ON THE VALIDITY OF RANDOM MATRIX MODELS IN PROBABILISTIC STRUCTURAL DYNAMICS

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ABSTRACT. An accurate and efficient uncertainty quantification of the dynamic response of complex structural systems is crucial for their design and analysis. Among the many approaches proposed, the random matrix approach has received significant attention over the past decade. In this paper two new random matrix models, namely (a) generalized scalar Wishart distribution and (b) generalized diagonal Wishart distribution have been proposed. The central aims behind the proposition of the new models are to (a) improve the accuracy of the statistical predictions, (b) simplify the analytical formulations and (c) improve computational efficiency. Identification of the parameters of the newly proposed random matrix models has been discussed. Closed-form expressions have been derived using rigorous analytical approaches. It is considered that the dynamical system is proportionally damped and the mass and stiffness properties of the system are random. The newly proposed approaches are compared with the existing Wishart random matrix model. In total, three possible random matrix approaches are applied to the vibration problem of a cantilever plate with random properties. Relative merits and demerits of different random matrix formulations are discussed and based on extensive numerical studies, recommendations for the best models for different scenarios have been outlined.

KEYWORDS: Unified Uncertainty Quantification, Random matrix theory, Wishart distribution, model validation

1 INTRODUCTION

Finite element codes implementing physics based models are used extensively for the dynamic analysis of complex systems. Laboratory based controlled tests are often performed to gain insight into some specific physics of a problem. Such tests can indeed lead to new physical laws improving the computational models. Test data can also be used to calibrate a known model. However, neither of these activities may be enough to produce a credible numerical tool because of several types of uncertainties which exist in the physics based computational framework. Such uncertainties include, but are not limited to (a) parameter uncertainty (e.g, uncertainty in geometric parameters, friction coefficient, strength of the materials involved and joints); (b) model uncertainty (arising from the lack of scientific knowledge about the model which is *a priori* unknown); (c) experimental error (uncertain and unknown errors percolate into the model when they are calibrated against experimental results). These uncertainties must be assessed and managed for credible computational predictions.

The predictions from high resolution numerical models may sometimes exhibit significant differences with the results from physical experiments due to uncertainty. When substantial statistical information exists, the theory of probability and stochastic processes offer a rich mathematical framework to represent such uncertainties. In a probabilistic setting, the data (parameter) uncertainty associated with the system parameters, such as the geometric properties and constitutive relations (i.e. Young's modulus, mass density, Poisson's ratio, damping coefficients), can be modeled as random variables or stochastic processes using the so-called parametric approach. These uncertainties can be quantified and propagated, for example, using the stochastic finite element method [1–14]. The companion paper addressed this issue experimentally where one hundred realizations of a beam with random mass distribution was created and tested. Recently, the uncertainty due to modelling error has received attention as this is crucial for model validation [15–23]. The model uncertainty problem poses serious challenges as the parameters contributing to the modelling errors are not available *a priori* and therefore precludes the application of a parametric approach to address such issues.

Model uncertainties do not explicitly depend on the system parameters. For example, there can be unquantified errors associated with the equation of motion (linear or non-linear), in the damping model (viscous or non-viscous [24, 25]), in the model of structural joints. The model uncertainty may be tackled by the so-called non-parametric method pioneered by Soize [26–28] and adopted by others [29–33].

The equation of motion of a damped n-degree-of-freedom linear dynamic system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{f}(t) \in \mathbb{R}^n$ is the forcing vector, $\mathbf{q}(t) \in \mathbb{R}^n$ is the response vector and $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times n}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass, damping and stiffness matrices respectively. In order to completely quantify the uncertainties associated with system (1) we need to obtain the probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. Using the parametric approach, such as the stochastic finite element method, one usually obtains a problem specific covariance structure for the elements of system matrices. The nonparametric approach [26–28, 31, 32] on the other hand results in a central Wishart distribution for the system matrices. In a recent paper [34] it was shown that Wishart matrix with properly selected parameters can be used for systems with both parametric uncertainty and nonparametric uncertainty. The main conclusions arising from this study were:

