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# Steady-state response of a random dynamical system described with Padé approximants and random eigenmodes

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**Abstract**

Designing a random dynamical system requires the prediction of the statistics of the response, knowing the random model of the uncertain parameters. Direct Monte Carlo simulations (MCS) is the reference method for propagating uncertainties but its main drawback is the high numerical cost. A surrogate model based on a polynomial chaos expansion (PCE) can be built as an alternative to MCS. However, some previous studies have shown poor convergence properties around the deterministic eigenfrequencies. In this study, an extended Padé approximant approach is proposed not only to accelerate the convergence of the PCE but also to have a better representation of the exact frequency response, which is a rational function of the uncertain parameters. A second approach is based on the random mode expansion of the response, which is widely used for deterministic dynamical systems. A PCE approach is used to calculate the random modes. Both approaches are tested on an example to check their efficiency.

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*Keywords:* Random dynamical systems; polynomial chaos expansion; multivariate Padé approximants; random modes.

## 1. Introduction

Uncertainty propagation aims to find the statistics of the model outputs. The classical method is the Monte Carlo simulation (MCS) approach. As this method requires a large computation cost, alternative methods have been developed. One of the most popular method is based on a surrogate model built with a polynomial chaos expansion (PCE) [1]. The PCE has been recently used by the authors to study the steady-state response of a linear dynamical system [2]. This method has been very successful far from the "deterministic eigenfrequencies", but it turns out that the polynomial expansion converges very slowly around these frequencies. The Padé approximant (PA) approach [3,4] is a well-known method to improve the convergence acceleration of a Taylor expansion by calculating a sequence

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of rational functions. In this study the probability density function of the steady-state response is estimated with a generalization of the Padé approximants [5], the “extended” Padé approximants (XPA). The modal approach is widely used to solve deterministic structural dynamics problems, but has been very rarely used to study uncertain dynamical system. The main issue is to calculate the random modes. In this study the PCE random mode approach proposed by Dessombz [6,7] is used to derive the pdf of the steady-state response of a linear dynamical system.

The paper is organized as follows: first the linear uncertain dynamical systems is presented as well as the PCE; second the XPA is developed; third the random modes are described according to a PCE approach; finally the methods are applied to an example.

## 2. Random dynamical system

### 2.1. Motion equation

A linear random $N$-dof dynamical system, which is excited with harmonic force vector $\mathbf{F}$ with frequency $\omega$, is investigated. The uncertain dynamical system is defined by the mass, stiffness, and damping matrices ($\mathbf{M}$, $\mathbf{K}$, and $\mathbf{D}$). These matrices are random and depend on an $r$-element uncertain parameter vector, $\Xi$: element $i$ of this vector, $\xi_i$, is the $i$-th zero mean random parameter. The dynamical response, $\mathbf{X}(\omega, \Xi) \in \mathbb{R}^N$, is the solution of the motion equation

$$(-\,\omega^2\,\mathbf{M} \;+\; \imath\,\omega\,\mathbf{D} \;+\; \mathbf{K})\,\mathbf{X}(\omega,\Xi) \;=\; \mathbf{F}(\omega) \tag{1}$$

where $\imath^2 = -1$, and each of the uncertain matrices can be expanded as $\mathbf{A}(\Xi) \;=\; \mathbf{A}_0 + \sum_{i=1}^{r} \xi_i \mathbf{A}_i$ The related so-called deterministic dynamical system is characterized by the mean matrices ($\mathbf{M}_0$, $\mathbf{K}_0$, and $\mathbf{D}_0$).

### 2.2. Polynomial chaos expansion

A polynomial chaos $\Psi_j(\Xi)$ is an element of a multivariate orthogonal polynomial family [1]. The response of a random dynamical system may be expanded in terms of polynomial chaos $\Psi_j$ as

$$\mathbf{X}^P(\omega,\Xi) \;=\; \sum_{i=0}^{P} \mathbf{Y}_i^P(\omega)\,\Psi_j(\Xi) \tag{2}$$

where $P$ depends on the number of random variables and the PC degree [1]. In this study, the polynomial chaos set is based on a product of normalized Hermite or Legendre polynomials. Coefficients $\mathbf{Y}_i^P$ are determined by replacing $\mathbf{X}^P$ by its expansion in Eq. (1) and by using the orthogonality properties of the polynomials chaos [2]. Once the coefficients of the PCE are calculated, the probability density function (pdf) can then be estimated with an MCS directly applied to Eq. (2).

