DATA ASSIMILATION IN STRUCTURAL DYNAMICS: EXTENDED-, ENSEMBLE KALMAN AND PARTICLE FILTERS

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ABSTRACT. Combined state and parameter estimation of dynamical systems plays a crucial role in extracting system response from noisy measurements. A wide variety of methods have been developed to deal with the joint state-parameter estimation of nonlinear dynamical systems. The Extended Kalman Filter method is a popular approach for the joint system-parameter estimation of nonlinear systems. This method combines the traditional Kalman filtering techniques with the linearisation tools to tackle nonlinear problems and its formulation is based on the assumption that the probability density function of the state vector can be reasonably approximated to be Gaussian. Recent research has been focused on non-Gaussian models. Of particular interest is the Ensemble Kalman Filter and the Particle Filter. These methods are capable of handling various forms of nonlinearities as well as non-Gaussian noise models. This paper examines and contrasts the feasibility of joint state and parameter identification in non-linear dynamical systems using the Extended Kalman, Ensemble Kalman and Particle filters.

KEYWORDS: Nonlinear Dynamical Systems, Parameter Estimation, Nonlinear Filtering

1 INTRODUCTION

Combined state and parameter estimation of dynamical systems plays a crucial role in extracting system response from noisy measurements. A wide variety of methods have been developed to deal with the joint state-parameter estimation of nonlinear dynamical systems. The Extended Kalman Filter method is a popular approach for the joint system-parameter estimation of nonlinear systems. This method combines the traditional Kalman filtering techniques with the linearisation tools to tackle nonlinear problems and its formulation is based on the assumption that the probability density function of the state vector can be reasonably approximated to be Gaussian. Recent research has been focused on non-Gaussian models. Of particular interest is the Ensemble Kalman Filter and the Particle Filter. These methods are capable of handling various forms of nonlinearities as well as non-Gaussian noise models. This paper examines and contrasts the feasibility of joint state and parameter identification in non-linear dynamical systems using the Extended Kalman, Ensemble Kalman and Particle filters.

The most widely applied technique is the Kalman Filter (KF) [1, 2]. The KF is a recursive solution of discrete-data linear filtering problem. It provides an efficient computational means to estimate the state of a process, in a way that minimises the error variance of the estimate. The filter is very powerful in that it can estimate the system state even in the presence of model uncertainties. However, the KF is based on assumptions of linear system dynamics with additive Gaussian model and measurement errors.

In the framework filtering theory applied to parameter estimation, one or more unknown system parameters are appended to the original state vector. In general, the augmented state vector evolves nonlinearly, and thus KF can not be applied directly. As a result, various types of nonlinear state estimators have been proposed (see [3] for a summary). Examples of these techniques include the Extended Kalman Filter (EKF), the Ensemble Kalman Filter (EnKF), the Particle Filter (PF) and variational data assimilation techniques [4]. Such state estimators attempt to reconstruct the conditional pdf of the state vector given sparse and noisy measurements of certain state variables.

EKF provides a forecast estimate of the state vector through linearisation of the nonlinear state evolution equation [4]. In the context of structural dynamics, the EKF has been widely applied to problems of parameter identification (see references [5–9] for examples). Although it has received wide-spread attention, the EKF has two main drawbacks: (a) the memory requirement for storing the covariance matrix grows quadratically with the number of degrees-of-freedom (DOFs); (b) linearisation leads, in some cases, to poor error covariance evolution and/or unstable growth of the error covariance matrix [4, 10]. The EnKF address the latter issue.

EnKF inherits the same analysis step from the KF and EKF. However, the measurement error covariance matrix in EnKF is estimated from the ensemble of state vectors [11]. It is in the forecast step where the EnKF differs from KF and EKF: EnKF propagates the samples using the full nonlinear model. It has been shown in many applications (for example, see [11]) that EnKF resolves the issue of poor error covariance evolution associated with EKF. In the context of structural dynamics, the EnKF has just recently been introduced as a parameter identification tool by Ghanem and Ferro [12].

The EnKF is a particularly efficient ensemble-based data assimilation method because it assumes a Gaussian prior pdf (albeit the non-Gaussian effect considered in the forecast step). This assumptions greatly simplifies the analysis step. But this simplification can also limit the filters ability to provide accurate distributional information for highly non-Gaussian systems. Similar to EnKF, PF is also an ensemble-based estimation technique. PF however goes one step further in making no assumption of Gaussianity [13]. Manohar and Roy [14] applied the PF to identify a nonlinear stiffness parameter of a Duffing oscillator.