- Wishart random matrix distribution can be obtained either using the maximum entropy approach or using the matrix factorization approach.
- The maximum entropy approach gives a natural selection for the parameters. Through numerical examples it was shown that the parameters obtained using the maximum entropy approach may yield non-physical results. In that the 'mean of the inverse' and the 'inverse of the mean' of the random matrices can be significantly different.
- The parameters of the pdf of a Wishart random matrix obtained using the maximum entropy approach are not unique since they depend on what constraints are used in the optimization approach.
- Considering that the available 'data' is the mean and (normalized) standard deviation of a system matrix, it was shown that when the mean of the inverse equals to the inverse of the mean of the system matrices, the predictive accuracy is maximum

The aim of this paper is to further explore the idea of proper parameter selection of Wishart matrices. Specifically, we ask the question 'what is the simplest possible Wishart random matrix model can be used without compromising the predictive accuracy?'. The definition of 'simplest' and 'accuracy' will be explained later in the paper.

2 FREQUENCY RESPONSE FUNCTION OF STOCHASTIC SYSTEMS

Assuming all the initial conditions are zero and taking the Laplace transform of (1) we have

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{2}$$

where $(\bar{\bullet})$ denotes the Laplace transform of respective quantities. The undamped eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2\mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{3}$$

where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system. We define the matrices

$$\boldsymbol{\Omega} = \mathrm{diag}\left[\omega_1, \omega_2, \ldots, \omega_n\right] \quad \text{and} \quad \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right]. \tag{4}$$

so that

$$\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 \quad \text{and} \quad \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_n \tag{5}$$

where $\mathbf{I}_n$ is an N dimensional identity matrix. Using these, Eq. (2) can be transformed into the modal coordinates as

$$\left[s^2\mathbf{I}_n + s\mathbf{C}' + \mathbf{\Omega}^2\right]\bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{6}$$

where and $(\bullet)'$ denotes the quantities in the modal coordinates:

$$\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}, \quad \bar{\mathbf{q}} = \mathbf{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}}. \tag{7}$$

For simplicity let us assume that the system is proportionally damped with deterministic modal damping factors $\zeta_1, \zeta_2, \ldots, \zeta_n$. Therefore, when we consider random systems, the matrix of eigenvalues $\mathbf{\Omega}^2$ in equation (6) will be a random matrix of dimension N. Suppose this random matrix is denoted by $\mathbf{\Xi} \in \mathbb{R}^{n\times n}$:

$$\mathbf{\Omega}^2 \sim \mathbf{\Xi} \tag{8}$$

Since $\mathbf{\Xi}$ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\mathbf{\Psi}_r$ such that

$$\mathbf{\Psi}_r^T\mathbf{\Xi}\mathbf{\Psi}_r = \mathbf{\Omega}_r^2 \tag{9}$$

Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\mathbf{\Xi}$. Recalling that $\mathbf{\Psi}_r^T\mathbf{\Psi}_r = \mathbf{I}_n$, from equation (6) we obtain

$$\bar{\mathbf{q}}' = \left[s^2\mathbf{I}_n + s\mathbf{C}' + \mathbf{\Omega}^2\right]^{-1}\bar{\mathbf{f}}' \tag{10}$$

$$= \mathbf{\Psi}_r\left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\mathbf{\Omega}_r + \mathbf{\Omega}_r^2\right]^{-1}\mathbf{\Psi}_r^T\bar{\mathbf{f}}' \tag{11}$$

where

$$\boldsymbol{\zeta} = \mathrm{diag}\left[\zeta_1, \zeta_2, \ldots, \zeta_n\right] \tag{12}$$

The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \mathbf{\Phi}\mathbf{\Psi}_r\left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\mathbf{\Omega}_r + \mathbf{\Omega}_r^2\right]^{-1}(\mathbf{\Phi}\mathbf{\Psi}_r)^T\bar{\mathbf{f}}(s) = \sum_{j=1}^{n}\frac{\mathbf{x}_{r_j}^T\bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2}\mathbf{x}_{r_j}. \tag{13}$$

where

$$\mathbf{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_n}\right] \quad \text{and} \quad \mathbf{X}_r = \mathbf{\Phi}\mathbf{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_n}\right] \tag{14}$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system. In the next section we discuss the derivation of the random matrix $\mathbf{\Xi}$.

