THE NATURE OF EPISTEMIC UNCERTAINTY IN LINEAR DYNAMICAL SYSTEMS

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ABSTRACT

In the modeling of complex dynamical systems, high-resolution finite element models are routinely adopted to reduce the discretization error. This is often implemented by exploiting cost-effective computing hardware through parallel processing to solve the resulting large scale linear systems. Such an approach fails to enhance confidence in simulation-based predictions when the dynamical systems exhibit significant variability in their data and model leading to so-called data and modeling uncertainty. When substantial statistical information is available, data uncertainty can be tackled using probabilistic methods by modeling the parameters of the data as random variables or stochastic processes. Model uncertainty poses significant challenges as no parameter is available *a priori* as opposed to the case of data uncertainty. In modelling complex systems such marine (e.g. ships, submarines) and aerospace systems (e.g. helicopters and space shuttles) modeling uncertainty arises naturally due to the lack of complete knowledge and even presence of many subsystems attached to the main structural components. In the low frequency regions, effect of such substructure may be modeled by rigid masses attached to the primary structures. In higher frequency, the mechanics of energy flow among the primary and secondary systems may not be captured by these rigid masses alone as dynamics of the subsystems becomes more important. The additional degree-of-freedom arising from the subsystems should be incorporated to model the entire system. A sprung-mass models are adopted in the current study to investigate the effect of such subsystem on the vibration of a thin steel plate. The location of the attachments of these sprung-mass systems and their natural frequencies are assumed to be uncertain while the constitutive and geometric properties of the steel plate (e.g. the primary structure) are known. In contrast to the case of data uncertainty (traditionally modeled in the framework of stochastic finite element method), the model uncertainty arising from the sprung-masses (attached randomly to the plate) gives rise to entirely different variety of dynamical system for each sample. This can be observed from the variation in *sparsity structure* of the mass, stiffness and damping matrices of the total system from sample to sample. Clearly such change in sample-wise sparsity pattern can not be modeled by data uncertainty alone. In the case of data uncertainty, the actual configuration of dynamical system remains unchanged, just its local parameters change from sample to sample and therefore, sparsity structure of the system matrices for each sample remains the same. In this study, we investigate the feasibility of adopting a global probabilistic model to represent such entire ensemble of different dynamical systems derived from perturbing the *model* of a baseline system. In the current study, the baseline dynamical system is just the thin plate (without any sprung-mass attachment). A range of dynamical systems is then generated from the random attachment topology of the sprung-masses with the thin plate. As mentioned before, each of these dynamical systems possesses mass, stiffness and damping matrices for which sparsity pattern differ from sample to sample. The objective of this investigation is to represent uncertainty arising from model perturbation. We explore the possibility of stochastic representation of this entire variety derived from the baseline

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system with model perturbation (in contrast to data perturbation). More specifically, in the framework of random matrix theory, we fit the parameters of Wishart random matrices to model uncertainty in the mass, stiffness and damping matrices of the total system, namely the plate having randomly attached sprung-masses.

1 INTRODUCTION

High-resolution finite element models (FEM) are routinely adopted for accurate predictions of dynamical response of complex systems such as aerospace, marine and automotive vehicles. Recent availability of cost-effective high performance computing platforms (e.g. linux clusters) offer practical means to solve such large-scale linear systems using parallel processing. Although such high-resolution numerical models can reduce discretization errors, in numerous cases, experimental results and numerical predictions exhibit significant differences stemming from the uncertainties in the data and models. When substantial statistical information is available, the data uncertainty can be *parametrically* modeled within the framework of probability theory. However, such *parametric* approaches may not be suitable to tackle model uncertainty as the parameters contributing to the modeling errors are a priori unknown.

For complex dynamical systems, the main structural parts (often known as the primary or master structure) are often modeled *determinististically* using the finite element method. Here the underlying assumption is that the constitutive and geometric properties including the boundary conditions are known with sufficient accuracy leading to a well-defined boundary value problem. On the other hand, the substructures (often known as the secondary systems) attached to the primary structure may not be practically accessible for conventional finite element modeling due to the lack of knowledge of such subsystems. Such 'lack of knowledge' may arise due to, but not restricted to,

- the lack of knowledge of the presence of such subsystems,
- the lack of knowledge of their spatial locations with respect to the primary structures
- imprecise and incomplete information about their constitutive and geometric properties, and
- unknown coupling characteristics.