This investigation compares the feasibility of using the EKF, EnKF and PF for joint state and parameter identification. The system we will be investigating will be a highly nonlinear mass-spring oscillator. This investigation compares the performance of the EKF, EnKF and PF as system identification tools in structural dynamics. To the best of the authors' knowledge, such comparative study has not been widely reported in the literature of structural dynamics. The paper is organised in the following manner. A brief background of KF is given in Section 2. The application of EKF to nonlinear state-evolution equations is briefly presented in Section 3. A brief introduction to the EnKF is presented in Section 4. In Section 5, the formulation of the PF is reviewed. Section 6 reports the results from a numerical example elucidating the capability of various filtering techniques. The paper concludes in Section 7 where a summary and findings of the current investigation are detailed.

2 THE KALMAN FILTER

The Kalman Filter assimilates data into linear systems with Gaussian model and measurement noise. In addition, KF also serves as the mathematical foundation for the two nonlinear filters, EKF and EnKF. This Section presents a brief overview of the KF. For a detailed formulation, please refer to the references [4, 10, 15, 16] for example.

Discretising a continuous system model in both space and time, one can obtain the discrete state-space representation

$$\mathbf{x}_{k+1} = \boldsymbol{\Phi}_k \mathbf{x}_k + \mathbf{f}_k + \mathbf{q}_k \tag{1}$$

$$\mathbf{d}_k = \mathbf{C}_k \mathbf{x}_k + \boldsymbol{\epsilon}_k \tag{2}$$

where $\mathbf{x}$ is the state vector, $\mathbf{d} \in \mathbb{R}^m$ is the measurement vector related to the true state $\mathbf{x}$ through the measurement matrix $\mathbf{C} \in \mathbb{R}^{m \times n}$, m is the dimension of the measurement vector and n is the dimension of the state vector. $\mathbf{q}_k$ and $\boldsymbol{\epsilon}_k$ are independent zero-mean Gaussian processes with covariance matrices $\mathbf{Q}_k$ and $\mathbf{\Gamma}_k$ at time t_k, respectively. It is also assumed that $\mathbf{x}_k$ has a Gaussian prior pdf given by $\mathbf{x}_k \sim \mathcal{N}(\mathbf{x}_k^f, \mathbf{P}_k)$. KF first estimates the conditional mean and covariance of $\mathbf{x}_k$ given the measurement vector $\mathbf{d}_k$, denoted by $\mathbf{x}_k^a$ and $\widehat{\mathbf{P}}_k$, respectively. This constitutes the analysis step of the filter:

Analysis step:

$$\mathbf{K}_k = \mathbf{P}_k \mathbf{C}_k^T \left[\mathbf{\Gamma}_k + \mathbf{C}_k \mathbf{P}_k \mathbf{C}_k^T\right]^{-1}, \tag{3}$$

$$\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k \left(\mathbf{d}_k - \mathbf{C}_k \mathbf{x}_k^f\right), \tag{4}$$

$$\widehat{\mathbf{P}}_k = \left[\mathbf{I} - \mathbf{K}_k \mathbf{C}_k\right] \mathbf{P}_k. \tag{5}$$

$\mathbf{K}_k$ is known as the Kalman gain. The estimate $\mathbf{x}_k^a$ given in Eq. (4) is the optimal linear unbiased estimate. Optimal in the sense that its error covariance given in Eq. (5) is minimum. In the above equations, the superscript f denotes the results of the propagation step since they are interpreted as a *forecast* of the system variables [17]. The data-updated forecasts are labeled with superscript a signifying *analysis*.

The next step in KF is to estimate the forecast mean and covariance of the state vector for the next time step. This forms the forecast step:

Forecast step:

$$\mathbf{x}_{k+1}^f = \mathbf{\Phi}_k \mathbf{x}_k^a + \mathbf{f}_k, \tag{6}$$

$$\mathbf{P}_{k+1} = \mathbf{\Phi}_k \widehat{\mathbf{P}}_k \mathbf{\Phi}_k^T + \mathbf{Q}_k. \tag{7}$$

The KF assumes that the system is linear and that the model and measurement noises are Gaussian. These two assumptions are the major limitations of this technique. EKF is an extension of KF to deal with nonlinear evolution equations as will be seen in the next section.

3 THE EXTENDED KALMAN FILTER

For nonlinear temporal evolution of the state vector $\mathbf{x}$, the prior pdf for $\mathbf{x}_k$ is non-Gaussian even if the noise is additive and Gaussian. For weakly non-Gaussian behaviour, one can approximate the posterior pdf by an equivalent Gaussian process through linearisation.

