions of a random parameter vector $\mathbf{x} \in \mathbb{R}^m$. Statistical properties of the system are completely described by the joint probability density function $p(\mathbf{x})$. In general the random variables are expected to be non-Gaussian and correlated. However, these random variables can be numerically transformed to a set of uncorrelated Gaussian random variables via the Rosenblatt transformation (Rosenblatt, 1952). Further, each random variable can be scaled to have zero mean and unit variance so that their joint probability density function (pdf) is given by

$$p(\mathbf{x}) = (2\pi)^{-m/2} e^{-\mathbf{x}^T\mathbf{x}/2}. \tag{2}$$

In this paper statistics of the eigenvalues of linear dynamic systems are considered. It is assumed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices so that all the eigenvalues are real and positive.

The final goal of studying random eigenvalue problems is to obtain the joint probability density function of the eigenvalues and the eigenvectors. Under very special circumstances when the matrix $\mathbf{M}^{-1}\mathbf{K}$ is GUE (Gaussian unitary ensemble) or GOE (Gaussian orthogonal ensemble) an exact closed-form expression can be obtained for the joint

pdf of the eigenvalues (Mehta, 1991). In general the system matrices for real structures are not GUE or GOE and consequently some kind of approximate analysis is required. The current literature on random eigenvalue problems is dominated by the mean-centered perturbation methods. These methods work well when the uncertainties are small and the resulting distribution is Gaussian, or close to Gaussian. Here two new approaches are proposed to obtain the moments, cumulants and probability density function of the eigenvalues. The first approach is based on a perturbation expansion of the eigenvalues about a point in the **x**-space which is 'optimal' is some sense (in general different from the mean). The second approach is based on asymptotic approximation of multidimensional integrals. In Section 2.1 mean-centered perturbation methods are briefly reviewed. Perturbation method based on the optimal point is discussed in Section 2.2. Attention is limited up to second-order perturbation only. In Section 3 asymptotic methods are used to obtain various moments of the eigenvalues. Some closed-form expressions of approximate pdf of the eigenvalues are derived in Section 4. The proposed analytical methods are applied to a simple two-dof problem and results are compared with Monte Carlo simulation.

2 STATISTICS OF THE EIGENVALUES USING PERTURBATION METHODS

2.1 *Mean-centered perturbation method*

The mass and the stiffness matrices are in general non-linear functions of the random vector **x**. Because **x** is assumed to be the standard Gaussian variable its mean is **0**. Assume that

$$\mathbf{M}(\mathbf{0}) = \mathbf{M}_0, \quad \text{and} \quad \mathbf{K}(\mathbf{0}) = \mathbf{K}_0 \tag{3}$$

are respectively the 'deterministic parts' of the mass and stiffness matrices. In general $\mathbf{M}_0$ and $\mathbf{K}_0$ are different from the mean matrices. The eigenvalues, $\lambda_j(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ are also non-linear functions of the parameter vector **x**. The deterministic part of the eigenvalues, $\lambda_j(\mathbf{0}) = \lambda_{j_0}$ is obtained from the deterministic eigenvalue problem $\mathbf{K}_0\boldsymbol{\phi}_j = \lambda_{j_0}\mathbf{M}_0\boldsymbol{\phi}_{j_0}$. In mean-centered perturbation approach the function $\lambda_j(\mathbf{x})$ is replaced by its Taylor series about the point $\mathbf{x} = 0$ as

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\mathbf{0}) + \mathbf{d}^T_{\lambda_j}(\mathbf{0})\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{D}_{\lambda_j}(\mathbf{0})\mathbf{x}. \tag{4}$$

Here $(\bullet)^T$ implies matrix transpose and $\mathbf{d}_{\lambda_j}(\mathbf{0}) \in \mathbb{R}^m$ and $\mathbf{D}_{\lambda_j}(\mathbf{0}) \in \mathbb{R}^{m\times m}$ are respectively the gradient vector and the Hessian matrix of $\lambda_j(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$, that is

$$\left\{\mathbf{d}_{\lambda_j}(\mathbf{0})\right\}_k = \frac{\partial\lambda_j(\mathbf{x})}{\partial x_k}|_{\mathbf{x}=\mathbf{0}} \tag{5}$$

$$\text{and} \quad \left\{\mathbf{D}_{\lambda_j}(\mathbf{0})\right\}_{kl} = \frac{\partial^2\lambda_j(\mathbf{x})}{\partial x_k\,\partial x_l}|_{\mathbf{x}=\mathbf{0}}. \tag{6}$$

Expressions of the elements of the gradient vector and the Hessian matrix is given in the Appendix. Equation (4) implies that the eigenvalues are effectively expanded about their corresponding deterministic value λ_{j_0}. Equation (4) represents a quadratic form in Gaussian random variables. Extensive discussions on quadratic forms in Gaussian random variables can be found in the books by Johnson & Kotz (1970, Chapter 29) and Mathai & Provost (1992). Using the methods outlined in these references it is possible to obtain moments, cumulants and pdf of $\lambda_j(\mathbf{x})$.

2.2 *Perturbation method based on an optimal point*

In the mean-centered approach $\lambda_j(\mathbf{x})$ is expanded in a Taylor series about $\mathbf{x} = \mathbf{0}$. Here we are looking for a point $\mathbf{x} = \boldsymbol{\alpha}$ in the **x**-space such that the Taylor series expansion of $\lambda_j(\mathbf{x})$ about this point

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}^T_{\lambda_j}(\boldsymbol{\alpha})\,(\mathbf{x}-\boldsymbol{\alpha}) + \frac{1}{2}\,(\mathbf{x}-\boldsymbol{\alpha})^T\,\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\,(\mathbf{x}-\boldsymbol{\alpha}) \tag{7}$$

is optimal in some sense. This optimal point is obtained by using the concepts borrowed from structural reliability analysis. However, unlike reliability theory, the criteria behind the choice of the optimal point is not unique because we are trying to obtain the complete distribution of the eigenvalues as opposed to calculating the probability of crossing a given level. The optimal point $\boldsymbol{\alpha}$ is selected such that the mean or the first moment of each eigenvalue is calculated most accurately.

