Uncertainty Propagation in Linear Systems: An Exact Solution Using random Matrix Theory

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In many stochastic mechanics problems the solution of a system of coupled linear random algebraic equations is needed. This problem in turn requires the calculation of the inverse of a random matrix. Over the past four decades several approximate analytical methods and simulation methods have been proposed for the solution of this problem in the context of probabilistic structural mechanics. In this paper, for the first time, we present an exact analytical method for the inverse of a real symmetric (in general non-Gaussian) random matrix of arbitrary dimension. The proposed method is based on random matrix theory and utilizes the Jacobian of the underlying nonlinear matrix transformation. Exact expressions for the mean and covariance of the response vector is obtained in closed-form. Numerical examples are given to illustrate the use of the expressions derived in the paper.

Nomenclature

$\mathbf{B}_n$	$n^2 \times n(n+1)/2$-dimensional translation matrix
$\mathbf{\Delta K}$	random part of the stiffness matrix
$\mathbf{f}$	forcing vector
$\mathbf{H}$	inverse of the stiffness matrix
$\mathbf{K}$	random symmetric stiffness matrix
$\mathbf{R}$	selection matrix
$\mathbf{u}$	response vector
$\mathbf{y}$	response quantify of interest
δ_{ij}	Kronecker's delta function
$\Gamma_n(a)$	multivariate gamma function
ξ_j	set of random variables
J	the Jacobian of a transformation
$m, \mathbf{\Psi}$	scalar and matrix parameters of the inverted Wishart distribution
n	number of degrees of freedom
$p, \mathbf{\Sigma}$	scalar and matrix parameters of the Wishart distribution

Conventions

$(\bullet)^\dagger$	Moore-Penrose generalized inverse of a matrix
$(\bullet)^T$	matrix transposition
$\mathbb{C}$	space of complex numbers
$\mathbb{R}$	space of real numbers
$\mathbb{R}_n^+$	space $n \times n$ real positive definite matrices
$\mathbb{R}^{n \times m}$	space $n \times m$ real matrices
$\mathbf{I}_n$	an identity matrix of dimension $n \times n$
$\mathbf{O}_n$	a null matrix of dimension $n \times n$
$|\bullet|$	determinant of a matrix

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etr $\{\bullet\}$	$\exp\{\text{Trace}(\bullet)\}$
$\|\bullet\|_{\text{F}}$	Frobenius norm of a matrix, $\|\bullet\|_{\text{F}} = \left(\text{Trace}\left((\bullet)(\bullet)^T\right)\right)^{1/2}$
$\otimes$	Kronecker product (see[1])
$\sim$	distributed as
Trace $(\bullet)$	sum of the diagonal elements of a matrix
cdf	cumulative distribution function
PC	Polynomial Chaos
pdf	probably density function
RMT	Random Matrix Theory
SFEM	Stochastic Finite Element Method

I. Introduction

$\mathbf{I}$N many stochastic mechanics problems we need to solve a system of linear stochastic equations

$$\mathbf{Ku} = \mathbf{f}. \tag{1}$$

Here $\mathbf{K} \in \mathbb{R}^{n \times n}$ is a $n \times n$ real non-negative definite random matrix, $\mathbf{f} \in \mathbb{R}^n$ is a n-dimensional real deterministic input vector and $\mathbf{u} \in \mathbb{R}^n$ is a n-dimensional real uncertain output vector which we want to determine. Equation (1) typically arise due to the discretisation of stochastic partial differential equations using the the stochastic finite element method.[2–11] In the context of linear structural mechanics, $\mathbf{K}$ is known as the stiffness matrix, $\mathbf{f}$ is the forcing vector and $\mathbf{u}$ is the vector of structural displacements. The central aim of a stochastic structural analysis is to determine the probability density function (pdf) and consequently the cumulative distribution function (cdf) of $\mathbf{u}$. This will allow one to calculate the reliability of the system. It is often difficult to obtain the probably density function (pdf) of the response. As a consequence, engineers often intend to obtain only fist few moments of the response quantity.

In this paper we propose an exact method to obtain the first two moments of the vector $\mathbf{u}$ for the general case when $\mathbf{K}$ is a $n \times n$ real non-negative definite non-Gaussian random matrix. In section II some current methods to obtain the statistics of $\mathbf{u}$ in Eq. (1) are briefly reviewed to put the proposed method in perspective. The concept of a random matrix and the probability density function of random matrices are introduced in section III. The exact probably density function of the inverse of a real symmetric random matrix is derived in section IV. In section V, the second-order statistics of the output vector is obtained using the expression of the matrix inverse. The expressions derived in the paper is illustrated by a numerical example in section VI.

II. Current methods for response-statistics calculation

The solution of set of random algebraic equations (1) is fundamental to stochastic mechanics problems. As a result, several papers were written on this topic over the past four decades. The purpose of this section is to give a brief overview on some of the existing methods in order to put the proposed method into proper perspective. Authors are referred to the review papers[5, 12–16] for further details. After the discretisation of the random fields and displacement fields, the governing stochastic partial differential equations can be expressed in form of Eq. (1). The stiffness matrix can be represented as

$$\mathbf{K} = \mathbf{K}^0 + \boldsymbol{\Delta}\mathbf{K} \tag{2}$$

where $\mathbf{K}^0 \in \mathbb{R}^{n \times n}$ is the deterministic part and $\boldsymbol{\Delta}\mathbf{K} \in \mathbb{R}^{n \times n}$ is the random part. The random part is often expressed as

$$\boldsymbol{\Delta}\mathbf{K} = \sum_{j=1}^{m} \xi_j \mathbf{K}_j^I + \sum_{j=1}^{m}\sum_{l=1}^{m} \xi_j \xi_l \mathbf{K}_{jl}^{II} + \cdots \tag{3}$$

where m is the number of random variables, $\mathbf{K}_j^I, \mathbf{K}_{jl}^{II} \in \mathbb{R}^{n \times n}$, $\forall\, j, l$ are deterministic matrices and ξ_j, $\forall\, j$ are real random variables. These random variables may be Gaussian and othonormalized in many problems. Under these settings, the following approaches have been employed to obtain the probabilistic descriptions of the response vector $\mathbf{u}$.

A. Perturbation based approach

The perturbation method can be applied in various forms. Here we consider the expansion of the response vector as

$$\mathbf{u} = \mathbf{u}^0 + \xi_j \mathbf{u}_j^I + \sum_{j=1}^{m} \sum_{l=1}^{m} \xi_j \xi_l \mathbf{u}_{jl}^{II} + \cdots . \tag{4}$$

The vectors $\mathbf{u}^0, \mathbf{u}_j^I, \mathbf{u}_{jl}^{II} \in \mathbb{R}^n$ need to be determined. In an alternative formulation, $\mathbf{u}$ can be viewed as a function of the vector $\boldsymbol{\xi} = \{\xi_1, \xi_2, \cdots, \xi_m\}^T$ and can be expanded in a Taylor series about the mean of $\boldsymbol{\xi}$ or some other suitable point. The mathematical details to be outlined will be similar for both approaches. Substituting $\mathbf{K}$ and $\mathbf{u}$ from Eqs. (2) and (4) into the governing equation (1) and equating the corresponding coefficients associated with the random variables we have

$$\mathbf{K}^0 \mathbf{u}^0 = \mathbf{f} \tag{5}$$

$$\mathbf{K}_j^I \mathbf{u}^0 + \mathbf{K}^0 \mathbf{u}_j^I = \mathbf{0} \tag{6}$$

$$\text{and} \quad \mathbf{K}_{jl}^{II} \mathbf{u}^0 + \mathbf{K}_j^I \mathbf{u}_l^I + \mathbf{K}_l^I \mathbf{u}_j^I + \mathbf{K}^0 \mathbf{u}_{jl}^{II} = \mathbf{0}. \tag{7}$$

Solving these equations one has

$$\mathbf{u}^0 = \mathbf{K}^{0^{-1}} \mathbf{f} \tag{8}$$

$$\mathbf{u}_j^I = -\mathbf{K}^{0^{-1}} \mathbf{K}_j^I \mathbf{u}^0, \quad \forall\, j \tag{9}$$

$$\text{and} \quad \mathbf{u}_{jl}^{II} = -\mathbf{K}^{0^{-1}} [\mathbf{K}_{jl}^{II} \mathbf{u}^0 + \mathbf{K}_j^I \mathbf{u}_l^I + \mathbf{K}_l^I \mathbf{u}_j^I], \quad \forall\, j, l. \tag{10}$$

The above equations completely define the unknown vectors appearing in the perturbation expansion (4).