For general nonlinear dynamical systems, we have

$$\mathbf{x}_{k+1} = \mathbf{\Psi}_k\left(\mathbf{x}_k, \mathbf{f}_k\right) + \mathbf{q}_k. \tag{8}$$

Given some measurement vector $\mathbf{d}_k$, we obtain a Gaussian conditional pdf for the state vector with mean $\mathbf{x}_k^a$ and covariance matrix $\widehat{\mathbf{P}}_k$ using the same analysis step as in the traditional KF given by Eqs. (3)-(5). Linearising $\mathbf{\Psi}_k$ about $\mathbf{x}_k^a$, we have

$$\mathbf{\Psi}_k\left(\mathbf{x}_k, \mathbf{f}_k\right) \approx \mathbf{\Psi}_k\left(\mathbf{x}_k^a, \mathbf{f}_k\right) + \mathbf{\Psi}_k'\left(\mathbf{x}_k^a, \mathbf{f}_k\right)\left(\mathbf{x}_k - \mathbf{x}_k^a\right) \tag{9}$$

where $\mathbf{\Psi}_k'\left(\mathbf{x}_k, \mathbf{f}_k\right)$ denotes the Jacobian matrix of $\mathbf{\Psi}_k\left(\mathbf{x}_k, \mathbf{f}_k\right)$ with respect to $\mathbf{x}_k$.

From Eqs. (8) and (9), we obtain

$$\mathbf{x}_{k+1} = \mathbf{\Psi}_k\left(\mathbf{x}_k^a, \mathbf{f}_k\right) + \mathbf{\Psi}_k'\left(\mathbf{x}_k^a, \mathbf{f}_k\right)\left(\mathbf{x}_k - \mathbf{x}_k^a\right) + \mathbf{q}_k. \tag{10}$$

From Eq. (10), the posterior mean and error covariance [4, 10, 15, 16] can be expressed as:

$$\mathbf{x}_{k+1}^f = \mathbf{\Psi}_k\left(\mathbf{x}_k^a, \mathbf{f}_k\right), \tag{11}$$

$$\mathbf{P}_{k+1} = \left[\mathbf{\Psi}_k'\left(\mathbf{x}_k^a, \mathbf{f}_k\right)\right] \widehat{\mathbf{P}}_k \left[\mathbf{\Psi}_k'\left(\mathbf{x}_k^a, \mathbf{f}_k\right)\right]^T + \mathbf{Q}_k. \tag{12}$$

Due to nonlinearity, note that the Jacobian matrix is being used in Eq. (12) in EKF instead of $\mathbf{\Phi}_k$ in Eq. (6) in KF. Therefore, the major steps in EKF are:

1. Analysis step:

$$\mathbf{K}_k = \mathbf{P}_k \mathbf{C}_k^T \left[\mathbf{\Gamma}_k + \mathbf{C}_k \mathbf{P}_k \mathbf{C}_k^T \right]^{-1}, \tag{13}$$

$$\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k \left(\mathbf{d}_k - \mathbf{C}_k \mathbf{x}_k^f \right), \tag{14}$$

$$\widehat{\mathbf{P}}_k = \left[\mathbf{I} - \mathbf{K}_k \mathbf{C}_k \right] \mathbf{P}_k. \tag{15}$$

2. Forecast step:

$$\mathbf{x}_{k+1}^f = \mathbf{\Psi}_k \left(\mathbf{x}_k^a, \mathbf{f}_k \right), \tag{16}$$

$$\mathbf{P}_{k+1} = \left[\mathbf{\Psi}_k' \left(\mathbf{x}_k^a, \mathbf{f}_k \right) \right] \widehat{\mathbf{P}}_k \left[\mathbf{\Psi}_k' \left(\mathbf{x}_k^a, \mathbf{f}_k \right) \right]^T + \mathbf{Q}_k. \tag{17}$$

4 THE ENSEMBLE KALMAN FILTER

We have mentioned that the linearisation step in the EKF may lead to poor error covariance evolution [4, 10]. The EnKF partly alleviates this issue. For the EnKF, an ensemble of initial state vectors drawn from the prior pdf $p\left(\mathbf{x}_0\right)$ is created. Each sample is integrated forward in time independently using the full nonlinear evolution model defined in Eq. (8). The linear analysis step found in the KF and EKF is maintained. The EnKF has the following algorithm [18]:

1. Create an ensemble $\left\{ \mathbf{x}_{0,i}^f \right\}$ of size N with $i = 1, \ldots, N$, using the prior pdf of $\mathbf{x}_0$.

