A Statistical Analysis of Modal Parameters for Uncertainty Quantification in Structural Dynamics

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Abstract
This paper presents modal parameters which have been extracted from frequency response data from one hundred nominally identical systems. The experimental setup used is a fixed-fixed beam arrangement, with multiple randomly placed masses added to simulate mass matrix errors, published by Adhikari et. al. [1]. In that work, they present the probabilistic characteristics of the amplitude and phase of the measured frequency response functions. These FRFs are discussed in the low, medium and high frequency ranges. In the current paper the previous work is extended to include modal parameters estimated from each of the one hundred sets of frequency response data. Statistical information, including mean and standard deviation are compiled and discussed for the low frequency (up to 1000 Hz) range. The results obtained in this work may prove useful in uncertainty quantification investigations.

1 Introduction

Uncertainty Quantification (UQ) is becoming increasingly important in establishing confidence in results obtained from finite element models. Uncertainties may be broadly classified as aleatoric or epistemic. Aleatoric uncertainty arises from inherent variability in system parameters, whereas epistemic uncertainty arises from lack of knowledge of the system. In the low frequency range, stochastic finite element methods [2-10] involve parametric uncertainties. In this work a probabilistic approach is used with the goal of enhanced understanding of the effect that variation in system parameters has on the frequency response of a model.

2.1 Experimental Procedure

A complete description of the experimental procedure has been previously published [1]. In this section the relevant information from that work is summarized. The reader is referred to the original paper for further experimental details.

A steel beam with uniform rectangular cross section, clamped between two massive fixed supports, was instrumented with three accelerometers and excited by an electromagnetic shaker, as shown in Figure 1. Pertinent data for the beam is given in Table 1. Accelerometers are placed at 230 mm (point 1), 500 mm (point 2), and 1020 mm (point 3) from the left end of the beam. The driving point 2 is actuated by impacts generated by an electromagnetic shaker outfitted with a steel tip. The result is clean FRF Baseline data for each of the measuring points to an upper limit of 4.2 kHz, with a resolution of 1 Hz [1].

Property	Value
Length (L)	1200 mm
Width (b)	40.06 mm
Thickness (t_h)	2.05 mm
Mass Density (ρ)	7800 kg/m^3
Elastic Modulus (E)	2.0 x 10^5 MPa

Table 1: Pertinent Beam Data

Next, varying mass distribution was simulated using 12 magnets weighing 2 grams each, placed in random locations at distances between 200 mm and 1000 mm along the beam. The 12 magnets represent a total mass of 1.6 percent of the mass of the beam. One hundred different random mass distributions are generated, and FRF data is generated for each. The low frequency response for point 1 (H_{21}), from the original paper [1], showing the baseline response, 100 sample ensemble average, and the 5% and 95 % probability lines, is given in Figure 2. In the current work we look in more detail at the individual modes contained in this low frequency range.