Using Eq. (2) the mean of $\lambda_j(\mathbf{x})$ can be obtained as

$$\bar{\lambda}_j = \int_{\mathbb{R}^m} \lambda_j(\mathbf{x})p(\mathbf{x})\,d\mathbf{x} = (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{-h(\mathbf{x})}\,d\mathbf{x} \tag{8}$$

$$\text{where} \quad h(\mathbf{x}) = \mathbf{x}^T\mathbf{x}/2 - \ln\lambda_j(\mathbf{x}). \tag{9}$$

Evaluation of the integral (8), either analytically or numerically, is in general difficult because (a) $\lambda_j(\mathbf{x})$ is a complicated function of **x**, (b) explicit functional form $\lambda_j(\mathbf{x})$ is not easy to obtain except for very simple problems, and (c) dimension of the integral m is large. For these reasons some kind of approximation is required. Expand the function $h(\mathbf{x})$ in a Taylor series about a point where $h(\mathbf{x})$ attends its global minimum. By doing so the error in evaluating the integral (8) would be minimized. Therefore, the optimal point can be obtained as

$$\frac{\partial h(\mathbf{x})}{\partial x_k} = 0 \quad \text{or} \quad x_k = \frac{1}{\lambda_j(\mathbf{x})}\frac{\partial\lambda_j(\mathbf{x})}{\partial x_k}, \quad \forall k. \tag{10}$$

Combining the above equation for all k, at $\mathbf{x} = \boldsymbol{\alpha}$ we have

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\boldsymbol{\alpha}. \tag{11}$$

It implies that at the optimal point the gradient vector and the vector to the point itself are parallel. Rearrange (11) to obtain

$$\boldsymbol{\alpha} = \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})/\lambda_j(\boldsymbol{\alpha}). \tag{12}$$

This equation immediately gives a recipe for an iterative algorithm to obtain $\boldsymbol{\alpha}$. One starts with an initial $\boldsymbol{\alpha}$ in the right-hand side and obtains an updated $\boldsymbol{\alpha}$ in the left-hand side. This procedure can be continued till the difference between the values of $\boldsymbol{\alpha}$ obtained from both sides of (12) is

less than (any vector norm can be used to measure the difference) a predefined small value. A good value to start the iteration process is $\boldsymbol{\alpha} = \mathbf{0}$, as in the case of mean-centered approach. Due to the explicit analytical expression of $\mathbf{d}_{\lambda_j}$ in terms of the derivative of the mass and the stiffness matrices, much expensive numerical differentiation of $\lambda_j(\mathbf{x})$ at each iteration step is not needed. Substituting $\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})$ in Eq. (7) and rearranging one has

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha})\left(1-|\boldsymbol{\alpha}|^2\right) + \frac{1}{2}\boldsymbol{\alpha}^T\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\boldsymbol{\alpha} + \boldsymbol{\alpha}^T\left(\lambda_j(\boldsymbol{\alpha})\mathbf{I} - \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\right)\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\mathbf{x}. \quad (13)$$

Here $|\bullet|$ is the l_2 vector norm and $\mathbf{I}$ is an identity matrix of order m. Like the approximation in (4), this approximation also yields a quadratic form in Gaussian random variable. Next, theory of quadratic forms is used to obtain statistics of the eigenvalues.

2.3 *Eigenvalue statistics using theory of quadratic forms*

We can combine equations (4) and (13) in a single equation as

$$\lambda_j(\mathbf{x}) \approx c_j + \mathbf{a}_j^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}_j\mathbf{x} \quad (14)$$

with obvious definitions of $c_j \in \mathbb{R}$, $\mathbf{a}_j \in \mathbb{R}^m$ and $\mathbf{A}_j \in \mathbb{R}^{m\times m}$. Because $\mathbf{A}_j$ is a symmetric matrix, the matrix of its eigenvectors is orthogonal. Using this orthogonal matrix it is possible to rotate the coordinate so that

$$\lambda_j \approx \sum_{k=1}^{m}\sigma_k(y_k-\varrho_k)^2 \quad (15)$$

where σ_k are the eigenvalues of $\mathbf{A}_j$, $y_k \sim N(0,1)$ and the symbol $\sim$ implies 'distributed as'. Thus $\lambda_j(\mathbf{x})$ can be approximated by a weighted sum of independent non-central χ^2 random variables. If some eigenvalues of $\mathbf{A}_j$ are zero then $\lambda_j(\mathbf{x})$ can be expressed as a sum of independent Gaussian and independent non-central χ^2 random variables. The moment generating function of $\lambda_j(\mathbf{x})$, which, for any $s \in \mathbb{C}$ can be obtained from (14) as

$$M_{\lambda_j}(s) = \mathrm{E}\left[e^{s\lambda_j(\mathbf{x})}\right] \approx \frac{e^{sc_j+\frac{s^2}{2}\mathbf{a}_j^T[\mathbf{I}-s\mathbf{A}_j]^{-1}\mathbf{a}_j}}{\sqrt{\|\mathbf{I}-s\mathbf{A}_j\|}} \quad (16)$$

where $\|\bullet\|$ denotes the determinant of a matrix. To obtain the pdf of $\lambda_j(\mathbf{x})$, the inverse Laplace transform of (16) is required. Exact closed-form expression of the pdf can be obtained for very special cases. Approximate pdf of $\lambda_j(\mathbf{x})$ will be discussed in Section 4.

