MPP Based Correlated Function Expansion for Reliabilities and Moments of Uncertain Dynamical System

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ABSTRACT: Reliability analysis of uncertain dynamical system is considered in this study. The excitations and structural properties are modeled as non-stationary Gaussian processes and non-Gaussian random variables, respectively. The structural responses are therefore non-Gaussian processes, the statistics of which are not generally available in closed form. The limit state is formulated in terms of the extreme value distribution of the response random process. Developing these extreme value statistics analytically is not straight-forward, which makes failure probability estimations difficult. An alternative procedure, based on a newly developed MPP-based correlated function expansion method, is used for computing exceedance probabilities. This involves generating a full functional operational model, which approximates the limit surface in the neighborhood of MPP. Once the explicit form of original limit state is defined, the failure probability can be obtained by statistical simulation. Thus, the method can be integrated with commercial finite element software, which permits uncertainty analysis of large structures with complexities that include material and geometric nonlinear behavior.

KEYWORDS: Correlated function expansion, Uncertain dynamical system, Reliability

1. INTRODUCTION

A randomly vibrating structure with parametric uncertainty is deemed to safe if its response process lies well below the specified thresholds over a given duration of load process. Thus, extreme values of responses play a crucial role in the estimation of structural reliability. In this context, theory of asymptotic distributions [1–3] can be used to study the extreme values of random processes. As an alternative, extreme value theory of random processes over a given period can be related to the probability distribution of first passage times. For stationary Gaussian processes, above two approaches lead to Gumbel models for the extreme responses [4], provided the joint probability density function (PDF) of the process and its derivative at a given instant are known. It can also be noted that, determination of the joint PDF for Gaussian random responses is straightforward, while for non-Gaussian process evaluation of joint statistics is extremely difficult task.

Few studies on the exceedance probabilities of non-Gaussian random processes are reported in the literature. Von Mises stress is a commonly studied non-Gaussian process, the exceedance probabilities of which are often required for estimating reliabilities of ductile structure. Analytical expressions for the mean outcrossing rate of Von Mises stress are developed [5] for linear structures using outcrossing approximations. Several approaches are also developed [6–8] for computing the root mean square of Von Mises stress. However, in real-life problems, where the finite element (FE) method is an essential tool for handling modeling complexities, it is difficult to apply these methods as the performance function is defined in implicit form. For this class of problems, surrogate models provide best alternative solution scheme for estimating the exceedance probability of the response.

Recently, authors developed a new computational framework for stochastic FE analysis and time-invariant reliability analysis. In the present study we extend the scope of the correlated function expansion (CFE) approach, by considering the reliability analysis of nonlinear, randomly parametered dynamical systems, subjected to non-stationary Gaussian excitations. The structure is modeled and analyzed in commercial FE software (e.g. ANSYS). External software builds full

functional operational model using CFE and is interfaced with the FE model. The treatment of the problem includes one or more of the following features:

i) Parametric uncertainties in structural parameters–such as Young's modulus, density and strength characteristics–require non-Gaussian models.
ii) Possibility of geometric and/or material nonlinear structural behavior.
iii) Response variables that are nonlinear functions of displacement response, such as the Von Mises stress.
iv) Non-stationary random excitations.

The procedures developed are illustrated through a numerical example and are validated with the help of limited Monte Carlo simulations (MCS) and existing results.

2. PROBLEM STATEMENT

The governing equations of a randomly parametered vibrating structure are well known to be of the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t)+\boldsymbol{C}\dot{\boldsymbol{x}}(t)+\boldsymbol{K}\left[\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t)\right]=\boldsymbol{F}(t) \tag{1}$$

where, $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are, respectively, the global mass, damping and stiffness matrices. $\ddot{\boldsymbol{x}}(t),\dot{\boldsymbol{x}}(t)$ and $\boldsymbol{x}(t)$ are, respectively, vectors of nodal acceleration, velocity and displacements. $\boldsymbol{F}(t)$ represents a vector of random load processes characterized by the power spectral density (PSD) matrix, $\boldsymbol{S}_{\boldsymbol{FF}}(\omega)$. For stationary Gaussian process, $\boldsymbol{F}(t)$ is expressed as a sum of harmonic functions with random coefficients and is of the form

$$\boldsymbol{F}(t)=\sum_{i=1}^{P} a_p\cos(\omega_p t)+b_p\sin(\omega_p t) \tag{2}$$

