An Efficient Computational Solution Scheme of the Random Eigenvalue Problems

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This paper presents a practical solution for probabilistic characterization of real valued eigenvalues of positive semi-definite random matrices. The present method is founded on the concept of high dimensional model representation (HDMR) technique. The method involves HDMR that facilitates lower dimensional approximation of the eigenvalues, response surface generation of HDMR component functions, and efficient Monte Carlo simulation for probability density functions. HDMR is a general set of quantitative model assessment and analysis tools for capturing the high-dimensional relationships between sets of input and output model variables. It is a very efficient formulation of the system response, if higher-order variable correlations are weak, allowing the physical model to be captured by the first few lower-order terms. Results of two numerical examples indicate that the proposed method provides accurate and computationally efficiency. Compared with commonly-used perturbation and recently-developed asymptotic methods, no derivatives of eigenvalues are required in the present method.

Nomenclature

$\boldsymbol{x}$	=	N-dimensional random variables
$\bar{\boldsymbol{x}}$	=	reference points for HDMR expansion
μ_x	=	mean of random variable
σ_x	=	standard deviation of random variable
$E(\bullet)$	=	expectation operator
Σ_x	=	covariance of random variables
$\boldsymbol{k}(\bullet)$	=	stiffness matrix
$\boldsymbol{m}(\bullet)$	=	mass matrix
$\Lambda(\bullet)$	=	eigenvalues
$\Phi(\bullet)$	=	eigenvectors
$\phi_j(\bullet)$	=	MLS interpolation function
n	=	sample points in HDMR expansion
s	=	order of component function in HDMR expansion
l	=	order of HDMR expansion
N_s	=	simulation size in direct MCS

I. Introduction

RANDOM matrix theory is concerned with determining probabilistic characteristics of eigenvalues and eigenvectors of matrices defined by a statistical distribution. The study of probabilistic characterization of the

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eigenfunctions of random matrix and differential operators has emerged as an important research topic in the field of stochastic structural mechanics. In particular, several studies have been conducted on both self adjoint and non-self adjoint eigenvalue problems. Other issues are multiplicity of eigenvalues and the related problems.

Characterization of the natural frequencies and mode-shapes play an important descriptor of the dynamics and stability of structural systems. The determination of natural frequency and mode shapes require the solution of an eigenvalue problem. Eigenvalue problems also arise in the context of the stability analysis of structures. Description of real-life engineering structural systems is inevitably associated with some amount of uncertainty in specifying material properties, geometric parameters, boundary conditions and applied loads. Taking into account of these uncertainties, it is necessary to consider random eigenvalue problems.

The majority of recent studies employed the mean centered first- and second-order perturbation approach to estimate the first- and the second-order statistics of eigenvalues and mode shapes [1–3], the iteration method [1], the Ritz method [4], the crossing theory [5], the stochastic reduced basis [6], the subspace iteration [6], and the asymptotic method [7]. Among these methods, perturbation methods have dominated the current literature. A systematic account of perturbation approaches to random eigenvalue problems is well documented in a research monograph by Scheidt and Purkert [8].

Perturbation approach involves first- or second-order Taylor series expansions of the eigenvalue or eigenvector in terms of basic input random parameters. However, the perturbation methods require expensive calculation of derivatives of eigensvalues[9], and more importantly, are limited to small non-linearity of input–output mapping or small input uncertainties. But such limitations are rarely satisfied in engineering applications. On the other hand, simulation methods could be applied to solve any random eigenvalue problem. But the main disadvantage is that, simulation methods require tremendous computational effort due to large number of deterministic structural analysis for different realizations of the random variables. Consequently, the simulation method is useful only when alternative methods are inapplicable or inaccurate, and has been traditionally employed as a yardstick for evaluating approximate methods. Hence, new stochastic methods for characterizing eigenvalue problems that can manage highly nonlinear input-output mapping and arbitrarily large uncertainties of input are highly desirable.

This paper presents a novel computational method for predicting probability density functions of eigenvalues of real, positive semi-definite, stochastic matrices. The method involves HDMR technique in conjunction with moving least squares (MLS) technique that facilitates a multivariate eigenfunction with multiple low dimensional eigenfunctions.

