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Dynamic Stiffness and Eigenvalues of Nonlocal Nano-Beams

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Abstract

Free and forced bending vibration of damped nonlocal beams are investigated in this paper. Two types of nonlocal damping models, namely, strain-rate-dependent viscous damping and velocity-dependent viscous damping are considered. A frequency dependent dynamic finite element method is developed to obtain the forced vibration response. Frequency-adaptive complex-valued shape functions are used for the derivation of the dynamic stiffness matrix. The stiffness and mass matrices of the nonlocal beam are also obtained using the conventional finite element method. Results from the dynamic finite element method and conventional finite element method are compared. The frequency response function obtained using the proposed dynamic finite element method shows high modal density in the higher frequency ranges.

Keywords: bending vibration, nonlocal mechanics, dynamic stiffness, asymptotic analysis, frequency response.

1 Introduction

Nanoscale experiments demonstrate that the mechanical properties of nano dimensional materials are much influenced by size effects or scale effects [1, 2]. Size effects are related to atoms and molecules that constitute the materials. Further, atomistic simulations have also reported size-effects on the magnitudes of resonance frequency and buckling load of nanoscale objects such as nanotubes and graphene[3, 4]. The application of classical continuum approaches is thus questionable in the analysis of nanostructures such as nanobeams, nanobeams and nanoplates. Examples of nanobeams and nanobeams include carbon and boron nanotubes, while nanoplates can be graphene sheets and gold nanoplates.

One widely promising size-dependant continuum theory is the nonlocal elasticity theory pioneered by [5] which brings in the scale effects and underlying physics within the formulation. Nonlocal elasticity theory contains information related to the forces between atoms, and the internal length scale in structural, thermal and mechanical analyses. In the nonlocal elasticity theory, the small-scale effects are captured by assuming that the stress at a point is a function of the strains at all points in the domain. Nonlocal theory considers long-range inter-atomic interaction and yields results dependent on the size of a body [5]. Some drawbacks of the classical continuum theory could be efficiently avoided and the size-dependent phenomena can be reasonably explained by nonlocal elasticity. Recent literature shows that the theory of nonlocal elasticity is being increasingly used for reliable and fast analysis of nanostructures. Studies include nonlocal analysis of nanostructures viz. nanobeams [6, 7, 8], nanoplates [9], carbon nanotubes [10], graphenes [11], microtubules [12] and nanorings [13].

bending vibration experiments can be used for the determination of Youngs modulus of CNTs. Generally, the flexural modes occur at low frequencies. However vibrating nanobeams (CNTs) may also have longitudinal modes at relatively high frequencies and can be of very practical significance in high operating frequencies. Nanobeams when used as electromechanical resonators can be externally excited and exhibit bending vibrations. Furthermore for a moving nanoparticle inside a single-walled carbon nanotube (SWCNT), the SWCNT generally vibrates both in the transverse and longitudinal directions. The longitudinal vibration is generated because of the friction existing between the outer surface of the moving nanoparticle and the inner surface of the SWCNT. It is also reported [14] that transport measurements on suspended single-wall carbon nanotubes show signatures of phonon-assisted tunnelling, influenced by longitudinal vibration (stretching) modes. Chowdhury et al. [15] have reported sliding bending modes for multiwalled carbon nanotubes. Only limited work on nonlocal elasticity has been devoted to the bending vibration of nanobeams. Aydogdu [16] developed a nonlocal elastic beam model and applied it to investigate the small scale effect on the bending vibration of clamped-clamped and clamped-free nanobeams. Filiz and Aydogdu [17] applied the bending vibration of nonlocal beam theory to carbon nanotube heterojunction systems. Narendra and Gopalkrishnan [18] have studied the wave propagation of nonlocal nanobeams. Recently Murmu and Adhikari [19] have studied the bending vibration analysis of a double-nanobeam-system. In this paper, we will be refereeing nanobeam as nonlocal beam, so as to distinguish it from local beam.