Here, P denotes the number of terms required for discretizing the PSD function, ω_p are the discretized frequencies and σ_p^2 is the variance of the p^{th} segment of the discretized PSD. a_p and b_p are Gaussian random variables with zero mean and standard deviation σ_p. When the random process is modeled as non-stationary, $\boldsymbol{F}(t)$ is multiplied by a deterministic envelope function $e(t)$ [9] of the form

$$e(t)=\alpha\left[\exp(-\beta t)-\exp(-\chi t)\right] \tag{3}$$

where the parameters β and χ, determine the shape of the envelope function and α is a normalization factor such that $e_{\max}(t)=1$.

Let $V(t)$ be the response quantity of interest, which, in its most general form, is written as

$$V(t)=h\left[\ddot{\boldsymbol{x}}(t),\dot{\boldsymbol{x}}(t),\boldsymbol{x}(t)\right] \tag{4}$$

Thus, even in linear structures under Gaussian excitations, if $h[\cdot]$ is a nonlinear function, determining the probability distribution of $V(t)$ is not easy, even though the probability distributions of $\ddot{\boldsymbol{x}}(t),\dot{\boldsymbol{x}}(t)$ and $\boldsymbol{x}(t)$ are known exactly. Examples of such processes are the principal stress components and Von Mises stress. For structures which behave nonlinearly, or when $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are randomly parametered, the task is even more difficult as the distributions of $\ddot{\boldsymbol{x}}(t),\dot{\boldsymbol{x}}(t)$ and $\boldsymbol{x}(t)$ are

themselves not available explicitly. For computing the failure probability against yielding, the probability of Von Mises stress exceeding the specified threshold, $V_{\lim}$, over $\left[t_0, t_0+T\right]$, is defined as

$$P_F = P\left[V(t) \le V_{\lim};\ \forall t \in (t_0, t_0+T)\right] \tag{5}$$

where $V_{\lim} = 2\sigma_y/3$ is the permissible stress and $P[\cdot]$ is the probability measure. Since $V(t)$ is obtained as a nonlinear combination of the stress components, the probability distribution of $V(t)$ is unknown and hence, it is difficult to obtain analytical expressions for P_F. In this study, CFE approach scheme is used to estimates of the exceedance probability. The detail of the CFE methodology is outlined in the following section.

3. MPP-BASED CORRELATED FUNCTION EXPANSION

The key question in many scientific problems is to find the map between sets of high dimensional input and output system variables. That is the place where so called "curse of dimensionality" arises. Full space analysis without any a priori assumption has an exponentially growing computational complexity. In this context, Correlated Function Expansion (CFE) [10–12] could evolve as an efficient computational tool for solving high-dimensional scientific problems. In its background there stands the simple observation: only low-order correlations amongst the input variables have a significant impact upon the outputs. Such a presumption permits expressing single multidimensional mapping as a sum of many low dimensional mappings. One may show that system response $g(\boldsymbol{x}) = g(x_1, x_2, \ldots, x_N)$ can be expressed as summands of different dimensions:

$$\begin{aligned} g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{1\le i_1<i_2\le N} g_{i_1 i_2}(x_{i_1}, x_{i_2}) + \ldots \\ + \sum_{1\le i_1<\ldots<i_l\le N} g_{i_1 i_2 \ldots i_l}(x_{i_1}, x_{i_2}, \ldots, x_{i_l}) \quad + \ldots + g_{12\ldots N}(x_1, x_2, \ldots, x_N) \end{aligned} \tag{6}$$

where g_0 is a constant term representing the mean response of $g(\boldsymbol{x})$. The function $g_i(x_i)$ describes the independent effect of variable x_i acting alone, although generally nonlinearly, upon the output $g(\boldsymbol{x})$. The function $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ gives pair cooperative effect of the variables x_{i_1} and x_{i_2} upon the output $g(\boldsymbol{x})$. The last term $g_{12\ldots N}(x_1, x_2, \ldots, x_N)$ contains any residual correlated behaviour over all of the system variables.

