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Response statistics of linear stochastic systems: A joint diagonalisation approach

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In stochastic finite element problems the solution of a system of coupled linear random algebraic equations is needed. This problem in turn requires the calculation of the inverse of a random matrix. Over the past four decades several approximate analytical methods and simulation methods have been proposed for the solution of this problem in the context of probabilistic structural mechanics. In this paper we present a new solution method for stochastic linear equations. The proposed method is based on Neumann expansion and the recently developed joint diagonalisation solution strategy. Numerical examples are given to illustrate the use of the expressions derived in the paper.

I. Introduction

In many stochastic mechanics problems we need to solve a system of linear stochastic equations

$$\mathbf{K}\mathbf{u} = \mathbf{f}. \tag{1}$$

Here $\mathbf{K} \in \mathbb{R}^{n\times n}$ is a $n \times n$ real non-negative definite random matrix, $\mathbf{f} \in \mathbb{R}^n$ is a n-dimensional real deterministic input vector and $\mathbf{u} \in \mathbb{R}^n$ is a n-dimensional real uncertain output vector which we want to determine. Equation (1) typically arises due to the discretisation of stochastic partial differential equations using the stochastic finite element method.[1–23] In the context of linear structural mechanics, $\mathbf{K}$ is known as the stiffness matrix, $\mathbf{f}$ is the force vector and $\mathbf{u}$ is the vector of structural displacements. The central aim of a stochastic structural analysis is to determine the probability density function (pdf) and consequently the cumulative distribution function (cdf) of $\mathbf{u}$. This will allow one to calculate the reliability of the system. It is often difficult to obtain the probability density function (pdf) of the response. As a consequence, engineers often intend to obtain only the first few moments of the response quantity.

In this paper we propose a new solution method for stochastic linear systems. The proposed method is based on Neumann expansion and the recently developed joint diagonalisation solution strategy.[23] The expressions derived in the paper are illustrated by numerical examples in Section V.

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II. Discretisation of stochastic material parameters

For heterogeneous materials such as rocks, soils and composite materials with random inclusions, their material properties vary irregularly through the medium and they are often termed random media in engineering. In a practical problem with the presence of heterogeneous materials, the random material properties are usually modeled by stochastic fields. In most cases, the stochastic material parameters are initially defined by lower order statistical moments, e.g. expectation and covariance functions. However, in a stochastic finite element formulation, an explicit discretisation of the stochastic material parameters is required.

In the past few decades, different numerical techniques have been developed for this task and they include the middle point method,[15,24] the local averaging method,[25] the shape function method,[16–18] the least-squares discretisation method[26] and the trigonometric series approximation method.[27,28] However, some of these are either inefficient (measured by the total number of random variables required) or inaccurate (measured by the error between the interpolated stochastic field and the real one) for an advanced SFEM formulation. To date, the most significant step forward for the discretisation of stochastic fields of material parameters is the application of Karhunen-Loève (K-L) expansions,[29] which was first introduced into SFEM research by Spanos and Ghanem,[2] and since then has been widely used in various SFEM formulations to discretise random material properties.

For stationary random media, the classic K-L expansion method using background mesh is recently extended by Li, Feng and Owen[30,31] and a Fourier-Karhunen-Loéve (F-K-L) representation method is proposed. Briefly speaking, a stationary stochastic field $H(\mathbf{x},\omega)$ representing material parameter (e.g. Young's module or Poisson's ratio) can be explicitly expressed as

$$H(\mathbf{x},\omega) = H_0(\mathbf{x}) + \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\xi_j(\omega)\phi_j(\mathbf{x}) \tag{2}$$

where $H_0(\mathbf{x})$ is the expectation function of $H(\mathbf{x},\omega)$, $\xi_j(\omega)$ are uncorrelated random variables and λ_j and $\phi_j(\mathbf{x})$ are the eigen-values and eigen-functions of the stochastic field. In particular, due to the stationarity of the stochastic field, the eigen-pairs λ_j and $\phi_j(\mathbf{x})$ in the F-K-L expansion scheme can be explicitly obtained via fast Fourier transform (FFT) without solving any equation. Numerical examples have shown that comparing to the K-L expansion method using a background mesh, both the eigen-values and the eigen-functions can be obtained to a very high accuracy in the F-K-L expansion scheme. However, this advantage is compromised by the limit use of the later scheme, i.e. only applicable to stationary stochastic fields, while the former one is applicable to any stochastic field with finite second-order statistics. An important issue in the discretisitation (2) is how to determine the probability distribution of random variables $\xi_j(\omega)$. For Gaussian stochastic fields, these random variables are orthonormal Gaussian random variables, but for non-Gaussian stochastic fields, the determination of these random variables usually requires the information of higher-order statistical moments.