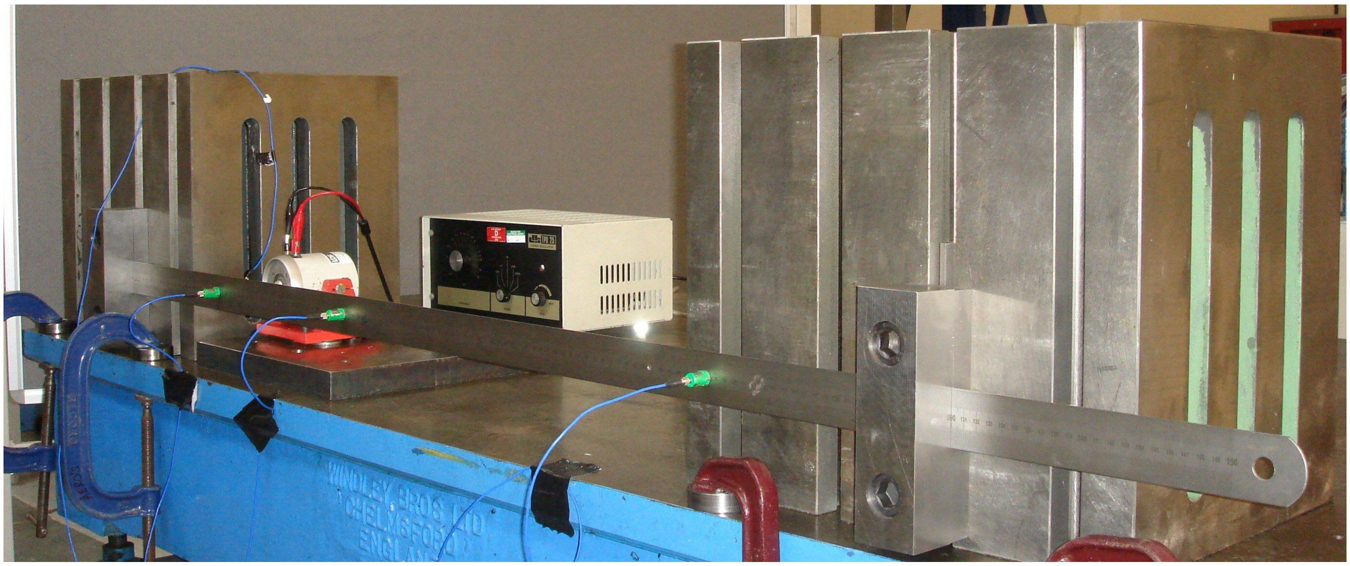


Figure 1: Experimental Setup for the Fixed-Fixed Beam

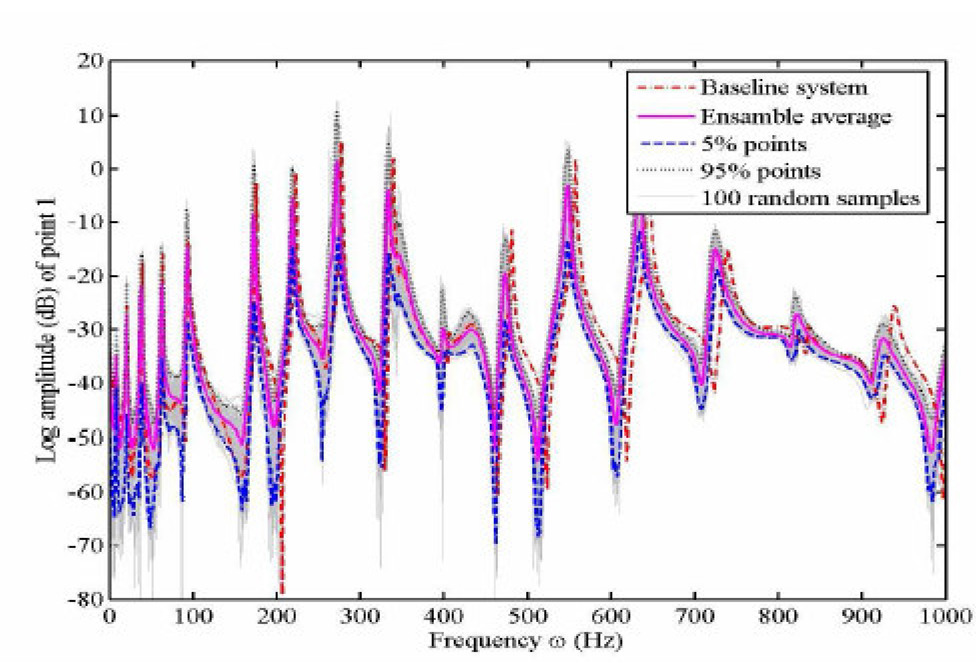


Figure 2: Low Frequency Response

2.2 Modal Parameter Extraction

There are numerous techniques available for extracting modal parameters from FRF data. In general, these methods may be classified as single degree of freedom (SDOF) or multiple degree of freedom (MDOF), depending upon whether one chooses fit a curve to a single mode or to multiple modes. Because of the large number of modes observed in the 0 – 1000 Hz frequency range, selection of a MDOF technique is indicated. In the present work the Rational Fraction Polynomial (RFP) method [11] is used. This method is based on a rational fraction formula for the FRF given by:

$$H(\omega) = \sum_{r=1,N} \frac{A_r}{(\omega_r^2 - \omega^2 + 2i\omega\omega_r\zeta_r)}$$

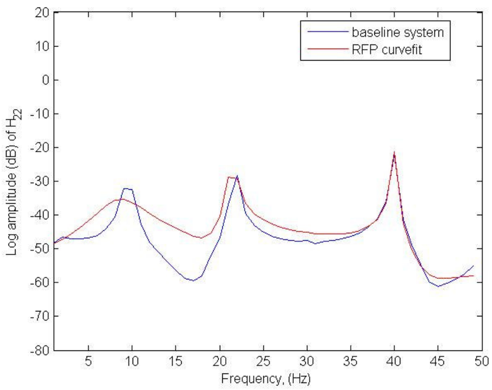
This expression is used to find the natural frequencies ω_r, the damping factors ζ_r, and the complex modal constants A_r. This method presupposes knowledge of the number of modes contained in the frequency range under investigation. In order to use this method a frequency range is selected and the number of modes contained therein is assumed. The analysis is repeated using different frequency ranges and numbers of modes until confidence in the results is obtained. Frequencies which remain relatively constant with changing input choices indicate actual, rather than computational, modes.