II. Random Eigenvalue Problem

Let the *N*–dimensional vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$, represent the input random variables with mean vector $\boldsymbol{\mu}_x \equiv E(\boldsymbol{x})$ and covariance matrix $\Sigma_x \equiv E\left[(\boldsymbol{x} - \mu_x)(\boldsymbol{x} - \mu_x)^T\right]$. For undamped or proportionally damped systems, let $\boldsymbol{k}(\boldsymbol{x})$ and $\boldsymbol{m}(\boldsymbol{x})$ denote real valued, symmetric, positive semi-definite random stiffness and mass matrices, respectively. The generalized linear random eigenvalue problem for a random matrix pair $\boldsymbol{k}(\boldsymbol{x})$ and $\boldsymbol{m}(\boldsymbol{x})$ is defined as

$$\boldsymbol{k}(\boldsymbol{x})\Phi(\boldsymbol{x}) = \Lambda(\boldsymbol{x})\boldsymbol{m}(\boldsymbol{x})\Phi(\boldsymbol{x}) \tag{1}$$

where $\left[\Lambda^i(\boldsymbol{x}), \Phi^i(\boldsymbol{x})\right]; i = 1, 2, \ldots, l$ is the *i*-th eigenpair that includes random eigenvalues $\Lambda^i(\boldsymbol{x})$ and eigenvector $\Phi^i(\boldsymbol{x})$. Based on the characteristics of random matrix pair $\boldsymbol{k}(\boldsymbol{x})$ and $\boldsymbol{m}(\boldsymbol{x})$, Equation 1 yields real eigenvalues $\Lambda^i(\boldsymbol{x}); i = 1, 2, \ldots, l$, where $\Lambda(\boldsymbol{x})$ depends on random input $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$ through the solution of characteristics equation

$$\det\left[\boldsymbol{k}(\boldsymbol{x}) - \Lambda(\boldsymbol{x})\boldsymbol{m}(\boldsymbol{x})\right] = 0 \tag{2}$$

A major objective in solving a random eigenvalue problem is to find probabilistic characteristics of eigenpair $\left[\Lambda^i(\boldsymbol{x}), \Phi^i(\boldsymbol{x})\right]; i = 1, 2, \ldots, l$.

III. Concept of HDMR

Let the *N*–dimensional vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$, represent the input random variables of the model under consideration, and real-valued eigenvalue as $\lambda(\boldsymbol{x})$. Since the influence of the input variables on the response

variable can be independent and/or cooperative, HDMR[10–13] expresses the response $\lambda(\boldsymbol{x})$ as a hierarchical correlated function expansion in terms of the input variables as

$$\lambda(\boldsymbol{x}) = \lambda_0 + \sum_{i=1}^{N} \lambda_i(x_i) + \sum_{1 \le i_1 < i_2 \le N} \lambda_{i_1 i_2}(x_{i_1}, x_{i_2}) + \ldots + \sum_{1 \le i_1 < \ldots < i_l \le N} \lambda_{i_1 i_2 \ldots i_l}(x_{i_1}, x_{i_2}, \ldots, x_{i_l}) + \ldots + \lambda_{12\ldots N}(x_1, x_2, \ldots, x_N) \tag{3}$$

where λ_0 is a constant term representing the zeroth-order component function. The function $\lambda_i(x_i)$ is a first-order term expressing the effect of variable x_i acting alone. The function $\lambda_{i_1 i_2}(x_{i_1}, x_{i_2})$ is a second-order term which describes the cooperative effects of the variables x_i and x_j upon the output $\lambda(\boldsymbol{x})$ and so on. Once all the relevant component functions in Equation 3 are determined and suitably represented, then the component functions constitute HDMR, thereby replacing the original computationally expensive method of calculating $\lambda(\boldsymbol{x})$ by the computationally efficient model.