Another variant of the perturbation of type approach is the so called Neumann expansion method.[17] Provided $\left\| \mathbf{K}^{0^{-1}} \boldsymbol{\Delta K} \right\|_{\mathrm{F}} < 1$, the inverse of the random matrix can be expanded in a binomial type of series as

$$\begin{aligned}
\mathbf{K}^{-1} &= \left[\mathbf{K}_0 (\mathbf{I}_n + \mathbf{K}^{0^{-1}} \boldsymbol{\Delta K}) \right]^{-1} \\
&= \mathbf{K}^{0^{-1}} - \mathbf{K}^{0^{-1}} \boldsymbol{\Delta K} \mathbf{K}^{0^{-1}} + \mathbf{K}^{0^{-1}} \boldsymbol{\Delta K} \mathbf{K}^{0^{-1}} \boldsymbol{\Delta K} \mathbf{K}^{0^{-1}} + \cdots .
\end{aligned} \tag{11}$$

From this expansion one has

$$\mathbf{u} = \mathbf{K}^{-1} \mathbf{f} = \mathbf{u}^0 - \mathbf{T} \mathbf{u}_0 + \mathbf{T}^2 \mathbf{u}_0 + \cdots \tag{12}$$

where $\mathbf{u}^0$ is as defined in Eq. (8) and $\mathbf{T} = \mathbf{K}^{0^{-1}} \boldsymbol{\Delta K} \in \mathbb{R}^{n \times n}$ is a random matrix. An improved version of the perturbation method was proposed by Elishakoff et al.[18] Later Adhikari and Manohar[7] extended the Neumann expansion method to complex symmetric random matrices arising in dynamic problems.

The first and second-order statistics of $\mathbf{u}$ can be calculated from Eq. (4) or (12). The following general points may be noted for all perturbation based approaches

- If the random variations are large, the higher-order terms may not be negligibly small.

- The calculation of response statistics become difficult if the elements of $\boldsymbol{\Delta K}$ are non-Gaussian random variables.

- Even if the elements of $\boldsymbol{\Delta K}$ are Gaussian random variables, the inclusion of higher-order terms (more than second-order) results in very messy calculations.

For the above reasons, other methods have been proposed in literature.

B. Projection methods

Any perturbation based solution can be viewed as a local approximation around the deterministic solution. The subspace projection schemes for stochastic finite element analysis on the other hand achieve global representation. The basic idea is simple, powerful and is based on solid theoretical foundations.[15, 16] Here one 'projects' the solution vector

on to a complete stochastic basis. Depending on how the basis is selected, several methods are proposed. Using the Polynomial Chaos (PC) projection scheme[3] one can express the solution vector as

$$\mathbf{u} = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{13}$$

where $\mathbf{u}_j \in \mathbb{R}^n$, $\forall j$ are unknown vectors and the polynomial chaoses $\Psi_j(\boldsymbol{\xi})$ are multidimensional Hermite polynomials in ξ_r. Instead of Hermite polynomials, Xiu and Karniadakis[19,20] used a generalized orthogonal basis which utilizes functions from the Askey family of hypergeometric polynomials. In the context of dynamic problems, Adhikari and Manohar[7] proposed the random eigenfunction expansion method where the random basis is selected to be the complete set of eigenfunctions of the undamped dynamic stiffness matrix. The number of terms in expansion (13) is given by

$$P = \sum_{j=0}^{r} \frac{(m+j-1)!}{j!(m-1)!} \tag{14}$$

where r is the order of the PC expansion. Substituting $\mathbf{u}$ in Eq. (1) and imposing the Galerkin condition, the unknown $\mathbf{u}_j$ vectors can be obtained by solving a set of nP dimensional deterministic linear algebraic equations. To reduce the computational effort, Nair and Keane[21] proposed the stochastic reduced basis method. Here the solution vector is represented using basis vectors spanning the preconditioned stochastic Krylov subspace and consequently the application of the Galerkin scheme leads to a reduced-order deterministic system of equations. A comparison of different PC projection schemes for stochastic finite element analysis is given by Sachdeva et. al.[22] Recently Sarkar et. al[23,24] have proposed a domain decomposition approach to obtain $\mathbf{u}_j$ which utilizes PC expansion and can be implemented in parallel to solve large scale problems. Once all $\mathbf{u}_j$ are known, the statistics of $\mathbf{u}$ can be obtained from (13) in a relatively straight-forward manner.

C. Monte Carlo simulation and other methods

Once the statistical properties of the random matrix $\mathbf{K}$ is known, the samples can be generated using the Monte Carlo simulation and the moments and pdf of $\mathbf{u}$ can be obtained by standard statistical methods. Several authors have also proposed other methods to obtain statistical properties of $\mathbf{u}$ and a partial summary of the solution techniques for coupled linear algebraic equations arising in stochastic mechanics problems is given in Table 1.

Table 1. A partial summary of the solution techniques for coupled linear algebraic equations arising in stochastic mechanics problems.

	Methods	Sub-methods
1.	Perturbation based methods	First and second order perturbation,[2,4] Neumann expansion,[7,17] improved perturbation method.[18]
2.	Projection methods	Polynomial chaos expansion,[3] random eigenfunction expansion,[7] stochastic reduced basis method,[21,22,25] Wiener−Askey chaos expansion,[19,20,26] domain decomposition method.[23,24]
3.	Monte carlo simulation and other methods	Simulation methods,[27,28] Analytical method in references,[29–33] Exact solutions for beams.[11,34]

III. Matrix variate probability density functions

In this section we introduce the concept of matrix variate probability density functions or random matrices. In the rest of the paper we aim to treat a random matrix as a generalization of the familiar univariate case. A random matrix can be considered as an observable phenomenon representable in the form of a matrix which under repeated observation yields different non-deterministic outcomes. A random matrix is therefore simply a collection of random variables which may satisfy certain rules (for example symmetry, positive definiteness etc). Random matrices were introduced by Wishart[35] in the late 1920s in the context of multivariate statistics. However, random matrix theory (RMT) was not used in other branches until 1950s when Wigner[36] published his works (leading to the Nobel prize in Physics in 1963) on the eigenvalues of random matrices arising in high-energy physics. Using an asymptotic theory for large dimensional matrices, Wigner was able to bypass the Schrödinger equation and explain the statistics of measured atomic energy levels in terms of the limiting eigenvalues of these random matrices. Since then research on random matrices has continued to attract interests in multivariate statistics, physics, number theory and more recently in mechanical and electrical engineering. We refer the books by Mezzadri and Snaith,[37] Tulino and Verdú,[38] Eaton,[39] Girko,[40] Muirhead[41] and Mehta[42] for the history and applications of random matrix theory.

The probability density function of a random matrix can be defined in a manner similar to that of a random variable or random vector. See the book by Gupta and Nagar[43] (see page 44) for a more formal definition of random matrices. If $\mathbf{A}$ is a $n \times m$ real random matrix, then the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}^{n \times m}$, denoted by $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}^{n \times m} \to \mathbb{R}$. Here we define probability density functions of few random matrices which are relevant to stochastic mechanics problems.