2. For each subsequent step, recursively, obtain perturbed measurements and estimated measurement error covariance matrix:

$$\mathbf{d}_{k,i} = \mathbf{d}_k + \boldsymbol{\epsilon}_{k,i}, \tag{18}$$

$$\mathbf{\Gamma}_k = \frac{1}{N-1} \sum_{j=1}^{N} \boldsymbol{\epsilon}_{k,j} \boldsymbol{\epsilon}_{k,j}^T \,. \tag{19}$$

3. Analysis step:

$$\mathbf{K}_k = \mathbf{P}_k \mathbf{C}_k^T \left[\mathbf{\Gamma}_k + \mathbf{C}_k \mathbf{P}_k \mathbf{C}_k^T \right]^{-1}, \tag{20}$$

$$\mathbf{x}_{k,i}^a = \mathbf{x}_{k,i}^f + \mathbf{K}_k \left(\mathbf{d}_{k,i} - \mathbf{C}_k \mathbf{x}_{k,i}^f \right), \tag{21}$$

$$\widehat{\mathbf{P}}_k = \left[\mathbf{I} - \mathbf{K}_k \mathbf{C}_k \right] \mathbf{P}_k \,. \tag{22}$$

4. Forecast step:

$$\mathbf{x}_{k+1,i}^f = \mathbf{\Psi}_k \left(\mathbf{x}_{k,i}^a, \mathbf{f}_k \right) + \mathbf{q}_{k,i}, \tag{23}$$

$$\widehat{\mathbf{x}}_{k+1}^f = \frac{1}{N} \sum_{j=1}^{N} \mathbf{x}_{k+1,j}^f, \tag{24}$$

$$\mathbf{P}_{k+1} = \frac{1}{N-1} \sum_{j=1}^{N} \left(\mathbf{x}_{k+1,j}^f - \widehat{\mathbf{x}}_{k+1}^f \right) \left(\mathbf{x}_{k+1,j}^f - \widehat{\mathbf{x}}_{k+1}^f \right)^T. \tag{25}$$

According to Burgers et. al. [19], without adding perturbations to the original measurement vector, one would obtain an updated ensemble with a variance which maybe too low. As alluded to previousely, the EnKF resolves two issues as encountered in the KF and EKF: (a) the full nonlinear model is integrated forward in time, i.e. there is no need to linearise the state equation as required in the EKF; (b) there is no need to store the entire covariance matrix from the previous step, as the forecast covariance matrix is estimated from the ensemble at each step. This results in a significant reduction of memory requirement in contrast to the KF and EKF.

5 THE PARTICLE FILTER

Consider a more general representation of the evolution and measurement equations:

$$\mathbf{x}_{k+1} = \mathbf{g}_k\left(\mathbf{x}_k, \mathbf{f}_k, \mathbf{q}_k\right) \tag{26}$$

$$\mathbf{d}_k = \mathbf{h}_k\left(\mathbf{x}_k, \boldsymbol{\epsilon}_k\right), \tag{27}$$

where $\mathbf{q}_k$ and $\boldsymbol{\epsilon}_k$ are independent zero-mean random vectors. We define the state and the measurement matrices as

$$\mathbf{X}_k = \left\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\right\}, \tag{28}$$

$$\mathbf{D}_k = \left\{\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_k\right\}. \tag{29}$$

By Bayes' Theorem, we have e.g. [4, 10, 14]

$$p\left(\mathbf{X}_k|\mathbf{D}_k\right) = \frac{p\left(\mathbf{D}_k|\mathbf{X}_k\right)p\left(\mathbf{X}_k\right)}{\int p\left(\mathbf{D}_k|\mathbf{X}_k\right)p\left(\mathbf{X}_k\right)\mathrm{d}\mathbf{X}_k}. \tag{30}$$

Thus,

$$p\left(\mathbf{x}_k|\mathbf{D}_k\right) = \frac{\int p\left(\mathbf{D}_k|\mathbf{X}_k\right)p\left(\mathbf{X}_k\right)\mathrm{d}\mathbf{X}_{k-1}}{\int p\left(\mathbf{D}_k|\mathbf{X}_k\right)p\left(\mathbf{X}_k\right)\mathrm{d}\mathbf{X}_k}. \tag{31}$$

One obtains the mean of the state vector by

$$\widehat{\mathbf{x}_k} = \frac{\int \mathbf{x}_k p\left(\mathbf{D}_k|\mathbf{X}_k\right)p\left(\mathbf{X}_k\right)\mathrm{d}\mathbf{X}_k}{\int p\left(\mathbf{D}_k|\mathbf{X}_k\right)p\left(\mathbf{X}_k\right)\mathrm{d}\mathbf{X}_k}. \tag{32}$$

As defined in Eq. (27), $\mathbf{d}_k$ depends only on $\mathbf{x}_k$. We therefore have

$$p\left(\mathbf{D}_k|\mathbf{X}_k\right) = \prod_{s=0}^{k} p\left(\mathbf{d}_s|\mathbf{x}_s\right). \tag{33}$$