For typical marine and aerospace structures, the cargo, piping, fuel, control cables, electronic systems and bulkheads constitute such subsystems. Soize introduced the concept of *fuzzy* structures [1] to address some of these issues whereby the dynamical effect of the secondary systems are incorporated through *random* impedance functions while analysing the entire dynamical systems. In the low frequency region, chracterized by predominantly modal response, the effect of such subsystems can simply be modeled by randomly distributed added-mass without increasing the total number of degree-of-freedom (DOF). Such approach however renders ineffectual in mid-frequency band whereupon the secondary systems acts as energy sinks by absorbing energy from the primary structures. For transient vibration, the energy thus stored in the secondary systems may transfer to the primary structures when the transient excitation stops. For steady-state excitations, the vibrations energy in the subsystems remains constant. For such energy transfer mechanism, it is necessary to model the subsystems by *sprung-mass systems* (as opposed to just modeled by pure added-mass) thereby increasing the total DOF of the complete system. By explicitly modeling the secondary systems by incorporating additional DOFs not only permits the analyst to accurately estimate the amplitude of the direct- and cross-frequency response functions, but also to predict the average phases, being significant to study *wave-propagation* effects.

In this study, we focuss on the effect of modeling uncertainty emerging from the presence of a set of randomly distributed sprung-masses on a vibrating plate. This system may be construed to simulate the effect of uncertain secondary systems whose spatial attachment locations is not available on the primary structures *a priori* . In particular, the feasibility of adopting Random Matrix Theory (RMT) to model the effect of such secondary systems whose dynamics may influence the response of the primary system. The inverse problem involving fitting the optimal parameters of Wishart random matrices representing the mass, stiffness and damping matrices of the systems from their samples are investigated. These random matrices preserve the positive-definiteness and their mean values are the mass, stiffness and damping matrices of the baseline system. By fitting an average global model, the identification of such statistical system circumvents the need of local parameterization of model uncertainty which may be impractical in many occasions.

2 DYNAMICAL SYSTEM WITH MODEL UNCERTAINTY

The equation of motion of a damped n-degree-of-freedom linear dynamical system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times n}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass, damping and stiffness matrices respectively. In the framework of probability theory, uncertainty in the data and model leads to random mass, stiffness and damping matrices. The importance of considering uncertainty largely depends on the frequency of excitation driving the system. For example, as the wavelengths of the vibration become increasingly smaller with higher frequency, the system response can be very sensitive to the small details of the system. Furthermore, the dynamics of secondary systems play important role as these subsystems can significantly affect both the amplitude and phases of the vibration. In addition to data uncertainty, modeling uncertainty arising from the effect of unknown or imprecisely known subsystems should be considered to improve the accuracy of numerical predictions. The parametric approach generally used to represent the uncertainty in the local system cannot be used to describe the model uncertainty. A non-parametric approach is therefore adopted. We refer to Soize [2–8] and Adhikari [9, 10] for comprehensive details. We only highlight some noteworthy points: (1) the approach circumvents the need for the detailed statistical characterization of local parameters, instead the statistical dispersions of global system properties (e.g. mass, stiffness and damping matrices) are addressed directly. (2) Each sample of such random matrix is symmetric and positive-definite and thus is appropriate model for mass, stiffness and damping matrix of a linear self-adjoint dynamical system. (3) The mean values of the random matrices are the system matrices (e.g. mass, stiffness and damping) of the *baseline* deterministic predictive model. (4) A single parameter controls the statistical dispersion of the model. Such dispersion parameters can be estimated from experimental measurements [11, 12]. Due to such single parametric dependence, it is therefore difficult to match the detailed correlations structure among the elements of the system matrices when such information is indeed available. (5) The approach can concurrently account for data and model uncertainty. However, open question still remains on the inability of the random matrix theory to consider the statistical correlations among the mass, stiffness and damping matrices as well as accounting the correlations among the elements of each of these matrices [13].