Several computational techniques have been used for solving the nonlocal governing differential equations. These techniques include Naviers Method [20], Differential Quadrature Method (DQM) [21] and the Galerkin technique [22]. Recently attempts have been made to develop a Finite Element Method (FEM) based on nonlocal elasticity for solving the same. The upgraded finite element method in contrast to these methods can effectively handle more complex geometry, material property, boundary and/or loading conditions. Pisano et al. [23] reported a finite element procedure for nonlocal integral elasticity. Recently some motivating work on a finite element approach based on nonlocal elasticity was reported [24]. The majority of the reported works consider free vibration studies where the effect of non-locality on the eigen-solutions has been studied. However, forced vibration response analysis of nonlocal systems received very little attention.

Based on the above discussion, in this paper we develop the dynamic finite element based on nonlocal elasticity with the aim of considering dynamic response analysis. The dynamic finite element method belongs to the general class of spectral methods for linear dynamical systems [25]. This approach, or approaches very similar to this, is known by various names such as the dynamic stiffness method [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36], spectral finite element method [25, 37] and dynamic finite element method [38, 39]. So far the dynamic finite element method has been applied to classical local systems only. In this paper we generalise this approach to nonlocal systems. One of the novel features of the analysis proposed here is the employment of frequency-dependent complex nonlocal shape functions for damped systems. This in turn enables us to obtain the element stiffness matrix using the usual weak form of the finite element method.

Overview of the paper is as follows. In 2 we describe the equation of motion for bending vibration of undamped and damped beams. The conventional and the dynamic finite element method are developed in 3. Closed form expressions are derived for the mass and stiffness matrices. In 4 the proposed methodology is applied to an armchair single walled carbon nanotube (SWCNT) for illustration. Theoretical results, including the asymptotic behaviours of the natural frequencies, are numerically illustrated. Finally, in 5 some conclusions are drawn based on the results obtained in the paper.

2 Bending vibration of undamped and damped nonlocal beams

2.1 Undamped system

For the bending vibration of a nonlocal damped beam, the equation of motion can be expressed by

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} + m\left(1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right)\left\{\frac{\partial^2 V(x,t)}{\partial t^2}\right\} = \left(1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right)\{F(x,t)\} \quad (1)$$

In the above equation EI is the bending rigidity, m is mass per unit length, e_0a is the nonlocal parameter, $V(x,t)$ is the transverse displacement and $F(x,t)$ is the applied force. Considering the free vibration, i.e., setting the force to zero, and assuming harmonic motion with frequency ω

$$V(x,t) = v(x)\exp[\mathrm{i}\omega t] \quad (2)$$

from (1) we have

$$EI\frac{d^4v}{dx^4} - m\omega^2\left(v - (e_0a)^2\frac{d^2v}{dx^2}\right) = 0 \tag{3}$$

$$\text{or} \quad \frac{d^4v}{dx^4} + b^4(e_0a)^2\frac{d^2v}{dx^2} - b^4v = 0 \tag{4}$$

where

$$b^4 = \frac{m\omega^2}{EI} \tag{5}$$

To obtain the characteristic equation, we assume

$$v(x) = \exp\left[\lambda x\right] \tag{6}$$

Substituting this in Eq. (4) we obtain

$$\lambda^4 + b^4(e_0a)^2\lambda^2 - b^4 = 0 \tag{7}$$

$$\text{or} \quad \lambda^2 = b^2\left(-b^2(e_0a)^2 \pm \sqrt{4 + b^4(e_0a)^4}\right)/2 \tag{8}$$

Defining

$$\gamma = b^2(e_0a)^2 \tag{9}$$

the two roots can be expressed as

$$\lambda^2 = -\alpha^2,\ \beta^2 \tag{10}$$

where

$$\alpha = b\sqrt{\left(\sqrt{4+\gamma^2}+\gamma\right)/2} \tag{11}$$

$$\text{and} \quad \beta = b\sqrt{\left(\sqrt{4+\gamma^2}-\gamma\right)/2} \tag{12}$$