3 RANDOM MATRIX MODELS AND ITS PARAMETERS

Recalling that $\mathbf{\Xi}$ is a symmetric and positive definite matrix, one can use the Wishart random matrix model [26–28, 31, 32]. Consider that a random symmetric and positive definite matrix $\mathbf{G}$ has mean $\overline{\mathbf{G}}$ and normalized standard deviation or dispersion parameter

$$\delta_G = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2} \tag{15}$$

Suppose that $\mathbf{G}$ can be modeled by a Wishart matrix with parameters p and $\mathbf{\Sigma}$ so that $\mathbf{G} \sim W_n(p, \mathbf{\Sigma})$. The probability density function [35–40] of $\mathbf{G}$ can be expressed as

$$p_{\mathbf{G}}(\mathbf{G}) = \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}|\mathbf{G}|^{\frac{1}{2}(p-n-1)}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{G}\right\} \tag{16}$$

where the multivariate Gamma function is given by

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \quad \text{for} \quad \Re(a) > \frac{1}{2}(n-1) \tag{17}$$

In the previous works [26–28, 31, 32] the system matrices were separately considered as Wishart matrices. Because the system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ have similar probabilistic characteristics, for notational convenience we will use the notation $\mathbf{G}$ which stands for any one of the system matrices.

The parameters p and $\mathbf{\Sigma}$ of the Wishart system matrices can be obtained based on what criteria we select. For this parameter estimation problem, the 'data' consist of the 'measured' mean ($\overline{\mathbf{G}}$) and dispersion parameter ($\widetilde{\delta}_G$) of a system matrix. Adhikari [34] investigated following three parameter selection methods:

1. *Parameter selection 1:* For each system matrices, one considers that the mean of the random matrix is same as the deterministic matrix and the dispersion parameter is same as the measured dispersion parameter. Mathematically this implies $\mathrm{E}[\mathbf{G}] = \overline{\mathbf{G}}$ and $\delta_G = \widetilde{\delta}_G$. This condition results

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \overline{\mathbf{G}}/p \tag{18}$$

where

$$\theta = \frac{1}{\widetilde{\delta}_G^2}\left\{1 + \gamma_G\right\} - (n+1) \tag{19}$$

and

$$\gamma_G = \frac{\left\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\right\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)} \tag{20}$$

This parameter selection was proposed by [26, 27].

2. *Parameter selection 2:* Here one considers that for each system matrices, the mean of the inverse of the random matrix equals to the inverse of the deterministic matrix and the dispersion parameter is same as the measured dispersion parameter. Mathematically this implies $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \overline{\mathbf{G}}^{-1}$ and $\delta_G = \widetilde{\delta}_G$. Using the theory of inverted Wishart distribution and after some simplifications one obtains

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \overline{\mathbf{G}}/\theta \tag{21}$$

where θ is defined in equation (19). This parameter selection was proposed by Adhikari [34].

3. *Parameter selection 3:* For each system matrices, one considers that the mean of the random matrix and the mean of the inverse of the random matrix are closest to the deterministic matrix and its inverse. Mathematically this implies $\left\|\overline{\mathbf{G}} - \mathrm{E}[\mathbf{G}]\right\|_\mathrm{F}$ and $\left\|\overline{\mathbf{G}}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_\mathrm{F}$ are minimum and $\delta_G = \widetilde{\delta}_G$. This condition results:

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \overline{\mathbf{G}}/\alpha \tag{22}$$

where $\alpha = \sqrt{\theta(n+1+\theta)}$ and θ is as defined in equation (19). This is known as the optimal Wishart distribution [31]. The rationale behind this approach is that a random system matrix and its inverse should be mathematically treated in a similar manner as both are symmetric and positive-definite matrices.

