

# TOWARDS IDENTIFICATION OF A GENERAL MODEL OF DAMPING

S. Adhikari and J. Woodhouse

Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ (U.K.)

## ABSTRACT

Characterization of damping forces in a vibrating structure has long been an active area of research in structural dynamics. In spite of a large amount of research, understanding of damping mechanisms is not well developed. A major reason for this is that unlike inertia and stiffness forces it is not in general clear what are the *state variables* that govern the damping forces. The most common approach is to use ‘viscous damping’ where the instantaneous generalized velocities are the only relevant state variables. However, viscous damping by no means the only damping model within the scope of linear analysis. Any model which makes the energy dissipation functional non-negative is a possible candidate for a valid damping model. This paper is devoted to develop methodologies for identification of such general damping models responsible for energy dissipation in a vibrating structure. The method uses experimentally identified complex modes and complex natural frequencies and does not a-priori assume any fixed damping model (eg. , viscous damping) but seeks to determine parameters of a general damping model described by the so called ‘relaxation function’. The proposed method and several related issues are discussed by considering a numerical example of a linear array of damped spring-mass oscillators.

## NOMENCLATURE

$\mathbf{M}$	mass matrix
$\mathbf{K}$	stiffness matrix
$\mathcal{G}(\tau)$	matrix of kernel functions
$\mathbf{y}(t)$	generalized coordinates
$\omega_j, \mathbf{x}_j$	$j$ -th undamped frequency and mode
$\lambda_j, \mathbf{z}_j$	$j$ -th complex frequency and mode
$\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_j$	real and imaginary parts of $\mathbf{z}_j$
$\mu$	relaxation parameter of damping
$\mathbf{C}$	damping coefficient matrix
$\Re(\bullet), \Im(\bullet)$	real and imaginary parts of $(\bullet)$
$(\bullet)^*$	complex conjugation
$(\hat{\bullet})$	measured quantity of $(\bullet)$

## 1 INTRODUCTION

In the context of experimental modal analysis, by far the most common damping model is so-called ‘viscous damping’, a linear model in which the instantaneous generalized velocities are the only relevant state variables that determine damping.

This model was first introduced by Rayleigh<sup>[1]</sup> via his famous ‘dissipation function’, a quadratic expression for the energy dissipation rate with a symmetric matrix of coefficients, the ‘damping matrix’. Complex modes arise with viscous damping when it is non-proportional<sup>[2]</sup>. Practical experience in modal testing also shows that most real-life structures possess complex modes – as Sestieri and Ibrahim<sup>[3]</sup> have put it ‘... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality!’

Although with viscous damping models, linear systems show complex modal behaviour, it is by no means the only damping model within the scope of linear analysis. Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model. Unfortunately, most of the studies on damping identification reported in the literature consider viscous damping only. Such a *a priori* selection of viscous damping in identification procedure rules out any possibility of detecting the need for a different damping model. In this paper we consider identification of non-viscous damping models in the context of general multiple degrees-of-freedom linear systems.

A key issue in identifying non-viscous damping is– what non-viscous damping model to consider whose parameters have to be identified? There have been detailed studies on material damping or specific structural components. Lazan<sup>[4]</sup>, Bert<sup>[5]</sup> and Ungar<sup>[6]</sup> have given excellent accounts of different mathematical methods for modeling damping in (solid) material and their engineering applications. The book by Nashif *et al* <sup>[7]</sup> presents more recent studies in this area. Currently a large body of literature can be found on damping in composite materials where many researchers have evaluated a materials specific damping capacity (SDC). Baburaj and Matsuzaki<sup>[8]</sup> and the references therein give an account of research in this area. Other than material damping a major source of energy dissipation in a vibrating structure is the structural joints<sup>[9, 10]</sup>. In many cases these damping mechanisms turn out be locally non-linear, requiring an equivalent linearization technique for a global analysis<sup>[11]</sup>.

One way to address this problem is to use non-viscous damping models which depend on the past history of motion via convolution integrals over kernel functions. The equations of motion of free vibration of a linear system with this type of

damping can be expressed by

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \int_{-\infty}^t \mathcal{G}(t-\tau) \dot{\mathbf{y}}(\tau) d\tau + \mathbf{K}\mathbf{y}(t) = \mathbf{0}. \quad (1.1)$$

The kernel functions  $\mathcal{G}(\tau)$ , or others closely related to them, are described under many different names in the literature of different subjects: for example, retardation functions, heredity functions, after-effect functions, relaxation functions. This model was originally introduced by Biot<sup>[12]</sup>. In the special case when  $\mathcal{G}(t) = C\delta(t)$ , where  $\delta(t)$  is the Dirac-delta function, equation (1.1) reduces to the case of viscous damping. Woodhouse<sup>[13]</sup> showed that for light damping, such damping models can be handled in a very similar way to viscous models, using a first-order perturbation method based on the undamped modes and natural frequencies. This motivates us in developing procedures for identification of more general linear damping models from standard vibration testing data. Several mathematical expressions can be used for the kernel functions  $\mathcal{G}(\tau)$  in equation (1.1). Out of several possible damping functions the exponential function has the most obvious physical plausibility: Cremer and Heckl<sup>[14]</sup> also have emphasized this fact 'Of the many after-effect functions that are possible in principle, only one – the so called relaxation function – is physically meaningful.' For this reason we focus our attention to fit an exponential model to measured data.