Gaussian random matrix: A rectangular random matrix $\mathbf{X} \in \mathbb{R}^{n \times p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}^{n \times p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X} - \mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X} - \mathbf{M})^T\right\}. \tag{15}$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Symmetric Gaussian random matrix: Let $\mathbf{Y} \in \mathbb{R}^{n \times n}$ be a symmetric random matrix and $\mathbf{M}, \mathbf{\Sigma}$ and $\mathbf{\Psi}$ are $n \times n$ constant matrices such that the commutative relation $\mathbf{\Sigma}\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Sigma}$ holds. If the $n(n+1)/2 \times 1$ vector $\operatorname{vecp}(\mathbf{Y})$ formed from $\mathbf{Y}$ is distributed as $N_{n(n+1)/2,1}\left(\operatorname{vecp}(\mathbf{M}), \mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n\right)$, then $\mathbf{Y}$ is said to have symmetric matrix variate Gaussian distribution with mean $\mathbf{M}$ and covariance matrix $\mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n$ and its pdf is given by

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-n(n+1)/4} \left|\mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n\right|^{-1/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{Y} - \mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{Y} - \mathbf{M})^T\right\}. \tag{16}$$

This distribution is usually denoted as $\mathbf{Y} = \mathbf{Y}^T \sim SN_{n,n}\left(\mathbf{M}, \mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n\right)$.

For a symmetric matrix $\mathbf{Y} \in \mathbb{R}^{n \times n}$, $\operatorname{vecp}(\mathbf{Y})$ is a $n(n+1)/2$-dimensional column vector formed from the elements above and including the diagonal of $\mathbf{Y}$ taken columnwise. The elements of the translation matrix $\mathbf{B}_n \in \mathbb{R}^{n^2 \times n(n+1)/2}$ are given by

$$(B_n)_{ij,gh} = \frac{1}{2}\left(\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg}\right), \quad i \le n, j \le n, g \le h \le n, \tag{17}$$

where δ_{ij} is the usual Kronecker's delta.

Wishart matrix: A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \ge n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np} \Gamma_n\left(\frac{1}{2}p\right) |\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\}. \tag{18}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$. Using a maximum entropy approach, Adhikari[44,45] proved that the system matrices arising in linear structural dynamics should be Wishart matrices.

Matrix variate gamma distribution: A $n \times n$ symmetric positive definite random matrix $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) |\mathbf{\Psi}|^{-a}\right\}^{-1} |\mathbf{W}|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\{-\mathbf{\Psi}\mathbf{W}\}; \quad \Re(a) > \frac{1}{2}(n-1). \tag{19}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \boldsymbol{\Psi})$. The matrix variate gamma distribution was used by Soize[46] for the random system matrices of linear dynamical systems.

In Eqs. (18) and (19), the function $\Gamma_n(a)$ is the multivariate gamma function, which can be expressed in terms of products of the univariate gamma functions as

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \quad \text{for} \quad \Re(a) > \frac{1}{2}(n-1). \tag{20}$$

For more details on the matrix variate distributions we refer to the books by Tulino and Verdú,[38] Gupta and Nagar,[43] Eaton,[39] Muirhead,[41] Girko[40] and references therein. Among the four types of random matrices introduced above, the distributions given by Eqs. (18) and (19) will always result in symmetric and positive definite matrices. Therefore, they can be possible candidates for modeling random system matrices arising in probabilistic structural mechanics.

IV. Exact inverse of a general real symmetric random matrix

A. The univariate case

Before considering the random matrix case, we first look at the univariate case. For a single-degree-of-freedom system ($n = 1$), Eq. (1) reduces to

$$ku = f \tag{21}$$

where $k, u, f \in \mathbb{R}$. Suppose the probability density function (may be non-Gaussian) of the random variable k is given by $p_k(k)$ and we are interested in deriving the pdf of

$$h = k^{-1}. \tag{22}$$

The Jacobian of the above transformation

$$J = \left|\frac{\partial h}{\partial k}\right| = \left|-k^{-2}\right| = |k|^{-2}. \tag{23}$$

Using the Jacobian, the probability density function of h can be obtained (see Chapter 5 of reference[47]) as

$$p_h(h)(dh) = p_k(k)(dk) \tag{24}$$

$$\text{or} \quad p_h(h) = \frac{1}{\left|\frac{\partial h}{\partial k}\right|} p_k(k) \tag{25}$$

$$\text{or} \quad p_h(h) = \frac{1}{J(k = h^{-1})} p_k\left(k = h^{-1}\right) = |h|^{-2} p_k\left(h^{-1}\right). \tag{26}$$

Example 1. *k has normal distribution:* The probability density function of k with mean k_0 and standard deviation σ_k is given by

$$p_k(k) = (2\pi)^{-1/2} |\sigma_k|^{-1} \exp\left\{-\frac{1}{2}\sigma_k^{-2}(k - k_0)^2\right\}. \tag{27}$$

Using Eq. (26) the pdf of $h = k^{-1}$ is obtained as

$$p_h(h) = |h|^{-2} p_k\left(h^{-1}\right) = (2\pi)^{-1/2} |\sigma_k|^{-1} |h|^{-2} \exp\left\{-\frac{1}{2}\sigma_k^{-2}(h^{-1} - k_0)^2\right\}. \tag{28}$$

The Gaussian model is not suitable for a strictly positive quantities such as the stiffness. The inverse Gaussian distribution derived in Eq. (28) has an essential singularity at $h = 0$ and the expressions of the moments cannot be obtained in closed-form.

Example 2. *k has χ^2 distribution with p degrees-of-freedom:* A random variable with χ^2 distribution is always positive. Therefore, it can be used to model strictly positive quantity such as the stiffness coefficient. The probability density function of k is given by

$$p_k(k) = \left\{2^{p/2}\Gamma(p/2)\right\}^{-1} k^{p/2-1}\exp\left\{-k/2\right\}, \quad k > 0. \tag{29}$$

From Eq. (18) observe that the χ^2 random variable is the univariate limit of the Wishart matrix. Using Eq. (26) the pdf of $h = k^{-1}$ is obtained as

$$p_h(h) = |h|^{-2} p_k\left(h^{-1}\right) = \left\{2^{p/2}\Gamma\left(p/2\right)\right\}^{-1} h^{-(p/2+1)}\exp\left\{-h^{-1}/2\right\}, \quad h > 0. \tag{30}$$

B. The general theory

Now we will extend the simple univariate case to the general $n \times n$ real symmetric random matrices. Suppose the matrix variate probability density function of the non-singular matrix $\mathbf{K}$ is given by $p_{\mathbf{K}}\left(\mathbf{K}\right) : \mathbb{R}^{n \times n} \to \mathbb{R}$. We are interested in the pdf of

$$\mathbf{H} = \mathbf{K}^{-1} \in \mathbb{R}^{n \times n}. \tag{31}$$

This implies that we want to obtain the joint probability density functions of *all* the elements of $\mathbf{H}$. The elements of $\mathbf{H}$ are complicated non-linear function of the elements of $\mathbf{K}$. Therefore, even if the elements of $\mathbf{K}$ have Gaussian distribution, in general the joint distribution of the elements of $\mathbf{H}$ will be non-Gaussian. Also note that $\mathbf{H}$ may not have any banded structure even if $\mathbf{K}$ is of banded nature. The key step to obtain the pdf of the inverse of a random matrix is the calculation of the Jacobian of the non-liner matrix transformation (31). For more details on the Jacobians of matrix transformations and matrix analytical calculus we refer the readers to the books by Mathai,[48] Girko,[40] Graham,[1] Eaton,[39] Muirhead,[41] Magnus and Neudecker[49] and Harville.[50]

From Eq. (31) one has

$$\mathbf{K}\mathbf{K}^{-1} = \mathbf{K}\mathbf{H} = \mathbf{I}_n. \tag{32}$$

Taking the matrix differential of this equation we have

$$(\mathbf{dK})\,\mathbf{H} + \mathbf{K}\,(\mathbf{dH}) = \mathbf{O}_n \quad \text{or} \quad (\mathbf{dH}) = -\mathbf{K}^{-1}\,(\mathbf{dK})\,\mathbf{K}^{-1}. \tag{33}$$

Now we treat $(\mathbf{dH})$, $(\mathbf{dK}) \in \mathbb{R}^{n \times n}$ as variables and $\mathbf{K}$ as constant since it does not contain $(\mathbf{dH})$ or $(\mathbf{dK})$. Define the operation $\mathrm{vec}\,(\bullet): \mathbb{R}^{m \times n} \to \mathbb{R}^{mn}$ which transforms a matrix to a long vector formed by stacking the columns of the matrix in a sequence one below another. It is known that[1,50] for any three matrices $\mathbf{A} \in \mathbb{C}^{k \times m}$, $\mathbf{X} \in \mathbb{C}^{m \times n}$, and $\mathbf{C} \in \mathbb{C}^{n \times l}$

$$\mathrm{vec}\,(\mathbf{AXC}) = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\mathrm{vec}\,(\mathbf{X}). \tag{34}$$

