Proceedings of the Eighth International Conference
on Computational Structures Technology,
B.H.V. Topping, G. Montero and R. Montenegro,
(Editors), Civil-Comp Press, Stirlingshire, Scotland.

Paper 73

Identification of Damping Using Proper Orthogonal Decomposition

M. Khalil†, S. Adhikari‡ and A. Sarkar†
† Department of Civil and Environmental Engineering
Carleton University, Ottawa, Canada
‡ Department of Aerospace Engineering
University of Bristol, United Kingdom

Abstract

A new efficient method for identification of the damping matrix of general linear dynamic systems is proposed. For a given frequency band, the proposed method estimates the equivalent reduced-order damping matrix of a general linear dynamical system, with the mass and stiffness matrices known a priori. Proper orthogonal decomposition (POD) method is used to represent an optimal reduced order model in the frequency range of interest. To mitigate the ill-conditioned nature of this inverse problem, Tikhonov regularisation is applied. The proposed methodology circumvents, or at least alleviates, the difficulties encountered in applying conventional modal analysis techniques in the high and mid frequency range.

Keywords: system identification, proper orthogonal decomposition, Tikhonov regularisation, damping matrix identification, least squares estimation, constrained optimisation, Kronecker algebra.

1 Introduction

System identification plays a crucial role in the validation of numerical models. In the context of damped multiple-degree-of-freedom linear dynamical systems, the process of system identification involves identification of the mass, damping and stiffness matrices. In spite of extensive research, the knowledge regarding damping forces is least developed compared to the other forces acting on a structure. There are two basic reasons for this. Firstly, from theoretical point of view, it is not in general clear which state variables are relevant to determine the damping forces. This fact makes it difficult to choose on what mathematical form of damping model should be used at the first place, let alone how to identify its parameters from experimental measurements. Secondly, from experimental point of view, unlike the mass and stiffness properties, the damping properties can be identified only by a dynamic testing. Traditionally this has been achieved using the experimental modal analysis [1, 2, 3].

The most common method to model damping in multiple-degree-of-freedom linear systems is to assume the so called *viscous damping*. Many researches have proposed methods to identify viscous damping matrix from experimental measurements (see Pilkey and Inman [4] for a survey). These methods can be divided into two broad categories [5]: (a) damping identification from modal testing and analysis [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], and (b) direct damping identification from the forced response measurements in the frequency or time domain [5, 19, 20, 21, 22, 23, 24, 25]. The modal method entails estimating the modal parameters, such as natural frequencies, damping ratio and mode shapes, from the measured transfer functions, and consequently the damping matrix are reconstructed from the estimated modal parameters. Direct methods on the other hand bypass the modal parameter extraction step and fits the damping matrix to the measured response.

Each of the above methods have their own advantages and disadvantages. The common issues regarding the identification of the system matrices using conventional modal analysis are (a) the accuracy of the identified modal parameters, and consequently the system matrices, relies on the number of distinct 'peaks' in the measured frequency response functions (FRFs), and (b) if the damping is non-proportional, the identification of complex modes poses a serious challenge [26, 27]. The first problem is inherent to conventional modal analysis. If the peaks in the measured FRFs are not distinct or are closely spaced, the modal parameter extraction procedure is difficult to apply [1]. As a consequence, the identified system matrices using the extracted modal parameters become erroneous. It is therefore difficult to extend the modal identification procedure in the mid-frequency range, or for periodic systems (such as bladed disks in turbomachineries) inherently containing closely spaced modes. The second problem arises for systems with high damping materials such as a panel with viscoelastic damping. In this paper a new approach using proper orthogonal decomposition is proposed which can potentially circumvent, or at least alleviate, these difficulties.

The POD method, also known as Principle Component Analysis (PCA), entails the extraction of the dominant eigen-subspace of the response correlation matrix, namely *proper orthogonal modes* (POMs), over a given frequency band of interest. These dominant eigenvectors, or POMs, span the system response *optimally* on the prescribed frequency range of interest. The methodology permits the construction of an optimal and adaptive reduced-order model of the dynamical system. The POD method has been used for the model reduction of linear as well as non-linear dynamical systems, for example see [28, 29, 30]. POD has also been used for the identification of a non-linear dynamical system whereby the non-linear stiffness parameter was identified with a reduced-order model [31].

The primary objective of this investigation is the identification of the damping matrix of a general linear dynamical system in the medium frequency range, with the mass and stiffness matrices being known a priori with reasonable assumption.

To achieve this objective, the POD method is adopted as a tool for model reduction strategy to solve the inverse problem involving system identification over a frequency range of interest.

2 Outline of the Proposed Method

The system of equations describing the forced vibration of a viscously damped linear discrete system with n degrees of freedom can be represented by

$$\mathbf{M}_n\ddot{\mathbf{q}}(t) + \mathbf{C}_n\dot{\mathbf{q}}(t) + \mathbf{K}_n\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}_n \in \mathbb{R}^{n\times n}$ is the mass matrix, $\mathbf{C}_n \in \mathbb{R}^{n\times n}$ is the damping matrix, $\mathbf{K}_n \in \mathbb{R}^{n\times n}$ is the stiffness matrix, $\mathbf{q}(t) \in \mathbb{R}^n$ is the displacement vector, and $\mathbf{f}(t) \in \mathbb{R}^n$ is the forcing vector at time t. With the knowledge of $\mathbf{M}_n$ and $\mathbf{K}_n$, our aim is to identify $\mathbf{C}_n$ from the measured dynamic response.