If mean-centered first-order perturbation is used then $\mathbf{A}_j = \mathbf{O}$ and $M_{\lambda_j}(s) \approx e^{s\lambda_{j_0}+\frac{s^2}{2}\mathbf{d}_{\lambda_j}^T\mathbf{d}_{\lambda_j}}$, that is, $\lambda_j(\mathbf{x})$ is a Gaussian random variable with mean λ_{j_0} and variance $\mathbf{d}_{\lambda_j}^T\mathbf{d}_{\lambda_j}$. However, for second-order perturbation in general the mean of the eigenvalues is not same as the deterministic value. The cumulants of $\lambda_j(\mathbf{x})$ can be obtained from (16) as

$$\kappa_r = \frac{d^r}{ds^r}\ln M_{\lambda_j}(s)|_{s=0}. \quad (17)$$

Here κ_r is the rth order cumulant. From this equation it can be shown that

$$\kappa_r = \begin{cases} c_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j\right) & \text{if} \quad r=1, \\ \frac{r!}{2}\mathbf{a}_j^T\mathbf{A}_j^{r-2}\mathbf{a}_j + \frac{(r-1)!}{2}\mathrm{Trace}\left(\mathbf{A}_j^r\right) & \text{if} \quad r\geq 2. \end{cases} \quad (18)$$

So, the mean and first few cumulants of the eigenvalues can be explicitly obtained as

$$\bar{\lambda}_j = \kappa_1 = c_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j\right) \quad (19)$$

$$\mathrm{Var}\left[\lambda_j\right] = \kappa_2 = \mathbf{a}_j^T\mathbf{a}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j^2\right), \quad (20)$$

$$\kappa_3 = 3\mathbf{a}_j^T\mathbf{A}_j\mathbf{a}_j + \mathrm{Trace}\left(\mathbf{A}_j^3\right), \quad (21)$$

$$\text{and } \kappa_4 = 12\mathbf{a}_j^T\mathbf{A}_j^2\mathbf{a}_j + 3\mathrm{Trace}\left(\mathbf{A}_j^4\right) \quad (22)$$

From the cumulants, the raw moments $\mu_r' = \mathrm{E}\left[\lambda_j^r\right]$ and the central moments $\mu_r = \mathrm{E}\left[(\lambda_j-\bar{\lambda}_j)^r\right]$ can be obtained using standard formulae (Chapter 26, Abramowitz & Stegun, 1965).

3 ASYMPTOTIC STATISTICS

In this section the moments of the eigenvalues are obtained by an asymptotic approximation. Consider a function function $f(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ which is smooth and at least twice differentiable. We want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} f(\mathbf{x})p(\mathbf{x})\,d\mathbf{x} = (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{\widetilde{h}(\mathbf{x})}\,d\mathbf{x} \quad (23)$$

$$\text{where} \quad \widetilde{h}(\mathbf{x}) = \ln f(\mathbf{x}) - \mathbf{x}^T\mathbf{x}/2. \quad (24)$$

Assume that this function reaches its global maximum at an unique point $\boldsymbol{\theta} \in \mathbb{R}$. Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial\widetilde{h}(\mathbf{x})}{\partial x_k} = 0 \text{ or } x_k = \frac{\partial}{\partial x_k}\ln f(\mathbf{x}), \forall k, \text{ or } \boldsymbol{\theta} = \frac{\partial}{\partial\mathbf{x}}\ln f(\boldsymbol{\theta}). \quad (25)$$

Further assume that $\widetilde{h}(\boldsymbol{\theta})$ is so large that

$$\left|\frac{1}{\widetilde{h}(\boldsymbol{\theta})}\mathcal{D}^j(\widetilde{h}(\boldsymbol{\theta}))\right| \to 0 \quad \text{for} \quad j>2 \quad (26)$$

where $\mathcal{D}^j(\widetilde{h}(\boldsymbol{\theta}))$ is jth order derivative of $\widetilde{h}(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\theta}$. Under such assumptions, using second-order Taylor series of $\widetilde{h}(\mathbf{x})$ and using (25) it is possible to approximate the integral (23) as

$$\mathcal{J} \approx (2\pi)^{-m/2}e^{\widetilde{h}(\boldsymbol{\theta})}\int_{\mathbb{R}^m} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T\widetilde{\mathbf{H}}(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta})}\,d\mathbf{x} \quad (27)$$

$$\text{where} \quad \widetilde{H}_{kl}(\boldsymbol{\theta}) = -\frac{\partial^2\widetilde{h}(\mathbf{x})}{\partial x_k\,\partial x_l}|_{\mathbf{x}=\boldsymbol{\theta}}. \quad (28)$$

Using a transformation $\boldsymbol{\xi} = \mathbf{x}-\boldsymbol{\theta}$ so that the Jacobian $\|\mathbf{J}\| = \mathbf{I}$, the integral (27) can be evaluated as

$$\mathcal{J} \approx \frac{e^{\widetilde{h}(\boldsymbol{\theta})}}{\sqrt{\|\widetilde{\mathbf{H}}(\boldsymbol{\theta})\|}} = f(\boldsymbol{\theta})e^{-\left(\boldsymbol{\theta}^T\boldsymbol{\theta}/2\right)}\|\widetilde{\mathbf{H}}(\boldsymbol{\theta})\|^{-1/2}. \quad (29)$$

A similar approach was outlined by Papadimitriou et al. (1997). This asymptotic analysis is somewhat different from the widely used Laplace's method of asymptotic approximation of integrals (Chapter IX, Theorem 3, Wong, 2001). Here we assume that $\widetilde{h}(\boldsymbol{\theta})$ itself is a large quantity, unlike a free parameter λ, as in the case of Laplace type integrals. Equation (29) will now be used to obtain the moments of the eigenvalues.