Suppose $\mathbf{X}$ and $\mathbf{C} = \mathbf{A} = \mathbf{A}^T$ are both symmetric matrices. Then from the preceding equation one has

$$\mathrm{vec}\,(\mathbf{AXA}) = \mathrm{vec}\,\left(\mathbf{AXA}^T\right) = (\mathbf{A} \otimes \mathbf{A})\,\mathrm{vec}\,(\mathbf{X}). \tag{35}$$

Therefore, taking the vec operation on Eq. (33) we have

$$\mathrm{vec}\,(\mathbf{dH}) = -\mathrm{vec}\,\left(\mathbf{K}^{-1}\,(\mathbf{dK})\,\mathbf{K}^{-1}\right) = -\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathrm{vec}\,(\mathbf{dK}). \tag{36}$$

Because $(\mathbf{dH})$ and $(\mathbf{dK})$ are symmetric matrices we need to eliminate the 'duplicate' variables appearing in the above *linear* transformation. This can be achieved in a systematic manner by using the translation matrix $\mathbf{B}_n$ defined in Eq. (17). First denote the Moore-Penrose inverse of the translation matrix $\mathbf{B}_n$ as

$$\mathbf{B}_n^{\dagger} = \left(\mathbf{B}_n{}^T\mathbf{B}_n\right)^{-1}\mathbf{B}_n{}^T \in \mathbb{R}^{n(n+1)/2 \times n^2}. \tag{37}$$

It can be shown that for any symmetric matrix (see Chapter 16 of reference[50]) $\mathbf{X}$

$$\mathrm{vecp}\,(\mathbf{X}) = \mathbf{B}_n^{\dagger}\mathrm{vec}\,(\mathbf{X}) \tag{38}$$
$$\text{and} \quad \mathrm{vec}\,(\mathbf{X}) = \mathbf{B}_n\mathrm{vecp}\,(\mathbf{X}). \tag{39}$$

Using these relationships, from Eq. (36) we have

$$\mathrm{vecp}\,(\mathbf{dH}) = \mathbf{B}_n^{\dagger}\mathrm{vec}\,(\mathbf{dH}) = -\left[\mathbf{B}_n^{\dagger}\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathbf{B}_n\right]\mathrm{vecp}\,(\mathbf{dK}). \tag{40}$$

The Jacobian associated with the above linear transformation is simply the determinant of the matrix $\mathbf{B}_n^{\dagger}\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathbf{B}_n$. Using Theorem 16.4.2 of[50] one obtains

$$\left|\mathbf{B}_n^{\dagger}\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathbf{B}_n\right| = |\mathbf{K}|^{-(n+1)}. \tag{41}$$

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Therefore the Jacobian associated with the inverse transformation in Eq. (31) is given by

$$J = |\mathbf{K}|^{-(n+1)}. \tag{42}$$

This is the matrix variate generalization of the simple univariate case obtained in Eq. (23). This expression of the Jacobian of the inverse matrix transformation plays the central role in the determination of the response of a linear stochastic system. Next the expression in Eq. (42) is illustrated by an example.

Example 3. *The Jacobian of the inverse of a* 3×3 *symmetric matrix:* In this example we will verify the formula (42) by direct calculation for a 3×3 symmetric matrix. Suppose the 3×3 symmetric matrix is given by

$$\mathbf{K} = \begin{bmatrix} x & u & v \\ u & y & w \\ v & w & z \end{bmatrix}. \tag{43}$$

Here x, u, v, y, w and z are real scalar variables which can be random. From the direct matrix inversion we have

$$\mathbf{H} = \mathbf{K}^{-1} = \frac{\mathrm{adj}\,(\mathbf{K})}{|\mathbf{K}|} = \frac{1}{|\mathbf{K}|} \begin{bmatrix} yz - w^2 & -uz + vw & uw - vy \\ -uz + vw & xz - v^2 & -xw + uv \\ uw - vy & -xw + uv & xy - u^2 \end{bmatrix} \tag{44}$$

where adj $(\mathbf{K})$ is the adjoint of the matrix $\mathbf{K}$ and the determinant is given by

$$|\mathbf{K}| = xyz - xw^2 - u^2z + 2\,uvw - v^2y. \tag{45}$$

Observe that Eq. (44) represents complex coupled nonlinear transformations of the original (random) variables in Eq. (43). Differentiating the upper triangular part of the $\mathbf{H}$ matrix with respect to the independent elements of the $\mathbf{K}$ matrix one has

$$\begin{Bmatrix} dH_{1,1} \\ dH_{1,2} \\ dH_{1,3} \\ dH_{2,2} \\ dH_{2,3} \\ dH_{3,3} \end{Bmatrix} = \frac{1}{|\mathbf{K}|^2} \mathbf{B} \begin{Bmatrix} dx \\ du \\ dv \\ dy \\ dw \\ dz \end{Bmatrix} \tag{46}$$

where the elements of the $\mathbf{B}$ matrices are obtained as
$B_{1,1} = -\left(yz - w^2\right)^2$, $B_{1,2} = -2\left(yz - w^2\right)(-uz + vw)$, $B_{1,3} = 2\left(yz - w^2\right)(-uw + vy)$, $B_{1,4} = -u^2z^2 + 2\,zuvw - w^2v^2$, $B_{1,5} = 2\,wu^2z - 2\,uvw^2 + 2\,wv^2y - 2\,yzuv$, $B_{1,6} = 2\,yuvw - v^2y^2 - w^2u^2$,
$B_{2,1} = -\left(yz - w^2\right)(-uz + vw)$, $B_{2,2} = -xyz^2 + zxw^2 - u^2z^2 + 2\,zuvw + zv^2y - 2\,w^2v^2$, $B_{2,3} = wxyz - xw^3 + wu^2z + wv^2y - 2\,yzuv$, $B_{2,4} = (-uz + vw)\left(-xz + v^2\right)$, $B_{2,5} = vxyz + vxw^2 + vu^2z - v^3y - 2\,uzxw$, $B_{2,6} = uxw^2 - u^2vw + uv^2y - vwxy$,
$B_{3,1} = \left(yz - w^2\right)(-uw + vy)$, $B_{3,2} = wxyz - xw^3 + wu^2z + wv^2y - 2\,yzuv$, $B_{3,3} = -xy^2z + yxw^2 + yu^2z + 2\,yuvw - v^2y^2 - 2\,w^2u^2$, $B_{3,4} = vxw^2 + vu^2z - uv^2w - uzxw$, $B_{3,5} = uxyz + uxw^2 - u^3z + uv^2y - 2\,vwxy$, $B_{3,6} = -(-uw + vy)\left(-xy + u^2\right)$,
$B_{4,1} = -u^2z^2 + 2\,zuvw - w^2v^2$, $B_{4,2} = 2\left(-uz + vw\right)\left(-xz + v^2\right)$, $B_{4,3} = -2\,uzxw + 2\,vu^2z + 2\,vxw^2 - 2\,uv^2w$, $B_{4,4} = -\left(-xz + v^2\right)^2$, $B_{4,5} = 2\left(-xz + v^2\right)(-xw + uv)$, $B_{4,6} = -x^2w^2 + 2\,xuvw - v^2u^2$,
$B_{5,1} = wu^2z - uvw^2 + wv^2y - yzuv$, $B_{5,2} = vxyz + vxw^2 + vu^2z - v^3y - 2\,uzxw$, $B_{5,3} = uxyz + uxw^2 - u^3z + uv^2y - 2\,vwxy$, $B_{5,4} = \left(-xz + v^2\right)(-xw + uv)$, $B_{5,5} = -x^2yz - x^2w^2 + xu^2z + 2\,xuvw + xv^2y - 2\,v^2u^2$, $B_{5,6} = (-xw + uv)\left(-xy + u^2\right)$,
$B_{6,1} = 2\,yuvw - v^2y^2 - w^2u^2$, $B_{6,2} = 2\,uxw^2 - 2\,u^2vw + 2\,uv^2y - 2\,vwxy$, $B_{6,3} = -2\left(-uw + vy\right)\left(-xy + u^2\right)$, $B_{6,4} = -x^2w^2 + 2\,xuvw - v^2u^2$, $B_{6,5} = 2\left(-xw + uv\right)\left(-xy + u^2\right)$, $B_{6,6} = -\left(-xy + u^2\right)^2$.