The results based on first-order perturbation theory give a firm basis for further analysis, to use the details of the measured complex modes to learn more about the underlying damping mechanisms. There are several general questions of interest:

1. From experimentally determined complex modes can one identify the underlying damping mechanism? Is it viscous or non-viscous? Can the correct model parameters be found experimentally?
2. Is it possible to establish experimentally the *spatial distribution* of damping?
3. Is it possible that more than one damping model with corresponding correct sets of parameters may represent the system response equally well, so that the identified model becomes non-unique?
4. Does the selection of damping model matter from an engineering point of view? Which aspects of behaviour are wrongly predicted by an incorrect damping model?

This paper begin to address these questions. The analysis is restricted to linear systems with light damping: we assume throughout the validity of the first-order perturbation results. The initial aim is to consider what can be learned about these questions in principle, so procedures will be illustrated by applying them to simulated transfer functions, with no noise. The issue of how the usefulness of any procedure might be limited in practice by measurement noise will be deferred to later studies.

## 2 COMPLEX FREQUENCIES AND MODES OF NON-VISCOUSLY DAMPED SYSTEMS

Complex modes and frequencies of system (1.1) can be obtained using the first-order perturbation method. Substituting  $\mathbf{y}(t) = \mathbf{z} \exp[i \lambda t]$  and rewriting equation (1.1) in the frequency domain, the eigenvalue equation can be expressed as

$$-\lambda_j^2 \mathbf{M}\mathbf{z}_j + i \lambda_j \mathbf{G}(\lambda_j) \mathbf{z}_j + \mathbf{K}\mathbf{z}_j = \mathbf{0}, \quad \forall i = 1, \dots, N. \quad (2.1)$$

Following the procedure outlined by Woodhouse<sup>[13]</sup> the complex eigenvalues and modes are expressed as

$$\lambda_j \approx \pm \omega_j + i G'_{jj}(\pm \omega_j)/2 \quad (2.2)$$

$$\mathbf{z}_j \approx \mathbf{x}_j + i \sum_{\substack{k=1 \\ k \neq j}}^N \frac{\omega_j G'_{kj}(\omega_j)}{(\omega_j^2 - \omega_k^2)} \mathbf{x}_k. \quad (2.3)$$

Here  $G'_{kl}(\omega_j) = \mathbf{x}_k^T \mathbf{G}(\omega_j) \mathbf{x}_l$  are the elements of the frequency dependent damping matrix at the  $j$ -th natural frequency. Since the inverse Fourier transform of  $\mathbf{G}(\omega)$  must be real they satisfy the condition  $\mathbf{G}(-\omega) = \mathbf{G}(\omega)^*$ . In view of this relationship the eigenvalues appear in pairs  $\lambda$  and  $-\lambda^*$  (unless  $\lambda$  is purely imaginary), and eigenvectors in pairs  $\mathbf{z}_j$  and  $\mathbf{z}_j^*$ .

## 3 IDENTIFICATION OF NON-VISCOUS DAMPING MODELS

In this section we outline a general method to fit an exponential model to measured data. It is assumed that the system has only one relaxation parameter, that is the damping model is of the form

$$\mathcal{G}(t) = \mu e^{-\mu t} \mathbf{C} \quad (3.1)$$

where  $\mu$  is the relaxation parameter and  $\mathbf{C}$  is the associated coefficient matrix. Complex natural frequencies and mode shapes for a system with this kind of damping can be obtained from equations (2.2) and (2.3). The term  $G'_{kj}(\omega_j)$  appearing in these equations can be expressed as

$$G'_{kj}(\omega_j) = \frac{\mu}{\mu + i \omega_j} C'_{kj} = \left[ \frac{\mu^2}{\mu^2 + \omega_j^2} - i \frac{\mu \omega_j}{\mu^2 + \omega_j^2} \right] C'_{kj} \quad (3.2)$$

where  $C'_{kj} = \mathbf{x}_k^T \mathbf{C} \mathbf{x}_j$ . Now suppose that  $\hat{\lambda}_j$  and  $\hat{\mathbf{z}}_j$  for all  $j = 1, 2, \dots, m$  are the *measured* complex natural frequencies and modes. Write

$$\hat{\mathbf{z}}_j = \hat{\mathbf{u}}_j + i \hat{\mathbf{v}}_j. \quad (3.3)$$

Here  $\hat{\mathbf{z}}_j \in \mathbb{C}^N$  and  $N$  denotes the number of measurement points on the structure. Suppose that the number of modes to considered in the study is  $m$ : in general  $m \neq N$ , usually  $N \geq m$ . Assume that  $\hat{\mathbf{x}}_j$  are the corresponding *undamped modes* and  $\hat{\mu}$  is the relaxation parameter to be estimated from the experiment. From (2.2)