An arbitrary rth order moment of the eigenvalues can be obtained as

$$\mu'_r = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots \tag{30}$$

Comparing this with Eq. (23) it is clear that

$$f(\mathbf{x}) = \lambda_j^r(\mathbf{x}) \quad \text{and} \quad \widetilde{h}(\mathbf{x}) = r \ln \lambda_j(\mathbf{x}) - \mathbf{x}^T\mathbf{x}/2. \tag{31}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from (25) as

$$\boldsymbol{\theta} = r\, \mathbf{d}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}). \tag{32}$$

This equation is very similar to Eq. (12) and again an iterative procedure can be used to obtain $\boldsymbol{\theta}$. The elements of $\widetilde{\mathbf{H}}(\boldsymbol{\theta})$ can be obtained as follows:

$$-\frac{\partial \widetilde{h}(\mathbf{x})}{\partial x_k} = x_k - \frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}. \tag{33}$$

Differentiating this with respect to x_l and rearranging we have

$$-\frac{\partial^2 \widetilde{h}(\mathbf{x})}{\partial x_k\, \partial x_l} = \delta_{kl} + \frac{1}{r}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}\right\}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\right\} - \frac{r}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k\, \partial x_l} \tag{34}$$

where δ_{kl} is the Kroneker delta function. Because at $\mathbf{x} = \boldsymbol{\theta}$, $\frac{\partial \widetilde{h}(\mathbf{x})}{\partial x_k} = 0\, \forall k$, from (33) we have

$$x_k = \theta_k = \frac{r}{\lambda_j(\boldsymbol{\theta})}\frac{\partial \lambda_j(\boldsymbol{\theta})}{\partial x_k}\ \forall k. \tag{35}$$

Using this, Eq. (34) reduces to

$$-\frac{\partial^2 \widetilde{h}(\mathbf{x})}{\partial x_k\, \partial x_l}\Big|_{\mathbf{x}=\boldsymbol{\theta}} = \delta_{kl} + \frac{1}{r}\theta_k\theta_l - \frac{r}{\lambda_j(\boldsymbol{\theta})}\frac{\partial^2 \lambda_j(\boldsymbol{\theta})}{\partial x_k\, \partial x_l}. \tag{36}$$

Combining this equation for all k and l we have

$$\widetilde{\mathbf{H}}(\boldsymbol{\theta}) = \mathbf{I} + \frac{1}{r}\boldsymbol{\theta}\boldsymbol{\theta}^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \tag{37}$$

where $\mathbf{D}_{\lambda_j}(\bullet)$ is defined in Eq. (6). Using the asymptotic approximation (29), the rth moment of the eigenvalues can be obtained as

$$\mu'_r = \lambda_j^r(\boldsymbol{\theta}) e^{-\frac{|\boldsymbol{\theta}|^2}{2}} \left\| \mathbf{I} + \frac{1}{r}\boldsymbol{\theta}\boldsymbol{\theta}^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}. \tag{38}$$

This formula can also be used to obtain fractional order moments provided the asymptotic condition in (26) is satisfied and an unique $\boldsymbol{\theta}$ is found. If the optimal point $\boldsymbol{\theta}$ is not unique then it is required to sum the contributions arising from all such optimal points separately. The mean of the eigenvalues can be obtained by substituting $r = 1$ in Eq. (38), that is

$$\bar{\lambda}_j = \lambda_j(\boldsymbol{\theta}) e^{-\frac{|\boldsymbol{\theta}|^2}{2}} \left\| \mathbf{I} + \boldsymbol{\theta}\boldsymbol{\theta}^T - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}) \right\|^{-1/2}. \tag{39}$$

Once the mean in known, the central moments $\mu_r = \mathrm{E}\left[(\lambda_j - \bar{\lambda}_j)^r\right]$ can be expressed in terms of the raw moments μ'_r using the binomial transform

$$\mathrm{E}\left[(\lambda_j - \bar{\lambda}_j)^r\right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu'_k \bar{\lambda}_j^{r-k}. \tag{40}$$

Probability density function of the eigenvalues is considered in the next section.

4 APPROXIMATE PDF OF THE EIGENVALUES

Using second-order approximations, in Section 2.3 it was shown that $\lambda_j(\mathbf{x})$ can be expressed as a sum of independent non-central χ^2 random variables. Under very special conditions $\lambda_j(\mathbf{x})$ itself can be a non-central χ^2 random variable. From the representation (14), such conditions can be described in the following (Theorem 5.1.4, Mathai & Provost, 1992) theorem

Theorem 1 *$\lambda_j(\mathbf{x})$ is distributed as a non-central χ^2 random variable with noncentrality parameter δ^2 and degrees-of-freedom m' if and only if (a) $\mathbf{A}_j^2 = \mathbf{A}_j$, (b) $\mathrm{Trace}\,(\mathbf{A}_j) = m'$ and (c) $\mathbf{a}_j = \mathbf{A}_j\mathbf{a}_j$, $\delta^2 = c_j = \mathbf{a}_j^T\mathbf{a}_j/4$.*

This implies that the the Hessian matrix $\mathbf{A}_j$ should be an idempotent matrix. In general this requirement is not expected to be satisfied for eigenvalues of real structural systems.

Derivation of probability density and distribution functions of quadratic forms in Gaussian random variables like (14) has received extensive attention in literature. Johnson & Kotz (1970, Chapter 29) and Mathai & Provost (1992, Chapter 4) have summarized significant contributions in this area. Several representations of the *exact* pdf of a quadratic form like (14) are available in the literature (see Mathai & Provost, 1992, Chapter 4). Some examples include, power series expansion, series expansion in terms of central χ^2 random variables, Laguerre series expansion, series expansion in terms of confluent hypergeometric function and series expansion in zonal polynomials. All of these expansions are in terms of an *infinite* series. Here we have focused our attention to two simple approximations which result in simple closed-form expression of the pdf of $\lambda_j(\mathbf{x})$.