Extraction of baseline modes

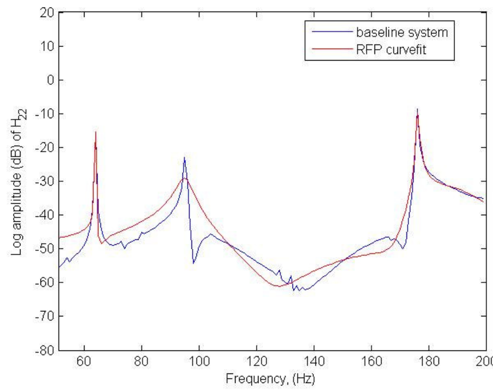
Initially, the baseline FRF data was analyzed to determine natural frequencies in the low frequency (0-1000 Hz) range. Appropriate intermediate frequency spans for use with the RFP algorithm were found through trial and error, and these spans are used for all three sets of FRF data (H_{21}, H_{22}, H_{23}). Figure 2 shows the results for each of the frequency ranges used. In total, seventeen frequencies were found, and are listed in Table 2 for each of the three sets of FRF data.

Mode Number	Frequency Response Function		
	H_{21}	H_{22}	H_{23}
1	11.40	8.861	9.159
2	21.89	21.44	21.64
3	39.88	39.88	39.85
4	64.35	64.03	64.35
5	96.19	95.10	95.40
6	176.1	176.2	176.2
7	180.4	188.9	178.4
8	223.7	223.2	232.4
9	278.4	278.1	277.7
10	340.0	339.5	339.0
11	405.2	406.0	405.8
12	479.3	480.6	480.9
13	557.5	557.5	557.5
14	645.4	645.3	644.6
15	741.7	735.9	738.6
16	834.9	831.2	833.6
17	940.6	930.7	937.5

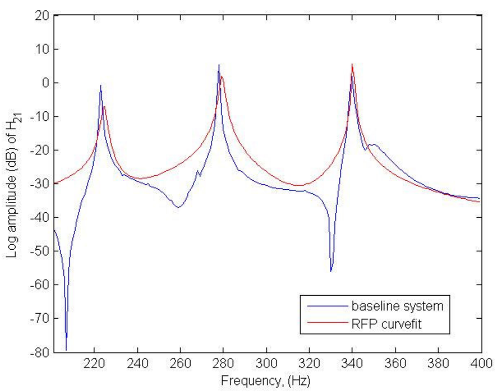
Table 2: Baseline Modal Frequencies (Hz) for each FRF.