Therefore, the four roots of the characteristic equation can be expressed as

$$\lambda = \mathrm{i}\alpha, -\mathrm{i}\alpha,\ \beta, -\beta \tag{13}$$

where $\mathrm{i} = \sqrt{-1}$. The displacement function within the beam can be expressed by linear superposition as

$$v(x) = \sum_{j=1}^{4} c_j \exp[\lambda_j x] \tag{14}$$

Here the unknown constants c_j need to be obtained from the boundary conditions.

Using Eq. (14), the natural frequency of the system can be obtained by imposing the necessary boundary conditions [40]. For example, the bending moment and shear force are given by:

- Bending moment at $x = 0$ or $x = L$:

$$EI \frac{d^2 v(x)}{dx^2} = 0 \tag{15}$$

- Shear force at $x = 0$ or $x = L$:

$$EI \frac{d^3 v(x)}{dx^3} + m\omega^2 (e_0 a)^2 \frac{dv(x)}{dx} = 0 \quad \text{or} \quad \frac{d^3 v(x)}{dx^3} + b^4 (e_0 a)^2 \frac{dv(x)}{dx} = 0 \tag{16}$$

Boundary conditions involving displacements and rotations can be applied in the conventional manner.

2.2 Damped system

The equation of motion of bending vibration for a damped nonlocal beam can be expressed as

$$EI \frac{\partial^4 V(x,t)}{\partial x^4} + m\left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right)\left\{\frac{\partial^2 V(x,t)}{\partial t^2}\right\} + \widehat{c}_1 \left(1 - (e_0 a)_1^2 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^5 V(x,t)}{\partial x^4 \partial t}$$
$$+ \widehat{c}_2 \left(1 - (e_0 a)_2^2 \frac{\partial^2}{\partial x^2}\right) \frac{\partial V(x,t)}{\partial t} = \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \{F(x,t)\} \tag{17}$$

Here $\widehat{c}_1$ is the strain-rate-dependent viscous damping coefficient and $\widehat{c}_2$ is the velocity-dependent viscous damping coefficient. The parameters $(e_0 a)_1$ and $(e_0 a)_2$ are nonlocal parameters related to the two damping terms respectively. For simplicity we have not taken into account any nonlocal effect related to the damping. Although this can be mathematically incorporated in the analysis, the determination of these nonlocal parameters is beyond the scope of this work and therefore only local interaction for the damping is adopted. Thus, in the following analysis we consider $(e_0 a)_1 = (e_0 a)_2 = 0$.

Assuming harmonic response as in (2) and considering free vibration, from Eq. (17) we have

$$EI \frac{d^4 v}{dx^4} - m\omega^2 \left(v - (e_0 a)^2 \frac{d^2 v}{dx^2}\right) + \mathrm{i}\omega \widehat{c}_1 \frac{d^4 v}{dx^4} + \mathrm{i}\omega \widehat{c}_2 v = 0 \tag{18}$$

Following the damping convention in dynamic analysis [40], we consider stiffness and mass proportional damping. Therefore, we express the damping constants as

$$\widehat{c}_1 = \zeta_1 (EI) \quad \text{and} \quad \widehat{c}_2 = \zeta_2 (m) \tag{19}$$

where ζ_1 and ζ_2 are stiffness and mass proportional damping factors. Substituting these, from Eq. (18) we have

$$EI\left(1 + \mathrm{i}\omega\zeta_1\right) \frac{d^4 v}{dx^4} + m\omega^2 (e_0 a)^2 \frac{d^2 v}{dx^2} - m\omega^2 \left(1 - \mathrm{i}\zeta_2/\omega\right) v = 0 \tag{20}$$

$$\text{or} \quad \frac{d^4 v}{dx^4} + \bar{b}^4 (e_0 a)^2 \frac{d^2 v}{dx^2} - \bar{b}^4 \theta v = 0 \tag{21}$$

where we define $\bar{b}$ and introduce θ as

$$\bar{b}^4 = \frac{m\omega^2}{EI\,(1 + \mathrm{i}\omega\zeta_1)} \quad \text{and} \quad \theta = (1 - \mathrm{i}\zeta_2/\omega) \tag{22}$$