4. *Parameter selection 4:* This criteria arises from the idea that the mean of the eigenvalues of the distribution is same as the 'measured' eigenvalues of the mean matrix and the dispersion parameter is same as the measured dispersion parameter. Mathematically one can express this as

$$\mathrm{E}\left[\mathbf{M}^{-1}\right] = \overline{\mathbf{M}}^{-1},\ \mathrm{E}[\mathbf{K}] = \overline{\mathbf{K}},\ \delta_M = \widetilde{\delta}_M \quad \text{and} \quad \delta_K = \widetilde{\delta}_K \tag{23}$$

Based on extensive numerical studies it was shown [34] that the parameter selection 2 produces least error when the results were compared to direct Monte Carlo simulation. Even for proportionally damped systems, following this approach, two correlated random Wishart matrices (for the mass and the stiffness matrices) need to be simulated and their joint eigenvalue problem need to be solved. Here we consider the possibility of considering the matrix $\boldsymbol{\Xi}$ as the only equivalent random Wishart matrix. The motivations behind this approach are

- Because $\boldsymbol{\Omega}^2$ is a diagonal matrix, $\boldsymbol{\Xi}$ is an uncorrelated Wishart matrix which is very easy to simulate.
- Only one matrix needs to be simulated for each sample.
- Instead of solving a generalized eigenvalue problem involving two matrices for each sample, one now needs to solve only a standard eigenvalue problem involving just one symmetric and positive definite matrix.
- Because the eigenvalues of random system is now governed by one one random matrix, it is possible to develop further insights into the distribution of the eigenvalues.
- Due to the simplicity of the expression of the frequency response function (13), it may form the essential basis for analytical derivation of response moments using the theory of Wishart matrices.

It is obvious that for the above points to be realized, one must (1) derive suitable parameters for the random matrix $\boldsymbol{\Xi} \sim W_n(p, \boldsymbol{\Sigma})$ from the available data, namely $\overline{\mathbf{M}}$, $\overline{\mathbf{K}}$, δ_M and δ_K, and (2) numerically verify that this approach results acceptable fidelity with respect to direct monte carlo simulation.

The mean of $\boldsymbol{\Xi}$ is the diagonal matrix containing the squared undamped natural frequencies, that is,

$$\mathrm{E}\left[\boldsymbol{\Xi}\right] = \boldsymbol{\Omega}_0^2 \tag{24}$$

Note that the eigenvalues of $\boldsymbol{\Xi}$ are same as the eigenvalues of $\mathbf{H} = \mathbf{M}^{-1}\mathbf{K}$. Therefore, the dispersion parameter of $\boldsymbol{\Xi}$ is same as that of $\mathbf{H} = \mathbf{M}^{-1}\mathbf{K}$. Using the theory of inverted Wishart random matrices [39], after some algebra it can be shown that the dispersion parameters of $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{K}$ are related as

$$\delta_H = \frac{n_H}{\left(1+\gamma_K\right)\left(-1-\gamma_M+n{\delta_M}^2\right)\left(-1-\gamma_M+n{\delta_M}^2+{\delta_M}^2\right)} \tag{25}$$