In the frequency domain Eq. (1) can be expressed as

$$\mathbf{A}(\omega)\bar{\mathbf{q}}(\omega) = \bar{\mathbf{f}}(\omega) \tag{2}$$

where $\mathbf{A}(\omega) = -\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K}$ is known as the dynamic stiffness matrix, and $(\bar{\bullet})$ denotes Fourier transform of $(\bullet)$. Suppose $\bar{\mathbf{q}}_m$ denotes the degrees-of-freedom of baseline system and $\bar{\mathbf{q}}_u$ denotes the degrees-of-freedom of secondary systems. Then, omitting ω for notational convenience, Eq. (2) can be partitioned as

$$\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{mu} \\ \mathbf{A}_{um} & \mathbf{A}_{uu} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{q}}_m \\ \bar{\mathbf{q}}_u \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{Bmatrix}. \tag{3}$$

In reality, one only knows $\bar{\mathbf{q}}_m$ and $\mathbf{A}_{mm}$ using the conventional finite element method. In most cases, no information regarding $\bar{\mathbf{q}}_u$ is available. The uncertainty associated with these 'unknown' DOFs include their dimension, nature and locations. As a result $\mathbf{A}_{uu}$ and the coupling matrix $\mathbf{A}_{um}$ are also unknown. For example, we consider a plate having a set of randomly attached sprung-mass oscillators. Then $\bar{\mathbf{q}}_m$ denotes DOFs of the plate and $\bar{\mathbf{q}}_u$ represents the additional DOFs arising from the sprung-mass oscillators. Depending on the attachment configuration of sprung-masses, the sparsity of $\mathbf{A}_{um}$ will differ from sample to sample.

Eliminating $\bar{\mathbf{q}}_u$ from Eq. (3) by condensation, one has

$$\left[\mathbf{A}_{mm} - \mathbf{A}_{mu}\mathbf{A}_{uu}^{-1}\mathbf{A}_{um}\right]\bar{\mathbf{q}}_m = \bar{\mathbf{f}} \tag{4}$$

$$\text{or} \quad \left[\mathbf{A}_{mm} + \mathbf{\Delta A}\right]\bar{\mathbf{q}}_m = \bar{\mathbf{f}}, \quad \text{where} \quad \mathbf{\Delta A} = -\mathbf{A}_{mu}\mathbf{A}_{uu}^{-1}\mathbf{A}_{um} \in \mathbb{R}^{n\times n}. \tag{5}$$

This equation shows that whatever may be the nature of uncertainty associated with the DOFs arising from the secondary systems, they randomly perturb the condensed 'baseline' matrix $\mathbf{A}_{mm}$ by $\mathbf{\Delta A}$. Moreover, from Eq. (4) it is clear that sparsity structures associated with deterministic matrix $\mathbf{A}_{mm}$ and $\mathbf{A}_{mm} + \mathbf{\Delta A}$ are different. From this analysis, the following two fundamental facts regarding the uncertainty associated with the*model perturbation* in a linear system have emerged:

- The model perturbation manifests as a *random* perturbation to the condensed baseline dynamic stiffness matrix obtained using conventional finite element method. This uncertainty includes, but not limited to, the number, locations, and dynamic properties associated with attached subsystems.

- The *sparsity structure* associated with the condensed baseline finite element matrices *are not preserved*. Recall that parametric uncertainty quantification methods such as stochastic finite element methods always preserve the sparsity structure in the system matrices. We conjecture that a fundamental consequence of considering uncertainty arising from model perturbation is the loss of the sparsity structure of the condensed system matrices. Clearly no parametric method can address this issue in its present form.

These two conclusions leads us to consider random matrix theory to model these effects.

3 BACKGROUND OF THE RANDOM MATRIX THEORY

The probability density function of a random matrix can be defined in a manner similar to that of a random variable or random vector. If $\mathbf{A}$ is a $n \times m$ real random matrix, then the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}^{n\times m}$, denoted by $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}^{n\times m} \to \mathbb{R}$. Here we define probability density functions of few random matrices which are relevant to stochastic mechanics problems.

Gaussian random matrix: A rectangular random matrix $\mathbf{X} \in \mathbb{R}^{n\times p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}^{n\times p}$ and covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}$, where $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \left|\boldsymbol{\Sigma}\right|^{-p/2} \left|\boldsymbol{\Psi}\right|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\}. \tag{6}$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}\left(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}\right)$.

Symmetric Gaussian random matrix: Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ be a symmetric random matrix and $\mathbf{M}, \boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$ are $n \times n$ constant matrices such that the commutative relation $\boldsymbol{\Sigma}\boldsymbol{\Psi} = \boldsymbol{\Psi}\boldsymbol{\Sigma}$ holds. If the $n(n+1)/2 \times 1$ vector $\operatorname{vecp}(\mathbf{Y})$ formed from $\mathbf{Y}$ is distributed as $N_{n(n+1)/2,1}\left(\operatorname{vecp}(\mathbf{M}), \mathbf{B}_n^T(\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})\mathbf{B}_n\right)$, then $\mathbf{Y}$ is said to have symmetric matrix variate Gaussian distribution with mean $\mathbf{M}$ and covariance matrix $\mathbf{B}_n^T(\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})\mathbf{B}_n$ and its pdf is given by

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-n(n+1)/4} \left|\mathbf{B}_n^T(\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})\mathbf{B}_n\right|^{-1/2} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{Y}-\mathbf{M})^T\right\}. \tag{7}$$

This distribution is usually denoted as $\mathbf{Y} = \mathbf{Y}^T \sim SN_{n,n}\left(\mathbf{M}, \mathbf{B}_n^T(\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})\mathbf{B}_n\right)$.