The expansion functions are determined by evaluating the input-output responses of the system relative to the defined MPP $\bar{\boldsymbol{x}}^* = \{\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_N^*\}$ in the input variable space. This process reduces to the following relationship for the component functions in Eq. (6)

$$g_0 = g(\bar{\boldsymbol{x}}^*) \tag{7}$$

$$g_i(x_i) = g(x_i, \bar{\boldsymbol{x}}^{*i}) - g_0 \tag{8}$$

$$g_{i_1 i_2}(x_{i_1}, x_{i_2}) = g(x_{i_1}, x_{i_2}, \bar{\boldsymbol{x}}^{*i_1 i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0 \tag{9}$$

where the notation $g(x_i, \bar{\boldsymbol{x}}^{*i}) = g(\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_{i-1}^*, x_i, \bar{x}_{i+1}^*, \ldots, \bar{x}_N^*)$ denotes that all the input variables are at their MPP except x_i. The g_0 term is the output response of the system evaluated at the MPP

$\bar{\boldsymbol{x}}^*$. The higher order terms are evaluated as cuts in the input variable space through the MPP. Therefore, each first-order term $g_i\left(x_i\right)$ is evaluated along its variable axis through the MPP. Each second-order term $g_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right)$ is evaluated in a plane defined by the binary set of input variables x_{i_1}, x_{i_2} through the MPP, etc. The process of subtracting off the lower order expansion functions removes their dependence to assure a unique contribution from the new expansion function.

Considering terms up to first- and second-order in Eq. (6) yields, respectively

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i\left(x_i\right) + \mathcal{R}_2 \tag{10}$$

and

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i\left(x_i\right) + \sum_{1 \le i_1 < i_2 \le N} g_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right) + \mathcal{R}_3 \tag{11}$$

Substituting Eq. (7)–(9) into Eq. (10) and (11) leads to

$$g(\boldsymbol{x}) = \sum_{i=1}^{N} g\left(\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_{i-1}^*, x_i, \bar{x}_{i+1}^*, \ldots, \bar{x}_N^*\right) - (N-1)\, g\left(\bar{\boldsymbol{x}}^*\right) + \mathcal{R}_2 \tag{12}$$

and

$$\begin{aligned} g(\boldsymbol{x}) = & \sum_{\substack{i_1=1, i_2=1 \\ i_1 < i_2}}^{N} g\left(\bar{x}_1^*, \ldots, \bar{x}_{i_1-1}^*, x_{i_1}, \bar{x}_{i_1+1}^*, \ldots, \bar{x}_{i_2-1}^*, x_{i_2}, \bar{x}_{i_2+1}^*, \ldots, \bar{x}_N^*\right) \\ & - (N-2) \sum_{i=1}^{N} g\left(\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_{i-1}^*, x_i, \bar{x}_{i+1}^*, \ldots, \bar{x}_N^*\right) + \frac{(N-1)(N-2)}{2} g\left(\bar{\boldsymbol{x}}^*\right) + \mathcal{R}_3 \end{aligned} \tag{13}$$

Now consider first- and second-order approximation of $g(\boldsymbol{x})$, denoted respectively by

$$\tilde{g}^{I}(\boldsymbol{x}) \cong \sum_{i=1}^{N} g\left(\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_{i-1}^*, x_i, \bar{x}_{i+1}^*, \ldots, \bar{x}_N^*\right) - (N-1)\, g\left(\bar{\boldsymbol{x}}^*\right) \tag{14}$$

and

$$\begin{aligned} \tilde{g}^{II}(\boldsymbol{x}) \cong & \sum_{\substack{i_1=1, i_2=1 \\ i_1 < i_2}}^{N} g\left(\bar{x}_1^*, \ldots, \bar{x}_{i_1-1}^*, x_{i_1}, \bar{x}_{i_1+1}^*, \ldots, \bar{x}_{i_2-1}^*, x_{i_2}, \bar{x}_{i_2+1}^*, \ldots, \bar{x}_N^*\right) \\ & - (N-2) \sum_{i=1}^{N} g\left(\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_{i-1}^*, x_i, \bar{x}_{i+1}^*, \ldots, \bar{x}_N^*\right) + \frac{(N-1)(N-2)}{2} g\left(\bar{\boldsymbol{x}}^*\right) \end{aligned} \tag{15}$$