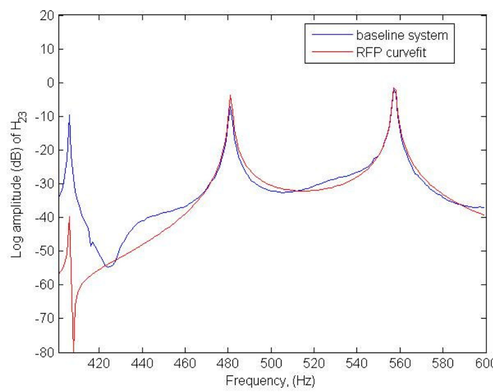
(a) 0-50 Hz Response



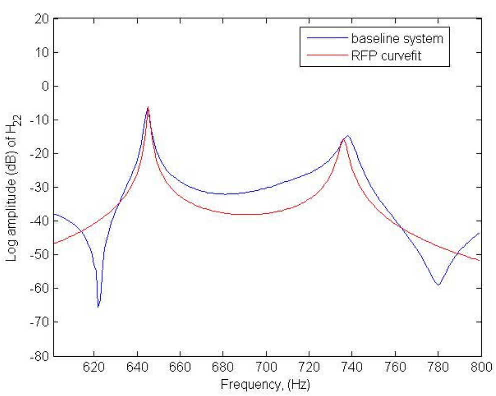
(b) 51-200 Hz Response



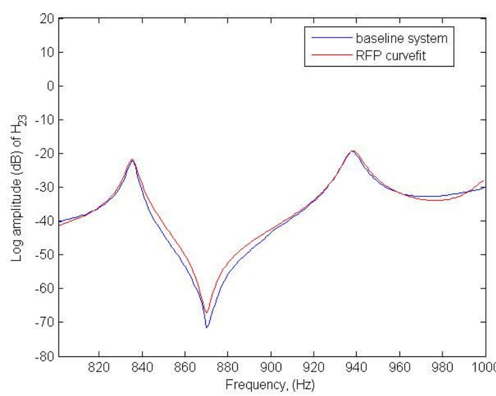
(c) 201-400 Hz Response



(d) 401-600 Hz Response



(e) 601-800 Hz Response



(f) 801-1000 Hz Response

Figure 3: Amplitude of the FRF for the Baseline System and RFP Curve-fit for Various Sensor Locations.

2.3 Statistical Analysis

The next step is to extract the natural frequencies from each FRF generated from the entire distribution of 100 FRF's. Once again, the frequency range of interest was divided into intermediate spans and the RFP method was used. The natural frequencies within each span for each of the three frequency response functions for the entire 100 sample distribution were found. The minimum, maximum, mean, and standard deviation of each modal frequency, for all three measuring point locations, is computed and given in Table 3.

	H_{21}				H_{22}				H_{23}			
Mode	min.	mean	max.	σ	min.	mean	max.	σ	min.	mean	max.	σ
1	6.384	8.725	9.558	0.641	8.607	9.200	9.693	0.221	4.196	7.392	9.623	1.385
2	19.84	21.06	22.36	0.580	19.81	21.07	22.38	0.561	19.80	21.06	22.36	0.591
3	37.36	38.96	40.33	0.662	37.39	38.97	40.30	0.649	37.32	38.95	40.83	0.685
4	72.53	76.57	83.92	2.493	61.49	63.29	64.98	0.663	61.54	63.43	64.91	0.719
5	95.73	101.9	127.8	5.744	94.17	98.21	106.5	2.343	91.64	93.76	95.91	0.856
6	170.9	173.4	175.7	0.964	170.4	173.1	175.5	1.026	166.9	172.6	175.1	1.377
7					177.5	188.9	196.6	4.177	172.0	175.5	182.7	2.266
8	218.7	221.5	227.5	1.569	217.7	219.4	221.7	0.855	268.5	272.5	275.0	1.294
9	271.4	274.5	277.4	1.485	270.4	273.3	275.9	1.339	282.1	321.0	334.1	10.523
10	333.0	336.9	342.9	1.890	330.7	333.9	337.0	1.219	331.0	335.4	345.4	2.267
11	392.9	408.0	422.9	6.339	395.7	399.4	403.3	1.484	395.4	399.1	403.2	1.445
12	463.8	473.8	482.9	3.540	489.3	511.3	538.2	12.096	468.7	473.2	477.6	1.850
13	543.3	548.3	555.7	1.876	543.2	548.1	555.5	1.843	543.2	548.2	555.8	1.908
14	629.6	634.1	639.7	2.066	629.8	634.4	640.0	2.172	629.1	633.5	639.3	2.038
15	720.8	726.7	733.0	2.615	718.6	723.9	728.7	1.965	721.0	728.4	736.6	2.919
16	795.4	822.9	838.5	6.007	815.4	820.7	824.9	2.101	815.1	820.4	824.8	2.108
17	877.7	896.0	909.8	7.209	910.3	921.3	955.8	5.270	914.2	922.5	933.4	3.558

Table 3: Statistical Frequency Data (Hz) for 100 Sample FRFs.

The data from each measurement point FRF is combined to obtain pooled natural frequency data, which is listed in Table 4. Also listed is the pooled standard deviation of the natural frequency, and the standard deviation as a percent of mean frequency, providing absolute and relative measures of the spread of the data.

Mode Number	Mean Frequency (Hz)	Standard Deviation σ (Hz)	σ (as Percent of Mean Frequency)
1	8.439	0.749	8.9
2	21.06	0.577	2.7
3	38.96	0.665	1.7
4	67.76	1.292	1.9
5	97.96	2.981	3.0
6	173.0	1.122	0.6
7	182.2	3.221	1.8
8	237.8	1.239	0.5
9	289.6	4.449	1.5
10	335.4	1.792	0.5
11	402.1	3.089	0.8
12	486.1	5.829	1.2
13	548.2	1.876	0.3
14	634.0	2.092	0.3
15	726.3	2.500	0.3
16	821.3	3.405	0.4
17	913.3	5.345	0.6

Table 4: 100 Sample Pooled Mean Frequency, Standard Deviation, and Standard Deviation as a Percent of Mean.