4.1 *Non-central χ^2 approximation*

Assuming that $\mathbf{A}_j^{-1}$ exists we rewrite Eq. (14) as

$$\lambda_j(\mathbf{x}) = \eta_j + \frac{1}{2}(\mathbf{x} + \boldsymbol{\rho}_j)^T\mathbf{A}_j(\mathbf{x} + \boldsymbol{\rho}_j) \tag{41}$$

$$\text{where } \eta_j = c_j - \frac{1}{2}a_j^T\mathbf{A}_j^{-1}a_j, \text{ and } \boldsymbol{\rho}_j = \mathbf{A}_j^{-1}a_j. \tag{42}$$

From theorem 1 it is known that for $(\mathbf{x}+\boldsymbol{\rho}_j)^T\mathbf{A}_j(\mathbf{x}+\boldsymbol{\rho}_j)$ to be χ^2 distributed the matrix $\mathbf{A}_j$ has to be an idempotent matrix. Since in general $\mathbf{A}_j$ is not an idempotent matrix it is approximated as

$$\mathbf{A}_j \approx \frac{\text{Trace}\,(\mathbf{A}_j)}{m}\mathbf{I}. \tag{43}$$

The term $\frac{\text{Trace}(\mathbf{A}_j)}{m}$ is the mean of the eigenvalues of $\mathbf{A}_j$. The error introduced by this approximation is expected to be less if the spacing between the eigenvalues of $\mathbf{A}_j$ are close to uniform. With this approximation, from Eq. (41) one has

$$\lambda_j(\mathbf{x}) \approx \eta_j + \gamma_j Q_j \tag{44}$$

$$\text{where } \gamma_j = \frac{\text{Trace}\,(\mathbf{A}_j)}{2m}, \tag{45}$$

$$\text{and } Q_j = (\mathbf{x}+\boldsymbol{\rho}_j)^T(\mathbf{x}+\boldsymbol{\rho}_j) \tag{46}$$

is a non-central χ^2 random variable with noncentrality parameter $\delta_j^2 = \boldsymbol{\rho}_j^T\boldsymbol{\rho}_j$ and degrees-of-freedom m, that is, $Q_j \sim \chi^2_m(\delta_j^2)$. From the transformation in Eq. (44) the pdf of $\lambda_j(\mathbf{x})$ can be obtained as

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{Q_j}\left(\frac{u-\eta_j}{\gamma_j}\right) \tag{47}$$

where

$$p_{Q_j}(u) = \frac{e^{-(\delta_j+u/2)}u^{m/2-1}}{2^{m/2}} \sum_{r=0}^{\infty} \frac{(\delta u)^r}{r!\,2^r\Gamma(m/2+r)}. \tag{48}$$

If some eigenvalues of $\mathbf{A}_j$ are zero then a slight variation of this formulation is required. In this case $\lambda_j(\mathbf{x})$ can be approximated by sum of independent Gaussian and non-central χ^2 random variables. This pdf cannot be used in conjunction with the asymptotic moments derived before. A different approach is proposed to use the asymptotic moments.

4.2 *Pearson's approximation*

We use an approximation analogous to Pearson's three moment central χ^2 approximation to the distribution of a non-central χ^2 (Pearson, 1959). The eigenvalues are approximated as

$$\lambda_j(\mathbf{x}) \approx U_j = \widetilde{\eta} + \widetilde{\gamma}\chi^2_\nu. \tag{49}$$

For notational convenience the subscript j is omitted. The constants $\widetilde{\eta}, \widetilde{\gamma}$ and ν are obtained such that the first three cumulants/moments of $\lambda_j(\mathbf{x})$ and U_j are the same. The moment generating function of U_j is given by

$$\text{E}\left[e^{-s\left(\widetilde{\eta}+\widetilde{\gamma}\chi^2_\nu\right)}\right] = e^{-s\widetilde{\eta}}\left(1+2s\widetilde{\gamma}\right)^{-\nu/2}. \tag{50}$$

Equating the first three cumulants of $\lambda_j(\mathbf{x})$ and U_j we obtain

$$\widetilde{\eta} + \widetilde{\gamma}\,\nu = \kappa_1,\ 2\widetilde{\gamma}^2\nu = \kappa_2 \text{ and } 8\widetilde{\gamma}^3\nu = \kappa_3. \tag{51}$$

By solving these three equations simultaneously the parameters $\widetilde{\eta}, \widetilde{\gamma}$ and ν can be obtained as

$$\widetilde{\eta} = \frac{-2{\kappa_2}^2 + \kappa_1\kappa_3}{\kappa_3},\ \widetilde{\gamma} = \frac{\kappa_3}{4\kappa_2},\ \text{and } \nu = 8\frac{{\kappa_2}^3}{{\kappa_3}^2}. \tag{52}$$

Cumulants of $\lambda_j(\mathbf{x})$, obtained in Eq. (18), can be used directly in these equations. These constants can alternatively be obtained by equating the first three moments of $\lambda_j(\mathbf{x})$ and U_j as

$$\widetilde{\eta} + \nu\widetilde{\gamma} = \mu'_1, \tag{53}$$

$$\widetilde{\eta}^2 + 2\widetilde{\eta}\nu\widetilde{\gamma} + \nu^2\widetilde{\gamma}^2 + 2\nu\widetilde{\gamma}^2 = \mu'_2, \tag{54}$$

and

$$\widetilde{\eta}^3 + 3\widetilde{\eta}^2\nu\widetilde{\gamma} + 3\widetilde{\eta}\nu^2\widetilde{\gamma}^2 + 6\widetilde{\eta}\nu\widetilde{\gamma}^2 + \nu^3\widetilde{\gamma}^3 + 6\nu^2\widetilde{\gamma}^3 + 8\nu\widetilde{\gamma}^3 = \mu'_3.$$

Again by solving these non-linear equations we obtain

$$\widetilde{\eta} = \frac{{\mu'_1}^2\mu'_2 - 2\,{\mu'_2}^2 + \mu'_1\mu'_3}{2\,{\mu'_1}^3 - 3\,\mu'_1\mu'_2 + \mu'_3} \tag{55}$$

$$\widetilde{\gamma} = \frac{2\,{\mu'_1}^3 - 3\,\mu'_1\mu'_2 + \mu'_3}{4\left(\mu'_2 - {\mu'_1}^2\right)}, \tag{56}$$

$$\text{and } \nu = 8\,\frac{\left(\mu'_2 - {\mu'_1}^2\right)^3}{\left(2\,{\mu'_1}^3 - 3\,\mu'_1\mu'_2 + \mu'_3\right)^2}. \tag{57}$$