Comparing this with the equivalent expression for the undamped case in Eq. (5) we notice that $\bar{b}^4$ is a complex function of the frequency parameter ω as opposed to a real function. In the special case when damping coefficients ζ_1 and ζ_1 go to zero, $\bar{b}^4$ in Eq. (22) reduces to the expression of b^4 in Eq. (5) and θ becomes 1. The characteristics equation can be obtained in manned similar to the undamped system with the expressions of α and β being

$$\alpha = \bar{b}\sqrt{\left(\sqrt{4\theta + \gamma^2} + \gamma\right)/2} \tag{23}$$

$$\text{and} \quad \beta = \bar{b}\sqrt{\left(\sqrt{4\theta + \gamma^2} - \gamma\right)/2} \tag{24}$$

where $\gamma = \bar{b}^2(e_0 a)^2$.

3 Dynamic finite element matrix

3.1 Classical finite element of nonlocal beams

We first consider standard finite element analysis of the nonlocal beam. Recently Phadikar and Pradhan [24] proposed a variational-formulation-based finite element approach for nanobeams and nanoplates. We consider an element of length ℓ_e with bending stiffness EI and mass per unit length m. An element of the beam is shown in 1. This element has four degrees of freedom and there are four shape functions. The

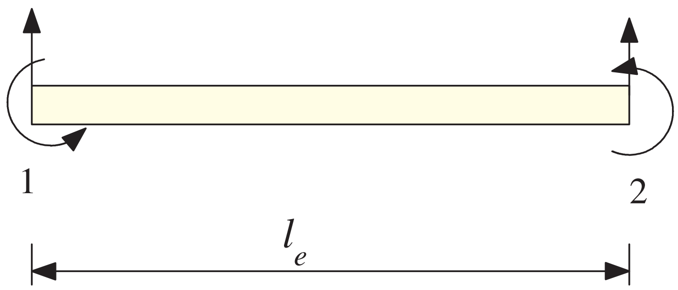


Figure 1: A nonlocal element for the bending vibration of a beam. It has two nodes and four degrees of freedom. The displacement field within the element is expressed by cubic shape functions.

shape function matrix for the bending deformation [41] can be given by

$$\mathbf{N}(x) = [N_1(x), N_2(x), N_3(x), N_4(x)]^T \tag{25}$$

where

$$
\begin{aligned}
N_1(x) &= 1 - 3\tfrac{x^2}{\ell_e^2} + 2\tfrac{x^3}{\ell_e^3}, & N_2(x) &= x - 2\tfrac{x^2}{\ell_e} + \tfrac{x^3}{\ell_e^2}, \\
N_3(x) &= 3\tfrac{x^2}{\ell_e^2} - 2\tfrac{x^3}{\ell_e^3}, & N_4(x) &= -\tfrac{x^2}{\ell_e} + \tfrac{x^3}{\ell_e^2}
\end{aligned}
\tag{26}
$$

Using this, the stiffness matrix can be obtained following the conventional variational formulation [42] as

$$
\mathbf{K}_e = EI \int_0^{\ell_e} \frac{d^2\mathbf{N}(x)}{dx^2}\frac{d^2\mathbf{N}^T(x)}{dx^2}\mathrm{d}x = \frac{EI}{\ell_e^3}\begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e^2 \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix} \tag{27}
$$