In order to further examine the spread of the data, the extreme values for each of the seventeen frequencies, defined as the maximum and minimum found from any of the three sets of FRF data, are found. Assuming that the data is normally distributed, virtually all points should lie within a span of plus or minus three standard deviations of the mean frequency. Points outside that range are considered outliers. These extreme values and the range of plus or minus three standard deviations are non-dimensionalized by dividing by the modal frequency and plotted in Figure 4.

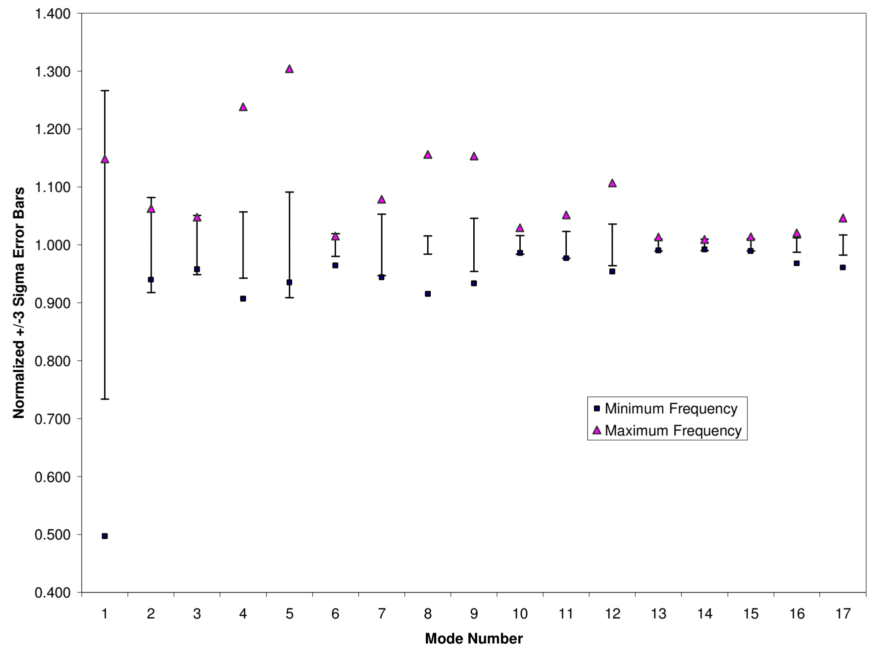
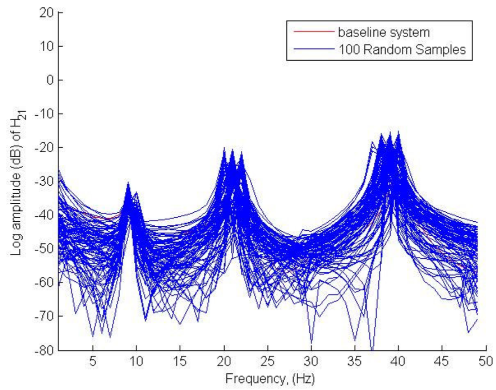


Figure 4: Minimum and Maximum Frequency Points Compared to +/- 3 Sigma Error Bars for Each Mode

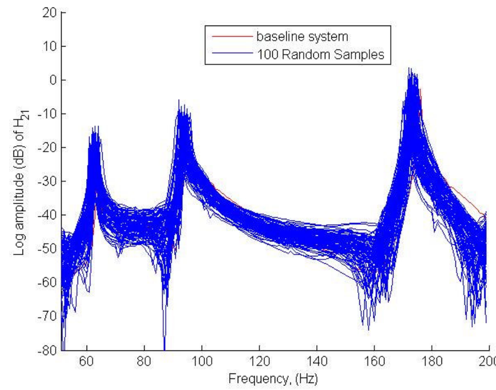
Finally the baseline frequency results are compared to the 100 sample distribution data. Table 5 shows the baseline frequencies and the pooled frequencies, along with the percent difference between the two. Figure 5 shows the H_{21} FRF for the baseline system compared to the 100 sample distribution, illustrating the spread of the 100 sample data and the shifting of the frequency peak.