## 3. Extended Padé approximant

A Padé approximant (PA) of response vector $\mathbf{X}(\Xi)$ is a rational function derived from the Taylor series of $\mathbf{X}(\Xi)$. The Padé approximant converges much faster than the Taylor expansion [3,4] when the function has poles. PA has been extended to the case of multivariate functions [8–12]. In the random dynamical system context, PC expansion is much more interesting than a Taylor series. Such a generalization had been defined and studied in many papers [3,5,13–16]. The extended Padé approximants can be defined as

$$[M_k/N_k]_{\mathbf{X}_k^{PC}}(\Xi) \;=\; \frac{\sum_{j=0}^{n_k} N_{k,j}^{XPA}(\omega)\,\Psi_j(\Xi)}{\sum_{j=0}^{d_k} D_{k,j}^{XPA}(\omega)\,\Psi_j(\Xi)} \tag{3}$$

where $n_k = \sharp M_k - 1$ and $d_k = \sharp N_k - 1$ ((where $\sharp m$ denotes the maximum number of coefficients of a multivariate polynomial of degree $m$); $k$ refers to the $k$-th dof; usually $D_{k,0}^{XPA}$ is equal to unity.

$N_{k,i}^{XPA}$ and $D_{k,i}^{XPA}$ are derived by comparing Eq. (2) to Eq. (3), and then by projecting the resulting equation on $\Psi_l(\Xi)$. More details can be found in [17].

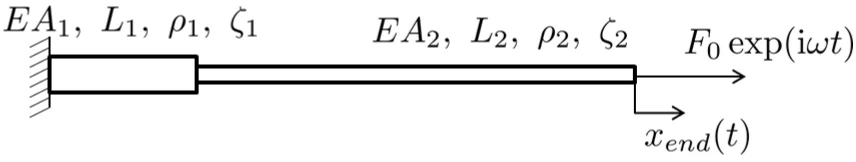


Fig. 1. Uncertain bar

## 4. Random modes

A classical way to obtain the response of an $N$-dof deterministic dynamical system is to expand the solution on the deterministic eigenvectors. This can be done with a random system, and the steady-state response described with the random mode reads

$$\mathbf{X}(\omega) = \sum_{n=1}^{N} \widetilde{q}_n(\omega)\, \widetilde{\boldsymbol{\phi}}_n \tag{4}$$

where $\{\widetilde{\omega}_k,\ \widetilde{\boldsymbol{\phi}}_k\}$ is the $k$-th random mode. Modal coordinate $\widetilde{q}_n$ is derived by substituting Eq. (4) in Eq. (1) and by projecting this latter equation on each $\widetilde{\boldsymbol{\phi}}_n$. Then the modal coordinate reads

$$\widetilde{q}_n(t) \ = \frac{\widetilde{\boldsymbol{\phi}}_n^T\ \mathbf{F}}{\widetilde{m}_n\ (\widetilde{\omega}_n^2 - \omega^2 + 2\widetilde{\eta}_n\ \widetilde{\omega}_n \omega)} \tag{5}$$

where $\widetilde{\eta}_n$ (resp. $\widetilde{m}_n$) is the damping ratio (resp. the generalized modal mass) of mode $n$.