2.1 Reduced Order Modelling using Proper Orthogonal Decomposition

The dynamic response due to external excitations are normally recorded by piezoelectric accelerometers and these analogue signals are then converted to digital signals using an analogue to digital converter card (ADC). The response vector $\mathbf{q}(t)$ due to this forcing is normally stored in the discrete time format so that

$$\mathbf{q}_d(j) = \mathbf{q}(j\,\Delta t), j = 1, 2, 3, \cdots, J \tag{2}$$

where Δt is the uniform time step used in data acquisition card and J is the number of steps used in the measurement. We can form the matrix

$$\widehat{\mathbf{Q}} = [\mathbf{q}_d(1), \mathbf{q}_d(2), \cdots, \mathbf{q}_d(J)] \in \mathbb{R}^{n\times J} \tag{3}$$

In the continuous time domain, we obtain the correlation matrix, $\mathbf{C}_{qq} \in \mathbb{R}^{n\times n}$, to be used in the proper orthogonal decomposition method, as

$$\mathbf{C}_{qq} = \langle \mathbf{q}(t)\mathbf{q}(t)^T \rangle \tag{4}$$

where $\langle\cdot\rangle$ is the time averaging operator. In the discrete time domain, equation (4) becomes

$$\mathbf{C}_{qq} = \frac{1}{J}\widehat{\mathbf{Q}}\widehat{\mathbf{Q}}^T \tag{5}$$

The above matrix is symmetric and positive definite. The correlation matrix can also be expressed in the continuous frequency domain [32] by

$$\mathbf{C}_{qq} = \int_B \Re\left\{\mathbf{H}^\dagger(\omega)\mathbf{H}(\omega)\right\} d\omega \tag{6}$$

where $\mathbf{H}(\omega)$ is the system transfer matrix, $(\cdot)^\dagger$ is the is the complex conjugate transpose (Hermitian) operator, and B is the frequency bandwidth of interest. In equation (6), it is implied that the forcing used to excite the system is an incoherent band-limited stationary vector white noise of unit strength. In the discrete frequency domain, the correlation matrix can be obtained using

$$\mathbf{C}_{qq} = \frac{1}{J} \sum_{i=1}^{J} \Re \left\{ \mathbf{H}^\dagger(\omega_i) \mathbf{H}(\omega_i) \right\} \tag{7}$$

where $\omega_1, \omega_2, \ldots, \omega_J \in B$ are the discrete frequencies at which $H(\omega)$ is evaluated.

Using the spectral decomposition of $\mathbf{C}_{qq}$ one obtains

$$\mathbf{C}_{qq} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \tag{8}$$

where $\mathbf{v}_i \in \mathbb{R}^n$ are the eigenvectors of $\mathbf{C}_{qq}$, and λ_i are the corresponding eigenvalues, arranged such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Due to the symmetry and positive definiteness of $\mathbf{C}_{qq}$, all eigenvalues are positive and the set of eigenvectors forms an orthonormal basis. For the POD method, it turns out that the first few modes are the most important, i.e. $\mathbf{C}_{qq}$ can be approximated by

$$\mathbf{C}_{qq} \approx \sum_{i=1}^{m} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \tag{9}$$

where m is the number of dominant POD modes. The optimal dimension, m, of the reduced-order system is determined such that

$$\frac{\sum_{i=1}^{m} \lambda_i}{\sum_{i=1}^{n} \lambda_i} \geq 99.5\% \tag{10}$$

assuming that POD is required to capture 99.5% of the energy of the response signal. In general, m is much smaller than n for a narrow frequency band.

For convenience we define the transformation matrix as

$$\mathbf{U} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m] \in \mathbb{R}^{n \times m} \tag{11}$$

Using this, it is possible to transform the n-dimensional displacement vector $\mathbf{q}(t)$ and the forcing vector $\mathbf{f}(t)$ into a much reduced dimension as

$$\mathbf{q}_m(t) = \mathbf{U}^T \mathbf{q}(t) \in \mathbb{R}^m \tag{12}$$

$$\text{and} \quad \mathbf{f}_m(t) = \mathbf{U}^T \mathbf{f}(t) \in \mathbb{R}^m \tag{13}$$

Consequently, the original equilibrium equation (1) reduces to

$$\mathbf{M}_m \ddot{\mathbf{q}}_m(t) + \mathbf{C}_m \dot{\mathbf{q}}_m(t) + \mathbf{K}_m \mathbf{q}_m(t) = \mathbf{f}_m(t) \tag{14}$$

where

$$\mathbf{M}_m = \mathbf{U}^T\mathbf{M}_n\mathbf{U} \in \mathbb{R}^{m\times m} \tag{15}$$

$$\mathbf{C}_m = \mathbf{U}^T\mathbf{C}_n\mathbf{U} \in \mathbb{R}^{m\times m} \tag{16}$$

$$\mathbf{K}_m = \mathbf{U}^T\mathbf{K}_n\mathbf{U} \in \mathbb{R}^{m\times m} \tag{17}$$