III. Current methods for response-statistics calculation

After explicitly representing the random material parameters with discretised stochastic fields and discretising the unknown functions (e.g. the displacement field $\mathbf{u}$) with a finite element mesh, the stochastic linear system (1) can be readily constructed following a standard procedure of finite element formulation.[2,21] The solution of the random linear algebraic system (1) is fundamental to stochastic mechanics problems. As a result, a number of papers were written on this topic over the past four decades. The purpose of this section is to give a brief overview on some of the existing methods in order to put the proposed method into proper perspective. Readers are referred to the review papers[4,19–21,32,33] for further details. In a SFEM formulation, the stiffness matrix can always be represented as

$$\mathbf{K} = \mathbf{K}^0 + \mathbf{\Delta K} \tag{3}$$

where $\mathbf{K}^0 \in \mathbb{R}^{n\times n}$ is the deterministic part and $\mathbf{\Delta K} \in \mathbb{R}^{n\times n}$ is the random part. The random part is often expressed as

$$\mathbf{\Delta K} = \sum_{j=1}^{m} \xi_j \mathbf{K}_j^I + \sum_{j=1}^{m}\sum_{l=1}^{j} \xi_j \xi_l \mathbf{K}_{jl}^{II} + \cdots \tag{4}$$

where m is the number of random variables, $\mathbf{K}_j^I, \mathbf{K}_{jl}^{II} \in \mathbb{R}^{n\times n}$, $\forall\, j,l$ are deterministic matrices and ξ_j, $\forall\, j$ are real random variables. These random variables may be Gaussian and othonormalized in many problems. Under these settings, the following approaches have been employed to obtain the probabilistic descriptions of the response vector **u**.

A. Perturbation based approach

The perturbation method can be applied in various forms. Here we consider the expansion of the response vector as

$$\mathbf{u} = \mathbf{u}^0 + \sum_{j=1}^{m} \xi_j \mathbf{u}_j^I + \sum_{j=1}^{m}\sum_{l=1}^{j} \xi_j \xi_l \mathbf{u}_{jl}^{II} + \cdots . \tag{5}$$

The vectors $\mathbf{u}^0, \mathbf{u}_j^I, \mathbf{u}_{jl}^{II} \in \mathbb{R}^n$ need to be determined. In an alternative formulation, **u** can be viewed as a function of the vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_m)^T$ and can be expanded in a Taylor series about the mean of $\boldsymbol{\xi}$ or some other suitable point. The mathematical details to be outlined will be similar for both approaches. Substituting **K** and **u** from Eqs. (3) and (5) into the governing equation (1) and equating the corresponding coefficients associated with the random variables we have

$$\mathbf{K}^0\mathbf{u}^0 = \mathbf{f} \tag{6}$$

$$\mathbf{K}_j^I\mathbf{u}^0 + \mathbf{K}^0\mathbf{u}_j^I = \mathbf{0} \tag{7}$$

$$\text{and} \quad \mathbf{K}_{jl}^{II}\mathbf{u}^0 + \mathbf{K}_j^I\mathbf{u}_l^I + \mathbf{K}^0\mathbf{u}_{jl}^{II} = \mathbf{0}. \tag{8}$$

Solving these equations one has

$$\mathbf{u}^0 = \mathbf{K}^{0^{-1}}\mathbf{f} \tag{9}$$

$$\mathbf{u}_j^I = -\mathbf{K}^{0^{-1}}\mathbf{K}_j^I\mathbf{u}^0, \quad \forall\, j \tag{10}$$

$$\text{and} \quad \mathbf{u}_{jl}^{II} = -\mathbf{K}^{0^{-1}}[\mathbf{K}_{jl}^{II}\mathbf{u}^0 + \mathbf{K}_j^I\mathbf{u}_l^I + \mathbf{K}_l^I\mathbf{u}_j^I], \quad \forall\, j,l. \tag{11}$$

The above equations completely define the unknown vectors appearing in the perturbation expansion (5).