Furthermore, $\mathbf{x}_{k+1}$ depends on $\mathbf{x}_k$ (as in Eq. (26)) leading to

$$p\left(\mathbf{X}_k\right) = p\left(\mathbf{x}_0\right)\prod_{s=1}^{k} p\left(\mathbf{x}_s|\mathbf{x}_{s-1}\right). \tag{34}$$

Next, we generate N random samples of $\mathbf{X}_k$ from $p\left(\mathbf{X}_k\right)$ using Eq. (34). The estimate of the mean in Eq. (32) can be approximated by [10]

$$\widehat{\mathbf{x}_k} \approx \frac{\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_{k,i}p\left(\mathbf{D}_k|\mathbf{X}_{k,i}\right)}{\frac{1}{N}\sum_{j=1}^{N}p\left(\mathbf{D}_k|\mathbf{X}_{k,j}\right)} = \sum_{i=1}^{N} w_{k,i}\mathbf{x}_{k,i} \tag{35}$$

where

$$w_{k,i} = \frac{p\left(\mathbf{d}_k|\mathbf{x}_{k,i}\right)w_{k-1,i}}{\sum_{j=1}^{N}p\left(\mathbf{d}_k|\mathbf{x}_{k,j}\right)w_{k-1,j}}. \tag{36}$$

Eq. (36) implies the need for a choice for the values $w_{0,i}$, $i = 1, \ldots, N$. One can start with $w_{0,i} = 1/N$. This leads to the following algorithm for the particle filter:

1. Draw N samples $\mathbf{x}_{0,i}$ using $p\left(\mathbf{x}_0\right)$, $i = 1, \ldots, N$
2. Set $w_{0,i} = 1/N$
3. Perform the following steps recursively:
   (a) Obtain $\mathbf{x}_{k,i}$ from $\mathbf{x}_{k-1,i}$ for each value of i using Eq. (26)
   (b) Obtain $w_{k,i}$ for each value of i using Eq. (36)
   (c) Compute the estimate of $\mathbf{x}_k$ using Eq. (35)

Resampling: In most practical applications of the particle filter, after a certain number of recursive steps, all but one particle (sample) will have negligible weights $w_{k,i}$. This is known as the degeneracy phenomenon [13]. Effectively, as a result of degeneracy, a large computational effort is wasted in updating particles which make little coontribution to the state vector estimate. A suitable measure of degeneracy is the effective sample size given by

$$N_{eff} = \frac{1}{\sum_{i=1}^{N} (w_{k,i})^2}. \tag{37}$$

When $w_{k,i} = 1/N$, we obtain $N_{eff} = N$. On the other hand, when all but one weight is zero, i.e. when $w_{k,i} = \Delta_{ij}$, we have $N_{eff} = 1$, and that indicates degeneracy.

Thus, degeneracy can be detected when $N_{eff} < N_{thr}$, i.e. when N_{eff} falls below a threshold N_{thr}. In that case, we perform resampling to reduce the degree of degeneracy:

1. Draw N particles from the current particle set with probabilities proportional to their weights $w_{k,i}$, replacing the current particle set with the new one.

2. Set $w_{k,i} = 1/N$ for $i = 1, \ldots, N$.

6 APPLICATION TO NON-LINEAR DYNAMICAL SYSTEMS

In this section we consider a single-degree-of-freedom non-linear mass-spring oscillator for parameter identification using the aforementioned nonlinear filters. This simple model is well-suited to demonstrate the capabilities of the various filtering methods to parameter estimation in non-linear dynamical systems. In this example, third and fifth order stiffness nonlinearities are present and estimates for the corresponding coefficients will be obtained.

Original System Model: We assume the differential equation describing the system behaviour to have the following form:

$$m\ddot{u}(t) + c\dot{u}(t) + k_1 u(t) + k_3 u^3(t) + k_5 u^5(t) = f(t) + q(t), \tag{38}$$

where m is the mass coefficient, c is the damping coefficient, k_1, k_3 and k_5 are the stiffness coefficients, $u(t)$ is the displacement, $f(t)$ is a deterministic force and $q(t)$ is a random forcing term describing modelling error. Our aim is to estimate the displacement u as well as the nonlinear stiffness coefficients k_3 and k_5 from some noisy measurements of u.