Transforming equation (14) into the frequency domain one has

$$\left[-\omega^2\mathbf{M}_m + \mathrm{i}\omega\mathbf{C}_m + \mathbf{K}_m\right]\mathbf{Q}_m(\omega) = \mathbf{F}_m(\omega) \tag{18}$$

where $\mathbf{Q}_m(\omega) \in \mathbb{C}^m$ and $\mathbf{F}_m(\omega) \in \mathbb{C}^m$ are the Fourier transforms of $\mathbf{q}_m(t)$ and $\mathbf{f}_m(t)$, respectively.

Equation (18) is the reduced order model to be used for system identification. Recall that $\mathbf{Q}_m(\omega)$ and $\mathbf{F}_m(\omega)$ are known from the original measurements. Furthermore, knowledge of the original mass and stiffness matrices, $\mathbf{M}_n$ and $\mathbf{K}_n$, allows us to compute $\mathbf{M}_m$ and $\mathbf{K}_m$ using (15) and (17), respectively. Thus $\mathbf{C}_m$ is the only unknown in (18). The advantage of using equation (18) is that the number of unknowns are now greatly reduced compared to the original system; $\mathbf{C}_m$ has $m(m+1)/2$ number of unknowns, which is much smaller compared to $n(n+1)/2$ of unknowns present in $\mathbf{C}_n$.

2.2 Reduced Order System Identification

With the aid of Kronecker Algebra [33], equation (18) can be rewritten as

$$\left[\mathrm{i}\omega\mathbf{Q}_m(\omega)^T \otimes \mathbf{I}_m\right] vec\left(\mathbf{C}_m\right) = \mathbf{F}_m(\omega) + \omega^2\mathbf{M}_m\mathbf{Q}_m(\omega) - \mathbf{K}_m\mathbf{Q}_m(\omega) \tag{19}$$

where $\otimes$ denotes the Kronecker matrix product, or tensor product, and $vec\left(\cdot\right)$ denotes the vectorisation operator (see Appendix for details). $\mathbf{I}_m$ is the identity matrix of order m. Equation (19) has only one unknown, namely the reduced order damping matrix, $\mathbf{C}_m$. Having measured the system's response vector at J different frequencies in the frequency band of interest, we can rewrite the generalised form of equation (19) as follows

$$\begin{bmatrix} \mathrm{i}\omega_i\mathbf{Q}_m(\omega_1)^T \otimes \mathbf{I}_m \\ \mathrm{i}\omega_2\mathbf{Q}_m(\omega_2)^T \otimes \mathbf{I}_m \\ \vdots \\ \mathrm{i}\omega_J\mathbf{Q}_m(\omega_J)^T \otimes \mathbf{I}_m \end{bmatrix} vec\left(\mathbf{C}_m\right) = \begin{Bmatrix} \mathbf{F}_m(\omega_1) + \omega_1^2\mathbf{M}_m\mathbf{Q}_m(\omega_1) - \mathbf{K}_m\mathbf{Q}_m(\omega_1) \\ \mathbf{F}_m(\omega_2) + \omega_2^2\mathbf{M}_m\mathbf{Q}_m(\omega_2) - \mathbf{K}_m\mathbf{Q}_m(\omega_2) \\ \vdots \\ \mathbf{F}_m(\omega_J) + \omega_J^2\mathbf{M}_m\mathbf{Q}_m(\omega_J) - \mathbf{K}_m\mathbf{Q}_m(\omega_J) \end{Bmatrix}. \tag{20}$$

Equation (20) can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{y} \tag{21}$$

where $\mathbf{x} \in \mathbb{R}^{m^2}$ is equal to $vec\left(\mathbf{C}_m\right)$, $\mathbf{A} \in \mathbb{C}^{mJ\times m^2}$ is equal to the matrix on the left side of equation (20), and $\mathbf{y} \in \mathbb{C}^m J$ is the vector on the right side of equation (20). The above system of equations is overdetermined in the case where $J > m$. The vectorised equivalent of the reduced order damping matrix, $\mathbf{x}$, and consequently the reduced order damping matrix itself, $\mathbf{C}_m$, can be solved in the least-square sense using the least-square inverse of the matrix $\mathbf{A}$, as follows

$$\mathbf{x} = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{y}. \tag{22}$$

2.3 Regularisation

In equation (21), the estimated vector $\widetilde{\mathbf{x}}$ is good provided that it gives rise to $\mathbf{A}\widetilde{\mathbf{x}}$ that is close to $\mathbf{y}$. We proposed, in the last section, to solve for $\mathbf{x}$ in the least square sense. In the case of matrix $\mathbf{A}$ having rank less than m^2, there exist zero singular value(s) of $\mathbf{A}$. The solution vector $\mathbf{x}$, in that case, will have two components. One component lies in the subspace spanned by the singular vectors of $\mathbf{A}$ with nonzero singular values. The other non-zero component exists in the subspace spanned by the singular vectors with zero singular values. Only the first component can be reasonably estimated from the data set $\mathbf{y}$.