To develop the response surface, first a reference point $\bar{\boldsymbol{x}} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N\}$ is defined in the variable space. This process reduces to the following relationship for the component functions in Equation 3

$$\lambda_0 = \lambda(\bar{\boldsymbol{x}}) \tag{4}$$

$$\lambda_i(x_i) = \lambda(x_i, \bar{\boldsymbol{x}}^i) - \lambda_0 \tag{5}$$

where the notation $\lambda(x_i, \bar{\boldsymbol{x}}^i) = \lambda(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N)$ denotes that all the input variables are at their reference point values except x_i.

The notion of 0th, 1st order, etc. in the HDMR expansion should not be confused with the terminology used either in the Taylor series or in the conventional least-squares based response surface model. It can be shown that[14, 15], the first order component function $\lambda_i(x_i)$ is the sum of all the Taylor series terms which contain and only contain variable x_i. Hence first-order HDMR approximations should not be viewed as first-order Taylor series expansions nor do they limit the nonlinearity of $\lambda(\boldsymbol{x})$. Therefore, provide in general higher-order representation of eigenvalues than those by commonly employed first- or second-order perturbation methods.

IV. Marginal Densities of Eigenvalues

HDMR in Equation 3 is exact along any of the cuts, and the output response $\lambda(\boldsymbol{x})$ at a point $\boldsymbol{x}$ off of the cuts can be obtained by following the procedure in step 1 and step 2 below:

Step 1: Interpolate each of the low dimensional HDMR expansion terms with respect to the input values of the point $\boldsymbol{x}$. If for $x_i = x_i^j$, n function values $\lambda(x_i^j, \bar{\boldsymbol{x}}^i) = \lambda(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N)$; $j = 1, 2, \ldots, n$, are given at n $(= 3, 5, 7 \text{ or } 9)$ regularly spaced sample points $\mu_i - (n-1)\sigma_i/2$, $\mu_i - (n-3)\sigma_i/2$, …, μ_i, …, $\mu_i + (n-3)\sigma_i/2$, $\mu_i + (n-1)\sigma_i/2$ along the variable axis x_i with mean μ_i and standard deviation σ_i, the function value for arbitrary x_i can be obtained using MLS[16] interpolation as

$$\lambda(x_i, \bar{\boldsymbol{x}}^i) = \sum_{j=1}^{n} \phi_j(x_i)\ \lambda(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N) \tag{6}$$

where $\phi_j(x_i)$ is the interpolation function obtained using the MLS interpolation. By using Equation 6, $\lambda_i(x_i)$ can be generated if n function values are given at corresponding sample points. The same procedure shall be repeated for all the first-order component functions (as defined in Equation 5), i.e., $\lambda_i(x_i)$; $i = 1, 2, \ldots, N$.

Step 2: Sum the interpolated values of HDMR expansion terms from zeroth-order to the highest order retained in keeping with the desired accuracy. This leads to first-order approximation of the function $\lambda(\boldsymbol{x})$ as

$$\tilde{\lambda}(\boldsymbol{x}) = \sum_{i=1}^{N} \sum_{j=1}^{n} \phi_j(x_i)\, \lambda(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N) - (N-1)\lambda_0 \tag{7}$$

If n is the number of sample points taken along each of the variable axis and s is the order of the component function considered, starting from zeroth-order to l-th order, then total number of function evaluation for interpolation purpose is given by, $\sum_{s=0}^{l}\left(N!(n-1)^s\right)\Big/\left((N-s)!s!\right)$ which grows polynomially with n and s. As a few low order component functions of HDMR are used, the sample savings due to HDMR are significant compared to traditional sampling. Hence uncertainty analysis using HDMR relies on an accurate reduced model being generated with a small number of full model simulations. An arbitrarily large sample Monte Carlo analysis can be performed on the outputs approximated by HDMR which result in the same distributions as obtained through the Monte Carlo analysis of the full model, if higher order cooperative effects are negligible. The tremendous computational savings result from just having to perform interpolation instead of full model simulations for output determination.