$$\hat{\omega}_j \approx \Re(\hat{\lambda}_j). \quad (3.4)$$

In view of equations (2.2)–(3.3) and considering that only  $m$  modes are measured, separating real and imaginary parts of  $\hat{z}_j$  one can write

$$\begin{aligned}\hat{\mathbf{u}}_j &= \hat{\mathbf{x}}_j + \sum_{\substack{k=1 \\ k \neq j}}^m \tilde{A}_{kj} \hat{\mathbf{x}}_k; & \tilde{A}_{kj} &= \frac{\hat{\mu} \hat{\omega}_j}{(\hat{\mu}^2 + \hat{\omega}_j^2)} B_{kj} \\ \hat{\mathbf{v}}_j &= \sum_{\substack{k=1 \\ k \neq j}}^m \tilde{B}_{kj} \hat{\mathbf{x}}_k; & \tilde{B}_{kj} &= \frac{\hat{\mu}^2}{(\hat{\mu}^2 + \hat{\omega}_j^2)} B_{kj}.\end{aligned}\quad (3.5)$$

Here the unknown constants  $B_{kj} = \frac{\hat{\omega}_j C'_{kj}}{\hat{\omega}_j^2 - \hat{\omega}_k^2}$ . Note that in addition to  $B_{kj}$ , the relaxation constant  $\hat{\mu}$  and the undamped modes  $\hat{\mathbf{x}}_k$  are also unknown. Combining these equations,

$$\hat{\mathbf{u}}_j = \hat{\mathbf{x}}_j + \frac{\hat{\omega}_j}{\hat{\mu}} \hat{\mathbf{v}}_j. \quad (3.6)$$

Because the undamped modes are orthonormal with respect to the mass matrix, from equation (3.6) it may be observed that the imaginary part of the complex mode  $\hat{\mathbf{v}}_j$  is M-orthogonal to its corresponding undamped mode so that  $\hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{x}}_j = 0$ . Premultiplying equation (3.6) by  $\hat{\mathbf{v}}_j^T \mathbf{M}$  and summing  $\forall j = 1, \dots, m$  one can write

$$\sum_{j=1}^m \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{u}}_j = \sum_{j=1}^m \left\{ \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{x}}_j + \frac{\hat{\omega}_j}{\hat{\mu}} \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{v}}_j \right\} \quad (3.7)$$

Use of the orthogonality property of  $\hat{\mathbf{v}}_j$  and  $\hat{\mathbf{x}}_j$  leads to

$$\hat{\mu} = \frac{\sum_{j=1}^m \hat{\omega}_j \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{v}}_j}{\sum_{j=1}^m \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{u}}_j}. \quad (3.8)$$

This  $\hat{\mu}$  can be substituted in (3.6) to obtain the undamped modes. Now,  $\forall j = 1, \dots, m$  the second equation of (3.5) can be arranged in a matrix form and the  $\tilde{B}_{kj}$  can be calculated using the Moore-Penrose pseudoinverse. From these constants  $C'_{kj}$  and consequently, using the undamped modes, the coefficient matrix  $\mathbf{C}$  can be obtained. This coefficient matrix together with the relaxation parameter  $\hat{\mu}$  completely defines the damping mechanism in the structure. In summary, this procedure can be described by the following steps:

1. Measure a set of transfer functions  $H_{ij}(\hat{\omega})$ . Fix the number of modes to be retained in the study, say  $m$ . Determine the complex natural frequencies  $\hat{\lambda}_j$  and complex mode shapes  $\hat{\mathbf{z}}_j$  from the transfer function, for all  $j = 1, \dots, m$ . Denote  $\hat{\mathbf{Z}} = [\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_m] \in \mathbb{C}^{N \times m}$  the complex mode shape matrix.
2. Evaluate the ‘undamped natural frequencies’ as  $\hat{\omega}_j = \Re(\hat{\lambda}_j)$ .
3. Set  $\hat{\mathbf{U}} = \Re[\hat{\mathbf{Z}}] = [\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \dots, \hat{\mathbf{u}}_m]$  and  $\hat{\mathbf{V}} = \Im[\hat{\mathbf{Z}}] = [\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \dots, \hat{\mathbf{v}}_m]$ .
4. Obtain the relaxation parameter  $\hat{\mu} = \frac{\sum_{j=1}^m \hat{\omega}_j \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{v}}_j}{\sum_{j=1}^m \hat{\mathbf{v}}_j^T \mathbf{M} \hat{\mathbf{u}}_j}$ .
5. For all  $j = 1, \dots, m$  calculate the ‘undamped modes’  $\hat{\mathbf{x}}_j = \left\{ \hat{\mathbf{u}}_j - \frac{\hat{\omega}_j}{\hat{\mu}} \hat{\mathbf{v}}_j \right\}$ . Set  $\hat{\mathbf{X}} = [\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_m] \in \mathbb{R}^{N \times m}$ .