Using the symbolic mathematical software `Maple`®, the Jacobian associated with the linear transformation in Eq. (46) is obtained as

$$J = \frac{1}{|\mathbf{K}|^2} |\mathbf{B}| = \frac{1}{\left(xyz - xw^2 - u^2z + 2\,uvw - v^2y\right)^4} = |\mathbf{K}|^{-(3+1)}. \tag{47}$$

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This detailed analytical calculation verifies the expression of the Jacobian derived in Eq. (42). The Jacobian formula in Eq. (42) therefore eliminates the need for the detailed complex calculations arising in larger matrices. This expression of the Jacobian of the inverse transformation allows us to relate the probability content of a random matrix and its inverse in a simplified manner as described next.

Once the Jacobian is obtained, the rest of the procedure to obtain the pdf of $\mathbf{H}$ is very similar to that of the univariate case. In particular, we have

$$p_{\mathbf{H}}(\mathbf{H})(\mathbf{dH}) = p_{\mathbf{K}}(\mathbf{K})(\mathbf{dK}) \tag{48}$$

$$\text{or} \quad p_{\mathbf{H}}(\mathbf{H}) = \frac{1}{\left|\dfrac{\mathbf{dH}}{\mathbf{dK}}\right|} p_{\mathbf{K}}(\mathbf{K}) \tag{49}$$

$$\text{or} \quad p_{\mathbf{H}}(\mathbf{H}) = \frac{1}{J\left(\mathbf{K} = \mathbf{H}^{-1}\right)} p_{\mathbf{K}}\left(\mathbf{K} = \mathbf{H}^{-1}\right) = |\mathbf{H}|^{-(n+1)} p_{\mathbf{K}}\left(\mathbf{H}^{-1}\right). \tag{50}$$

For the univariate case $n = 1$ and it is easy to see that Eq. (50) reduces to the familiar equivalent expression obtained in Eq. (26). The expression in Eq. (50) is surprisingly simple, yet a powerful result because it is applicable to *any* symmetric square random matrices. As long as the pdf of $\mathbf{K}$ is available (regardless of whether it is Gaussian or not), one can use this expression to obtain the pdf of the inverse random matrix exactly in closed-form! As opposed to the perturbation based methods or projection methods, no series expansion is involved in this expression. In the next two subsections we apply the formula in Eq. (50) to some well known random matrices for illustration.

C. The inverse of a symmetric Gaussian matrix

Suppose $\mathbf{K}$ is a symmetric Gaussian random matrix whose pdf is given by Eq. (16). The pdf $\mathbf{H} = \mathbf{K}^{-1}$, that is, the joint pdf of all the elements of $\mathbf{H}$ can be obtained from Eq. (50) as

$$p_{\mathbf{H}}(\mathbf{H}) = |\mathbf{H}|^{-(n+1)} p_{\mathbf{K}}\left(\mathbf{H}^{-1}\right) = (2\pi)^{-n(n+1)/4} \left|\mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n\right|^{-1/2} |\mathbf{H}|^{-(n+1)}$$
$$\text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{H}^{-1} - \mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{H}^{-1} - \mathbf{M})^T\right\}. \tag{51}$$

In the special univariate case when $n = 1$, Eq. (51) reduces to its corresponding univariate case in Eq. (28). A symmetric Gaussian random matrix is not positive definite with probability one. Therefore, it is not suitable to model strictly positive definite structural matrices and consequently symmetric Gaussian random matrices are not discussed further in this paper.

D. The inverse of a Wishart matrix

Wishart matrices are symmetric and positive definite with probability one. Soize[46] proved that the system matrices of a linear system should follow the matrix variate Gamma distribution, which in turn is related to the Wishart distribution. In a recent paper Adhikari[44] showed that if only the mean of a system matrix is known, then its maximum entropy probability distribution follows the Wishart distribution with proper parameters. Wishart matrices are therefore the most important class of random matrices arising in linear mechanical systems. As a result, in this paper significant emphasis is given on the Wishart distribution.

When $\mathbf{K}$ is a Wishart matrix, its pdf given by Eq. (18). The pdf $\mathbf{H} = \mathbf{K}^{-1}$, that is, the joint pdf of all the elements of $\mathbf{H}$ can be obtained from Eq. (50) as

$$p_{\mathbf{H}}(\mathbf{H}) = |\mathbf{H}|^{-(n+1)} p_{\mathbf{K}}\left(\mathbf{H}^{-1}\right)$$
$$= |\mathbf{H}|^{-(n+1)} \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1} \left|\mathbf{H}^{-1}\right|^{\frac{1}{2}(p-n-1)} \text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{H}^{-1}\right\} \tag{52}$$
$$= |\mathbf{H}|^{-(n+1+p)/2} \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1} \text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{H}^{-1}\right\}.$$

Using this exact pdf, the moments of the inverse matrix is obtained in the next section.

V. Exact response moments of linear systems

A. Moments of the inverse of a random matrix

Due to the reason mentioned earlier, here we consider only the Wishart matrix model for the random system matrix. Adhikari[44] showed that if the mean of $\mathbf{K}$ is $\overline{\mathbf{K}}$, then $\mathbf{K} \sim W_n(p, \boldsymbol{\Sigma})$, where

$$p = n + 1 + \theta \tag{53}$$

$$\text{and} \quad \boldsymbol{\Sigma} = \overline{\mathbf{K}}/\alpha. \tag{54}$$

The constants θ and α are obtained as

$$\theta = \frac{1}{\delta_K^2} \left\{ 1 + \frac{\{\text{Trace}\,(\overline{\mathbf{K}})\}^2}{\text{Trace}\,(\overline{\mathbf{K}}^2)} \right\} - (n+1) \tag{55}$$

$$\text{and} \quad \alpha = \sqrt{\theta(n+1+\theta)}. \tag{56}$$

Here δ_K is known as the dispersion parameter which characterize the uncertainty in the random matrix $\mathbf{K}$. The parameter δ_K is defined as

$$\delta_K^2 = \frac{\text{E}\left[\|\mathbf{K} - \text{E}[\mathbf{K}]\|_{\text{F}}^2\right]}{\|\text{E}[\mathbf{K}]\|_{\text{F}}^2}. \tag{57}$$

From this expression observe that δ_K can be viewed as the mean-normalized standard deviation of the random matrix $\mathbf{K}$. For most practical applications δ_K usually varies between 0 and 1 such that θ is no smaller than 4. The inverse of a Wishart matrix is also known as the inverted Wishart matrix[43] and defined as follows:

Inverted Wishart matrix: A $n \times n$ symmetric positive definite random matrix $\mathbf{V}$ is said to have an inverted Wishart distribution with parameters m and $\boldsymbol{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{V}}(\mathbf{V}) = \frac{2^{-\frac{1}{2}(m-n-1)n}|\boldsymbol{\Psi}|^{\frac{1}{2}(m-n-1)}}{\Gamma_n\left(\frac{1}{2}(m-n-1)\right)|\mathbf{V}|^{m/2}} \text{etr}\left\{-\frac{1}{2}\mathbf{V}^{-1}\boldsymbol{\Psi}\right\}; \quad m > 2n, \boldsymbol{\Psi} > 0. \tag{58}$$

This distribution is usually denoted as $\mathbf{V} \sim IW_n(m, \boldsymbol{\Psi})$.