V. Simulation

Equations 7 provides first-order HDMR approximation $\tilde{\lambda}(\boldsymbol{x})$ of the original random eigenvalues $\lambda(\boldsymbol{x})$ using the MLS interpolation functions, constant $\lambda(\bar{\boldsymbol{x}})$ term, first-order $\lambda\left(\bar{x}_1,\ldots,\bar{x}_{i-1},x_i^j,\bar{x}_{i+1},\ldots,\bar{x}_N\right)$ terms. Therefore, any probabilistic characteristics of eigenvalues, including the marginal or the joint probability density function of $\lambda(\boldsymbol{x})$ can be easily estimated by performing Monte Carlo simulation (MCS) on first-order approximation $\tilde{\lambda}(\boldsymbol{x})$ of the original random eigenvalues $\lambda(\boldsymbol{x})$.

Since first-order HDMR approximation leads to explicit representation of the original matrix characteristic equation, the MCS can be conducted for any sampling size. The total cost of original function evaluation entails a maximum of $(n-1)\times N+1$ by the present method using first-order HDMR approximation. It is worth noting that moments of eigenvalues can also be determined from samples generated using Equations 7.

VI. Numerical Example

Two numerical examples involving linear dynamics of spring-mass systems and a ten bar truss structure are presented to illustrate the performance of the present method. Whenever possible, comparisons have been made with the direct MCS to evaluate the accuracy and efficiency of the proposed method. The Gaussian assumption of random input is adopted mainly for comparing the proposed approach with existing methods that employ Gaussian input with readily available results. However, present method does not limit to the Gaussian assumption only. For the first example, the eigenvalues are calculated by an IMSL subroutine, which employs a hybrid double-shifted LR-QR algorithm[17]. A Lanczos algorithm embedded in the finite element code ADINA (Version 8.3.1)[18] is employed to obtain the fundamental eigenvalue for the second example. In all examples, five sample points (n = 5) are deployed along each of the variable axis for first-order HDMR approximation.

A. Example 1: Two degrees of freedom undamped system

Consider a two degree of freedom undamped[9] system as shown in Figure 1, with mass matrix

$$\boldsymbol{m}=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{8}$$

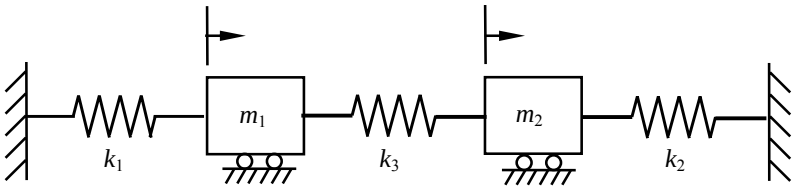


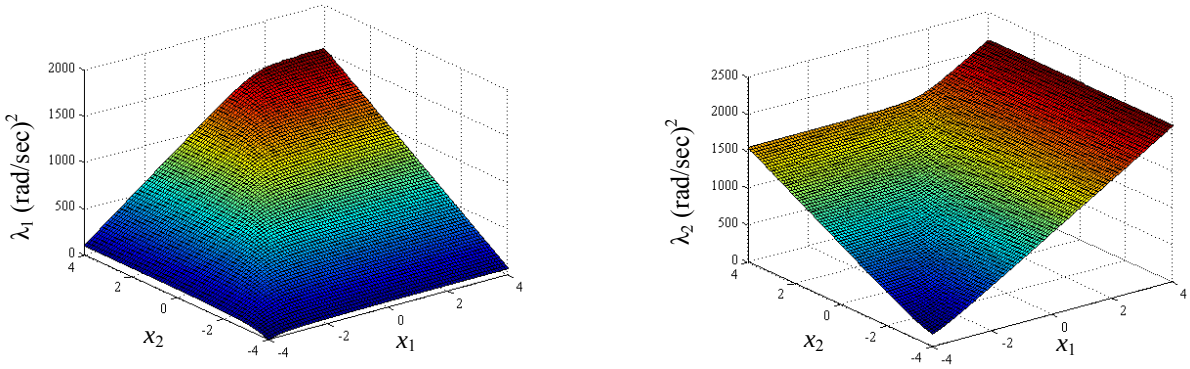
Figure 1: Two degree of freedom system

and stiffness matrix

$$\boldsymbol{k}(\boldsymbol{x})=\begin{bmatrix} k_1(\boldsymbol{x})+k_3 & -k_3 \\ -k_3 & k_2(\boldsymbol{x})+k_3 \end{bmatrix} \tag{9}$$