The mass matrix for the nonlocal element can be obtained as

$$
\begin{aligned}
\mathbf{M}_e &= m\int_0^{\ell_e} \mathbf{N}(x)\mathbf{N}^T(x)\mathrm{d}x + m(e_0a)^2\int_0^{\ell_e} \frac{d\mathbf{N}(x)}{dx}\frac{d\mathbf{N}^T(x)}{dx}\mathrm{d}x \\
&= \frac{m\ell_e}{420}\begin{bmatrix} 156 & 22\ell_e & 54 & -13\ell_e \\ 22\ell_e & 4\ell_e^2 & 13\ell_e & -3\ell_e^2 \\ 54 & 13\ell_e & 156 & -22\ell_e \\ -13\ell_e & -3\ell_e^2 & -22\ell_e & 4\ell_e^2 \end{bmatrix} \\
&+ \left(\frac{e_0a}{\ell_e}\right)^2 \frac{m\ell_e}{30}\begin{bmatrix} 36 & 3\ell_e & -36 & 3\ell_e \\ 3\ell_e & 4\ell_e^2 & -3\ell_e & -\ell_e^2 \\ -36 & -3\ell_e & 36 & -3\ell_e \\ 3\ell_e & -\ell_e^2 & -3\ell_e & 4\ell_e^2 \end{bmatrix}
\end{aligned}
\tag{28}
$$

For the special case when the beam is local, the mass matrix derived above reduces to the classical mass matrix [41, 42] as $e_0a = 0$.

3.2 Dynamic finite element for damped nonlocal beam

The first step for the derivation of the dynamic element matrix is the generation of dynamic shape functions. The dynamic shape functions are obtained such that the equation of dynamic equilibrium is satisfied exactly at all points within the element. Similarly to the classical finite element method, assume that the frequency-dependent displacement within an element is interpolated from the nodal displacements as

$$
v_e(x,\omega) = \mathbf{N}^T(x,\omega)\widehat{\mathbf{v}}_e(\omega) \tag{29}
$$

Here $\widehat{\mathbf{v}}_e(\omega) \in \mathbb{C}^n$ is the nodal displacement vector $\mathbf{N}(x,\omega) \in \mathbb{C}^n$ is the vector of frequency-dependent shape functions and $n = 4$ is the number of the nodal degrees-of-freedom. Suppose the $s_j(x,\omega) \in \mathbb{C}, j = 1, \cdots, 4$ are the basis functions which exactly satisfy Eq. (21). It can be shown that the shape function vector can be expressed as

$$
\mathbf{N}(x,\omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(x,\omega) \tag{30}
$$

where the vector $\mathbf{s}(x,\omega) = \{s_j(x,\omega)\}^T, \forall j = 1,\cdots,4$ and the complex matrix $\mathbf{\Gamma}(\omega) \in \mathbb{C}^{4\times4}$ depends on the boundary conditions. The elements of $\mathbf{s}(x,\omega)$ constitutes $\exp[\lambda_j x]$ where the values of λ_j are obtained from the solution of the characteristics equation as given in Eq. (13). An element for the damped beam under bending vibration is shown in 1. The degrees-of-freedom for each nodal point include a vertical and a rotational degrees-of-freedom.

In view of the solutions in Eq. (13), the displacement field with the element can be expressed by linear combination of the basic functions $e^{-\mathrm{i}\alpha x}$, $e^{\mathrm{i}\alpha x}$, $e^{\beta x}$ and $e^{-\beta x}$ so that in our notations $\mathbf{s}(x,\omega) = \left\{e^{-\mathrm{i}\alpha x}, e^{\mathrm{i}\alpha x}, e^{\beta x}, e^{-\beta x}\right\}^T$. We can also express $\mathbf{s}(x,\omega)$ in terms of trigonometric functions. Considering $e^{\pm\mathrm{i}\alpha x} = \cos(\alpha x) \pm \mathrm{i}\sin(\alpha x)$ and $e^{\pm\beta x} = \cosh(\beta x) \pm \mathrm{i}\sinh(\beta x)$, the vector $\mathbf{s}(x,\omega)$ can be alternatively expressed as

$$\mathbf{s}(x,\omega) = \left\{\begin{array}{c}\sin(\alpha x)\\ \cos(\alpha x)\\ \sinh(\beta x)\\ \cosh(\beta x)\end{array}\right\} \in \mathbb{C}^4 \tag{31}$$

The displacement field within the element can be expressed as

$$v(x) = \mathbf{s}(x,\omega)^T \mathbf{v}_e \tag{32}$$

where $\mathbf{v}_e \in \mathbb{C}^4$ is the vector of constants to be determined from the boundary conditions.