Another variant of the perturbation type approach is the so called Neumann expansion method.[15] Provided $\left\|\mathbf{K}^{0^{-1}}\mathbf{\Delta K}\right\|_{\mathrm{F}} < 1$, the inverse of the random matrix can be expanded in a binomial type of series as

$$\begin{aligned}\mathbf{K}^{-1} &= \left[\mathbf{K}^0(\mathbf{I}_n + \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K})\right]^{-1} \\ &= \mathbf{K}^{0^{-1}} - \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} + \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} + \cdots .\end{aligned} \tag{12}$$

From this expansion one has

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f} = \mathbf{u}^0 - \mathbf{T}\mathbf{u}_0 + \mathbf{T}^2\mathbf{u}^0 + \cdots \tag{13}$$

where $\mathbf{u}^0$ is as defined in Eq. (9) and $\mathbf{T} = \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K} \in \mathbb{R}^{n\times n}$ is a random matrix. An improved version of the perturbation method was proposed by Elishakoff et al.[34] Later Adhikari and Manohar[7] extended the Neumann expansion method to complex symmetric random matrices arising in dynamic problems.

The first and second-order statistics of $\mathbf{u}$ can be calculated from Eq. (5) or (13). The following general points may be noted for all perturbation based approaches

- If the random variations are large, the higher-order terms may not be negligibly small.
- The calculation of response statistics become difficult if the elements of $\boldsymbol{\Delta}\mathbf{K}$ are non-Gaussian random variables.
- Even if the elements of $\boldsymbol{\Delta}\mathbf{K}$ are Gaussian random variables, the inclusion of higher-order terms (more than second-order) results in very messy calculations.

For the above reasons, other methods have been proposed in literature.

B. Projection methods

The perturbation based solutions can be viewed as a local approximation around the deterministic solution. The subspace projection schemes for stochastic finite element analysis on the other hand achieve global representation. The basic idea is simple, powerful and is based on solid theoretical foundations.[20,21] Here one 'projects' the solution vector onto a complete stochastic basis. Depending on how the basis is selected, several methods are proposed. Using the Polynomial Chaos (PC) projection scheme[2] one can express the solution vector as

$$\mathbf{u} = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{14}$$

where $\mathbf{u}_j \in \mathbb{R}^n$, $\forall j$ are unknown vectors and the polynomial chaoses $\Psi_j(\boldsymbol{\xi})$ are multidimensional Hermite polynomials with respect to $\boldsymbol{\xi}$. Instead of choosing Hermite polynomials, Xiu and Karniadakis[22,35] used a generalized orthogonal basis which utilizes functions from the Askey family of hypergeometric polynomials. In the context of dynamic problems, Adhikari and Manohar[7] proposed the random eigenfunction expansion method where the random basis is selected to be the complete set of eigenfunctions of the undamped dynamic stiffness matrix. The number of terms in expansion (14) is given by[2,23]

$$P = \frac{(m+r)!}{m!r!} \tag{15}$$

where r is the order of the PC expansion. Substituting $\mathbf{u}$ from Eq. (14) into Eq. (1) and imposing the Galerkin condition, the unknown $\mathbf{u}_j$ vectors can be obtained by solving a set of nP dimensional deterministic linear algebraic equations. To reduce the computational effort, Nair and Keane[11] proposed the stochastic reduced basis method. Here the solution vector is represented using basis vectors spanning the preconditioned stochastic Krylov subspace and consequently the application of the Galerkin scheme leads to a reduced-order deterministic system of equations. A comparison of different PC projection schemes for stochastic finite element analysis is given by Sachdeva et. al.[13] Recently Sarkar et. al[36,37] have proposed a domain decomposition approach to obtain $\mathbf{u}_j$ which utilizes PC expansion and can be implemented in parallel to solve large scale problems. Once all $\mathbf{u}_j$ are known, the statistics of $\mathbf{u}$ can be obtained from (14) in a relatively straight-forward manner.

C. Monte Carlo simulation and other methods

Table 1. A partial summary of the solution techniques for coupled linear algebraic equations arising in stochastic mechanics problems.

	Methods	Sub-methods
1.	Perturbation based methods	First and second order perturbation,[1,3,16]
		Neumann expansion,[7,15]
		Improved perturbation method.[34]
2.	Projection methods	Polynomial chaos expansion,[2]
		Random eigenfunction expansion,[7]
		Stochastic reduced basis method,[11,13,14]
		Wiener−Askey chaos expansion,[22,35,38]
		Domain decomposition method.[36,37]
3.	Monte carlo simulation and other methods	Simulation methods,[39,40]
		Analytical method in references,[41–45]
		Exact solutions for beams.[12,46]

Once the statistical properties of the random matrix $\mathbf{K}$ is known, the samples can be generated using Monte Carlo simulation and the moments and pdf of $\mathbf{u}$ can be obtained by standard statistical methods. Several authors have also proposed other methods to obtain statistical properties of $\mathbf{u}$ and a partial summary of the solution techniques for coupled linear algebraic equations arising in stochastic mechanics problems is given in table 1.