Moments of $\lambda_j(\mathbf{x})$, asymptotically obtained in Eq. (38), can be used directly in these equations. Using the transformation in (49) the approximate probability density function of $\lambda_j(\mathbf{x})$ is given by

$$p_{\lambda_j}(u) \approx \frac{1}{\widetilde{\gamma}} p_{\chi^2_\nu}\left(\frac{u-\widetilde{\eta}}{\widetilde{\gamma}}\right) = \frac{(u-\widetilde{\eta})^{\nu/2-1}e^{-(u-\widetilde{\eta})/2\widetilde{\gamma}}}{(2\widetilde{\gamma})^{\nu/2}\Gamma(\nu/2)}. \tag{58}$$

5 NUMERICAL EXAMPLE

A simple two-degree-of-freedom undamped system has been considered to illustrate a possible application of the expressions developed so far. Fig. 1 shows the example, together with the numerical values of the masses and spring stiffnesses. The system matrices for the example are given

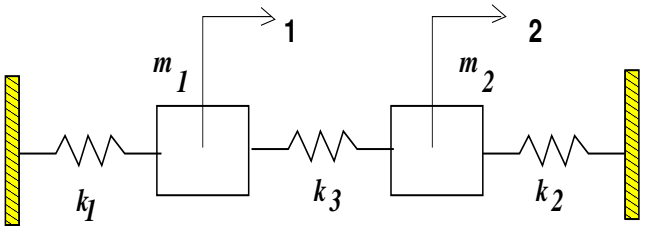

Figure 1: The undamped two degree-of-system system, $m_1 = 1$ Kg, $m_2 = 1.5$ Kg, $\bar{k}_1 = 1000$ N/m, $\bar{k}_2 = 1100$ N/m and $k_3 = 100$ N/m.

by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}. \tag{59}$$

It is assumed that only the stiffness parameters k_1 and k_2 are uncertain so that $k_i = \bar{k}_i(1 + \epsilon_i x_i)$, $i = 1, 2$ and $\bar{k}_i$ denote the deterministic values of the spring constants. Here $\mathbf{x} = \{x_1, x_2\}^T \in \mathbb{R}^2$ is a vector of Gaussian random variables. The numerical values of the 'strength parameters' are considered as $\epsilon_1 = \epsilon_2 = 0.25$. The strength parameters are selected so that the system matrices are almost surely positive definite.

The derivative of the system matrices with respect to the random vector $\mathbf{x}$ may be obtained as

$$\frac{\partial \mathbf{K}}{\partial x_1} = \epsilon_1 \begin{bmatrix} \bar{k}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_2} = \epsilon_2 \begin{bmatrix} 0 & 0 \\ 0 & \bar{k}_2 \end{bmatrix}, \tag{60}$$

$$\frac{\partial \mathbf{M}}{\partial x_i} = \mathbf{O} \quad \text{and} \quad \frac{\partial^2 \mathbf{K}}{\partial x_i \, \partial x_j} = \mathbf{O} \tag{61}$$

since $\mathbf{M}$ is independent of $\mathbf{x}$ and $\mathbf{K}$ is a linear function of $\mathbf{x}$.

We calculate the raw moments and the probability density functions of the two eigenvalues of the system. Recall that the eigenvalues obtained from Eq. (1) are square of the natural frequencies ($\lambda_j = \omega_j^2$). Following six methods are used to obtain the moments and the pdfs

1. *Mean-centered first-order perturbation:* This case arises when $\mathbf{D}_{\lambda_j}(\mathbf{0})$ in the Taylor series expansion (4) is assumed to be a null matrix so that only the first-order terms are retained. This is the simplest approximation, and as mentioned earlier, results Gaussian distribution of the eigenvalues. Mean-centered first-order perturbation can be obtained from the general expression in Eq. (14) by substituting $c_j = \lambda_j(\mathbf{0})$, $\mathbf{a}_j = \mathbf{d}_{\lambda_j}^T(\mathbf{0})$ and $\mathbf{A}_j = \mathbf{O}$. The cumulants, and hence the moments, can be obtained from Eq. (18).

2. *Mean-centered second-order perturbation:* In this case all the terms in Eq. (4) are retained. This is a special case of the general expression in Eq. (14) when $c_j = \lambda_j(\mathbf{0})$, $\mathbf{a}_j = \mathbf{d}_{\lambda_j}^T(\mathbf{0})$ and $\mathbf{A}_j = \mathbf{D}_{\lambda_j}(\mathbf{0})$.

3. *α-centered first-order perturbation:* This case arises when $\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})$ in the Taylor series expansion (7) is assumed to be a null matrix so that only the first-order terms are retained. This approach, also known as the advanced mean value method, results in Gaussian distribution of the eigenvalues. For this method $c_j = \lambda_j(\boldsymbol{\alpha})\left(1 - |\boldsymbol{\alpha}|^2\right)$, $\mathbf{a}_j = \lambda_j(\boldsymbol{\alpha})\boldsymbol{\alpha}$ and $\mathbf{A}_j = \mathbf{O}$, in Eq. (14).

4. *α-centered second-order perturbation:* In this case all the terms in Eq. (7) are retained and can be obtained from the general expression in Eq. (14) by substituting $c_j = \lambda_j(\boldsymbol{\alpha})\left(1 - |\boldsymbol{\alpha}|^2\right) + \frac{1}{2}\boldsymbol{\alpha}^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\boldsymbol{\alpha}$, $\mathbf{a}_j = \left[\lambda_j(\boldsymbol{\alpha})\mathbf{I} - \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\right]\boldsymbol{\alpha}$ and $\mathbf{A}_j = \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})$.