The random mode statistics can be determined with a MCS. However, an alternative consists in expanding the random modes on polynomial chaoses [6,7]

$$\widetilde{\omega}_k^2 = \omega_k^2 \left( \sum_{p=0}^{P} a_p^k\ \Psi_p(\Xi) \right) \tag{6}$$

$$\widetilde{\boldsymbol{\phi}}_k = \boldsymbol{\phi}_k\ +\ \sum_{\substack{n=1\\ n\neq k}}^{N} \sum_{p=0}^{P} \lambda_{np}^k\ \Psi_p(\Xi)\ \boldsymbol{\phi}_n \tag{7}$$

where $(\omega_k,\ \boldsymbol{\phi}_k)$ denotes the $k$-eigenmode of the deterministic system. The $N \times (P+1)$ unknowns that defined the random modes (see Eqs. (6) and (7)) are calculated by projecting the random eigenproblem on each deterministic eigenmode $\{\boldsymbol{\phi}_n\}_{n=1\cdots N}$ and each PC $\{\Psi_p(\Xi)\}_{p=0\cdots P}$ [6,7].

## 5. Example

### *5.1. Bar with uncertain stiffness*

The uncertain system is shown in Fig. 1. The quantity $EA = E \times A$ ($E$: Young's modulus; $A$: cross-section area) is assumed to be uncertain. In fact, the bar is divided in two parts and $EA_1$ (resp. $EA_2$) is constant along part 1 (resp. part 2) but uncertain. The bar is modelled with a finite element model. $L_1$ and $n_1$ (resp. $L_2$ and $n_2$) are the length and the number of elements of part 1 (resp. part 2). The damping matrix is supposed to be deterministic and defined as $D = 2\ \zeta \mathbf{M}(\mathbf{M}^{-1}\mathbf{K}_0)^{1/2}$ ($\zeta = 0.25\,\%$). The bar is excited by a harmonic force, $F_0\exp(\mathrm{i}\omega t)$ located at the end of the bar. The requested frequency response is the displacement at the end of the bar, $x_{end}$. $EA_i$ is modelled by a random variable: normal laws were addressed in this example, but similar results have been obtained with a uniform law [17]. The characteristics of the uncertain system are listed in Tables 1 and 2.

The mean of the frequency response as well as the probability density function were derived with several methods, which include the MCS, the PCE, the XPA, and the random modes. The reference results are obtained with Monte Carlo simulations: a Latin Hypercube Sampling (LHS) method is used with 10,000 samples.

Table 1. Bar: uncertain parameters

| part 1 | | part 2 | |
|---|---|---|---|
| mean | st. dev. | mean | st. dev. |
| $EA_1^m$ | $\delta_{k1} \times EA_1^m$ | $EA_2^m$ | $\delta_{k2} \times EA_2^m$ |

Table 2. Bar characteristics

| $EA_1^m$ (MN) | $\delta_{k1}$ (%) | $EA_2^m$ (MN) | $\delta_{k2}$ (%) | $F_0$ (kN) | $L_1$ (m) | $L_2$ (m) | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|---|
| 88 | 10 | 22 | 10 | 10 | 1 | 3 | 2 | 6 |

Table 3. Kullback-Leibler divergence - $|x_{end}|$ pdf

| | PCE ($P$=495) | PCE 4 ($P$=14) | Padé [1/2] ($P$=14) | Mode + PCE ($P$=5) |
|---|---|---|---|---|
| $D_{KL}$ | 0.93 | 4.63 | 0.02 | 0.01 |

### 5.2. Frequency response

The mean of the frequency response was derived with a PCE of degree 4 and 30, a XPA [1/2], and the random mode approach. The results are plotted in Fig. 2. The results with a PCE of order 30 are still not very good around the second and the third deterministic eigenfrequencies, even if the results were better to the ones obtained with a PCE of order 4. However the XPA results are in very agreement, even if they are based on a PCE of order 4. The random mode method provided very accurate results. It is also interested to notice that the calculations are much faster in the latter case compared to the other ones. Indeed, in that case, the random modes are independent of the frequency, whereas the PCE as well as the XPA are frequency dependent, and then the PCE coefficients must be calculated for each frequency line. However, the random mode approach assumes that the random modes are orthogonal with respect to the damping matrix, which is not the case. In particular, the damping is proportional only when $\xi_1 = \xi_2 = 0$ (mean stiffness matrix). Then it is striking that the results are so excellent.