For a symmetric matrix $\mathbf{Y} \in \mathbb{R}^{n\times n}$, $\operatorname{vecp}(\mathbf{Y})$ is a $n(n+1)/2$-dimensional column vector formed from the elements above and including the diagonal of $\mathbf{Y}$ taken columnwise. The elements of the translation matrix $\mathbf{B}_n \in \mathbb{R}^{n^2\times n(n+1)/2}$ are given by

$$(B_n)_{ij,gh} = \frac{1}{2}\left(\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg}\right), \quad i \leq n, j \leq n, g \leq h \leq n, \tag{8}$$

where δ_{ij} is the usual Kronecker's delta.

Wishart matrix: A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)\left|\boldsymbol{\Sigma}\right|^{\frac{1}{2}p}\right\}^{-1} \left|\mathbf{S}\right|^{\frac{1}{2}(p-n-1)} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right\}. \tag{9}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$. Using a maximum entropy approach, Adhikari [9, 10] proved that the system matrices arising in linear structural dynamics should be Wishart matrices.

Matrix variate gamma distribution: A $n \times n$ symmetric positive definite random matrix $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\boldsymbol{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a)\left|\boldsymbol{\Psi}\right|^{-a}\right\}^{-1} \left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\boldsymbol{\Psi}\mathbf{W}\right\}; \quad \Re(a) > \frac{1}{2}(n-1). \tag{10}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \boldsymbol{\Psi})$. The matrix variate gamma distribution was used by Soize [2–8] for the random system matrices of linear dynamical systems.

In Eqs. (9) and (10), the function $\Gamma_n(a)$ is the multivariate gamma function, which can be expressed in terms of products of the univariate gamma functions as

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \quad \text{for} \quad \Re(a) > \frac{1}{2}(n-1). \tag{11}$$

For more details on the matrix variate distributions we refer to the books by Tulino and Verdú [14], Gupta and Nagar [15], Eaton [16], Muirhead [17], Girko [18] and references therein. Among the four types of random matrices introduced above, the distributions given by Eqs. (9) and (10) will always result in symmetric and positive definite matrices. Therefore, they can be possible candidates for modeling random system matrices arising in probabilistic structural mechanics.

3.1 Wishart matrices in structural dynamics

Suppose that the mean values of **M**, **C** and **K** are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. This information is likely to be available, for example, using the deterministic finite element method. However, there are uncertainties associated with our modelling so that **M**, **C** and **K** are actually random matrices. The distribution of these random matrices should be such that they are

(a) symmetric

(b) positive-definite, and

(c) the moments of the inverse of the dynamic stiffness matrix

$$\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \tag{12}$$

should exist $\forall\,\omega$. That is, if $\mathbf{H}(\omega)$ is the frequency response function (FRF) matrix

$$\mathbf{H}(\omega) = \mathbf{D}^{-1}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right]^{-1} \tag{13}$$

then the following condition must be satisfied for positive values of ν:

$$\mathrm{E}\left[\|\mathbf{H}(\omega)\|_{\mathrm{F}}^{\nu}\right] < \infty, \quad \forall\,\omega \tag{14}$$

Because the matrices **M**, **C** and **K** have similar probabilistic characteristics, for notational convenience we will use the notation **G** which stands for any one the system matrices. Using the maximum entropy approach Adhikari [9, 10] showed that the optimal distribution **G** is given by $\mathbf{G} \sim W_n(p, \boldsymbol{\Sigma})$, where

$$p = n + 1 + \theta \tag{15}$$

$$\text{and} \quad \boldsymbol{\Sigma} = \overline{\mathbf{G}}/\alpha. \tag{16}$$