where mass matrix is deterministic and stiffness matrix is random. Masses $m_1 = 1$ kg, $m_2 = 1.5$ kg, $k_1(\boldsymbol{x}) = 1000(1 + 0.25x_1)$ N/m, $k_2(\boldsymbol{x}) = 1100(1 + 0.25x_2)$ N/m, and $k_1(\boldsymbol{x}) = 100$ N/m. The input random variables $\boldsymbol{x} = \{x_1, x_2\}$ are standard Gaussian with mean $\boldsymbol{\mu}_x = 0$ and standard deviation $\boldsymbol{\sigma}_x = 1$.

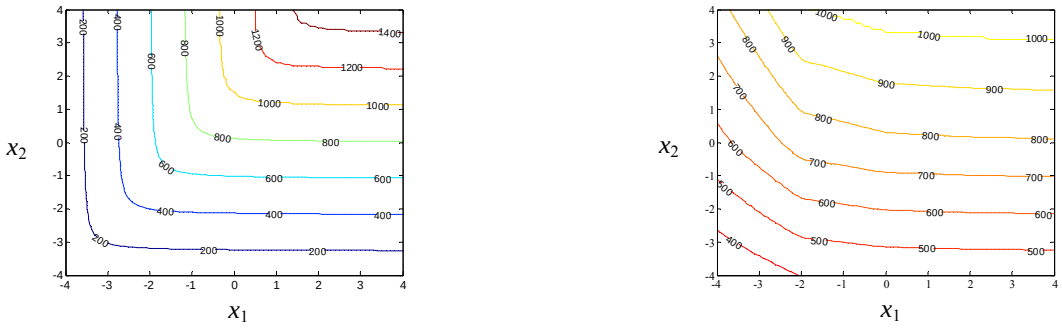
Figure 2(a) and (b) present the plots of eigensurface of two exact eigenvalues λ_1 and λ_2, respectively for $-4 \le x_i \le 4$; $i = 1, 2$, obtained by solving the matrix characteristic equation (refer Equation 2).



(a) **(b)**

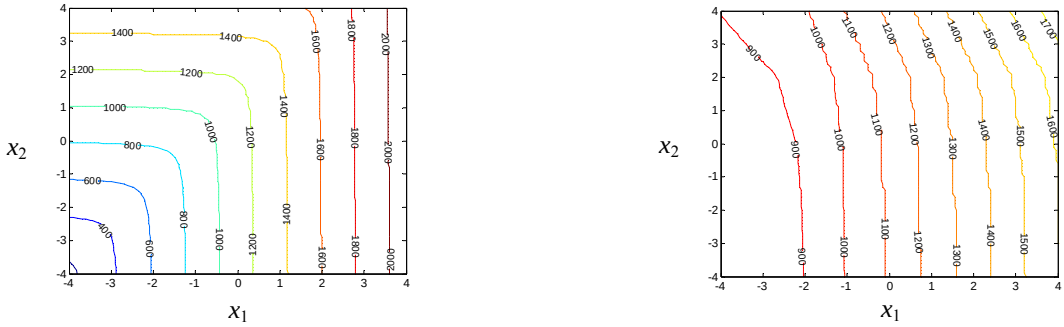
Figure 2: Plots of eigensurface

Approximate eigenvalues by the proposed method are evaluated using Equation 7. Figures 3 and 4 compare the effectiveness of the present method with exact solution in terms of contour plots of λ_1 and λ_2, respectively.