IV. Joint diagonalisation approach

For the solution of the stochastic linear system (1), a novel solution strategy, namely the joint diagonalisation method, was recently proposed by Li, Feng and Owen.[23] In this approach, Eq. (1)

is first reorganized as follows

$$(\mathbf{K}_0 + \xi_1(\omega)\mathbf{K}_1 + \xi_2(\omega)\mathbf{K}_2 + \cdots + \xi_m(\omega)\mathbf{K}_m)\mathbf{u} = \mathbf{f}, \tag{16}$$

where $\mathbf{K}_j,\ \forall\, j$ are real symmetric deterministic matrices and $\xi_j(\omega),\ \forall\, j$ are real random variables. The above stochastic linear system is commonly obtained in a SFEM formulation after discretising the random material parameters with the K-L (or F-K-L) expansion method.

It is shown by Li et al.[23] that by using a sequence of Givens transformations, stiffness matrices $\mathbf{K}_j$ in Eq. (16) can be simultaneously diagonalised such that

$$\mathbf{Q}^{-1}\mathbf{K}_j\mathbf{Q} = \mathbf{\Lambda}_j + \mathbf{\Delta}_j \approx \mathbf{\Lambda}_j, \quad j = 1, 2, \cdots, m \tag{17}$$

where $\mathbf{\Lambda}_j,\ \forall\, j$ are diagonal matrices and $\mathbf{\Delta}_j,\ \forall\, j$ are matrices with zero diagonal entries and small off-diagonal entries. The transform matrix $\mathbf{Q}$ is explicitly obtained as the product of the Givens rotation matrices

$$\mathbf{Q} = \mathbf{G}_1^T\mathbf{G}_2^T \cdots \mathbf{G}_k^T \tag{18}$$

where k is the total number of Givens transformations and the Givens rotation matrices $\mathbf{G}_j,\ \forall\, j$ take the following form

$$\mathbf{G} = \mathbf{G}(p, q, \theta) \triangleq \begin{array}{cc} & \begin{array}{ccccccccccc} & & & p & & & & q & & & \end{array} \\ \begin{array}{c} \\ \\ \\ p \\ \\ \\ \\ q \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\theta & & & & \sin\theta & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & -\sin\theta & & & & \cos\theta & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix} \end{array}. \tag{19}$$

The joint diagonalisation works in an iteration manner as shown in Eq. (18) and the algorithm is implemented to scan in turn all the off-diagonal entries (p, q), $p \neq q$ in the matrix system. In each iteration, a set of matrix elements $(\mathbf{K}_j)_{pq}$, $j = 1, \cdots, m$ are selected and to minimize the corresponding off-diagonal entries the Givens rotation matrix (19) is constructed. As matrices $\mathbf{K}_j,\ \forall\, j$ are real symmetric, the optimal Givens rotation angle θ_{opt} that in the current iteration maximizes the diagonal entries and minimizes the off-diagonal entries can be accurately obtained by solving the following characteristic equation

$$(\cos 2\theta_{opt} \quad \sin 2\theta_{opt})^T\, \mathbf{J} = \lambda_J\, (\cos 2\theta_{opt} \quad \sin 2\theta_{opt})^T \tag{20}$$

where $\mathbf{J}$ is a two by two matrix given by

$$\mathbf{J} = \sum_{j=1}^{m} \begin{pmatrix} 2\,(\mathbf{K}_j)_{pq}^2 & (\mathbf{K}_j)_{pq}\left((\mathbf{K}_j)_{qq} - (\mathbf{K}_j)_{pp}\right) \\ (\mathbf{K}_j)_{pq}\left((\mathbf{K}_j)_{qq} - (\mathbf{K}_j)_{pp}\right) & \frac{1}{2}\left((\mathbf{K}_j)_{qq} - (\mathbf{K}_j)_{pp}\right)^2 \end{pmatrix} \tag{21}$$

and λ_J is the smallest eigen-value of $\mathbf{J}$.