In order to estimate the component of $\mathbf{x}$ that lies in the null-space of $\mathbf{A}$, we need to exploit some prior information concerning the system. For example, we can decide to choose the solution that gives rise to an estimated damping matrix that is most symmetric, if we know beforehand that the damping matrix is symmetric. Mathematically, this problem can be posed as a constrained optimisation problem.

Consider the reduced order damping matrix $\mathbf{C}_m = [c_{ij}]$ having the scalar c_{ij} in the ith row and jth column. Let e_i be a column vector of order m with a unit in the ith position and zeros elsewhere. Similarly, e_i^T is the row vector of order m with a unit in the ith position and zeros elsewhere. Then $e_i \otimes e_j^T$ is an $m \times m$ matrix with a unit in the ijth position and zeros elsewhere. We can thus rewrite the reduced order damping matrix as

$$\mathbf{C}_m = \sum_{j=1}^{m} \sum_{i=1}^{m} c_{ij} \left(e_i \otimes e_j^T \right) \tag{23}$$

The transpose of the damping matrix is thus equivalent to

$$\mathbf{C}_m^T = \sum_{j=1}^{m} \sum_{i=1}^{m} c_{ij} \left(e_j \otimes e_i^T \right) \tag{24}$$

Applying the $vec\left(\cdot\right)$ operator on (24), we obtain

$$vec\left(\mathbf{C}_m^T\right) = vec\left(\sum_{j=1}^{m} \sum_{i=1}^{m} c_{ij} \left(e_j \otimes e_i^T \right) \right) \tag{25}$$

which is equivalent to

$$vec\left(\mathbf{C}_m^T\right) = \sum_{j=1}^{m} \sum_{i=1}^{m} c_{ij} vec\left(e_j \otimes e_i^T \right) \tag{26}$$

Equation (26) can be rewritten as

$$vec\left(\mathbf{C}_m^T\right) = \sum_{j=1}^{m} \sum_{i=1}^{m} c_{ij} \left(e_j \otimes e_i \right) \tag{27}$$

and simplified to

$$vec\left(\mathbf{C}_m^T\right) = \sum_{j=1}^{m}\sum_{i=1}^{m} c_{ij} p_{ij} \tag{28}$$

where each $p_{ij} = (e_j \otimes e_i)$ term is a column vector of order m^2.

In product form, $vec\left(\mathbf{C}_m^T\right)$ can be rewritten as

$$vec\left(\mathbf{C}_m^T\right) = \left[\begin{array}{c} p_{11}, p_{21}, \cdots, p_{m1} \cdots p_{1m}, p_{2m}, \cdots, p_{mm} \end{array}\right] \left\{\begin{array}{c} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \\ \vdots \\ c_{1m} \\ c_{2m} \\ \vdots \\ c_{mm} \end{array}\right\} \tag{29}$$

In equation (29), the vector containing the damping matrix elements is just simply $vec\left(\mathbf{C}_m\right)$. Therefore, $vec\left(\mathbf{C}_m^T\right)$ can be written in terms of $vec\left(\mathbf{C}_m\right)$ as

$$vec\left(\mathbf{C}_m^T\right) = \mathbf{P}\, vec\left(\mathbf{C}_m\right) \tag{30}$$

The transformation operator **P** has multiple nomenclature, such as the vec-permutation matrix [34], the commutation matrix [35], or the tensor commutator [36], having the form

$$\mathbf{P} = \left[\begin{array}{c} p_{11}, p_{21}, \cdots, p_{m1}, p_{12}, p_{22}, \cdots, p_{m2}, \cdots p_{1m}, p_{2m}, \cdots, p_{mm} \end{array}\right]. \tag{31}$$

To satisfy symmetry, we need to have

$$\mathbf{C}_m = \mathbf{C}_m^T \tag{32}$$

Combining the terms in (32) on one side, and applying the $vec\,()$ operator, we obtain

$$vec\left(\mathbf{C}_m\right) - vec\left(\mathbf{C}_m^T\right) = vec\left(\mathbf{0}_{m\times m}\right) \tag{33}$$

where $\mathbf{0}_{m\times m}$ is a matrix of order $m \times m$ whose elements are all zero. Thus, after applying identity (30) to (33), the symmetry constraint on the reduced order damping matrix can be restated as

$$\left[\mathbf{I}_{m\times m} - \mathbf{P}\right] vec\left(\mathbf{C}_m\right) = \mathbf{0}_{m^2} \tag{34}$$

where $\mathbf{I}_{m\times m}$ is the identity matrix of order m and $\mathbf{0}_{m^2}$ is a column vector of order m^2 whose elements are zero. The symmetry constraint (34) can be rewritten as

$$\mathbf{Lx} = \mathbf{0} \tag{35}$$

where $\mathbf{L} = \mathbf{I}_{m\times m} - \mathbf{P}$ and $\mathbf{x} = vec\left(\mathbf{C}_m\right)$. Combining equations (21) and (35), we obtain the following overall system of equations to be solved

$$\begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{L} \end{bmatrix} \mathbf{x} = \begin{Bmatrix} \mathbf{y} \\ \mathbf{0} \end{Bmatrix}. \tag{36}$$

whose solution, in the least square sense, is

$$\begin{aligned} \mathbf{x} &= \left[\begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{L} \end{bmatrix}^T \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{L} \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{A} \\ \lambda\mathbf{L} \end{bmatrix}^T \begin{Bmatrix} \mathbf{y} \\ \mathbf{0} \end{Bmatrix} \\ &= \left[\mathbf{A}^T\mathbf{A} + \lambda^2\mathbf{L}^T\mathbf{L}\right]^{-1} \mathbf{A}^T\mathbf{y} \end{aligned} \tag{37}$$