6. Evaluate the matrix  $\tilde{\mathbf{B}} = \left[ \hat{\mathbf{X}}^T \hat{\mathbf{X}} \right]^{-1} \hat{\mathbf{X}}^T \hat{\mathbf{V}}$ .

7. From the  $\tilde{\mathbf{B}}$  matrix get  $C'_{kj} = \frac{(\hat{\omega}_j^2 - \hat{\omega}_k^2) (\hat{\mu}^2 + \hat{\omega}_j^2)}{\hat{\omega}_j} \tilde{B}_{kj}$  for  $k, j = 1, 2, \dots, m; k \neq j$  and  $C'_{jj} = 2\Im(\hat{\lambda}_j) \frac{(\hat{\mu}^2 + \hat{\omega}_j^2)}{\hat{\mu}^2}$ .

8. Use  $\mathbf{C} = \left[ \left( \hat{\mathbf{X}}^T \hat{\mathbf{X}} \right)^{-1} \hat{\mathbf{X}}^T \right]^T \mathbf{C}' \left[ \left( \hat{\mathbf{X}}^T \hat{\mathbf{X}} \right)^{-1} \hat{\mathbf{X}}^T \right]$  to get the coefficient matrix in physical coordinates.

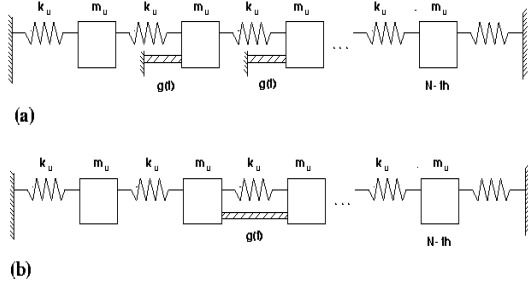
Using this procedure we need only the complex natural frequencies, mode shapes and mass matrix to identify the best exponential damping model associated with the measurements. The method is very simple and does not require much computational time. Another advantage is that a complete set of modal data is not necessary to estimate the relaxation parameter as well as the full coefficient matrix. It may be observed that even if the measured transfer functions are reciprocal, from the above mentioned procedure there is no reason why the fitted coefficient matrix  $\mathbf{C}$  will always be symmetric. If we indeed detect a non-symmetric  $\mathbf{C}$  then it may be guessed that the physical law behind the damping mechanism in the structure can not be described by an exponential model. This fact will be discussed later by considering an example in Section 4.

The procedure described above can also be used to identify the familiar viscous damping matrix as a special case by forcing  $\hat{\mu} \rightarrow \infty$ . In this case it is easy to see that  $\hat{\mathbf{u}}_j \rightarrow \hat{\mathbf{x}}_j$  i.e., the real part of the complex modes approach the undamped modes and all the steps described above remain valid. Interestingly, as step 4 is avoided, knowledge of the mass matrix is also not essential. Thus only complex natural frequencies and mode shapes are sufficient to identify the ‘best’ full (non-proportional) viscous damping matrix.

## 4 NUMERICAL EXAMPLES

There is a major difference in emphasis between this study and other related studies on damping identification reported in the literature. Most of the methods assume from the outset that the system is viscously damped (see the review paper by Pilkey and Inman<sup>[15]</sup>) and then formulate the theory to identify a viscous damping matrix. Here, we wish to investigate how much one can learn by fitting viscous and exponential damping models when the actual system is non-viscously damped, as one must expect to be the case for most practical systems. We study by simulation a system which has a known non-viscous damping model. Two different physically realistic non-viscous damping models are considered in this study. They are applied to a system consisting of a linear array of spring-mass oscillators and dampers shown in Figure 1. For the numerical values considered the resulting undamped natural frequencies range from near zero to approximately 200 Hz. Certain of the masses of the system shown in Figure 1(a) have dissipative elements connecting them to the ground. In this case the damping force depends only on the absolute motion of the individual masses. Such damping will be described

as ‘locally reacting’. For the system shown in Figure 1(b), by contrast, dissipative elements are connected between certain adjacent pairs of masses. In this case the damping force depends on the relative motion of the two adjacent masses, and will be called ‘non-locally reacting’.



**Figure 1: Linear array of  $N$  spring-mass oscillators,  $N = 30$ ,  $m_u = 1 \text{ Kg}$ ,  $k_u = 4 \times 10^3 \text{ N/m}$ .**

The dissipative elements shown in Figure 1 are taken to be linear non-viscous dampers so that the equations of motion are described by (1.1). Complex natural frequencies and modes can be calculated for the model system using equations (2.2) and (2.3), then treated like experimental data obtained from a modal testing procedure. Note that in a true experimental environment the measured complex natural frequencies and mode shapes will be contaminated by noise. Since the simulated data are noise-free the results obtained using them are ‘ideal’, the best one can hope using this approach. Once promising algorithms have been identified in this way, the influence of noise in degrading the performance will have to be addressed.