(a) **(b)**

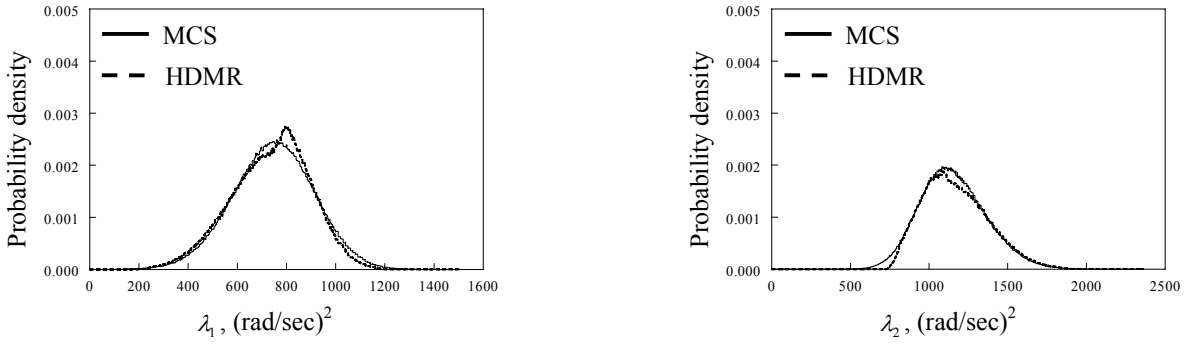
Figure 3: Contour plot of λ_1; (a) Exact; and (b) Approximate



(a) **(b)**

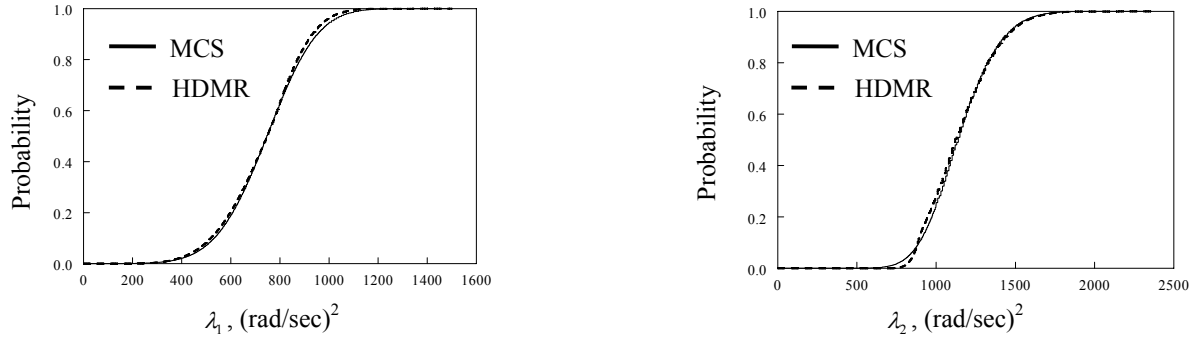
Figure 4: Contour plot of λ_2; (a) Exact; and (b) Approximate

The difference between contours by the exact method and by the proposed method using first-order approximation of HDMR is due to the absence of cooperative effect. Figures 5(a)–(b) compare predicted marginal densities of two eigenvalues obtained by the proposed method using first-order HDMR approximation and the direct MCS. Similarly, Figures 6(a)–(b) display the cumulative densities of two eigenvalues obtained by the proposed method using first-order HDMR approximation and the direct MCS.



(a) **(b)**

Figure 5: Plot of marginal densities; (a) λ_1 ; and (b) λ_2



(a) **(b)**

Figure 6: Plot of marginal densities; (a) λ_1 ; and (b) λ_2

In the present method MCS is conducted in conjunction with first-order HDMR approximation of eigenvalues. Here, although the same sampling size as in direct MCS is considered, the number of original matrix characteristic equation evaluation is very less. Compared with the direct MCS, present method using first-order HDMR approximation provide excellent estimates of the probability density of eigenvalues.

B. Example 2: Tenbar truss structure

To examine the accuracy and efficiency of the proposed method, a 10-bar, linear-elastic, truss structure[15], shown in Figure 7, is considered in this example. Young's modulus of the material is 10^7 psi. Cross-sectional area x_i, $i = 1,2,\ldots,10$ for each bar follows normal distribution and has mean $\mu = 2.5$ in^2 and standard deviation $\sigma = 0.5$ in^2. The reference point $\bar{\boldsymbol{x}}$ is taken as mean values of the random variables.