The constants θ and α are obtained as

$$\theta = \frac{1}{\delta_G^2}\left\{1 + \frac{\{\mathrm{Trace}\,(\overline{\mathbf{G}})\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}\right\} - (n+1) \tag{17}$$

$$\text{and} \quad \alpha = \sqrt{\theta(n+1+\theta)}. \tag{18}$$

Here δ_G is known as the dispersion parameter which characterize the uncertainty in the random matrix **G**. The parameter δ_G is defined as

$$\delta_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}[\mathbf{G}]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}[\mathbf{G}]\|_{\mathrm{F}}^2}. \tag{19}$$

From this expression observe that δ_G can be viewed as the mean-normalized standard deviation of the random matrix **G**. Soize [2–8] proved that the system matrices of a linear system should follow the matrix variate Gamma distribution, which in turn is related to the Wishart distribution as $W_n(p, \boldsymbol{\Sigma}) = G_n\left(p/2, \frac{1}{2}\boldsymbol{\Sigma}^{-1}\right)$. The first moment (mean) and the elements of the covariance tensor is given by [15]

$$\mathrm{E}[\mathbf{G}] = p\boldsymbol{\Sigma} = p\overline{\mathbf{G}}/\alpha \tag{20}$$

$$\mathrm{cov}\left(G_{ij}, G_{kl}\right) = p\left(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}\right) = \frac{1}{\theta}\left(\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk}\right). \tag{21}$$

Note that in Eq. (21), the values of $\overline{G}_{ik}$ etc. are fixed by the mean matrix. Therefore, the *only* parameter which controls the uncertainty in the distribution is θ.

4 A PLATE WITH DISORDERLY-ATTACHED SPRUNG-MASS OSCILLATORS HAVING RANDOM NATURAL FREQUENCIES

As a numerical illustration, we study the dynamics of a thin (Mindlin) plate. The following properties of the plate are considered: Young's modulus $E = 200 \times 10^9$ N/m, Poisson's ratio $\nu = 0.3$, density $\rho = 7800$ kg/m^3, thickness $t = 3$ mm, length $L_x = 0.998$ m and width $L_y = 0.53$ m. The plate is excited by a unit harmonic force and the response is calculated at the points shown in the diagram. The standard four-noded thin plate bending element (resulting 12 degrees of freedom per element) is used. The plate is divided into 25 elements in the x-axis and 15 elements in the y-axis for the numerical calculations. The resulting system has 1200 degrees of freedom so that $n = 1200$. Here we investigate the effect of randomly attached $n_r = 10$ sprung-mass oscillators having natural frequencies uniformly distributed between $(1 - 4)$ KHz. Fig. 1 shows the locations of the input and output (measurement) points. Fig. 2 shows the amplitude of cross and direct frequency response functions (FRF) of the plate

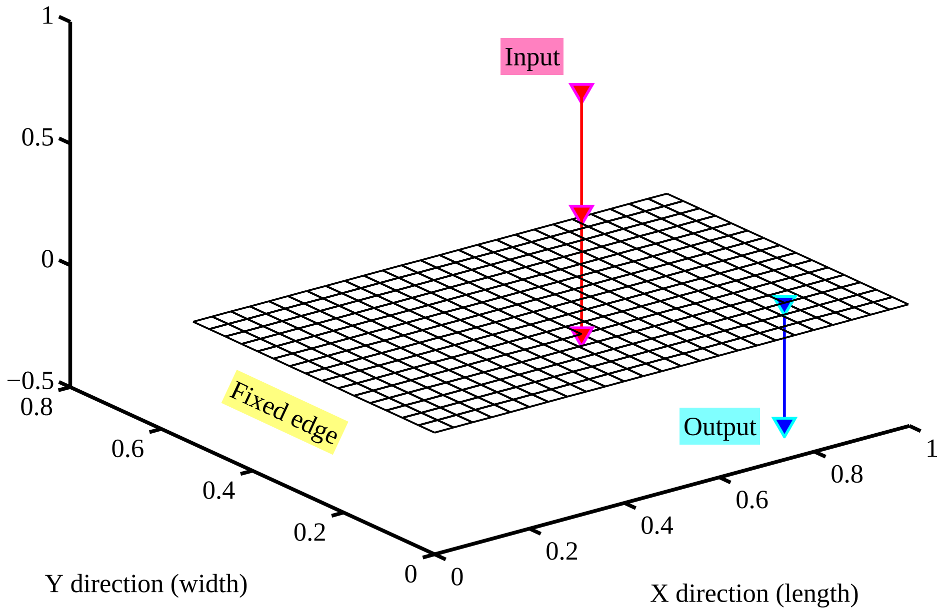
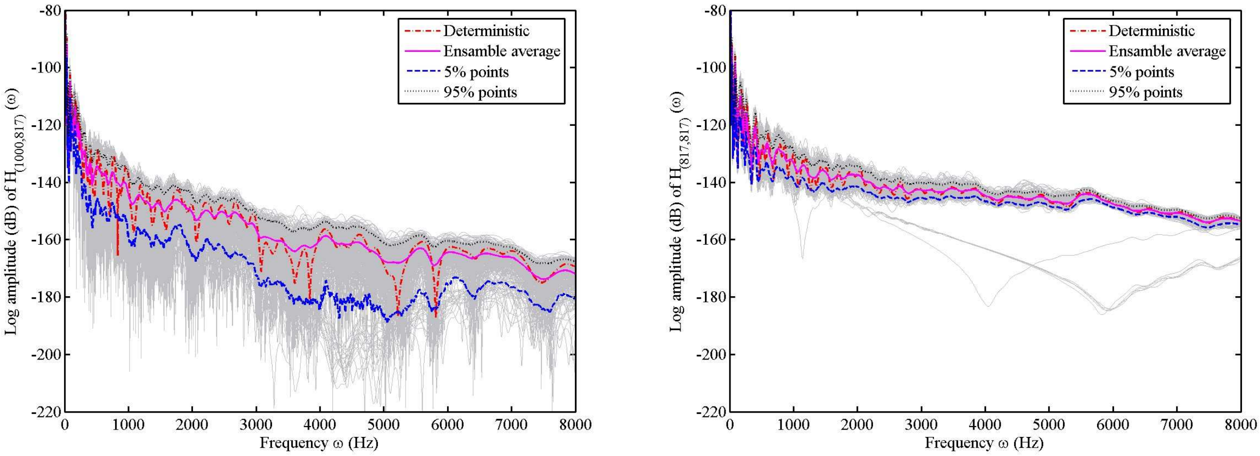