Comparison of Eq. (12) and (14) indicates that the first-order approximation leads to the residual error $g(\boldsymbol{x}) - \tilde{g}(\boldsymbol{x}) = \mathcal{R}_2$, which includes contributions from terms of two and higher order component functions. Similarly the second-order approximation leads to the residual error $g(\boldsymbol{x}) - \tilde{g}(\boldsymbol{x}) = \mathcal{R}_3$, which includes contributions from terms of three and higher order component functions. Computational flow diagram of the CFE-based approach is presented in Fig. 1.

The notion of 0th, 1st, 2nd order, etc. in CFE should not be confused with the terminology used either in the Taylor series or in the conventional least-squares based metamodel. It can be shown that, the first-order component function $g_i\left(x_i\right)$ is the sum of all the Taylor series terms which contain and only contain variable x_i. Similarly, the second order component function $g_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right)$ is the sum of all the Taylor series terms which contain and only contain variables x_{i_1} and x_{i_2}.

Hence first- and second-order approximations of CFE should not be viewed as first- or second-order Taylor series expansions nor do they limit the nonlinearity of $g(\boldsymbol{x})$.

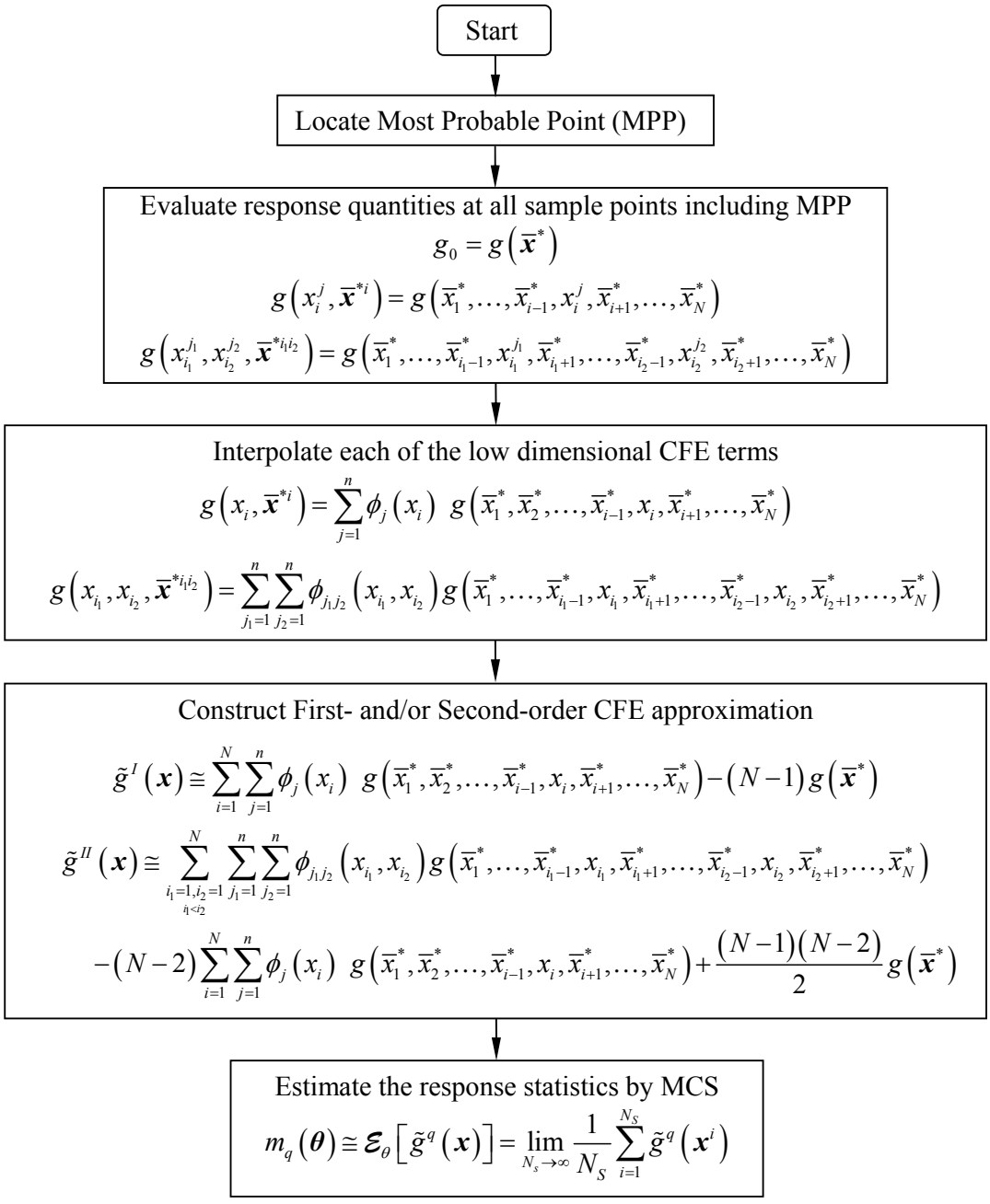

Figure 1. Flowchart of CFE-based Approximation

Furthermore, the approximations contain contributions from all input variables. Thus, the infinite number of terms in the Taylor series are partitioned into finite different groups and each group corresponds to one CFE component function. Therefore, any truncated CFE provides a better approximation and convergent solution of $g(\boldsymbol{x})$ than any truncated Taylor series because the latter only contains a finite number of terms of Taylor series. Furthermore, the coefficients associated with higher dimensional terms are usually much smaller than that with one-dimensional terms. As such, the impact of higher dimensional terms on the function is less, and therefore, can be neglected.

4. NUMERICAL EXAMPLE