Two specific damping models will be considered, defined by two different forms of  $g(t)$ :

$$\text{MODEL 1 (exponential): } g^{(1)}(t) = \mu_1 e^{-\mu_1 t} \quad (4.1)$$

$$\text{MODEL 2 (Gaussian): } g^{(2)}(t) = 2\sqrt{\frac{\mu_2}{\pi}} e^{-\mu_2 t^2} \quad (4.2)$$

where  $\mu_1$  and  $\mu_2$  are constants. It is convenient to normalize the functions to make comparisons between models meaningful. Both functions have been scaled so as to have unit area when integrated to infinity. This makes them directly comparable with the viscous model, in which the corresponding damping function would be a unit delta function,  $g(t) = \delta(t)$ , and the coefficient matrix  $C$  would be the usual dissipation matrix.

It is also convenient to define a characteristic time constant,  $\theta_j$  for each damping function, via the first moment of  $g^{(j)}(t)$ :

$$\theta_j = \int_0^\infty t g^{(j)}(t) dt \quad (4.3)$$

For the two damping models considered here, evaluating the above integral gives  $\theta_1 = \frac{1}{\mu_1}$  and  $\theta_2 = \frac{1}{\sqrt{\pi\mu_2}}$ . For viscous

damping  $\theta_j = 0$ . The characteristic time constant of a damping function gives a convenient measure of ‘width’: if it is close to zero the damping behaviour will be near-viscous, and vice versa. To establish an equivalence between the two damping models we can choose that they have the same time constant.

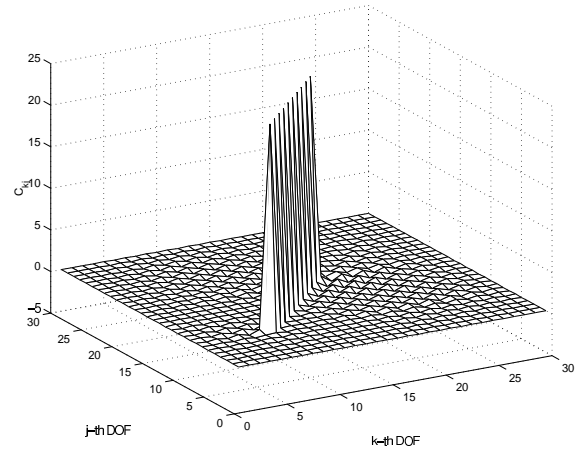
For both the systems shown in Figure 1, the dampers are associated only between the  $s$ -th and  $(s+l)$ -th masses. For the numerical calculations considered here, we have taken  $N = 30$ ,  $s = 8$  and  $(s+l) = 17$ . Various values of the time constant  $\theta$  have been tested for both the damping models. These are conveniently expressed as a fraction of the period of the highest undamped natural frequency:

$$\theta = \gamma T_{min} \quad (4.4)$$

When  $\gamma$  is small compared with unity the damping behaviour can be expected to be essentially viscous, but when  $\gamma$  is of order unity non-viscous effects should become significant.

### Results For Small $\theta$

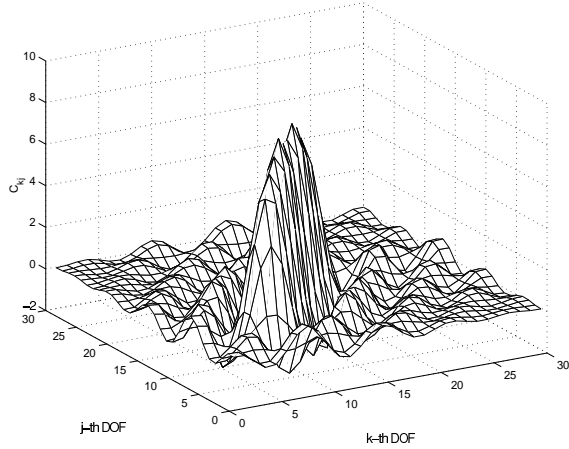
When  $\gamma = 0.02$  both damping models should show near-viscous behaviour. First consider the system shown in Figure 1(a) with locally reacting damping. Figure 2 shows the fitted viscous damping matrix for damping model 2, calculated using the complete set of 30 modes. The fitted matrix identifies the damping in the system very well. The high portion of the plot corresponds to the spatial location of the dampers. The off-diagonal terms of the identified damping matrix are very small compared to the diagonal terms, indicating correctly that the damping is locally reacting.