Fig. 1: A steel cantilever plate: $E = 200 \times 10^9$**N/m**^{2}**,** $\mu = 0.3$**,** $\rho = 7860$**kg/m**^{3}**,** $t = 3.0$**mm,** $L_x = 0.998$**m,** $L_y = 0.53$**m.**

alone (without attached sprung-masses) and 500 samples of the plate with random sprung-mass attachments. The sample frequency response functions show significant variability as clearly observed in Fig. 2 due to uncertainty stemming from both the randomness in the attachment points and natural frequencies of the sprung-mass oscillators.

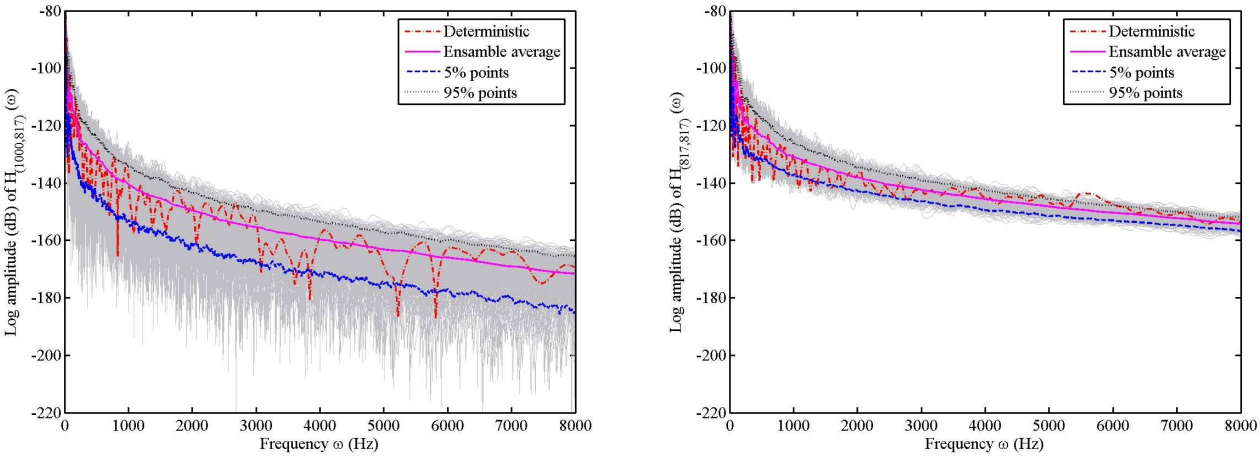
We fit the dispersion parameters δ_G in Eq. (19) of Wishart matrices from the samples of mass, stiffness and damping of the plate attached with sprung-mass oscillators. For this example a constant 2% modal damping is assumed so that $\delta_C = 0$. Here we consider that the attached springs have random stiffness values but the mass values are the same. By augmenting mass matrix to include the additional degree of freedom it can be seen that the mass matrix remains constant for different random samples. As a result $\delta_M = 0$. The value of δ_K is obtained directly from Eq. (19) as $\delta_K = 0.4754$. Using this, Monte Carlo simulation is used to generate samples from the Wishart random matrix distribution. The cross and direct FRFs generated from the sample mass, stiffness and damping of these Wishart matrices are presented in Fig. 3