A numerical example is presented here to illustrate the performance of the proposed approach and compared with existing results. The failure estimates have been compared through the following three procedures: (1) An estimate of P_F is obtained from full scale MCS on the exact performance function. This involves the analysis of an ensemble of response time histories obtained by direct integration of Eq. (1), for a set of sample time histories of excitation $F(t)$. (2) An estimate of P_F is obtained by adopting the improved response surface [13] procedure. (3) An estimate of P_F is obtained by Bucher and Bourgund algorithm [14].

Example 1: Linear Structure with Random Parameters

In this example, a 1.5 m long cantilever beam subjected to non-stationary random excitation, is studied. Cross sectional dimensions of the beam is taken as 0.15 × 0.03 m. Commercial FE software (ANSYS) is used to model and analyze this structure assuming plane stress condition. The beam is discretized into ten plane stress elements. The PSD of support excitation, $\boldsymbol{F}(t)$, is taken as

$$S_{\boldsymbol{FF}}(\omega) = S_0 \frac{1 + 4\eta_g^2 \left(\omega/\omega_g\right)^2}{\left(1 - \left(\omega/\omega_g\right)^2\right)^2 + 4\eta_g^2 \left(\omega/\omega_g\right)^2} \tag{16}$$

Where $\omega_g = 1500$ rad/s, $\eta_g = 1.2$ and ω is the frequency of excitation. Non-stationary load histories for support excitation are generated from Eq. (2) and (3), with $P = 5$. The parameters α, β and χ in Eq. (3) are taken to be 7.18, 60 and 41, respectively, to illustrate the performance of present approach with existing methods [13].

In this study, failure is defined as exceedance of Von Mises stress over material strength, which, for all practical purposes, denotes the limit of linear material behavior in ductile materials. The performance function is defined in terms of the Von Mises stress developed at the root of the cantilever beam and is of the form given in Eq. (5). Here, $V(t)$ is the Von Mises stress, given by $V(t) = \sqrt{\sigma(t)\boldsymbol{C}\sigma(t)}$, the time duration $T = 0.04$ s, $\sigma(t)$ is stress tensor and

$$\boldsymbol{C} = \begin{bmatrix} 1 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{17}$$