In the original joint diagonalisation solution,[23] the off-diagonal entries $\mathbf{\Delta}_j, \forall j$ in Eq. (17) are completely ignored and this significantly simplifies the random equation system which in turn leads to an explicit solution

$$\mathbf{u} \approx \mathbf{Q}(\mathbf{\Lambda}_0 + \xi_1(\omega)\mathbf{\Lambda}_1 + \cdots + \xi_m(\omega)\mathbf{\Lambda}_m)^{-1}\mathbf{Q}^{-1}\mathbf{f}. \tag{22}$$

The above solution relies on the assumption that the off-diagonal matrices $\mathbf{\Delta}_j, \forall j$ are negligible comparing to the diagonal matrices $\mathbf{\Lambda}_j, \forall j$. However, for larger matrices $\mathbf{K}_j, \forall j$, to reduce matrices $\mathbf{\Delta}_j, \forall j$ to a negligible level is extremely time consuming and with limited computational power available, the accuracy of the solution (22) decreases as the dimension of matrices increases. In order to reduce the computational load of joint diagonalisation and overcome the problem of inefficiency, the contribution from the off-diagonal matrices $\mathbf{\Delta}_j, \forall j$ must be taken into account in the solution. Substituting Eq. (17) into Eq. (16) yields

$$\mathbf{Q}\left[(\mathbf{\Lambda}_0 + \sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Lambda}_j) + (\mathbf{\Delta}_0 + \sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Delta}_j)\right]\mathbf{Q}^{-1}\mathbf{u} = \mathbf{f}. \tag{23}$$

Let

$$\begin{aligned} \mathbf{V} &= \mathbf{\Lambda}_0 + \sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Lambda}_j \\ \mathbf{A} &= \mathbf{\Delta}_0 + \sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Delta}_i, \end{aligned} \tag{24}$$

the solution of (23) can be expressed as

$$\mathbf{u} = \mathbf{Q}\left[\mathbf{V}(\mathbf{I}_n + \mathbf{V}^{-1}\mathbf{A})\right]^{-1}\mathbf{Q}^{-1}\mathbf{f}. \tag{25}$$

Noting that matrix $\mathbf{V}$ is a diagonal matrix whose inverse can be explicitly obtained, the above expression can be further simplified by using the Neumann expansion as

$$\mathbf{u} = \mathbf{Q}\left[\mathbf{V}^{-1} - (\mathbf{V}^{-1}\mathbf{A})\mathbf{V}^{-1} + (\mathbf{V}^{-1}\mathbf{A})^2\mathbf{V}^{-1} - \cdots\right]\mathbf{Q}^{-1}\mathbf{f}. \tag{26}$$

Comparing to the original joint diagonalisation solution (22), the above solution can be obtained more efficiently and without comprising the accuracy.

V. Numerical examples

A simple elastic plain-stress problem is computed to demonstrate the performance of the proposed solution method. It is assumed that the Young's modulus is modeled by a Gaussian stochastic field and Passion's ratio takes a deterministic value. The Gaussian stochastic field is discretised by using the F-K-L representation scheme.[30,31] Figure 1 shows two examples of the eigen-functions obtained in the F-K-L scheme and Figure 2 shows a specific realization of the Gaussian field reconstructed from the F-K-L scheme.

After the discretisation of the random material property, a stochastic finite element formulation can be readily constructed and the resulting equation system is a stochastic linear system. The proposed joint diagonalisation method is then used to solve the stochastic system. The result of the principal stresses are shown in Figure 3.

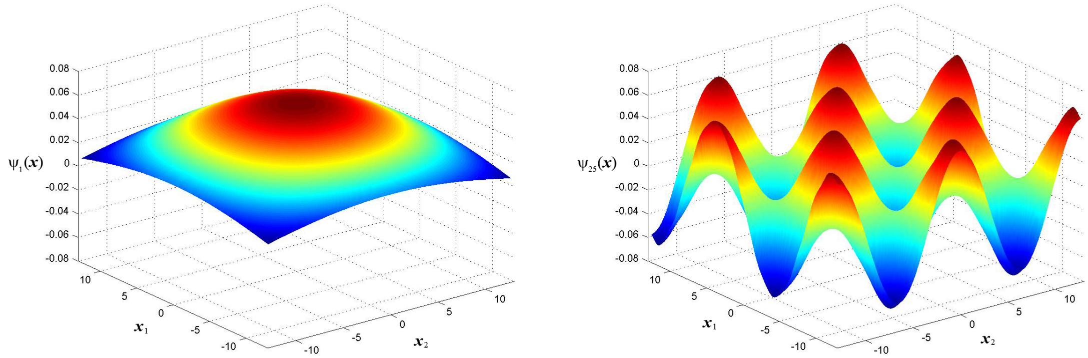

Figure 1. Eigen-functions of the stochastic material field (a) the first eigen-function, (b) the 25th eigen-function