Mode Number	Baseline Frequency (Hz)	100 sample Pooled Frequency (Hz)	Percent Difference
1	9.8	8.4	-14.3
2	21.7	21.1	-2.6
3	39.9	39.0	-2.3
4	64.2	67.8	5.5
5	95.6	98.0	2.5
6	176.2	173.0	-1.8
7	182.6	182.2	-0.2
8	226.4	237.8	5.0
9	278.1	289.6	4.1
10	339.5	335.4	-1.2
11	405.6	402.2	-0.9
12	480.3	486.1	1.2
13	557.5	548.2	-1.7
14	645.1	634.0	-1.7
15	738.7	726.3	-1.7
16	833.3	821.3	-1.4
17	936.3	913.3	-2.5

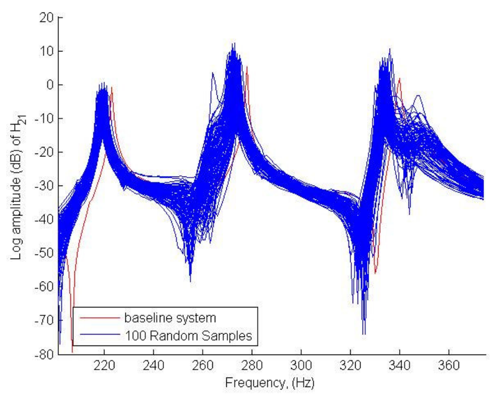
Table 5: Percent Difference between Baseline Frequency and 100 Sample Pooled Mean.



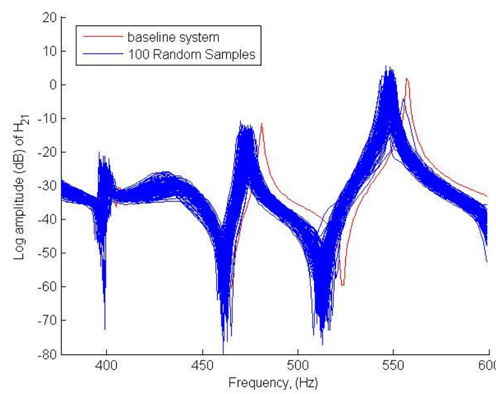
(a) 0-50 Hz Response



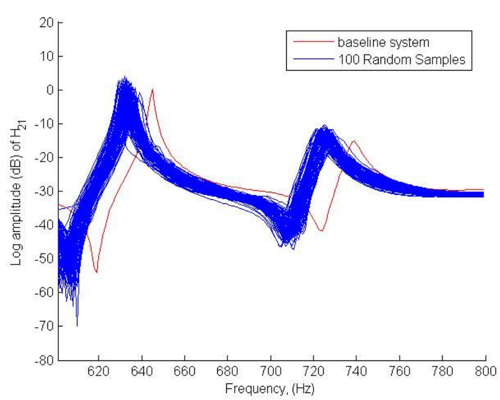
(b) 51-200 Hz Response



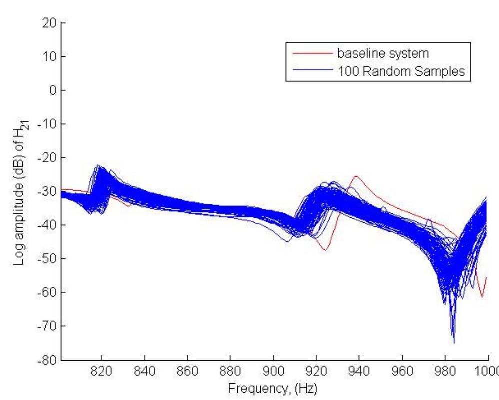
(c) 201-375 Hz Response



(d) 376-600 Hz Response



(e) 601-800 Hz Response



(f) 801-1000 Hz Response

Figure 5: Amplitude of H_{21} FRF Baseline System and 100 Random Sample Distribution.

2.4 Discussion

Extraction of natural frequency information from baseline data indicates good agreement between the three frequency response functions. However, for the 100 sample distribution, certain modal frequencies vary by significant amounts between the three measurement points. In addition, at some points individual modes have a pronounced variation from the mean value. Further investigation, including finite element modeling of the structure, will be undertaken to advance understanding the behaviors observed.

3 Conclusions

This paper has described the extraction and statistical analysis of natural frequency information from a set of experimental data previously generated for a fixed-fixed beam for the purpose of studying methods to quantify uncertainty in the dynamics of structures. Frequencies were extracted in the low frequency range (up to 1000 Hz) from frequency response functions measured at three points on the beam. Because the uncertainty in the response only arises from the randomness in the mass locations, it is possible to isolate and investigate the effect of these changes. Future work planned includes investigation of the higher frequency range, investigation of other modal properties, and finite element modeling of the structure.

References

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