where

$$\begin{aligned} n_H = {} & \left(1+\gamma_H\right){\delta_K}^2{\delta_M}^4 n^2 + \Big(\big(\left(4\,\gamma_H+2\right){\delta_K}^2 - \gamma_H\,\gamma_K + 3\,\gamma_K - \gamma_H + 3\big){\delta_M}^4 \\ & + \left(-2\,\gamma_M - 2\,\gamma_H - 2 - 2\,\gamma_H\,\gamma_M\right){\delta_K}^2{\delta_M}^2\Big)\,n + \big(\left(-3+3\,\gamma_H\right){\delta_K}^2 - 3\,\gamma_H\,\gamma_K + 3\,\gamma_K + 3 - 3\,\gamma_H\big){\delta_M}^4 \\ & + \big(\left(-4\,\gamma_H\,\gamma_M - 2 - 2\,\gamma_M - 4\,\gamma_H\right){\delta_K}^2 + \gamma_H\,\gamma_K + \gamma_H + \gamma_H\,\gamma_M - 3 - 3\,\gamma_K\,\gamma_M - 3\,\gamma_M \\ & \qquad -3\,\gamma_K + \gamma_H\,\gamma_K\,\gamma_M\big){\delta_M}^2 + \left(2\,\gamma_M + 2\,\gamma_H\,\gamma_M + {\gamma_M}^2 + \gamma_H\,{\gamma_M}^2 + 1 + \gamma_H\right){\delta_K}^2 \end{aligned} \tag{26}$$

and the ratio $\gamma_{(\bullet)}$ is as defined in Eq. (20). Note that

$$\gamma_H = \frac{\left\{\mathrm{Trace}\left(\overline{\mathbf{H}}\right)\right\}^2}{\mathrm{Trace}\left(\overline{\mathbf{H}}^2\right)} = \frac{\left\{\mathrm{Trace}\left(\boldsymbol{\Omega}_0^2\right)\right\}^2}{\mathrm{Trace}\left(\boldsymbol{\Omega}_0^4\right)} \tag{27}$$

Here two types of Wishart random matrix model is proposed for the system matrix $\boldsymbol{\Xi}$:

(a) *Scalar Wishart Matrix:* In this case it is assumed that

$$\boldsymbol{\Xi} \sim W_n\left(p, \frac{a^2}{n}\mathbf{I}_n\right) \tag{28}$$

Considering $\mathrm{E}[\boldsymbol{\Xi}] = \boldsymbol{\Omega}_0^2$ and $\delta_{\boldsymbol{\Xi}} = \delta_H$ the values of the unknown parameters can be obtained as

$$p = \frac{1+\gamma_H}{\delta_H^2} \quad \text{and} \quad a^2 = \mathrm{Trace}\left(\boldsymbol{\Omega}_0^2\right)/p \tag{29}$$

(b) *Diagonal Wishart Matrix with different entries:* In this case it is assumed that

$$\boldsymbol{\Xi} \sim W_n\left(p, \boldsymbol{\Omega}_0^2/\theta\right) \tag{30}$$

Considering $\mathrm{E}\left[\boldsymbol{\Xi}^{-1}\right] = \boldsymbol{\Omega}_0^{-2}$ and $\delta_{\boldsymbol{\Xi}} = \delta_H$ as proposed in [34] we have

$$p = n+1+\theta \quad \text{and} \quad \theta = \frac{(1+\gamma_H)}{\delta_H^2} - (n+1) \tag{31}$$

The samples of $\boldsymbol{\Xi}$ can be generated using the procedure outlined in the previous works [31, 34]. One need to solve the symmetric eigenvalue problem for every sample

$$\boldsymbol{\Xi}\boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2\boldsymbol{\Psi}_r, \quad \forall r = 1,2,\cdots,n_{\mathrm{samp}} \tag{32}$$

The eigenvalue and eigenvector matrices $\boldsymbol{\Omega}_r^2$ and $\boldsymbol{\Psi}_r$ can then be substituted in Eq. (13) to obtain the dynamic response. These two random matrix models together with the original approach proposed by Soize [26, 27] are numerically verified in the next section.

4 NUMERICAL INVESTIGATIONS

A rectangular cantilever steel plate is considered to illustrate the application of Wishart random matrices in probabilistic structural dynamics. The deterministic properties are assumed to be $\bar{E} = 200 \times 10^9 \mathrm{N/m}^2$, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860 \mathrm{kg/m}^3$, $\bar{t} = 3.0\mathrm{mm}$, $L_x = 0.998\mathrm{m}$, $L_y = 0.59\mathrm{m}$. Two different cases of uncertainties are considered. In the first case it is assumed that the material properties are randomly inhomogeneous. In the second case we consider that the plate is 'perturbed' by attaching spring-mass oscillators at random locations. The first case corresponds to a parametric uncertainty problem while the second case corresponds to a non-parametric uncertainty problem.