### 5.3. Probability density function

The pdf of $x_{end}$ has been estimated at the second deterministic eigenfrequency with the MCS, PCE, XPA, and random modes methods: the results are illustrated in Fig. 3. They were compared to the reference pdf with the Kullback-Leibler divergence [18–20], $D_{KL}$, defined as

$$D_{KL}(p_{ref}(x)||p(x)) = \int_{D_x} p_{ref}(x) \ln\left(\frac{p_{ref}(x)}{p(x)}\right) \mathrm{d}x \tag{8}$$

where $D_x$ is the domain of a random variable $x$. $D_{KL}$ is always nonnegative and is equal to zero when $p_{ref}(x) = p(x)$ almost everywhere. This indicator is listed in Table 3 for the PCE, XPA and random mode approach. The Kullback-Leibler divergence shows that the XPA and the random modes are much more efficient than the PCE approach.

The excellent agreement of the results obtained with the XPA and the random modes explains why the mean frequency responses obtained with these methods were excellent.

## 6. Conclusion

Two approaches were presented to estimated the probability density function of a random system response. The first one is based on a PCE and can be view as a numerical approach; the second one is based on the random modes and then is a more physical approach. Both methods have been very efficient not only to estimate the mean frequency response function, but also to evaluate the probability density function at critical frequencies (deterministic eigenfrequencies).

The main advantage of the random modes is that the random mode are independent of the frequency, which is not the case of the PCE coefficients and then of the XPA. As a consequence, the statistics of the frequency response function can be estimated very quickly. However, this method efficiency relies on the assumption that the modes