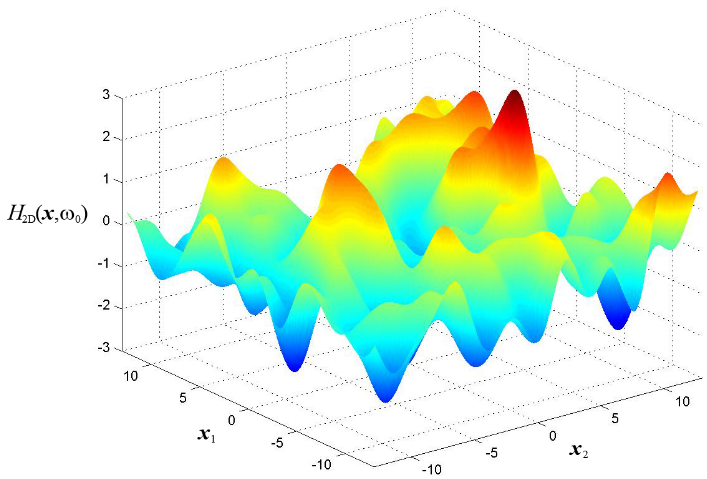

Figure 2. A specific realization of the stochastic material field

VI. Conclusion

For the solution of static and steady-state problems of random media, this paper presents an improved joint diagonalisation solution framework. Firstly, the random medial properties are discretised by using the Fourier-Karhunen-Loève expansion scheme. Then the resulting stochastic linear system is solved by using the improved joint diagonalization method. A demo numerical example is also included to show the performance of the proposed method.

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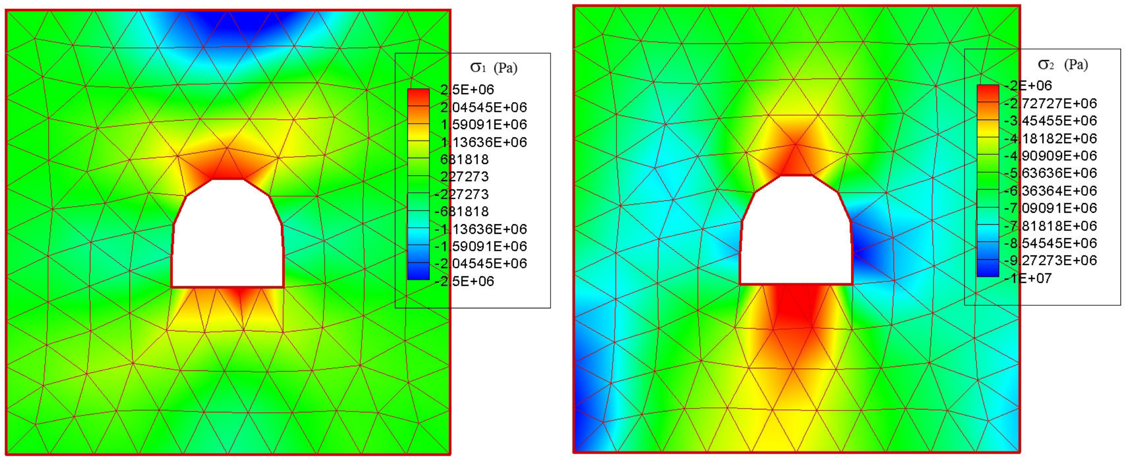

Figure 3. A specific realization of principal stresses

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References

[1]Shinozuka, M. and Yamazaki, F., "Stochastic finite element analysis: an introduction," *Stochastic structural dynamics: Progress in theory and applications*, edited by S. T. Ariaratnam, G. I. Schuëller, and I. Elishakoff, Elsevier Applied Science, London, 1998.

[2]Ghanem, R. and Spanos, P., *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, New York, USA, 1991.

[3]Kleiber, M. and Hien, T. D., *The Stochastic Finite Element Method*, John Wiley, Chichester, 1992.

[4]Matthies, H. G., Brenner, C. E., Bucher, C. G., and Soares, C. G., "Uncertainties in probabilistic numerical analysis of structures and solids - Stochastic finite elements," *Structural Safety*, Vol. 19, No. 3, 1997, pp. 283–336.

[5]Manohar, C. S. and Adhikari, S., "Dynamic stiffness of randomly parametered beams," *Probabilistic Engineering Mechanics*, Vol. 13, No. 1, January 1998, pp. 39–51.

[6]Manohar, C. S. and Adhikari, S., "Statistical analysis of vibration energy flow in randomly parametered trusses," *Journal of Sound and Vibration*, Vol. 217, No. 1, 1998, pp. 43–74.