4.1 Plate With Randomly Inhomogeneous Material Properties: It is assumed that the Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

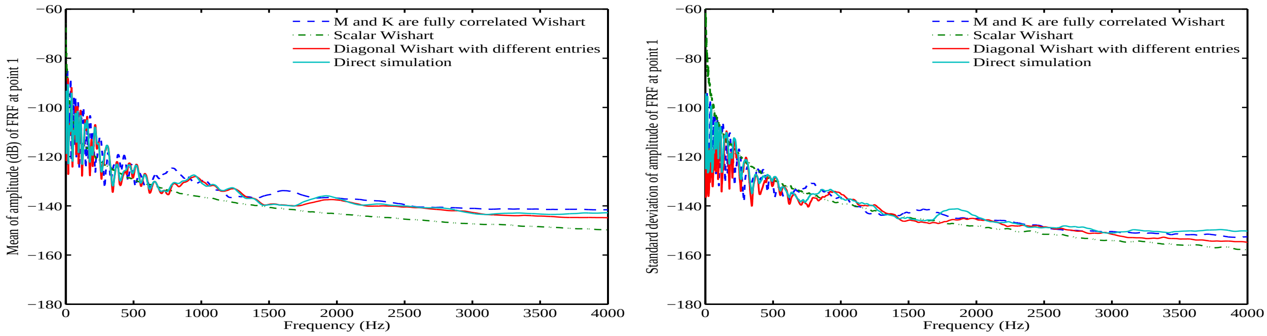
$$E(\mathbf{x}) = \bar{E}\left(1+\epsilon_E f_1(\mathbf{x})\right), \quad \mu(\mathbf{x}) = \bar{\mu}\left(1+\epsilon_\mu f_2(\mathbf{x})\right) \tag{33}$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1+\epsilon_\rho f_3(\mathbf{x})\right) \quad \text{and} \quad t(\mathbf{x}) = \bar{t}\left(1+\epsilon_t f_4(\mathbf{x})\right) \tag{34}$$

The two dimensional vector $\mathbf{x}$ denotes the spatial coordinates. The strength parameters are assumed to be $\epsilon_E = 0.10$, $\epsilon_\mu = 0.10$, $\epsilon_\rho = 0.08$ and $\epsilon_t = 0.12$. The random fields $f_i(\mathbf{x}), i = 1,\cdots,4$ are assumed to be correlated homogenous Gaussian random fields. An exponential correlation function with correlation length 0.2 times the lengths in each direction has been considered. A 5000-sample Monte Carlo simulation is performed to obtain the FRFs.

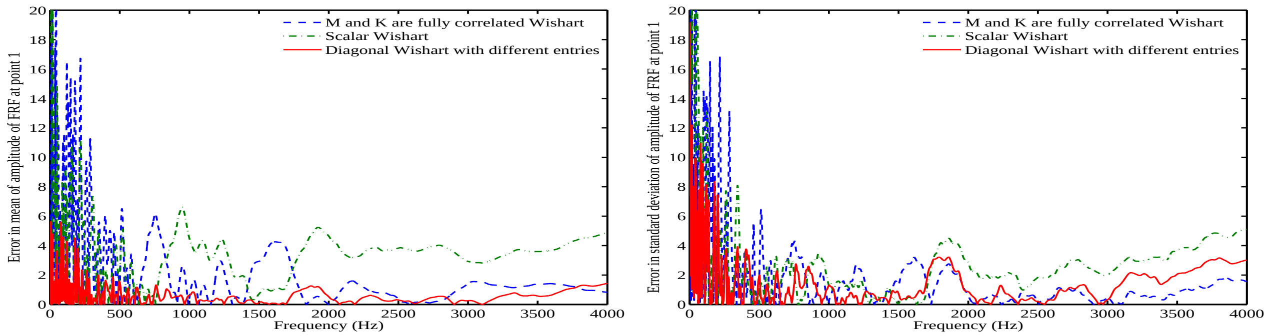
We want to identify which of the three Wishart matrix fitting approach considered here would produce highest fidelity with direct stochastic finite element Monte Carlo Simulation results. From the simulated random mass and stiffness matrices we obtain $n = 1200$, $\delta_M = 0.1133$ and $\delta_K = 0.2916$. Since 2% constant modal damping factor is assumed for all the modes, $\delta_C = 0$. The *only* uncertainty related information used in the random matrix approach are the values of δ_M and δ_K. The information regarding which element property functions are random fields, nature of these random fields (correlation structure, Gaussian or non-Gaussian) and the amount of randomness are *not* used in the Wishart matrix approach. This is aimed to depict a realistic situation when the detailed information regarding uncertainties in a complex engineering system is not available to the analyst.