Present method and MCS using the commercial finite element code (ADINA)[17] are employed for evaluating probabilistic characteristics of fundamental eigenvalue of the structure. In both methods, the calculation of the matrix characteristic equation for a given input is equivalent to performing a finite element analysis. Therefore, computational efficiency, even for this simple truss structure, is a major practical requirement in solving random eigenvalue problems. Five uniformly distributed sample points n = 5 along each of the variable axis are deployed for first-order HDMR approximation.

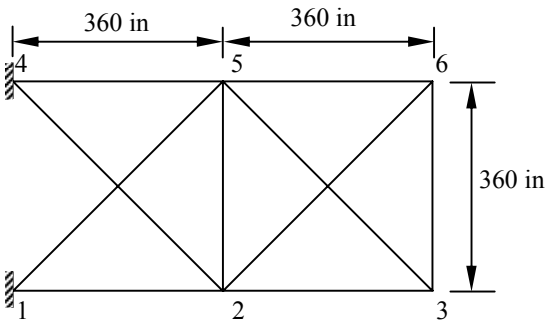


Figure 7. Ten bar truss structure for Example 2

Figures 8(a) and (b) show marginal probability density of the fundamental eigenvalue by the first-order HDMR approximation and the direct MCS. A sampling size $N_S = 10^5$ (finite element analyses) is considered in direct MCS, while only 41 finite element analyses are required by the first-order HDMR. However, since the first-order HDMR yields explicit eigenvalue approximations, an arbitrarily large sample size, e.g. 10^6 in this particular example, is selected to perform the embedded Monte Carlo analysis. It can be observed that, the agreement between the results of the first-order HDMR and the direct MCS is fairly accurate.

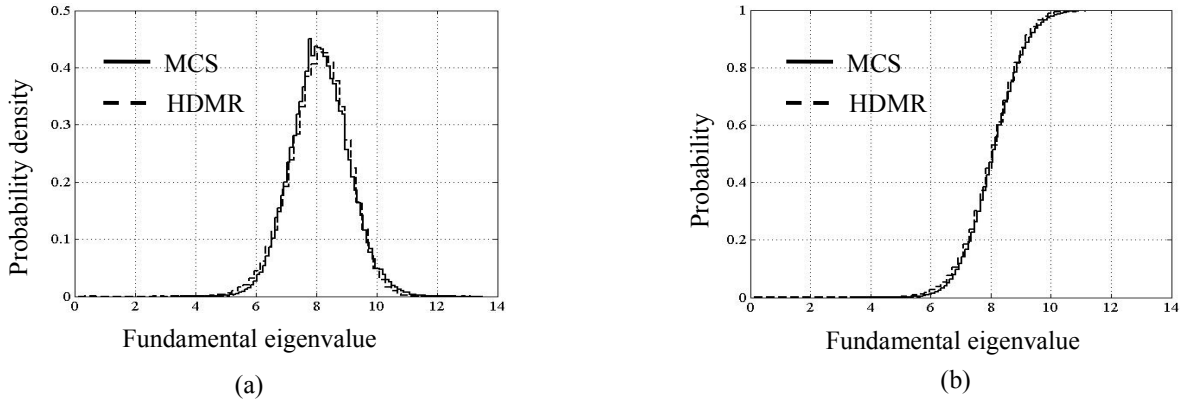


Figure 8: Characterization of eigenvalue; (a) Marginal density; and (b) Cumulative density

VII. Conclusion

A new method, called High Dimensional Model Representation (HDMR), was developed for probabilistic characterization of real-valued positive semi-definite random matrices. The method is based on: (a) HDMR which allows lower order approximations of eigenvalues of multivariate eigenfunction, and (b) MLS interpolation facilitating efficient MCS for marginal densities. The computational effort in probabilistic characterization of eigenvalues can be viewed as performing eigenvalue analyses at selected sample points deployed along each of the variable axis. Results of numerical examples involving linear dynamics of spring-mass systems and finite element analysis of a truss structure indicate that the developed method provides fairly accurate estimates probability densities of eigenvalues.

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