The above formulation leads to a family of solutions parametrised by the weighing factor λ, popularly known as the *regularisation parameter* [37]. If the regularisation parameter is very large, the constraint involving the observed data **y** weakly influences the solution **x** and the constraint enforcing the symmetry condition predominates in the solution of **x**. On the other hand, if λ is chosen to be small, the symmetry constraint is less satisfied and the solutions depends more heavily on the observed data. Of course, if λ is set to zero, the problem reverts back to solving equation (21), posed as an unconstrained optimisation problem. Thus, the value for λ is chosen depending on how strongly one would like to enforce the symmetry constraint. This regularisation method is generally known as Tikhonov Regularisation [37].

3 Numerical Validation

A proportionally damped linear array of mass-spring oscillators is considered to illustrate the application of the proposed system identification method. The system, together with the numerical values assumed for different parameters, is shown in Figure 1. The mass matrix of the system has the form

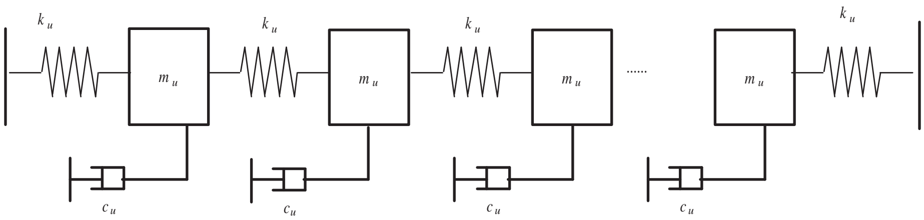

Figure 1: Linear array of mass-spring oscillators.

$$\mathbf{M}_n = \begin{bmatrix} m_1\mathbf{I}_{n/2} & \mathbf{0}_{n/2} \\ \mathbf{0}_{n/2} & m_2\mathbf{I}_{n/2} \end{bmatrix} \tag{38}$$

where $\mathbf{I}_{n/2}$ is the $n/2 \times n/2$ identity matrix and $\mathbf{0}_{n/2}$ is the $n/2 \times n/2$ null matrix. n, m_1, and m_2 were chosen to be 200 DOFs, 0.1kg, and 1kg respectively. The stiffness matrix of the system is given by

$$\mathbf{K}_n = k_u \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}, \tag{39}$$

with $k_u = 3.95 \times 10^5$ N/m.

The system is assumed to have Rayleigh damping, that is the damping matrix is given by $\mathbf{C}_n = \alpha_0 \mathbf{M}_n + \alpha_1 \mathbf{K}_n$, where $\alpha_0 = 0.946$ and $\alpha_1 = 6.69 \times 10^{-5}$. For a preliminary illustration, we considered the computer-simulated FRFs of this system as if they were experimentally measured. A typical FRF of the system is shown in Figure 2. In the same figure, the frequency range considered for the construction of

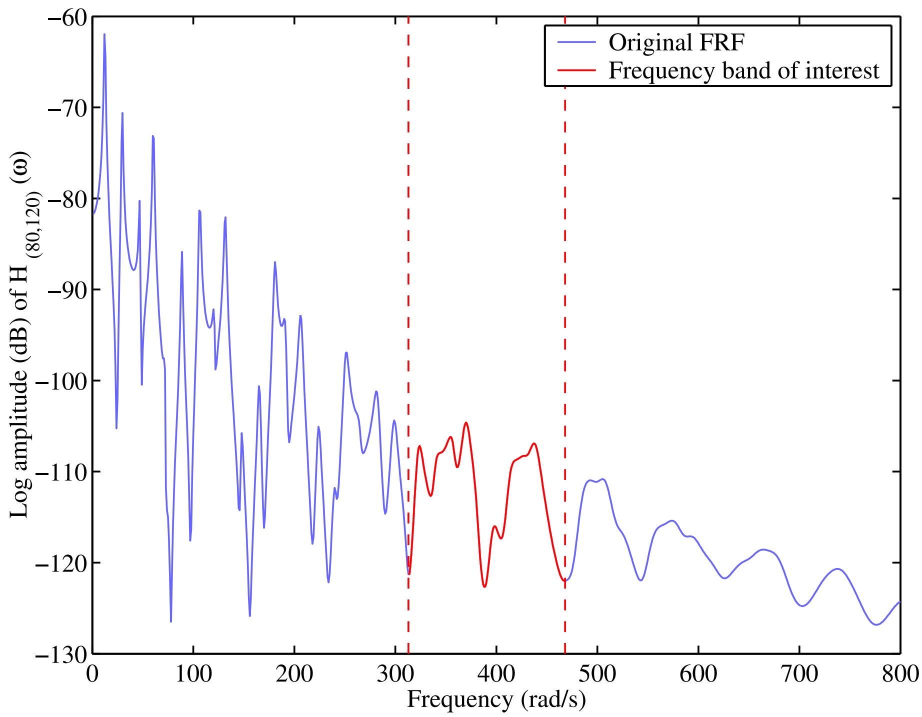

Figure 2: A typical FRF of the original system and the frequency band of interest.