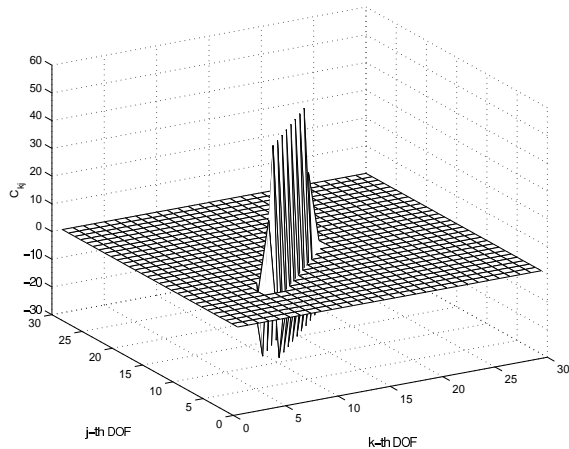
**Figure 2: Fitted viscous damping matrix for local case,  $\gamma = 0.02$ , damping model 2**

It is useful to understand the effect of modal truncation on the damping identification procedure. In practice, one might expect to be able to use only the first few modes of the system to identify the damping matrix. Figure 3 shows the fitted viscous damping matrix using the first 10 modes only. The quality of the fitted damping matrix deteriorated as the number of modes used to fit the damping matrix is reduced, but still the

identified damping matrix shows a reasonable approximation to the true behaviour. The spatial resolution of the identified damping is limited by that of the set of modes used, and some off-diagonal activity is seen in the fitted matrix. Since for this system the mode shapes are approximately sinusoidal, we can recognize the effects of modal truncation as analogous to Gibbs phenomenon in a truncated Fourier series.



**Figure 3: Fitted viscous damping matrix using first 10 modes,  $\gamma = 0.02$ , damping model 2**



**Figure 4: Fitted coefficient matrix of exponential model for non-local case,  $\gamma = 0.02$ , damping model 2**

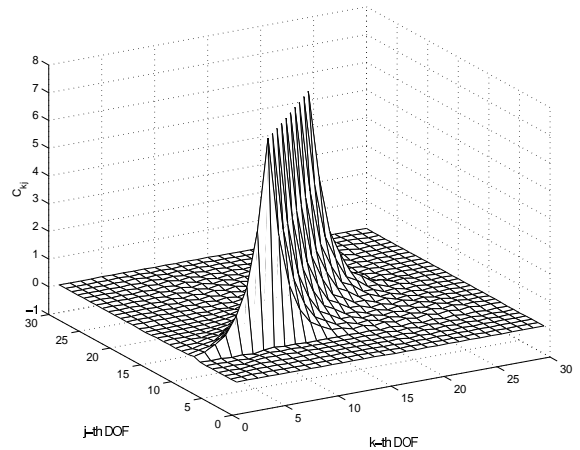
Now consider the system shown in Figure 1(b) with non-locally reacting damping. So far we have shown the results for fitting of viscous damping model. Figure 4 shows the fitted coefficient matrix of exponential model for damping model 2, using the full set of modes. The high portion of the plot corresponds to the spatial location of the dampers. The negative off-diagonal terms in the identified damping matrix indicate that the damping is non-locally reacting. We also observed that the fitted characteristic time constant  $\hat{\theta}$  is small, so that  $\hat{\gamma} = 0.0201$ . This implies that the identified (exponential) model is near to viscous. Thus the proposed method identifies the correct damping model quite well and gives a good

estimate of the correct characteristic time constant.

When the fitting procedure is repeated using any other damping model with a similarly short characteristic time constant, the results are very similar. The detailed difference in their functional behaviour does not influence the results significantly. In summary, we can say that when the time constant for a damping model is small the proposed identification method works quite well regardless of the functional form of the damping mechanism. The spatial location of damping is revealed clearly and associated relaxation parameter shows high value whether it is locally or non-locally reacting. Modal truncation blurs the results, but does not invalidate the identification process.

### Results For Larger $\theta$

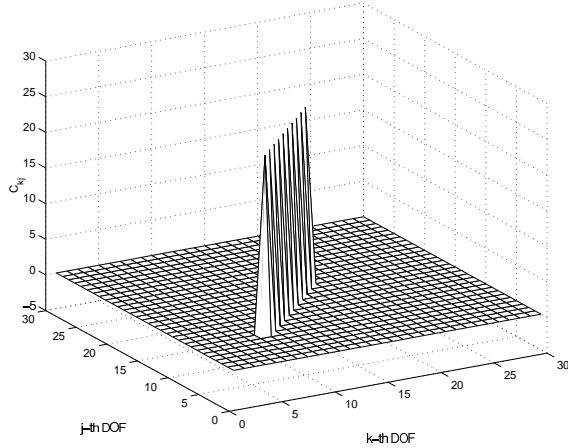
When  $\gamma$  is larger the two non-viscous damping models depart from the viscous damping model, each in its own way. For the value  $\gamma = 0.5$ , Figure 5(a) shows the result of fitting a viscous damping matrix for damping model 1 (equation (4.1)) with locally-reacting damping and the full set of modes. It may