Comparing Eqs. (52) and (58) and noting that $p = \theta + n + 1$ and $\boldsymbol{\Sigma} = \overline{\mathbf{K}}/\alpha$ we have

$$m - n - 1 = p = \theta + n + 1 \quad \text{or} \quad m = \theta + 2n + 2 \tag{59}$$

$$\text{and} \quad \boldsymbol{\Psi} = \boldsymbol{\Sigma}^{-1} = \alpha\overline{\mathbf{K}}^{-1}. \tag{60}$$

Therefore, the pdf of $\mathbf{K}^{-1}$ follows the inverted Wishart distribution with parameters $m = \theta + 2n + 2$ and $\boldsymbol{\Psi} = \alpha\overline{\mathbf{K}}^{-1}$, that is $\mathbf{K}^{-1} \sim IW_n(\theta + 2n + 2, \alpha\overline{\mathbf{K}}^{-1})$. The first moment (mean), second-moment and the elements of the covariance tensor of $\mathbf{K}^{-1}$ can be obtained[43] exactly in closed-form as

$$\text{E}\left[\mathbf{K}^{-1}\right] = \frac{\boldsymbol{\Psi}}{m - 2n - 2} = \frac{\alpha}{\theta}\overline{\mathbf{K}}^{-1} \tag{61}$$

$$\text{E}\left[\mathbf{K}^{-2}\right] = \frac{\text{Trace}\,(\boldsymbol{\Psi})\,\boldsymbol{\Psi} + (m - 2n - 2)\boldsymbol{\Psi}^2}{(m - 2n - 1)(m - 2n - 2)(m - 2n - 4)}$$

$$= \frac{\alpha^2\left(\text{Trace}\,(\overline{\mathbf{K}}^{-1})\,\overline{\mathbf{K}}^{-1} + \theta\overline{\mathbf{K}}^{-2}\right)}{\theta(\theta + 1)(\theta - 2)} \tag{62}$$

$$\text{cov}\left(K_{ij}^{-1}, K_{kl}^{-1}\right) = \frac{2(m - 2n - 2)^{-1}\psi_{ij}\psi_{kl} + \psi_{ik}\psi_{jl} + \psi_{il}\psi_{kj}}{(m - 2n - 1)(m - 2n - 2)(m - 2n - 4)}$$

$$= \frac{\left(\alpha^2(2\theta^{-1}\overline{K}_{ij}^{-1}\overline{K}_{kl}^{-1} + \overline{K}_{ik}^{-1}\overline{K}_{jl}^{-1} + \overline{K}_{il}^{-1}\overline{K}_{kj}^{-1})\right)}{\theta(\theta + 1)(\theta - 2)}. \tag{63}$$

In the above equations $\overline{K}_{ij}^{-1}$ implies ij-th element of the $\overline{K}^{-1}$ matrix (that is $\overline{K}_{ij}^{-1} \neq 1/\overline{K}_{ij}$). For any positive semi-definite $\mathbf{A} \in \mathbb{R}^{n \times n}$, another useful result is

$$
\begin{aligned}
\mathrm{E}\left[\mathbf{K}^{-1}\mathbf{A}\mathbf{K}^{-1}\right] &= \frac{\mathrm{Trace}\left(\mathbf{A}\boldsymbol{\Psi}\right)\boldsymbol{\Psi} + (m - 2n - 2)\boldsymbol{\Psi}\mathbf{A}\boldsymbol{\Psi}}{(m - 2n - 1)(m - 2n - 2)(m - 2n - 4)} \\
&= \frac{\alpha^2\left(\mathrm{Trace}\left(\mathbf{A}\overline{\mathbf{K}}^{-1}\right)\overline{\mathbf{K}}^{-1} + \theta\overline{\mathbf{K}}^{-1}\mathbf{A}\overline{\mathbf{K}}^{-1}\right)}{\theta(\theta+1)(\theta-2)}.
\end{aligned}
\tag{64}
$$

Eq. (62) is a special case of Eq. (64) when $\mathbf{A} = \mathbf{I}_n$. Now these expressions will be used to obtain the moments of the response vector in an exact manner.

B.   Moments of the response vector

The complete response vector is given by

$$
\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}.
\tag{65}
$$

In many practical problems only few elements of $\mathbf{u}$ or linear combinations of some elements of $\mathbf{u}$ may be of interest. Therefore, we are interested in the quantity

$$
\mathbf{y} = \mathbf{R}\mathbf{u} = \mathbf{R}\mathbf{K}^{-1}\mathbf{f}.
\tag{66}
$$

where the rectangular matrix $\mathbf{R} \in \mathbb{R}^{r \times n}$ is quite general. Indeed when $\mathbf{R} = \mathbf{I}_n$, then all the elements of $\mathbf{u}$ will be considered. If one wants to consider only few elements of $\mathbf{u}$, then the elements of $\mathbf{R}$ is equal to one corresponding to the desired elements of $\mathbf{u}$ and zero everywhere else. The matrix $\mathbf{R}$ can be also selected to 'extract' other physical quantities such as the stress components within one element or a group of elements. Equation (66) can therefore be considered as a further generalization of Eq. (65). Using Eq. (61), the mean of $\mathbf{y}$ can be obtained as

$$
\bar{\mathbf{y}} = \mathrm{E}\left[\mathbf{y}\right] = \mathrm{E}\left[\mathbf{R}\mathbf{K}^{-1}\mathbf{f}\right] = \mathbf{R}\,\mathrm{E}\left[\mathbf{K}^{-1}\right]\mathbf{f} = \frac{\alpha}{\theta}\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{f}.
\tag{67}
$$

The complete covariance matrix of $\mathbf{y}$ is given by

$$
\begin{aligned}
\mathrm{cov}\left(\mathbf{y},\mathbf{y}\right) &= \mathrm{E}\left[(\mathbf{y} - \bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}})^T\right] = \mathrm{E}\left[\mathbf{y}\mathbf{y}^T\right] - \bar{\mathbf{y}}\bar{\mathbf{y}}^T \\
&= \mathbf{R}\,\mathrm{E}\left[\mathbf{K}^{-1}\mathbf{f}\mathbf{f}^T\mathbf{K}^{-1}\right]\mathbf{R}^T - \bar{\mathbf{y}}\bar{\mathbf{y}}^T.
\end{aligned}
\tag{68}
$$

Considering $\mathbf{A} = \mathbf{f}\mathbf{f}^T$ and using the result in Eq. (64), the expectation operation in the first part of the preceding equation can be obtained as

$$
\begin{aligned}
\mathbf{R}\,\mathrm{E}\left[\mathbf{K}^{-1}\mathbf{f}\mathbf{f}^T\mathbf{K}^{-1}\right]\mathbf{R}^T &= \mathbf{R}\frac{\alpha^2\left(\mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\overline{\mathbf{K}}^{-1} + \theta\overline{\mathbf{K}}^{-1}\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)}{\theta(\theta+1)(\theta-2)}\mathbf{R}^T \\
&= \frac{\alpha^2\mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{R}^T + \alpha^2\theta\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\mathbf{R}^T}{\theta(\theta+1)(\theta-2)} \\
&= \frac{\alpha^2\mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{R}^T + \theta^3\bar{\mathbf{y}}\bar{\mathbf{y}}^T}{\theta(\theta+1)(\theta-2)}.
\end{aligned}
\tag{69}
$$

Substituting the expression in Eq. (69) into Eq. (68) and simplifying we have

$$
\begin{aligned}
\mathrm{cov}\left(\mathbf{y},\mathbf{y}\right) &= \frac{\alpha^2\mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{R}^T + \theta^3\bar{\mathbf{y}}\bar{\mathbf{y}}^T}{\theta(\theta+1)(\theta-2)} - \bar{\mathbf{y}}\bar{\mathbf{y}}^T \\
&= \frac{\alpha^2\mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{R}^T + \theta(\theta+2)\bar{\mathbf{y}}\bar{\mathbf{y}}^T}{\theta(\theta+1)(\theta-2)}.
\end{aligned}
\tag{70}
$$

The standard deviation of the elements of the response vector can be obtained by taking the square-root of the diagonal of the covariance matrix in Eq. (70). Equations (67) and (70) gives the exact closed-form expressions of the first two moments of the response of a general n-dimensional linear stochastic system. Unlike the perturbation based methods or projection methods (e.g., Eq. (12) or Eq. (13)), no series expansion is involved in these expression. The statistics of the complete response vector $\mathbf{u}$ can be obtained by substituting $\mathbf{R} = \mathbf{I}_n$ in Eqs. (67) and (70). By choosing the 'selection matrix' $\mathbf{R}$ suitably, the joint statistics of several output quantities of interest can be studied using these equations. The expression derived here are applied using a numerical example in the next section.