The structure material is assumed structural steel and proportional damping is assumed 5%. The randomness in Young's modulus (E) and yield strength of the material (σ_y) are expressed as $E = E_0(1 + \varepsilon_1 z_1)$ and $\sigma_y = \sigma_{y0}(1 + \varepsilon_2 z_2)$, where, E_0 and σ_{y0} respectively, denote the mean values of E and σ_y. ε_1 and ε_2 are small deterministic constants, taken to be equal to 0.05. z_1 and z_2 are assumed to be lognormal random variables with $\langle z_1 \rangle = 2.0$, $\langle z_2 \rangle = 0.1$, $\langle z_1 z_1 \rangle = 1.0$ and $\langle z_2 z_2 \rangle = 0.05$. The limit state/performance function is, thus, involved 12 random variables. Correlated function expansion of original limit state/performance function is generated using $n = 7$. The constant S_0 in Eq. (16) is varied from 40–80 m^2/s^3 and the corresponding failure probability is computed, presented in Fig. 2. A sample size of 3000 has been considered for direct MCS. Estimation of failure probabilities using other approaches (e.g. Bucher & Bourgund's algorithm

[14], Improved response surface [13]) are also presented in Fig. 2, to show the performance of the proposed methodology. Numbers inside parentheses represent the number of actual FE analysis required by each method.

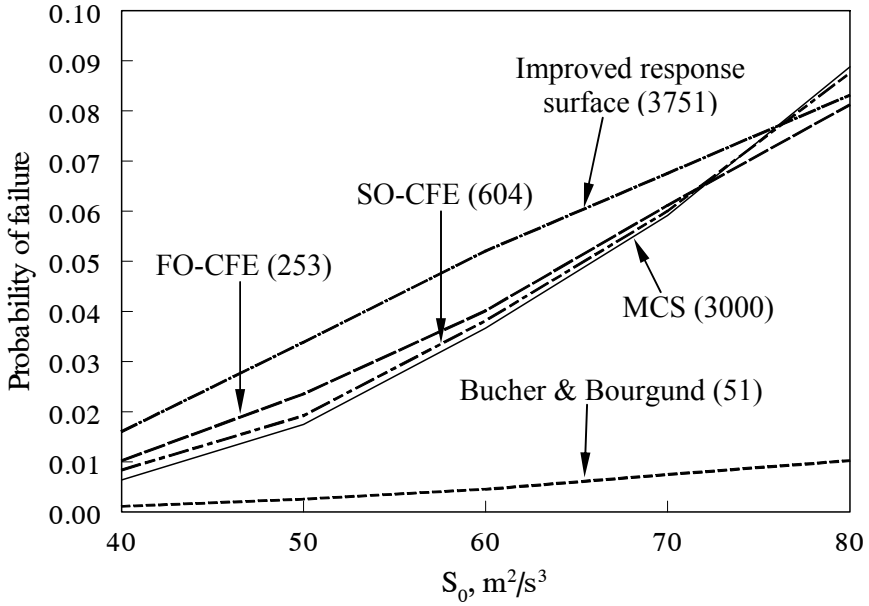

Figure 2. Estimation of exceedance probability for randomly parametered structure subjected to non-stationary excitations.

5. SUMMARY AND CONCLUSIONS

Computation of structural reliability under stochastic load process is obtained from the probability of exceedance of response processes during specified time evolution. Estimation of these exceedance probabilities needs an explicit knowledge of the mean outcrossing rate of the response process. This, in turn, requires knowledge of joint statistics of the response and its time derivative. In most of the cases, the response processes are non-Gaussian in nature. Determination of their joint PDF and mean outcrossing rates are difficult. This paper addressed a new computational method for obtaining exceedance probabilities, in an efficient manner. It utilized the excellent properties of correlated function expansion method for multivariate function approximation.

The present approach sidesteps the requirement of joint statistics of the response processes. As the method requires only conditional responses at selected sample points, this can easily be integrated with commercial FE software. Thus, complexities arising out of geometric and material nonlinear behavior, uncertainties in the structural properties and loads can be handled efficiently. The computational effort required in correlated function expansion is economical compared with full scale MCS, when estimates of low failure probabilities are desired and when the evaluation of the performance function requires significant computer time.

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