**Figure 5(a): Fitted viscous damping matrix for local case,  $\gamma = 0.5$ , damping model 1**

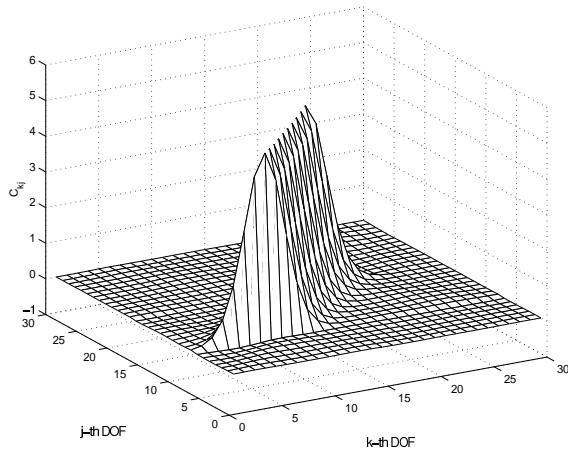
be noted that although we have started with a locally reacting damping model, which means the matrix is non-zero only along the diagonal, the non-zero values in the off-diagonal terms show that the fitted viscous damping is, in a sense, not locally reacting. Nevertheless, the spatial distribution of the damping is well identified, and perhaps one might be able to guess that the underlying mechanism was locally-reacting from the fact that the significantly non-zero elements all have positive values, with a clear peak centred on the diagonal of the matrix. Figure 5(b) shows the result of fitting the exponential model for this problem. Since the model is 'identified' correctly in this case, the correct value of the relaxation parameter is obtained, and the coefficient matrix corresponds to the exact coefficient matrix for the problem.

Figure 6(a) shows the corresponding fitted viscous damping matrix for damping model 2 (equation (4.2)). Features of the



**Figure 5(b): Fitted coefficient matrix of exponential model for local case,  $\gamma = 0.5$ , damping model 1**

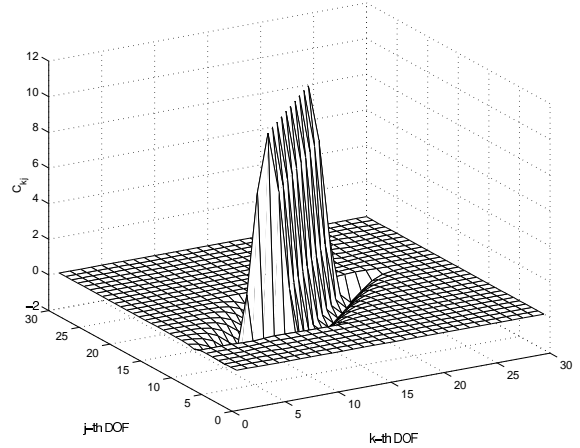
fitting are similar to those of Figure 5(a). The result of fitting



**Figure 6(a): Fitted viscous damping matrix for local case,  $\gamma = 0.5$ , damping model 2**

the exponential model for this problem is shown in Figure 6(b). For the characteristic time constant of the fitted exponential model,  $\hat{\gamma} = 0.6415$  which is not very different from the exact  $\gamma$  of the simulated model. However, since the 'identified' model is not correct, because the original model is Gaussian while the fitted model is exponential, the coefficient matrix does not correspond to the exact coefficient matrix for the problem. In fact the result of fitting of the exponential model is not very different from fitting a viscous model (Figure 6(a)). From this result we conclude that when the characteristic time constant of a damping model is long an incorrect damping model may not accurately indicate the actual damping behaviour of a structure.

A useful check on the accuracy of these procedures is to compare the 'measured' and reconstructed transfer functions. For the examples considered here it was observed (results not shown) that the reconstructed transfer function agreed remarkably well with the original one. This is to be expected:



**Figure 6(b): Fitted coefficient matrix of exponential model for local case,  $\gamma = 0.5$ , damping model 2**

the fitting procedure outlined in the previous section is exact, within the approximations of the small-damping perturbation theory, provided the full set of modes is used. The full set of poles and their residues are correctly reproduced — this is the essential contrast between this approach and one which fits only proportional damping, for which the poles can be correct but the residues cannot (because they will be real, not complex). This result has a far-reaching implication: an incorrect damping model (eg. , the fitted viscous damping) with a different spatial distribution from the true damping model can reproduce accurately the full set of transfer functions. This means that by measuring transfer functions it is not possible to identify uniquely the governing mechanism.

However, it should be noted that in cases like Figures 5(a), 6(a) and 6(b) the fitted coefficient matrix is not symmetric. This is, in some sense, a non-physical result. In view of this non-symmetry, the reciprocity of the transfer functions was checked and it was observed that they are indeed reciprocal within an acceptable accuracy. Thus the non-symmetry of the fitted coefficient matrix in the spatial coordinate does not necessarily affect the reciprocity of the transfer functions. Instead, we should regard non-symmetry of a fitted coefficient matrix as evidence that the true damping model is not the one which is fitted . To obtain a correct physical description of the damping, a symmetry preserving non-viscous model should be fitted instead.

From equation (2.3) we can deduce that, within the approximation of small damping, each frequency function  $G'_{kj}(\omega)$  can be observed at only two frequencies,  $\omega_j$  and  $\omega_k$ . When the fitted coefficient matrix turns out to be non-symmetric, this indicates that it was not possible to fit the assumed function through both 'measured' frequency points, and two different coefficients were needed. To correct this problem, it is necessary to fit a different damping model with sufficient free parameters to pass through both measured points while retaining symmetric coefficients. The function cannot be uniquely determined by this requirement, of course. Research is currently

in progress to explore models of this kind, and their associated fitting algorithms.