the POD is also shown. Using only the selected frequency range of the 'measured' FRFs, the correlation matrix is constructed and the POD eigensolutions are extracted. Normalised eigenvalues of the correlation matrix, that is $\lambda/\lambda_{\max}$ are shown in Figure 3. It is clear that the first few eigenvalues are significantly large compared to rest

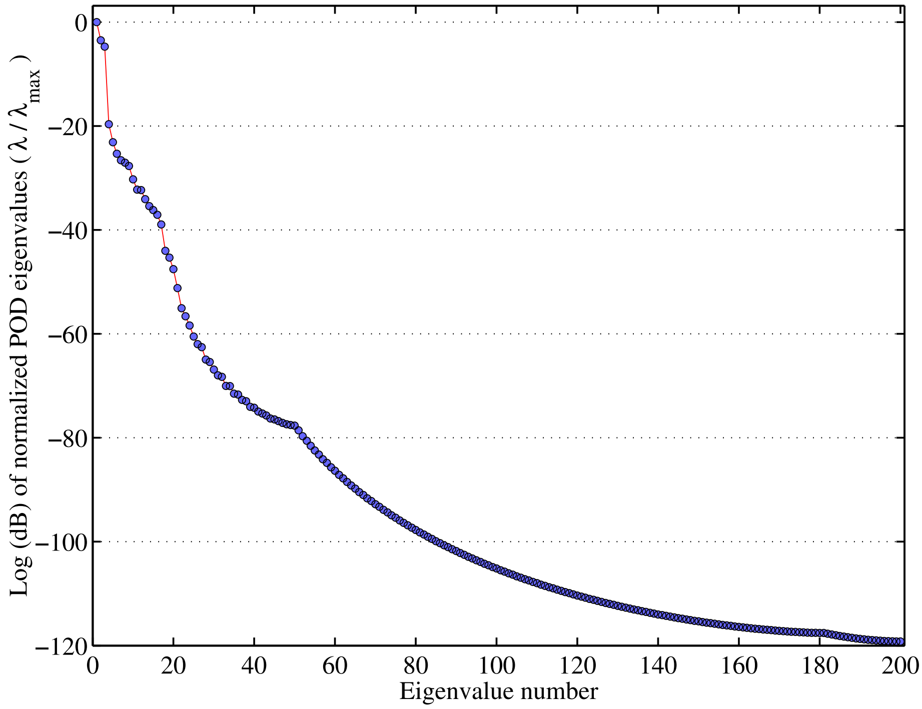

Figure 3: Normalised POD eigenvalues $(\lambda/\lambda_{\max})$.

of the eigenvalues. This justifies the approximation in equation (9). A typical FRF of the reconstructed system is compared with the original FRF in Figure 4. With only 22 POD modes, the reconstructed FRF matched reasonably well with the original FRF. The symmetry constraint was applied in the identification process, with the value for the regularisation parameter, λ, being 100. There are a number of factors that influence the reconstructed FRFs, for example (a) number of POD modes to retain, (b) level of damping, (c) the size and position of frequency window for the construction of POD. These issues will be investigated in the future work.

Furthermore, we have demonstrated the power of the proposed method assuming that our data is not contaminated by noise. In the low frequency range, the signal-to-noise ratio (SNR) is sufficiently high that our assumption is reasonable. However, in the mid-frequency range, and especially in the high-frequency range, the SNR decreases significantly that the assumption no longer holds. The presence of noise obviously affects the confidence level of the estimated parameters. Thus, in the future work, there is a need to apply noise sensitivity analysis to our proposed methodology in the mid-frequency range.