5. *Asymptotic method:* In this case the moments are obtained using Eq. (38). These moments are then used to obtain the constants in Eqs. (55)–(57). Using these constants the pdf of the eigenvalues are then obtained from Eq. (58).

6. *Monte Carlo Simulation:* The samples of two independent Gaussian random variables x_1 and x_2 are generated and the eigenvalues are computed directly from (1). A total of $10,000$ samples are used to obtain the statistical moments and pdf of both the eigenvalues. Results obtained from the Monte Carlo simulation is assumed to the benchmark for the purpose of comparing the five analytical methods described above.

Fig. 2 shows the percentage error for the first four raw moments of the first eigenvalue. The percentage error for

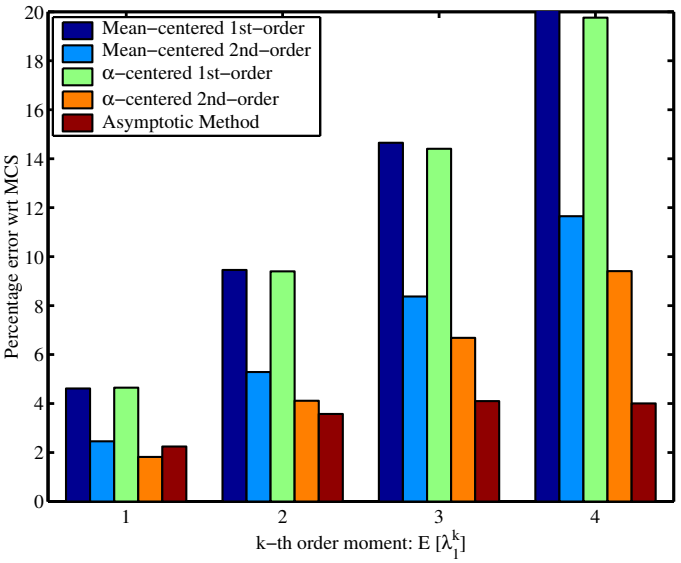

Figure 2: Percentage error for the first eigenvalue.

an arbitrary kth moment of an eigenvalue obtained using any one of the five analytical methods is given by

$$\text{Error}_{i\text{th method}} = \frac{\{\mu_k'\}_{i\text{th method}} - \{\mu_k'\}_{\text{MCS}}}{\{\mu_k'\}_{\text{MCS}}} \times 100 \qquad i = 1, \cdots 5. \tag{62}$$

Percentage error for the first four raw moments of the second eigenvalue is shown in Fig. 3. Note that for both eigenvalues error corresponding to the mean-centered first-order perturbation method is more than the other four methods. Error corresponding to the α-centered first-order perturbation method follows next. Moments obtained from mean-centered and α-centered second-order perturbation methods are more accurate compared to their corresponding first-order counterparts. In general the moments obtained from the asymptotic formula (38) turns out to be quite accurate. Absolute error for the second eigenvalue is less compared to the first eigenvalue. For the first eigenvalue, the moments obtained from the asymptotic formula turns out to be the most accurate, while for the second eigenvalue, mean-centered second-order perturbation method yields most accurate results.

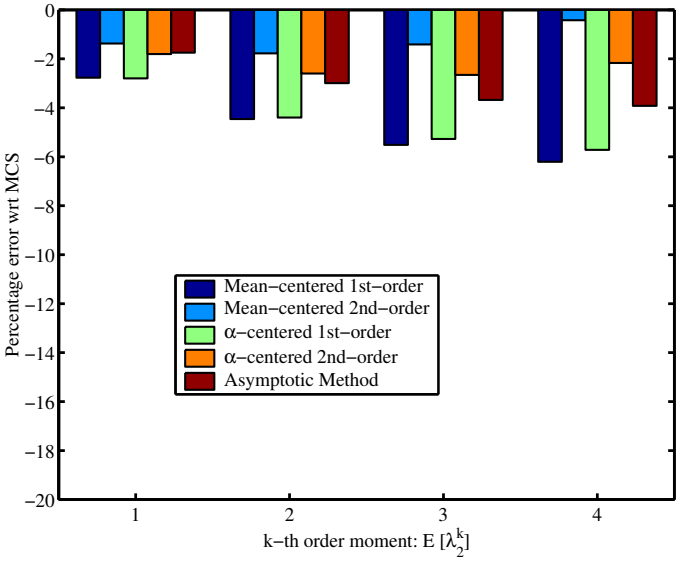

Figure 3: Percentage error for the second eigenvalue.

Now consider the probability density function of the eigenvalues. Figs. 4 and 5 respectively show the pdf of the first and the second eigenvalue obtained from the five methods described earlier. In the same plots, normalized histograms of the eigenvalues obtained from the Monte Carlo simulation are also plotted. The pdf corresponding to first five methods are obtained from Eq. (58). The constants appearing in this equation are calculated from the moments using Eqs. (55)–(57). For the first eigenvalue, pdf from the second-order perturbation methods are accurate in the lower and in the upper tail. For the second eigenvalue, pdf from the asymptotic moments is accurate over the whole curve.