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VI. A numerical example

A cantilever steel plate with a slot is considered in this section. The diagram of the plate together with the deterministic numerical values assumed for the system parameters are shown in Fig. 1. The standard four-noded thin plate
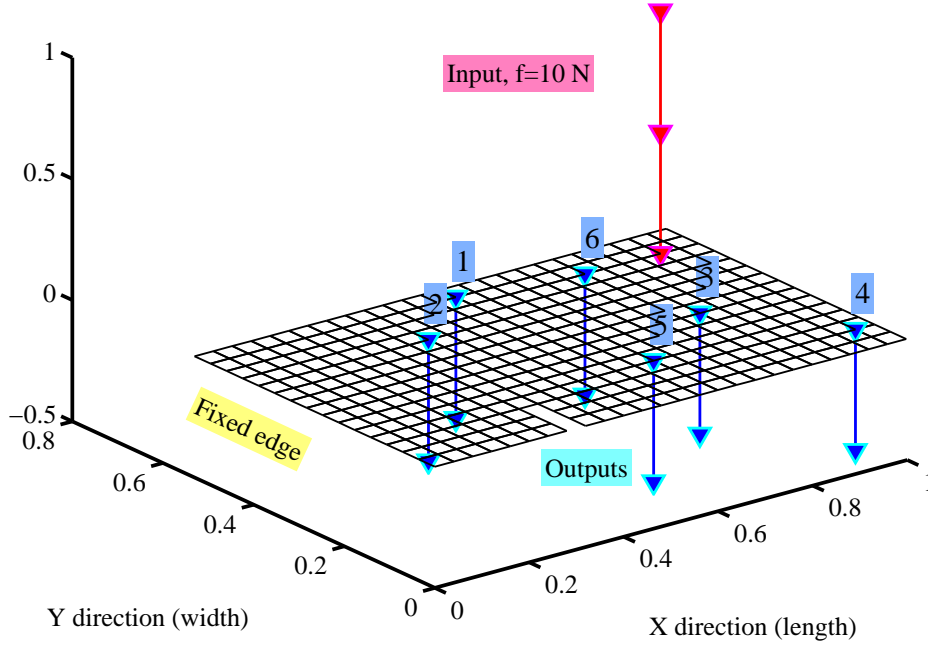

Fig. 1. A steel cantilever plate with a slot. The deterministic properties are: $\bar{E} = 200 \times 10^9$**N/m^2,** $\bar{\mu} = 0.3$**,** $\bar{t} = 7.5$**mm,** $L_x = 1.2$**m,** $L_y = 0.8$**m. The plate is deviled into** 25×15 **elements resulting** $n = 108$**.**

bending element (resulting 12 degrees of freedom per element) is used. The plate is divided into 25 elements in the x-axis and 15 elements in the y-axis for the numerical calculations. The resulting system has 1200 degrees of freedom so that $n = 1200$. A point force of $f = 10N$ is applied at the point shown in the Fig. 1. The degree-of-freedom corresponding to the applied force is 217. The response-points considered in study is also shown in the figure. The degrees-of-freedom corresponding to the six points are 112, 325, 658, 1045, 868 and 205 respectively. The force and the repone points are taken arbitrarily for the purpose of illustration only. The dimension of the matrix **R** is 6×1200. The non-zero elements (all equal to one) of the **R** matrix are $R_{1,112}$, $R_{2,325}$, $R_{3,658}$, $R_{4,1045}$, $R_{5,868}$ and $R_{6,205}$.

Now we consider uncertainties in the plate structure. It is assumed that the Young's modulus, Poissons ratio and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \tag{71}$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \tag{72}$$

$$\text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_3(\mathbf{x})\right). \tag{73}$$

Here the two dimensional vector **x** denotes the spatial coordinates. The strength parameters are assumed to be $\epsilon_E = 0.15$, $\epsilon_\mu = 0.10$, and $\epsilon_t = 0.15$. The random fields $f_i(\mathbf{x}), i = 1, 2, 3$ are assumed to be delta-correlated homogenous Gaussian random fields. The value of δ_k is obtained from Eq. (57) by using a 5000-sample Monte Carlo simulation of the random fields. From this problem we obtain $\delta_K = 0.2616$.

From the 1200×1200 stiffness matrix corresponding to the deterministic system we obtain

$$\text{Trace}\left(\overline{\mathbf{K}}\right) = 5.5225 \times 10^9 \quad \text{and} \quad \text{Trace}\left(\overline{\mathbf{K}}^2\right) = 9.6599 \times 10^{16}. \tag{74}$$

Using these values, from Eqs. (55) and (56) one obtains

$$\theta = 3.4274 \times 10^3 \quad \text{and} \quad \alpha = 3.9827 \times 10^3. \tag{75}$$

The samples of Wishart matrices are generated[45] using these values and the results are compared with the analytical expressions. Table 2 shows the mean and standard deviation of the response vector obtained using the analytical

Table 2. The mean and standard deviation of the response vector. The numbers in the parenthesis correspond to the percentage error in the Monte Carlo Simulation (with 1000 samples) results with respect to the exact analytical results.

Response quantity	Analytical mean (mm)	MSC mean (mm)	Analytical standard deviation (mm)	MSC standard deviation (mm)
$y_1 = u_{112}$	5.5058	5.5178 (0.218 %)	0.1438	0.1459 (1.436 %)
$y_2 = u_{325}$	2.6420	2.6475 (0.208 %)	0.0734	0.0740 (0.818 %)
$y_3 = u_{658}$	10.2265	10.2485 (0.216 %)	0.2537	0.2561 (0.972 %)
$y_4 = u_{1045}$	12.6039	12.6317 (0.221 %)	0.3294	0.3313 (0.570 %)
$y_5 = u_{868}$	5.9608	5.9725 (0.197 %)	0.1586	0.1604 (1.155 %)
$y_6 = u_{205}$	10.2951	10.3169 (0.212 %)	0.2507	0.2547 (1.580 %)

expressions derived in the paper and Monte Carlo simulation using 1000 samples. Using the values of θ and α in Eq. (75), the mean of the response vector is calculated using the analytical expression in Eq. (67). The standard deviation is calculated from the square-root of the diagonal elements of the analytical covariance matrix given by Eq. (70). In this case the results obtained using the analytical method is exact while those obtained using Monte Carlo simulation is approximate due to the limitation in the sample size. The percentage error associated with the Monte Carlo simulation results are also shown in Table 2. The percentage error associated with any quantity is calculated as

$$\text{Error}(\bullet) = \frac{|(\bullet)_{\text{MCS}} - (\bullet)_{\text{Analytical}}|}{(\bullet)_{\text{Analytical}}} \times 100. \tag{76}$$

The analytical covariance matrix of the response vector obtained using Eq. (70) is given by

$$\text{cov}(\mathbf{y}, \mathbf{y}) = 10^{-8} \times \begin{bmatrix} 2.0676 & 1.0063 & 3.2505 & 3.8112 & 1.9163 & 3.4473 \\ 1.0063 & 0.5381 & 1.6069 & 1.9047 & 0.9873 & 1.6590 \\ 3.2505 & 1.6069 & 6.4353 & 8.1640 & 3.9044 & 6.1530 \\ 3.8112 & 1.9047 & 8.1640 & 10.8500 & 5.0223 & 7.4765 \\ 1.9163 & 0.9873 & 3.9044 & 5.0223 & 2.5154 & 3.6114 \\ 3.4473 & 1.6590 & 6.1530 & 7.4765 & 3.6114 & 6.2867 \end{bmatrix}. \tag{77}$$