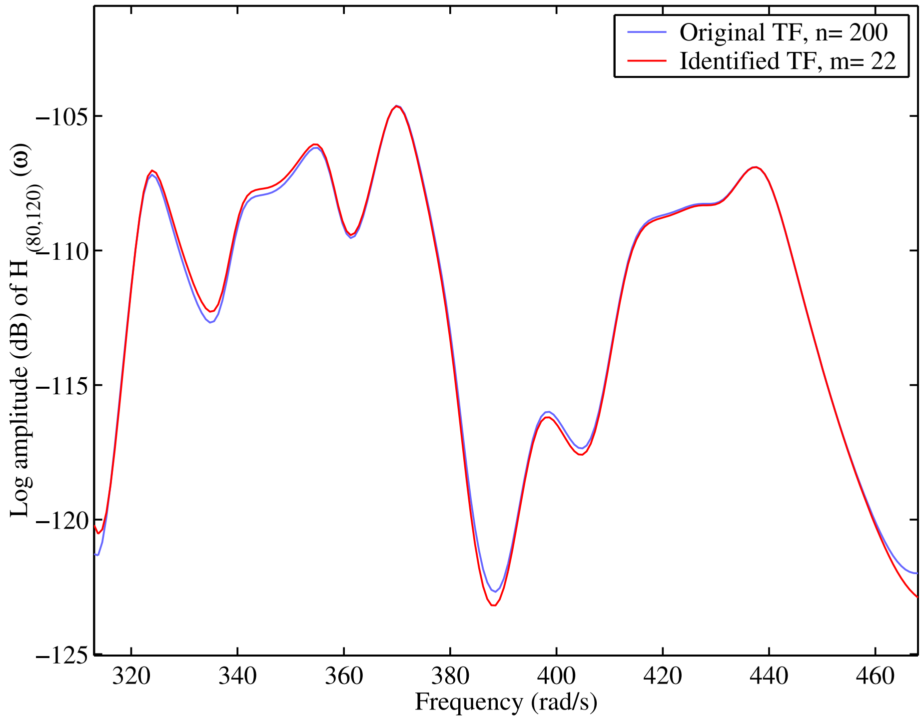

Figure 4: Original and estimated reconstructed FRF.

4 Conclusion

A new efficient method for identification of the damping matrix of general linear dynamic systems is proposed. The proposed method accurately reconstructs a reduced-order damping matrix with a POD-based reduced order model identification method. A novel mathematical framework is presented which exploits Kronecker Algebra. Prior information, such as symmetry of the damping matrix, is introduced as Tikhonov regularisation in a constrained optimisation framework. The proposed method has the potential to circumvent some of the difficulties encountered in the widely used modal analysis based identification methods. Such difficulties arise in (a) mid-frequency vibration problems or (b) systems with closely spaced modes or (c) non-proportionally or visco-elastically damped systems.

Acknowledgements

The first author acknowledges the support of the National Sciences and Engineering Research Council of Canada through the award of a Canada Graduate Scholarship. The second author acknowledges the support of the UK Engineering and Physical Sciences Research Council (EPSRC) through the award of an Advanced Research Fellowship. The third author acknowledges the support of a Discovery Grant from National Sciences and Engineering Research Council of Canada and the Canada Research Chair Program.

References

[1] D. J. Ewins, *Modal Testing: Theory and Practice*, Research Studies Press, Baldock, England, second edition, 2000.

[2] N. M. M. Maia and J. M. M. Silva, editors, *Theoretical and Experimental Modal Analysis*, Engineering Dynamics Series, Research Studies Press, Taunton, England, 1997, series Editor, J. B. Robetrs.

[3] J. M. M. E. Silva and N. M. M. Maia, editors, *Modal Analysis and Testing: Proceedings of the NATO Advanced Study Institute*, NATO Science Series: E: Applied Science, Sesimbra, Portugal, 1998.

[4] D. P. Pilkey and D. J. Inman, "Survey of damping matrix identification", in "Proceedings of the 16th International Modal Analysis Conference (IMAC)", volume 1, 104–110, 1998.

[5] J. Fabunmi, P. Chang and J. Vorwald, "Damping matrix identification using the spectral basis technique", Transaction of ASME, Journal of Vibration, Acoustics, Stress and Reliability in Design, 110, 332–337, 1988.

[6] P. Lancaster, "Expression of damping matrices in linear vibration problems", Journal of Aerospace Sciences, 28, 256, 1961.

[7] T. K. Hasselsman, "A method of constructing a full modal damping matrix from experimental measurements", AIAA Journal, 10(4), 526–527, 1972.

[8] J.-G. Béliveau, "Identification of viscous damping in structures from modal information", Transactions of ASME, Journal of Applied Mechanics, 43, 335–339, 1976.

[9] S. R. Ibrahim, "Dynamic modeling of structures from measured complex modes", AIAA Journal, 21(6), 898–901, 1983.

[10] C. Minas and D. J. Inman, "Identification of a nonproportional damping matrix from incomplete modal information", Transactions of ASME, Journal of Vibration and Acoustics, 113, 219–224, 1991.

[11] L. Starek and D. J. Inman, "A symmetric inverse vibration problem for non-proportional underdamped systems", Transactions of ASME, Journal of Applied Mechanics, 64, 601–605, 1997.

[12] D. F. Pilkey and D. J. Inman, "Iterative approach to viscous damping matrix identification", in "Proceedings of the 15th International Modal Analysis Conference (IMAC)", volume 2, 1152–1157, 1997.

[13] K. F. Alvin, L. D. Peterson and K. C. Park, "Extraction of normal modes and full modal damping from complex modal parameters", AIAA Journal, 35(7), 1187–1194, 1997.

[14] S. Adhikari and J. Woodhouse, "Identification of damping: part 1, viscous damping", Journal of Sound and Vibration, 243(1), 43–61, 2001.

[15] S. Adhikari and J. Woodhouse, "Identification of damping: part 2, non-viscous

damping”, Journal of Sound and Vibration, 243(1), 63–88, 2001.