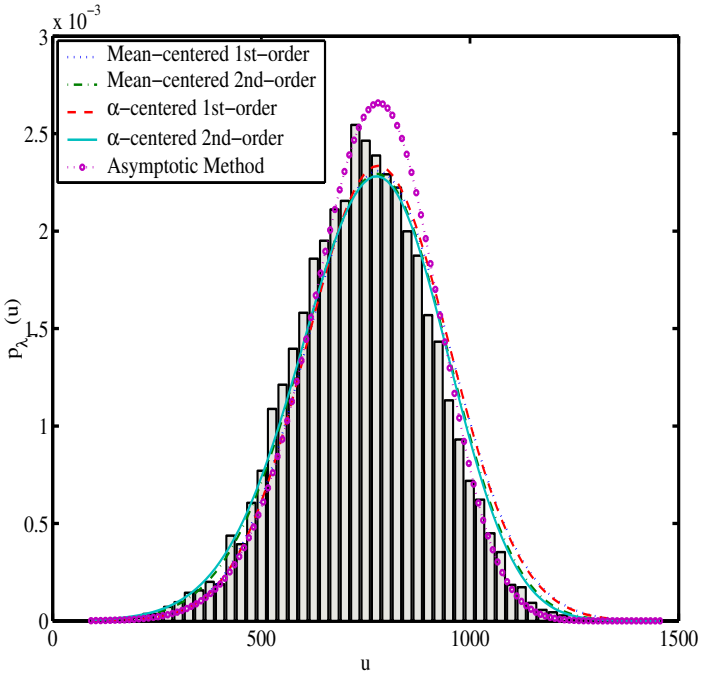

Figure 4: Probability density function of the first eigenvalue.

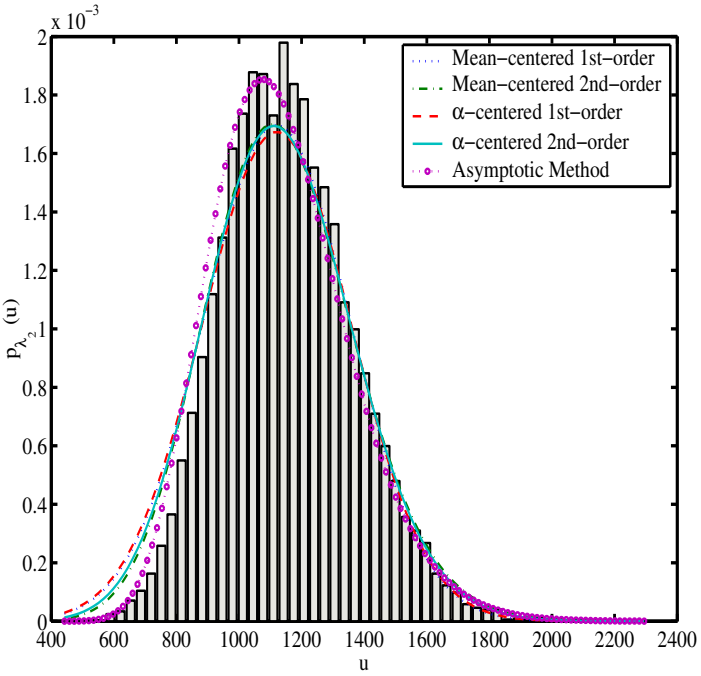

Figure 5: Probability density function of the second eigenvalue.

6 SUMMARY AND CONCLUSIONS

Statistics of the eigenvalues of linear dynamic system with parameter uncertainties are considered. It is assumed that the mass and the stiffness matrices are smooth and at least twice differentiable functions of a set of random variables which are assumed to be independent Gaussian or can be transformed to independent Gaussian random variables. The usual assumption of small randomness employed in most mean-centered based perturbation analysis is not employed in this study. Two methods, namely (a) optimal point expansion method, and (b) asymptotic moment method, are proposed. The optimal point is obtained so that the mean of the eigenvalues are estimated most accurately. Both methods are based on an unconstrained optimization problem. A simple iterative procedure is suggested for the optimization problem. Moments and cumulants of arbitrary orders are derived for both the approaches. Two simple approximations for the pdf of the eigenvalues are derived. These approximations are in terms of χ^2 random variables and yield closed-form expressions of the pdf which can be computed easily.

Proposed formulae are applied to a simple two-degree-of-freedom system. The moments and the pdf match encouragingly well with the corresponding Monte Carlo simulation results. Future studies will address the applicability of these formulary to large structural dynamic systems with many random variables. Further research is required to deal with systems with non-gaussian basic random variables and developing new methods to obtain joint statistics of the eigenvalues.

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A GRADIENT AND HESSIAN OF THE EIGENVALUES

The eigenvectors of symmetric linear systems are orthogonal with respect to the mass and the stiffness matrices. Normalize the eigenvectors so that they are unity mass normalized, that is,

$$\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j = 1. \tag{63}$$

Using this and differentiating Eq. (1) with respect to x_k it can be shown that (see for example Fox & Kapoor, 1968) for any $\mathbf{x}$

$$\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} = \boldsymbol{\phi}_j(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \tag{64}$$

where

$$\boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) = \left[\frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_k} - \lambda_j(\mathbf{x}) \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k} \right]. \tag{65}$$

Differentiating Eq. (1) with respect to x_k and x_l Plaut & Huseyin (1973) have shown that

$$\begin{aligned} \frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k \, \partial x_l} &= \boldsymbol{\phi}_j(\mathbf{x})^T \left[\frac{\partial^2 \mathbf{K}(\mathbf{x})}{\partial x_k \, \partial x_l} - \lambda_j(\mathbf{x}) \frac{\partial^2 \mathbf{M}(\mathbf{x})}{\partial x_k \, \partial x_l} \right] \boldsymbol{\phi}_j(\mathbf{x}) \\ &\quad - \left(\boldsymbol{\phi}_j(\mathbf{x})^T \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k} \boldsymbol{\phi}_j(\mathbf{x}) \right) \left(\boldsymbol{\phi}_j(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jl}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right) \\ &\quad - \left(\boldsymbol{\phi}_j(\mathbf{x})^T \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_l} \boldsymbol{\phi}_j(\mathbf{x}) \right) \left(\boldsymbol{\phi}_j(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right) \\ &\quad + 2 \sum_{r=1}^{N} \frac{\left(\boldsymbol{\phi}_r(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right) \left(\boldsymbol{\phi}_r(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jl}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x}) \right)}{\lambda_j(\mathbf{x}) - \lambda_r(\mathbf{x})}. \end{aligned} \tag{66}$$

Equations (64) and (66) completely define the elements of the gradient vector and Hessian matrix of the eigenvalues.