The predicted mean and standard deviation of the amplitude using the direct stochastic finite element simulation and three Wishart matrix approaches are compared in subfigure 1(a) and subfigure 1(b) for the driving-point-FRF. In subfigure 2(a) and subfigure 2(b) percentage errors in the mean and standard deviation of the amplitude of the driving-point-FRF obtained using the proposed



(a) Mean of amplitude (b) Standard deviation of amplitude

Figure 1: Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF obtained using the direct stochastic finite element simulation and proposed three Wishart matrix approaches for the plate with randomly inhomogeneous material properties.



(a) Error in mean (b) Error in standard deviation

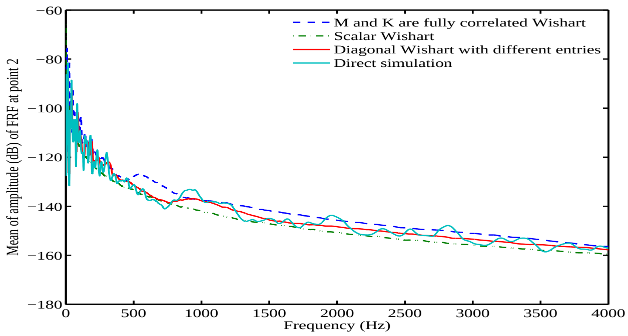
Figure 2: Comparison of percentage errors in the mean and standard deviation of the amplitude of the driving-point-FRF obtained using the proposed three Wishart matrix approaches for the plate with randomly inhomogeneous material properties.

three Wishart matrix approaches are shown. Percentage errors are calculated using the direct Monte Carlo Simulation results as the benchmarks. Error using any one of the random matrix approaches reduce with the increase in the frequency. Among the three Wishart matrix approaches discussed here diagonal Wishart matrix with different entries produces most accurate results across the frequency range. As expected, the scalar Wishart matrix model produces least accurate results. Note that this approach only need the simulation of one random matrix. From these results we conclude that Wishart random matrix corresponding to approach 3 should be used for a system with parametric uncertainty. In the next section we discuss the same system with non-parametric uncertainty.

4.2 Plate With Randomly Attached Spring-mass Oscillators: In this example we consider the same plate but with non-parametric uncertainty. We assume that 10 spring mass oscillators with random natural frequencies are attached at random nodal points in the plate. The nature of uncertainty in this case is different from the previous case because here the sparsity structure of the system matrices change with different realizations of the system. For numerical calculations we consider that the natural frequencies of the attached oscillators follow a uniform distribution between 0.2 kHz to 4.0 kHz. A 1000-sample Monte Carlo simulation is performed to obtain the FRFs.

From the simulated random mass and stiffness matrices we obtain $n = 1200$, $\delta_M = 0.1326$ and $\delta_K = 0.4201$. Since 2% constant modal damping factor is assumed for all the modes, $\delta_C = 0$. The *only* uncertainty related information used in the random matrix approach are the values of δ_M and δ_K. The information regarding the location and natural frequencies of the attached oscillators are *not* used in the Wishart matrix approach. This is aimed to depict a realistic situation when the detailed information regarding uncertainties in a complex engineering system is not available to the analyst.