The mean and standard deviations of the cross and direct FRFs functions estimated from actual samples (in Fig. 2) and those from the Wishart matrices (in Fig. 3) are plotted in Fig. 4. Means and standard deviations estimated from both the methods agree very well in the medium and high frequency ranges. The predicted 5% and 95% probability points using the direct Monte Carlo simulation and the random matrix approach are compared in Fig. 5. The essential feature of these plots are similar to the standard deviation plots shown before, that is, the results from both approach match well in the medium and high frequency regions. However, such agreement between the results is not so well in the low-frequency regions. Considering the fact that the *only* uncertainty related information used in the random matrix method is the value of δ_K, these results are encouraging.



(a) Cross FRF. (b) Direct FRF.

Fig. 2: Direct Monte Carlo Simulation of the amplitude of the FRFs of the plate with randomly placed oscillators.



(a) Cross FRF. (b) Direct FRF.

Fig. 3: Monte Carlo Simulation of the amplitude of FRFs of the plate using optimal Wishart stiffness matrix, $n = 1200$ and $\delta_K = 0.4754$.

5 CONCLUSIONS

In this study, we attempted to represent model uncertainty in linear dynamical system using Random Matrix Theory (RMT). The study explored practical means to represent the entire variety of dynamical systems derived from the model perturbation of a baseline system. In particular, we investigate the feasibility of adopting random matrices to account for uncertainty derived model perturbation. As an illustration, we considered model uncertainty in a vibrating plate due to disorderly attached sprung-mass systems having random natural frequencies. The encouraging agreements between the results obtained the from Wishart matrix model and direct Monte Carlo simulation suggest that it may be a practical method to represent the statistical dispersion observed in the response of dynamical systems arising from model uncertainty.

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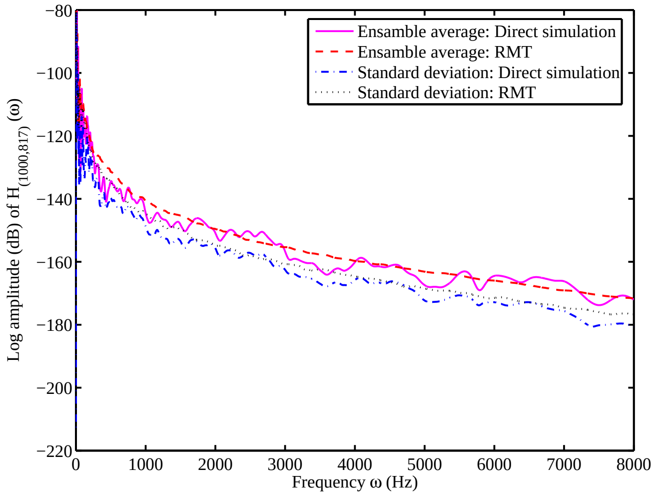
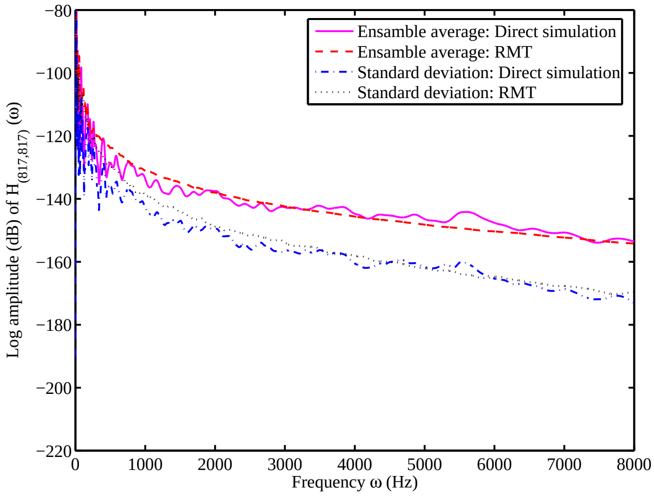
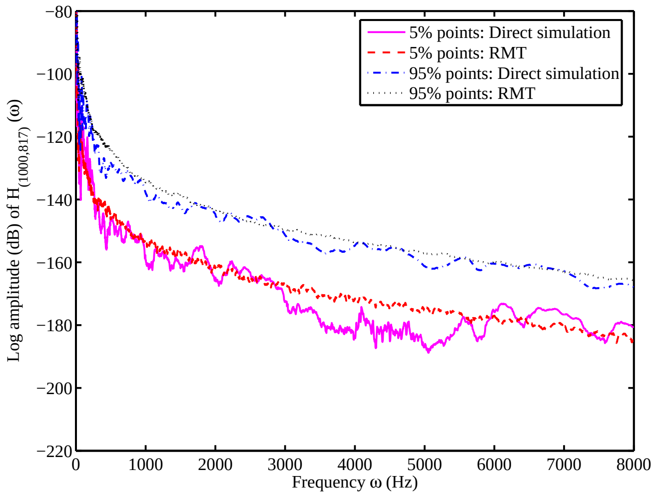