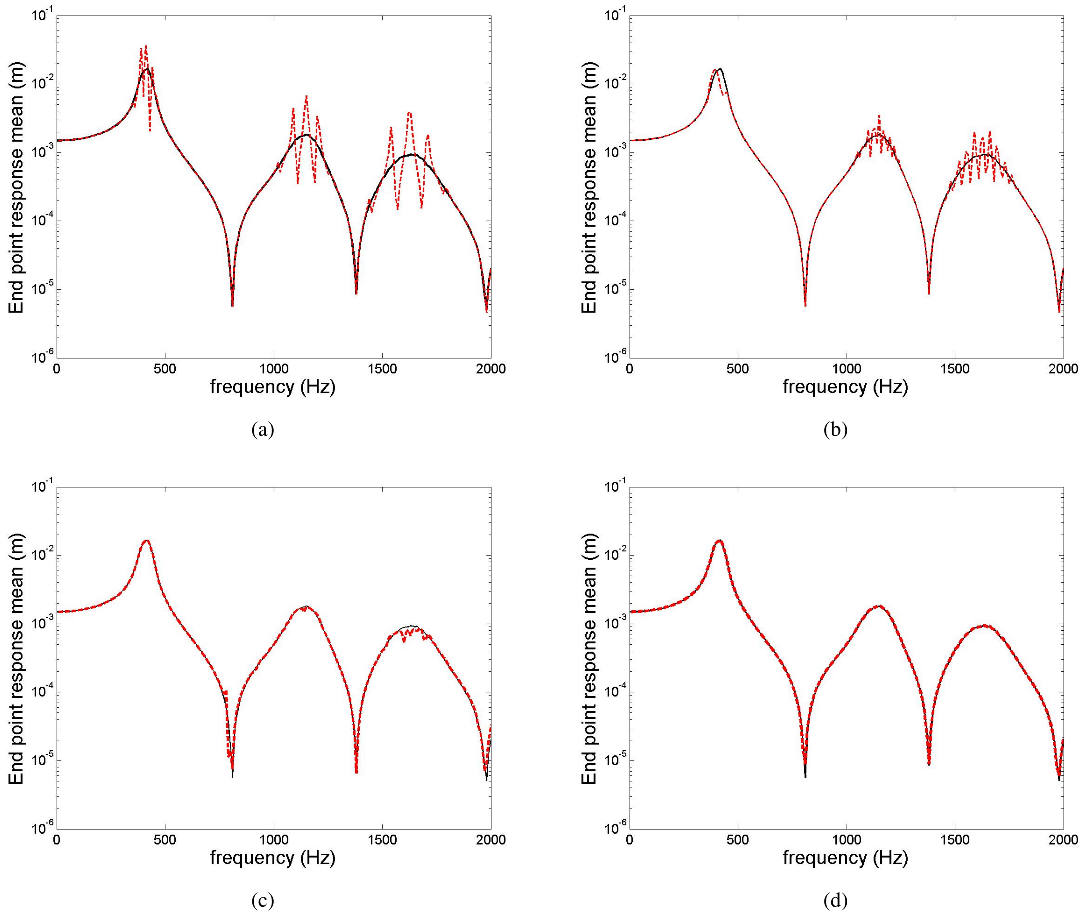


Fig. 2. Mean of the end displacement (m) for several approaches - MCS: black solid lines - red dashed line: (a): PC ($P$ = 14); (b): PC ($P$ = 495); (c): Padé ($P$ = 14); (d): Modal-PC ($P$ = 5)

are uncoupled through the damping contributions, which is true when the damping is proportional. This assumption may be violated for some uncertain systems: in particular this is the case in the example presented in this paper. Accordingly, it is important to verify that the coupling through the damping matrix is statistically negligible.

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## References

[1] R. G. Ghanem, P. D. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer-Verlag, New York, USA, 1991.
[2] E. Jacquelin, S. Adhikari, J.-J. Sinou, M. I. Friswell, The polynomial chaos expansion and the steady-state response of a class of random dynamic systems, Journal of Engineering Mechanics 141(4) (2015) 04014145.
[3] G. A. Baker, P. Graves-Morris, Padé approximants - second edition, Cambridge University press, 1996.
[4] C. Brezinski, Extrapolation algorithms and Padé approximations: a historical survey, Applied Numerical Mathematics 20 (1996) 299–318.
[5] A. C. Matos, Some convergence results for the generalized Padé-type approximants, Numerical Algorithms 11 (1996) 255–269.
[6] O. Dessombz, A. Diniz, F. Thouverez, L. Jézéquel, Analysis of stochastic structures: Pertubation method and projection on homogeneous chaos, in: 7th International Modal Analysis Conference IMAC-SEM, Kissimmee, Floride-USA, 1999.