The state-space representation of the above equation is

$$\dot{x}_1 = x_2 \tag{39}$$

$$\dot{x}_2 = -\frac{1}{m}\{cx_2\} - \frac{1}{m}\left\{k_1 x_1 + k_3 x_1^3 + k_5 x_1^5\right\} + \frac{1}{m}\{f\} + \frac{1}{m}\{q\}, \tag{40}$$

where $x_1 = u$ and $x_2 = \dot{u}$. Temporal discretisation with time step Δt leads to the discrete state-evolution equations

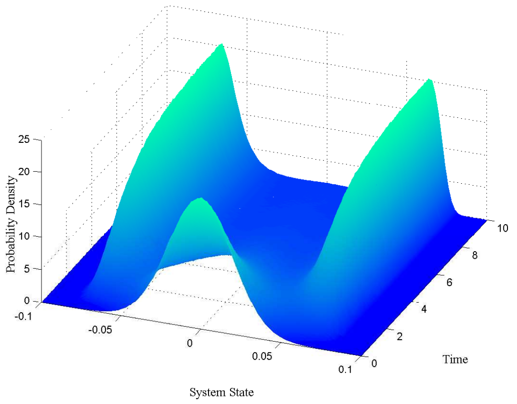
$$\{x_1\}_{k+1} = \{x_1\}_k + \Delta t \{x_2\}_k \tag{41}$$

$$\{x_2\}_{k+1} = \{x_2\}_k - \frac{\Delta t}{m}\left[c\{x_2\}_k + k_1\{x_1\}_k + k_3\{x_1\}_k^3 + k_5\{x_1\}_k^5 - (f_k + q_k)\right] \tag{42}$$

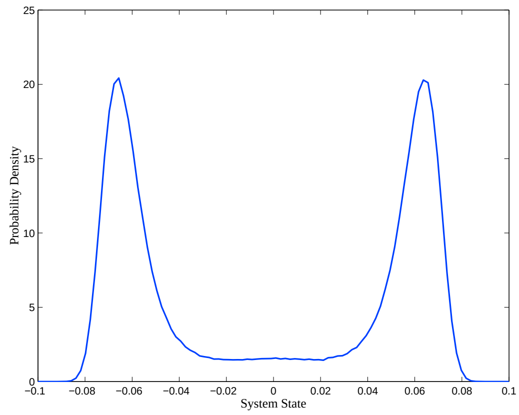
Assuming we obtain noisy measurements of u at specific times t_k, the measurement equation can be written as

$$d_k = \{x_1\}_k + \epsilon_k. \tag{43}$$

We assume that the perturbation term is Gaussian, i.e. $q_k \sim \mathcal{N}(0, \sigma_1^2)$. Fig. 1a shows the transitional pdf of the system under random excitation ($f_k = 0$) as a function of time and system state x_1. Initial conditions are $x_1 \sim \mathcal{N}(0, 4 \times 10^{-4})$ and $x_2 = 0$. The pdf evolves from a Gaussian initial pdf to a bimodal equilibrium distribution shown in Fig. 1b. The results are shown for $m = 1$kg, $c = 5$Ns/m, $k_1 = 1$N/m, $k_3 = -4.5 \times 10^3$N/m^3, $k_5 = 1 \times 10^6$N/m^5, $\Delta t = 5 \times 10^{-3}$sec and $\sigma_1^2 = 4$. The pdf results are obtained using 1 million realizations. Examining Figs. 1a and 1b, it is clear that the oscillator has a pdf that is highly non-Gaussian.



(a) transitional pdf



(b) transitional pdf at $t = 10$s

Figure 1: Transitional pdfs of the nonlinear mass-spring oscillator.

Note that the underlying autonomous system (i.e. $f(t)$ and $q(t)$ are zero) has five *fixed points* at $u = 0$, ± 0.0153 and ± 0.0653. The two dominant peaks of the pdf are centred around the stable fixed points $u = \pm 0.0653$.

State Estimation: Fig. 2 shows the response of the oscillator under a harmonic force $f(t) = \sin(2\pi t)$ with the same perturbation term $q_k \sim \mathcal{N}(0, \sigma_1^2 = 4)$. Also shown is the measured response d_k contaminated by Gaussian noise $\epsilon_k \sim \mathcal{N}(0, 2.05 \times 10^{-3})$. For the discrete