The covariance matrix of the response vector obtained using 1000-sample Monte Carlo simulation is obtained as

$$\{\text{cov}(\mathbf{y}, \mathbf{y})\}_{\text{MCS}} = 10^{-8} \times \begin{bmatrix} 2.1274 & 1.0274 & 3.3362 & 3.9043 & 1.9574 & 3.5610 \\ 1.0274 & 0.5470 & 1.6457 & 1.9598 & 1.0088 & 1.7034 \\ 3.3362 & 1.6457 & 6.5610 & 8.2978 & 3.9817 & 6.3012 \\ 3.9043 & 1.9598 & 8.2978 & 10.9741 & 5.1156 & 7.6186 \\ 1.9574 & 1.0088 & 3.9817 & 5.1156 & 2.5739 & 3.6891 \\ 3.5610 & 1.7034 & 6.3012 & 7.6186 & 3.6891 & 6.4868 \end{bmatrix}. \tag{78}$$

The percentage error in the Monte Carlo simulation result obtained in the above equation is given by

$$\begin{bmatrix} 2.8935 & 2.1053 & 2.6361 & 2.4419 & 2.1458 & 3.2978 \\ 2.1053 & 1.6420 & 2.4094 & 2.8961 & 2.1776 & 2.6781 \\ 2.6361 & 2.4094 & 1.9536 & 1.6388 & 1.9801 & 2.4085 \\ 2.4419 & 2.8961 & 1.6388 & 1.1430 & 1.8567 & 1.9016 \\ 2.1458 & 2.1776 & 1.9801 & 1.8567 & 2.3241 & 2.1513 \\ 3.2978 & 2.6781 & 2.4085 & 1.9016 & 2.1513 & 3.1842 \end{bmatrix}. \tag{79}$$

In order to verify whether increasing the number of samples in the Monte Carlo simulation will lead to a convergence in the relative error, we have performed Monte Carlo simulation with increasing number of samples. In Fig. 2
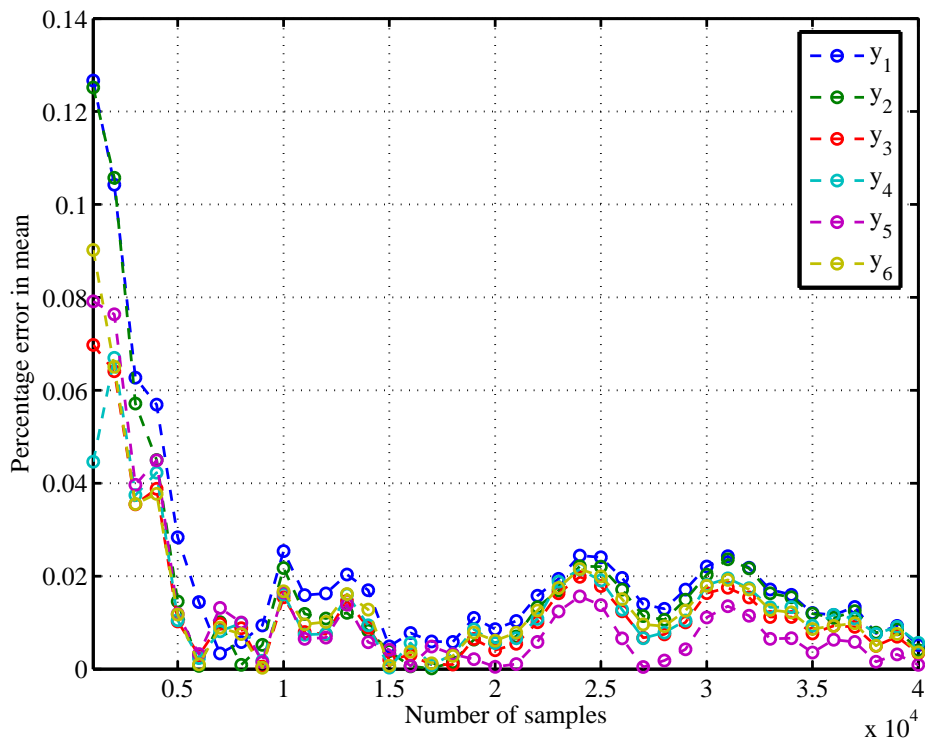
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Fig. 2. Variation of the relative error in MCS mean with respect to the number of samples.
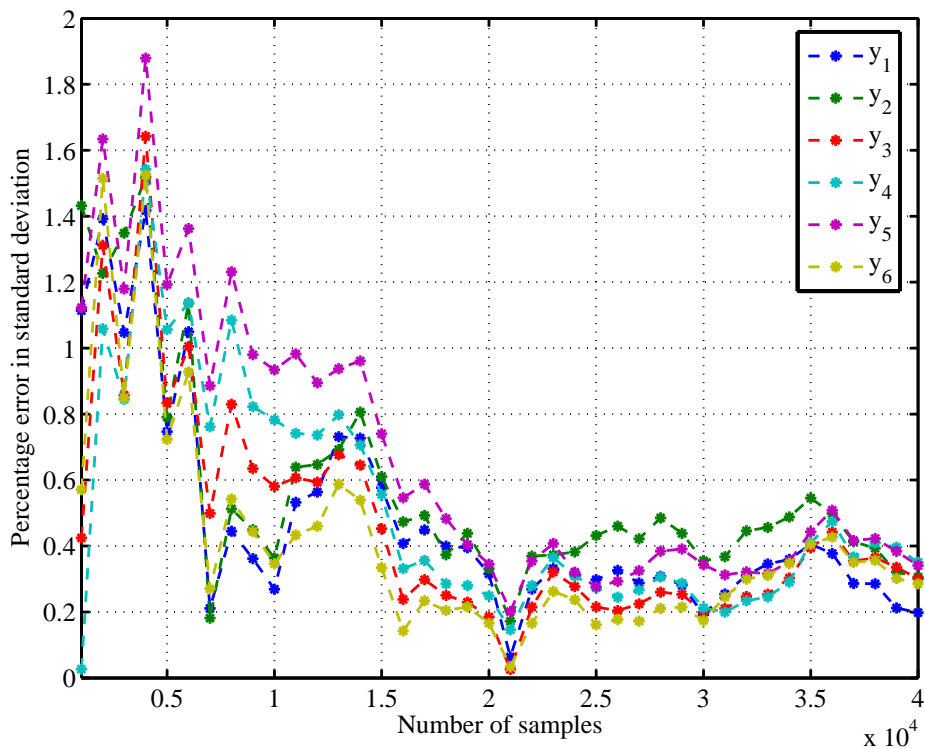

Fig. 3. Variation of the relative error in MCS standard deviation with respect to the number of samples.

and Fig. 3 the relative error in the mean and standard deviation obtained using Monte Carlo simulation with respect to

those obtained using the analytical results are plotted for sample size varying from $1,000$ to $40,000$. From these figures observe that as the sample size increases the relative error decreases (although not monotonically). This indicates that indeed when a large number of sample are used the Monte Carlo simulation, the results converge to the analytical results. This study shows that the exact analytical expressions in Eqs. (67) and (70) can be used to replace the Monte Carlo simulation and may serve as a benchmark to validate other numerical methods.

VII. Conclusions

The probabilistic characterization of the response of linear stochastic systems is considered. This problem requires the solution of a set of coupled algebraic equation, which in turn involves the inverse of a real symmetric random matrix. The inversion of a random matrix has been an outstanding problem for more than four decades. The approach taken in this paper is radically different from what has been considered so far in the stochastic mechanics literature. Here a matrix itself is treated like a variable, as opposed to view it as a collection of many variables. This outlook significantly simples the calculation of the Jacobian involved in the non-linear matrix transformation. An exact and simple closed-form expression of the joint probability density function of the elements of the inverse of a symmetric random matrix is derived in the paper. The random matrices considered here are in general non-Gaussian and of arbitrary dimensions. The derived expression is applied to Gaussian and Wishart random matrices for illustration. Because the Gaussian random matrices are not positive definite and invertible with probability one, the Wishart random matrix model is used to obtain the moments of the response. Exact closed-form expression of the mean and covariance matrix of the response vector is obtained. The analytical expressions are applied to an elastic plate problem involving a 1200×1200 random matrix. Agreement with independent Monte Carlo simulation results ensures the validity of the expressions derived in the paper. The result obtained in this paper is expected to put stochastic mechanics research into more stronger footing.

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