[7]Adhikari, S. and Manohar, C. S., "Dynamic analysis of framed structures with statistical uncertainties," *International Journal for Numerical Methods in Engineering*, Vol. 44, No. 8, 1999, pp. 1157–1178.

[8]Adhikari, S. and Manohar, C. S., "Transient dynamics of stochastically parametered beams," *ASCE Journal of Engineering Mechanics*, Vol. 126, No. 11, November 2000, pp. 1131–1140.

[9]Haldar, A. and Mahadevan, S., *Reliability Assessment Using Stochastic Finite Element Analysis*, John Wiley and Sons, New York, USA, 2000.

[10]Sudret, B. and Der-Kiureghian, A., "Stochastic Finite Element Methods and Reliability," Tech. Rep. UCB/SEMM-2000/08, Department of Civil & Environmental Engineering, University Of California, Berkeley, November 2000.

[11]Nair, P. B. and Keane, A. J., "Stochastic reduced basis methods," *AIAA Journal*, Vol. 40, No. 8, 2002, pp. 1653–1664.

[12]Elishakoff, I. and Ren, Y. J., *Large Variation Finite Element Method for Stochastic Problems*, Oxford University Press, Oxford, U.K., 2003.

[13]Sachdeva, S. K., Nair, P. B., and Keane, A. J., "Comparative study of projection schemes for stochastic finite element analysis," *Computer Methods in Applied Mechanics and Engineering*, Vol. 195, No. 19-22, 2006, pp. 2371–2392.

[14]Sachdeva, S. K., Nair, P. B., and Keane, A. J., "Hybridization of stochastic reduced basis methods with polynomial chaos expansions," *Probabilistic Engineering Mechanics*, Vol. 21, No. 2, 2006, pp. 182–192.

[15]Yamazaki, F., Shinozuka, M., and Dasgupta, G., "Neumann expansion for stochastic finite element analysis," *Journal of Engineering Mechanics-ASCE*, Vol. 114, No. 8, 1988, pp. 1335–1354.

[16]Liu, W. K., Belytschko, T., and Mani, A., "Random field finite-elements," *International Journal for Numerical Methods in Engineering*, Vol. 23, No. 10, 1986, pp. 1831–1845.

[17]Kaminski, M. and Kleiber, M., "Stochastic structural interface defects in fiber composites," *International Journal of Solids and Structures*, Vol. 33, No. 20-22, 1996, pp. 3035–3056.

[18]Hien, T. D. and Kleiber, M., "Stochastic finite element modelling in linear transient heat transfer," *Computer Methods in Applied Mechanics and Engineering*, Vol. 144, No. 1-2, 1997, pp. 111–124.

[19]Matthies, H. G. and Bucher, C., "Finite elements for stochastic media problems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 168, No. 1-4, 1999, pp. 3–17.

[20]Keese, A., "A review of recent developments in the numerical solution of stochastic PDEs (stochastic finite elements)," Tech. Rep. Informatikbericht 2003-6, Institute of Scientific Computing, Department of Mathematics and Computer Science, Technische Universitat Braunschweig, Brunswick,, June 2003, URL: http://opus.tu-bs.de/opus/volltexte/2003/504/.

[21]Matthies, H. G. and Keese, A., "Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations," *Computer Methods in Applied Mechanics and Engineering*, Vol. 194, No. 12-16, 2005, pp. 1295–1331.

[22]Xiu, D. B. and Karniadakis, G. E., "The Wiener-Askey polynomial chaos for stochastic differential equations," *Siam Journal on Scientific Computing*, Vol. 24, No. 2, 2002, pp. 619–644.

[23]Li, C. F., Feng, Y. T., and Owen, D. R. J., "Explicit solution to the stochastic system of linear algebraic equations $(\alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \ldots + \alpha_m\mathbf{A}_m)\mathbf{x} = \mathbf{b}$," *Computer Methods in Applied Mechanics and Engineering*, Vol. 195, No. 44-47, 2006, pp. 6560–6576.

[24]Shinozuka, M., "Monte Carlo solution of structural dynamics," *Computers & Structures*, Vol. 2, No. 5-6, 1972, pp. 855–874.

[25]Vanmarcke, E. H. and Grigoriu, M., "Stochastic finite element analysis of simple beams," *Journal of Engineering Mechanics-ASCE*, Vol. 109, No. 5, 1983, pp. 1203–1214.

[26]Li, C. C. and Derkiureghian, A., "Optimal discretization of random fields," *Journal of Engineering Mechanics-ASCE*, Vol. 119, No. 6, 1993, pp. 1136–1154.