[16] S. Adhikari and J. Woodhouse, “Identification of damping: part 3, symmetry-preserving method”, Journal of Sound and Vibration, 251(3), 477–490, 2002.

[17] S. Adhikari and J. Woodhouse, “Identification of damping: part 4, error analysis”, Journal of Sound and Vibration, 251(3), 491–504, 2002.

[18] S. Adhikari, “Damping modelling using generalized proportional damping”, Journal of Sound and Vibration, 293(1-2), 156–170, 2006.

[19] P. Caravani and W. T. Thomson, “Identification of damping coefficients in multidimensional linear systems”, Transactions of ASME, Journal of Applied Mechanics, 41, 379–382, 1974.

[20] C.-P. Fritzen, “Identification of mass, damping and stiffness matrices of mechanical systems”, Transaction of ASME, Journal of Vibration, Acoustics, Stress and Reliability in Design, 108, 9–16, 1986.

[21] J. E. Mottershead, “Theory for the estimation of structural vibration parameters from incomplete data”, AIAA Journal, 28(3), 559–561, 1990.

[22] M. J. Roemer and D. J. Mook, “Mass, stiffness and damping matrix identification: an integrated approach”, Transactions of ASME, Journal of Vibration and Acoustics, 114, 358–363, 1992.

[23] S. Y. Chen, M. S. Ju and Y. G. Tsuei, “Estimation of mass, stiffness and damping matrices from frequency response function”, Transactions of ASME, Journal of Vibration and Acoustics, 118, 78–82, 1996.

[24] M. Baruch, “Identification of the Damping Matrix”, Technical Report TAE No.803, Technion, Israel, Faculty of Aerospace Engineering, Israel Institute of Technology, Haifa, 1997.

[25] S. Adhikari, “Lancaster’s method of damping identification revisited”, Transactions of ASME, Journal of Vibration and Acoustics, 124(4), 617–627, 2002.

[26] A. Srikantha-Phani and J. Woodhouse, “The challenge of reliable identification of complex modes”, in “Proceedings of the 20th International Modal Analysis Conference (IMAC-XX)”, Los Angeles, CA, USA, 2002.

[27] S. Adhikari, “Optimal complex modes and an index of damping non-proportionality”, Mechanical System and Signal Processing, 18(1), 1–27, 2004.

[28] A. Sarkar and R. Ghanem, “Mid-frequency structural dynamics with parameter uncertainty”, Computer Methods in Applied Mechanics and Engineering, 191(47-48), 5499–5513, 2002.

[29] A. Sarkar and M. P. Paidoussis, “A compact limit-cycle oscillation model of a cantilever conveying fluid”, Jounal of Fluids and Strcutures, 17(4), 525–539, 2003.

[30] A. Sarkar and M. P. Paidoussis, “A cantilever conveying fluid: coherent modes versus beam modes”, International Jounal of Non-Linear Mechanics, 39(3), 467–481, 2004.

[31] V. Lenaerts, G. Kerschen and J.-C. Golinval, "Identification of a continuous structure with a geometrical non-linearity. Part II: Proper orthogonal decomposition", Journal of Sound and Vibration, 262(4), 907–919, 2003.

[32] A. Sarkar and R. Ghanem, "A substructure approach for the midfrequency vibration of stochastic systems", Journal of the Accoustical Society of America, 113(4), 1922–1934, 2003.

[33] A. Graham, *Kronecker Products and Matrix Calculus: With Applications*, Ellis Horwood, Chichester, England, 1981.

[34] H. V. Henderson and S. R. Searle, "The vec-permutation matrix, the vec operator and Kronecker products: A review", Linear and Multilinear Algebra, 9, 271–288, 1981.

[35] H. V. Henderson and S. R. Searle, "Some theorems on matrix differentiation with special reference to Kronecker matrix products", Journal of the American Statistical Association, 64(327), 953–963, 1969.

[36] D. S. G. Pollock, *The Algebra of Econometrics*, Wiley, New Yrok, 1979.

[37] A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-Posed Problems*, Winston-Wiley, Washington, 1977.

Appendix

A Background of Kronecker Algebra

A.1 Kronecker Product

Given a matrix $\mathbf{A}$ of order $m \times n$ and a matrix $\mathbf{B}$ of order $p \times q$ represented by

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{40}$$

and

$$\mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}, \tag{41}$$

the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ results in a matrix of order $mp \times nq$ given by [33]

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}. \tag{42}$$

A.2 Vectorisation

Given a matrix $\mathbf{A}$ of order $m \times n$ represented by

$$\mathbf{A} = \begin{bmatrix} A_{\cdot 1} & A_{\cdot 2} & \cdots & A_{\cdot n} \end{bmatrix} \tag{43}$$

in which $A_{\cdot i}$ denotes the ith column of $\mathbf{A}$, the vectorisation of $\mathbf{A}$ results in a column vector of order mn given by [33]

$$vec\,(\mathbf{A}) = \begin{Bmatrix} A_{\cdot 1} \\ A_{\cdot 2} \\ \vdots \\ A_{\cdot n} \end{Bmatrix}. \tag{44}$$