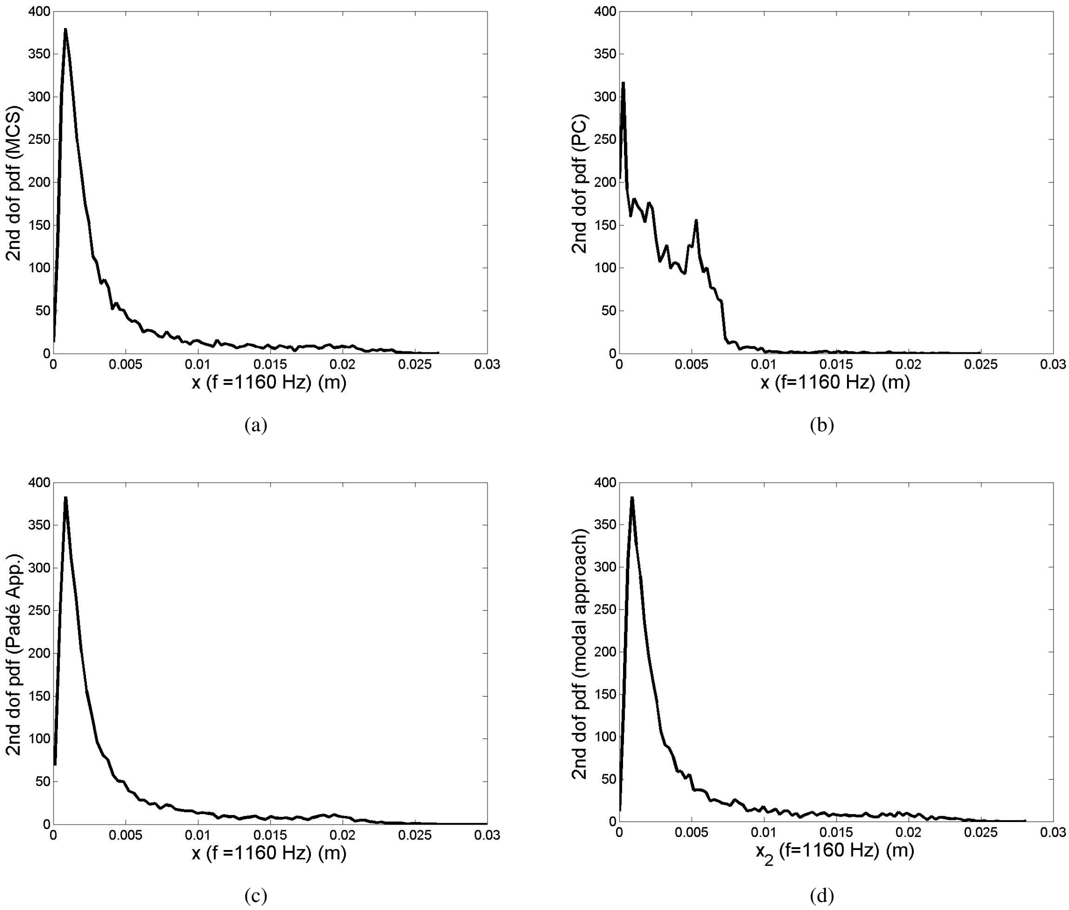


Fig. 3. Probability density function of the response, $x_{end}$, at the second deterministic eigenfrequency; (a): MCS (10,000 samples); (c): PCE ($P$ = 495); (b): XPA ([0/1], $P$ = 14); (d): Modal approach ($P$=5)

[7] O. Dessombz, Analyse dynamique de structures comportant des paramètres incertains (Dynamic Analysis of Structures with Uncertain Parameters), Ph.D. thesis, École Centrale de Lyon, 2000.
[8] J. S. R. Chisholm, Rational approximates defined from double power series, Mathematics of Computation 27 (124) (1973) 841–848.
[9] A. Cuyt, Multivariate Padé approximants revisited, BIT 26 (1986) 71–79.
[10] A. M. Cuyt, How well can the concept of Padé approximant be generalized to the multivariate case?, Journal of Computational and Applied Mathematics 105 (1999) 25–50.
[11] P. Guillaume, A. Huard, V. Robin, Multivariate Padé approximation, Journal of Computational and Applied Mathematics 121 (2000) 197–219.
[12] P. Guillaume, A. Huard, Generalized multivariate Padé approximants, Journal of Approximation Theory 95 (1998) 203–214.
[13] A. C. Matos, Recursive computation of Padé-Legendre approximants and some acceleration properties, Numerische Mathematik 89 (2001) 535–560.
[14] L. Emmel, S. M. Kaber, Y. Maday, Padé-Jacobi filtering for spectral approximations of discontinuous solutions, Numerical Algorithms 33 (2003) 251–264.
[15] A. C. Matos, J. Van Iseghem, Simultaneous Frobenius-Padé approximants, Journal of Computational and Applied Mathematics 176 (2005) 231–258.
[16] J. S. Hesthaven, S. M. Kaber, L. Lurati, Padé-Legendre interpolants for gibbs reconstruction, Journal of Scientific Computing 28 (2006) 337–359.
[17] E. Jacquelin, O. Dessombz, J.-J. Sinou, S. Adhikari, M. I. Friswell, Polynomial chaos based eXtended Padé expansion in structural dynamics, International Journal for Numerical Methods in Engineering Accepted for publication (2017).
[18] S. Kullback, R. A. Leibler, On information and sufficiency, Annals of Mathematical Statistics 22(1) (1951) 79–86.
[19] M. Basseville, Divergence measures for statistical data processing - An annotated bibliography, Signal Processing 93 (2013) 621–633.
[20] G. Greegar, C. Manohar, Global response sensitivity analysis of uncertain structures, Structural Safety 58 (2016) 94–104.