[27]Shinozuka, M. and Deodatis, G., "Simulation of stochastic processes by spectral representation," *Applied Mechanics Reviews-ASME*, Vol. 44, No. 4, 1991, pp. 191–204.

[28]Grigoriu, M., *Stochastic Calculus: Applications in Science and Engineering*, Birkhauser, New York, USA, 2002.

[29]Loève, M., *Probability Theory, 4th Edition*, Vol. 2, Springer-Verlag, Berlin, Germany, 1977.

[30]Li, C. F., Feng, Y. T., and Owen, D. R. J., "A Fourier-Karhunen-Loève discretization scheme for stationary random material properties in SFEM," *International Journal for Numerical Methods in Engineering*, Vol. 73, No. 13, 2008, pp. 1942–1965.

[31]Li, C. F., Feng, Y. T., Owen, D. R. J., and Davies, I. M., "Fourier representation of random media fields in stochastic finite element modelling," *Engineering Computations*, Vol. 23, No. 7-8, 2006, pp. 794–817.

[32]Manohar, C. S. and Ibrahim, R. A., "Progress in structural dynamics with stochastic parameter variations: 1987 to 1998," *Applied Mechanics Reviews, ASME*, Vol. 52, No. 5, May 1999, pp. 177–197.

[33]Manohar, C. S. and Gupta, S., "Modeling and evaluation of structural reliability: Current status and future directions," *Research reviews in structural engineering, Golden Jubilee Publications of Department of Civil Engineering, Indian Institute of Science, Bangalore*, edited by K. S. Jagadish and R. N. Iyengar, University Press, 2003.

[34]Elishakoff, I., Ren, Y. J., and Shinozuka, M., "Improved Finite-Element Method for Stochastic Problems," *Chaos Solitons & Fractals*, Vol. 5, No. 5, 1995, pp. 833–846.

[35]Xiu, D. B. and Karniadakis, G. E., "Modeling uncertainty in flow simulations via generalized polynomial chaos," *Journal of Computational Physics*, Vol. 187, No. 1, 2003, pp. 137–167.

[36]Sarkar, A., Benabbou, N., and Ghanem, R., "Domain decomposition of stochastic systems," *47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, Newport, Rhode Island, USA, May 2006.

[37]Sarkar, A., Benabbou, N., and Ghanem, R., "Stochastic PDES and its parallel implementation," *7th World Congress on Computational Mechanics*, Los Angeles, CA, USA, 2006.

[38]Wan, X. L. and Karniadakis, G. E., "Beyond wiener-askey expansions: Handling arbitrary pdfs," *Journal of Scientific Computing*, Vol. 27, No. (-3, 2006, pp. 455–464.

[39]Hurtado, J. E. and Barbat, A. H., "Monte Carlo techniques in computational stochastic mechanics," *Archives of Computational Methods in Engineering*, Vol. 5, No. 1, 1998, pp. 3–29.

[40]Papadrakakis, M. and Papadopoulos, V., "Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation," *Computer Methods in Applied Mechanics and Engineering*, Vol. 134, No. 3-4, 1996, pp. 325–340.

[41]Muscolino, G., Ricciardi, G., and Impollonia, N., "Improved dynamic analysis of structures with mechanical uncertainties under deterministic input," *Probabilistic Engineering Mechanics*, Vol. 15, No. 2, 2000, pp. 199–212.

[42]Impollonia, N. and Muscolino, G., "Static and dynamic analysis of non-linear uncertain structures," *Meccanica*, Vol. 37, No. 1-2, 2002, pp. 179–192.

[43]Falsone, G. and Impollonia, N., "A new approach for the stochastic analysis of finite element modelled structures with uncertain parameters," *Computer Methods in Applied Mechanics and Engineering*, Vol. 191, No. 44, 2002, pp. 5067–5085.

[44]Falsone, G. and Impollonia, N., "A new approach for the stochastic analysis of finite element modelled structures with uncertain parameters (vol 191, pg 5067, 2002)," *Computer Methods in Applied Mechanics and Engineering*, Vol. 192, No. 16-18, 2003, pp. 2187–2188.

[45]Impollonia, N. and Ricciardi, G., "Explicit solutions in the stochastic dynamics of structural systems," *Probabilistic Engineering Mechanics*, Vol. 21, No. 2, 2006, pp. 171–181.

[46]Elishakoff, I., Ren, Y. J., and Shinozuka, M., "Some Exact-Solutions for the Bending of Beams with Spatially Stochastic Stiffness," *International Journal of Solids and Structures*, Vol. 32, No. 16, 1995, pp. 2315–2327.