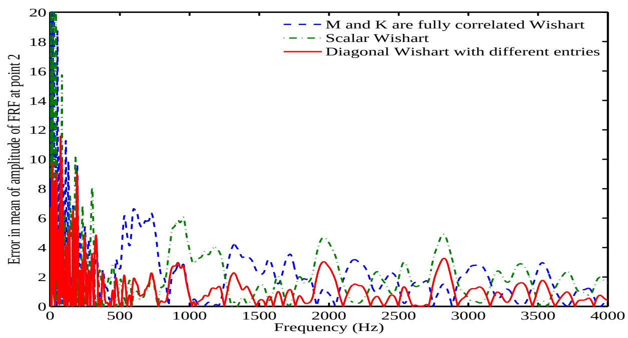
The predicted mean and standard deviation of the amplitude using the direct stochastic finite element simulation and three Wishart matrix approaches are compared in subfigure 3(a) and subfigure 3(b) for the driving-point-FRF. In subfigure 4(a) and subfigure 4(b) percentage errors in the



(a) Mean of amplitude (b) Standard deviation of amplitude

Figure 3: Comparison of the mean and standard deviation of the amplitude of the cross-FRF obtained using the direct stochastic finite element simulation and proposed three Wishart matrix approaches for the plate with randomly attached oscillators.

mean and standard deviation of the amplitude of the driving-point-FRF obtained using the proposed three Wishart matrix approaches are shown. Percentage errors are calculated using the direct Monte Carlo Simulation results as the benchmarks. Error using any one of the random matrix approaches



(a) Error in mean (b) Error in standard deviation

Figure 4: Comparison of percentage errors in the mean and standard deviation of the amplitude of the cross-FRF obtained using the proposed three Wishart matrix approaches for the plate with randomly attached oscillators.

reduce with the increase in the frequency. Among the three Wishart matrix approaches discussed here diagonal Wishart matrix with different entries produces most accurate results across the frequency range. Note that this approach only need the simulation of one random matrix. From these results we conclude that Wishart random matrix corresponding to approach 3 should be used for a system with parametric uncertainty. In the next section we discuss the same system with non-parametric uncertainty.

5 CONCLUSIONS

When uncertainties in the system parameters and modelling are considered, the discretized equation of motion of linear dynamical systems is characterized by random mass, stiffness and damping matrices. The possibility of using a single Wishart random matrix model for the system is investigated in the paper. The proposed Wishart random matrix models are applied to the forced vibration problem of a plate with stochastically inhomogeneous properties and randomly attached oscillators. For both cases it is possible to predict the variation of the dynamic response using the Wishart matrices across a wide range of driving frequency. The main conclusions are:

- The simulation of separate Wishart matrices corresponding to the mass and stiffness matrices may not be necessary. Instead, it is possible to use a generalized Wishart matrix to predict the response variability. This is not only computationally efficient, but can also give insights into the distribution of the eigenvalues of the system.

- Two new generalized Wishart matrix models, namely (a) generalized scalar Wishart distribution and (b) generalized diagonal Wishart distribution have been proposed and investigated. Model 'a' is very simple but inaccurate in the lower frequency ranges. Model 'b' turns out to be the most accurate across the frequency range.

- Considering that the available information is the mean and (normalized) standard deviation of mass and stiffness matrices, three different approaches are compared for a problem with parametric uncertainty and non-parametric uncertainty. It is shown that for model 'b', that is the generalized diagonal Wishart distribution, the calculated response statistics are in best agreements with the direct numerical simulation results.

- Numerical results show that the difference between three proposed approaches are more in the low frequency regions and less in the higher frequency regions.

These results suggest that a generalized diagonal Wishart distribution with suggested parameters may be used as a consistent and unified uncertainty quantification tool valid for medium and high frequency vibration problems.

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