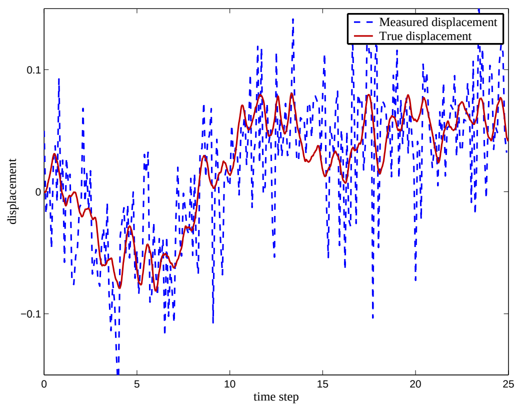
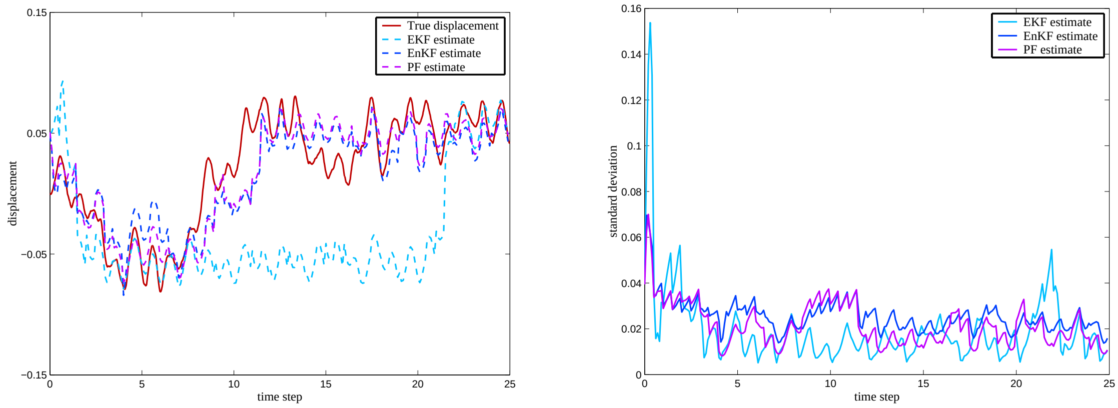


Figure 2: Measured response u of the mass-spring oscillator.

assimilation of data into this model, we consider the observations obtained at time intervals of one unit. The result of the state estimation using the various filtering techniques is shown in Fig. 3a. The same experimental setup was used from the previous Section. Initial conditions are $x_1 \sim \mathcal{N}(0, 1)$ and $x_2 \sim \mathcal{N}(0, 1)$. We can see that the EKF failed to track the displacement after approximately 8 seconds. Both the EnKF and PF track the transition successfully. For both the EnKF and PF, 200 samples are used. The threshold effective sample size for PF resampling is 150. The error in estimation is due to the relatively large amount of noise in the measurements as well as the unknown perturbations exiting the system. Fig. 3b shows the corresponding error standard deviation of the estimates.



(a) state x_1 estimate (b) estimate error standard deviation

Figure 3: EKF, EnKF and PF state estimates for the mass-spring oscillator.

Joint State and Parameter Estimation: Observing the poor performance of the EKF in the previous section, we now consider only the EnKF and PF for combined state and parameter estimation. We augment the coefficients k_3 and k_5 into the original state vector to obtain two new state variables $x_3 = k_3$ and $x_4 = k_5$. The new variables are assumed to evolve using the following model

$$\dot{x}_3 = e_3\,(t) \tag{44}$$
$$\dot{x}_4 = e_4\,(t)\,. \tag{45}$$

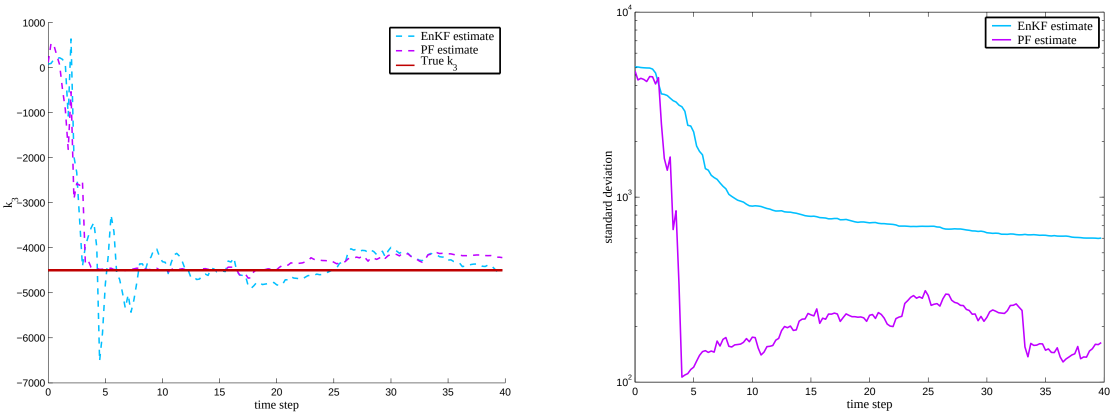
resulting in the following state-evolution equations

$$\{x_1\}_{k+1} = \{x_1\}_k + \Delta t\,\{x_2\}_k \tag{46}$$
$$\{x_2\}_{k+1} = \{x_2\}_k - \frac{\Delta t}{m}\left[c\,\{x_2\}_k + k_1\,\{x_1\}_k + \{x_3\}_k\,\{x_1\}_k^3 + \{x_4\}_k\,\{x_1\}_k^5 - (f_k + q_k)\right] \tag{47}$$
$$\{x_3\}_{k+1} = \{x_3\}_k + \Delta t\,\{e_3\}_k \tag{48}$$
$$\{x_4\}_{k+1} = \{x_4\}_k + \Delta t\,\{e_4\}_k\,. \tag{49}$$