(a) Cross FRF.

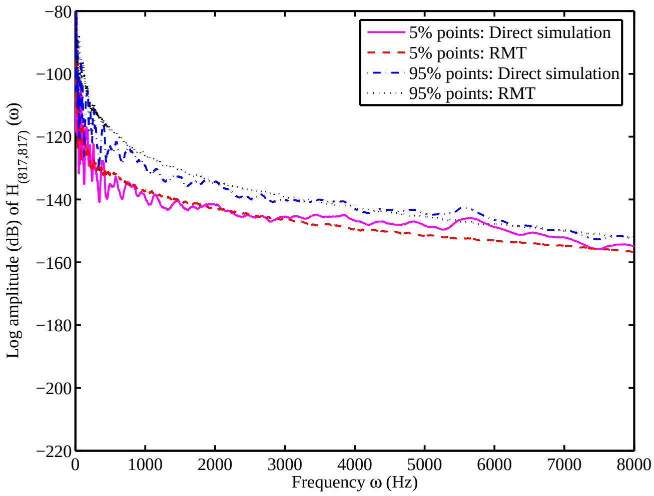


(b) Direct FRF.

Fig. 4: Comparison of the mean and standard deviation of the amplitude of the FRF obtained using the direct simulation simulation and proposed random matrix approach.



(a) Cross FRF.



(b) Direct FRF.

Fig. 5: Comparison of the 5% and 95% probability points of the amplitude of the FRF obtained using the direct simulation simulation and proposed random matrix approach.

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NOMENCLATURE

$\mathbf{A}(\omega)$	dynamic stiffness matrix
$\mathbf{B}_n$	$n^2 \times n(n+1)/2$-dimensional translation matrix
$\mathbf{D}(\omega)$	dynamic stiffness matrix
$\mathbf{f}(t)$	forcing vector
$\mathbf{H}(\omega)$	frequency response function (FRF) matrix
$\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$	mass, damping and stiffness matrices respectively
$\mathbf{q}(t)$	response vector
δ_{ij}	Kronecker's delta function
$\Gamma_n(a)$	multivariate gamma function
ν	order of the inverse-moment constraint
ω	excitation frequency
n	number of degrees of freedom
$p, \mathbf{\Sigma}$	scalar and matrix parameters of the Wishart distribution
$(\bullet)^T$	matrix transposition
$(\bar{\bullet})$	Fourier transform of $(\bullet)$
$\mathbb{R}$	space of real numbers
$\mathbb{R}_n^+$	space $n \times n$ real positive definite matrices
$\mathbb{R}^{n \times m}$	space $n \times m$ real matrices
$\lvert\bullet\rvert$	determinant of a matrix
$\mathrm{etr}\{\bullet\}$	$\exp\{\mathrm{Trace}(\bullet)\}$
$\lVert\bullet\rVert_{\mathrm{F}}$	Frobenius norm of a matrix, $\lVert\bullet\rVert_{\mathrm{F}} = \left(\mathrm{Trace}\left((\bullet)(\bullet)^T\right)\right)^{1/2}$
$\otimes$	Kronecker product (see [19])
$\sim$	distributed as
$\mathrm{Trace}(\bullet)$	sum of the diagonal elements of a matrix
cdf	cumulative distribution function
FRF	Frequency Response Function
pdf	probably density function
RMT	Random Matrix Theory
SFEM	Stochastic Finite Element Method