## 5 CONCLUSIONS

In this paper a method has been proposed to identify a non-proportional non-viscous damping model in vibrating systems. It is assumed that damping is light so that the first order perturbation method is applicable. The method is simple, direct, and compatible with conventional modal testing procedures. The complex modes and natural frequencies are used, but the method does not require the full set of modal data. Identification of the familiar viscous damping model is a special case of the general method proposed here. In that case, knowledge of the mass and stiffness matrices are also not necessary. The validity of the proposed method has been explored by applying it to simulated data from a simple test problem, in which a linear array of spring-mass oscillators is damped by non-viscous elements over part of its length.

Numerical experiments have been carried out with a wide range of parameter values and different damping models. The main features of the results have been illustrated by two particular damping models and representative parameter values. It has been shown that the method generally predicts the spatial location of the damping with good accuracy, and also gives a good indication of whether the damping is locally-reacting or not. Whatever the nature of the fitted coefficient matrix, the transfer functions obtained using the fitted damping agree well with the exact transfer functions of the simulated system. Reciprocity of the transfer functions remains preserved within an acceptable accuracy although in some cases the fitted coefficient matrix is not symmetric.

Symmetry breaking of the fitted coefficient matrix depends on the value of the characteristic time constant  $\theta$  of the damping model, defined by equation (4.3). When  $\theta$  is short compared with the natural periods of the vibration, the damping is effectively viscous and the fitting procedure gives a physically-sensible symmetric matrix. When  $\theta$  is larger, though, the memory of the damping function influences the detailed behaviour. In this case, if the fitted damping model is wrong, the procedure yields a non-physical result by fitting a non-symmetric coefficient matrix. That is, the procedure gives an indication that a wrong model is selected for fitting. Investigations are under way to fit symmetry-preserving models under such conditions.

## ACKNOWLEDGEMENTS

Funding provided by Trinity College, Cambridge and Cambridge Commonwealth Trust is greatly acknowledged.

## REFERENCES

- [1] **Rayleigh, L.**, Theory of Sound (two volumes), Dover Publications, New York, 1945th edn., 1877.
- [2] **Caughey, T. K. and O'Kelly, M. E. J.**, *Classical normal modes in damped linear dynamic systems*, Journal of Applied Mechanics, ASME, Vol. 32, pp. 583–588, September 1965.
- [3] **Sestieri, A. and Ibrahim, R.**, *Analysis of errors and approximations in the use of modal coordinates*, Journal of Sound and Vibration, Vol. 177, No. 2, pp. 145–157, 1994.
- [4] **Lazan, B. J.**, Damping of Materials and Members in Structural Mechanics, Pergamon Press, Oxford, 1968.
- [5] **Bert, C. W.**, *Material Damping: An introductory review of mathematical models, Measure and Experimental Techniques*, Journal of Sound and Vibration, Vol. 29, No. 2, pp. 129–153, 1973.
- [6] **Ungar, E. E.**, *The status of engineering knowledge concerning the damping of built-up structures*, Journal of Sound and Vibration, Vol. 26, pp. 141–154, 1973.
- [7] **Nashif, A. D., Jones, D. I. G. and Henderson, J. P.**, Vibration Damping, John Wiley, New York, 1985.
- [8] **Baburaj, V. and Matsukai, Y.**, *A study on the material damping of thin angle-ply laminated plates*, Journal of Sound and Vibration, Vol. 172, No. 3, pp. 415–419, 1994.
- [9] **Earls, S. W. E.**, *Theoretical estimation of frictional energy dissipation in a simple lap joint*, Journal of Mechanical Engineering Science, Vol. 8, No. 2, pp. 207–214, 1966.
- [10] **Beards, C. F. and Williams, J. L.**, *The damping of structural vibration by rotational slip in structural joint*, Journal of Sound and Vibration, Vol. 53, No. 3, pp. 333–340, 1977.
- [11] **Bandstra, J. P.**, *Comparison of equivalent viscous damping in discrete and continuous vibrating system*, Journal of Vibration Acoustic Stress and Reliability in Design, Vol. 105, pp. 382–392, July 1983.
- [12] **Biot, M. A.**, *Linear thermodynamics and the mechanics of solids*, Proceedings of the Third U. S. National Congress on Applied Mechanics, pp. 1–18, ASME, New York, 1958.
- [13] **Woodhouse, J.**, *Linear damping models for structural vibration*, Journal of Sound and Vibration, Vol. 215, No. 3, pp. 547–569, 1998.
- [14] **Cremer, L. and Heckl, M.**, Structure-Borne Sound, Springer-Verlag Berlin, Germany, 2nd edn., 1973, translated by E. E. Ungar.
- [15] **Pilkey, D. P. and Inman, D. J.**, *Survey of damping matrix identification*, Proceedings International Modal Analysis Conference - IMAC, Vol. 1, pp. 104–110, 1998.