The measurement equation remains as in Eq. (43) and the perturbations are assumed to be Gaussian, i.e. $e_3 \sim \mathcal{N}(0, \sigma_2^2)$ and $e_4 \sim \mathcal{N}(0, \sigma_3^2)$. The result of the parameter estimation using both the EnKF and PF methods are shown in Fig. 4. The system was exited by $f\,(t) = 0.25\sin(2\pi t)$ along with the random term $q_k \sim \mathcal{N}\,(0, \sigma_1^2 = 4)$. Furthermore, we used $\sigma_2 = 1\times10^3$, $\sigma_3 = 1\times10^4$ and the measurement error is given by $\epsilon_k \sim \mathcal{N}\,(0, 2.91\times10^{-3})$. Initial conditions are $x_1 \sim \mathcal{N}\,(0, 0.01)$, $x_2 \sim \mathcal{N}(0, 0.01)$, $x_3 \sim \mathcal{N}(0, 2.5\times10^7)$ and $x_4 \sim \mathcal{N}(0, 1\times10^{12})$. For both the EnKF and PF, 800 samples are used. The threshold effective sample size for PF resampling is 600. We can see that the EnKF and the PF both yield good estimates for the two nonlinear stiffness coefficients. When comparing the standard deviation of the error in the estimates, the PF results in smaller values for the standard deviation. This is partly attributed to the resampling step undertaken whenever degeneracy is encountered in the PF. The consequence of resampling is a reduction in the variance of our sample set.

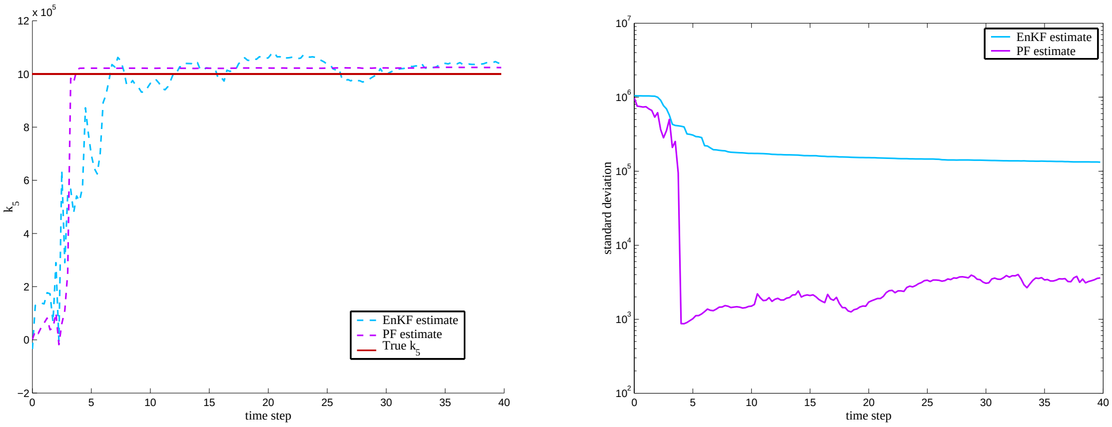
7 CONCLUSION

This paper explored the capabilities of the EKF, EnKF and PF for joint state-parameter estimation for a simple nonlinear structural dynamical system. Such methods can estimate the system state even in the presence of model uncertainties. The following features were brought out from the current investigation:



(a) k_3 estimate (b) estimate error standard deviation

Figure 4: EnKF and PF estimates of the nonlinear coefficient k_3.



(a) k_5 estimate (b) estimate error standard deviation

Figure 5: EnKF and PF estimates of the nonlinear coefficient k_5.

1. For the nonlinear system considered here, it is demonstrated that the EKF fails to track the true state of the system. This is due to the highly non-Gaussian nature of the state variables, caused by the presence of strong nonlinearities. The EnKF and PF performed well in tracking the true state of the system.

2. Even in the presence of relatively large model and measurement noise, the estimates of the nonlinear stiffness coefficients given by the EnKF and PF match reasonably well with the true values.

3. The error estimates of the nonlinear coefficients was smaller for the PF when compared to the EnKF. It is conjectured that the cause for such an observation is the resampling undertaken in the PF.

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