The relationship between the shape functions and the boundary conditions can be represented as in 1, where boundary conditions in each column give rise to the corresponding shape function. Writing Eq. (32) for the above four sets of boundary

	$N_1(x,\omega)$	$N_2(x,\omega)$	$N_3(x,\omega)$	$N_4(x,\omega)$
$y(0)$	1	0	0	0
$\frac{\mathrm{d}y}{\mathrm{d}x}(0)$	0	1	0	0
$y(L)$	0	0	1	0
$\frac{\mathrm{d}y}{\mathrm{d}x}(L)$	0	0	0	1

Table 1: The relationship between the boundary conditions and the shape functions for the bending vibration of beams.

conditions, one obtains

$$[\mathbf{R}]\left[\mathbf{y}_e^1, \mathbf{y}_e^2, \mathbf{y}_e^3, \mathbf{y}_e^4\right] = \mathbf{I} \tag{33}$$

where

$$\mathbf{R} = \begin{bmatrix} s_1(0) & s_2(0) & s_3(0) & s_4(0) \\ \frac{\mathrm{d}s_1}{\mathrm{d}x}(0) & \frac{\mathrm{d}s_2}{\mathrm{d}x}(0) & \frac{\mathrm{d}s_3}{\mathrm{d}x}(0) & \frac{\mathrm{d}s_4}{\mathrm{d}x}(0) \\ s_1(L) & s_2(L) & s_3(L) & s_4(L) \\ \frac{\mathrm{d}s_1}{\mathrm{d}x}(L) & \frac{\mathrm{d}s_2}{\mathrm{d}x}(L) & \frac{\mathrm{d}s_3}{\mathrm{d}x}(L) & \frac{\mathrm{d}s_4}{\mathrm{d}x}(L) \end{bmatrix} \tag{34}$$

and $\mathbf{y}_e^k$ is the vector of constants giving rise to the kth shape function. In view of the boundary conditions represented in 1 and equation (33), the shape functions for bending vibration can be shown to be given by Eq. (30) where

$$\boldsymbol{\Gamma}(\omega) = \left[\mathbf{y}_e^1, \mathbf{y}_e^2, \mathbf{y}_e^3, \mathbf{y}_e^4\right]^T = \left[\mathbf{R}^{-1}\right]^T \tag{35}$$

By obtaining the matrix $\boldsymbol{\Gamma}(\omega)$ from the above equation, the shape function vector can be obtained from Eq. (30).

The stiffness and mass matrices can be obtained similarly to the static finite element case discussed before. Note that for this case all the matrices become complex and frequency-dependent. It is more convenient to define the dynamic stiffness matrix as

$$\mathbf{D}_e(\omega) = \mathbf{K}_e(\omega) - \omega^2 \mathbf{M}_e(\omega) \tag{36}$$

so that the equation of dynamic equilibrium is

$$\mathbf{D}_e(\omega)\widehat{\mathbf{v}}_e(\omega) = \widehat{\mathbf{f}}(\omega) \tag{37}$$

In Eq. (36), the frequency-dependent stiffness and mass matrices can be obtained as

$$\mathbf{K}_e(\omega) = EI \int_0^L \frac{d^2\mathbf{N}(x,\omega)}{dx^2} \frac{d^2\mathbf{N}^T(x,\omega)}{dx^2} \mathrm{d}x \tag{38}$$

$$\text{and} \quad \mathbf{M}_e(\omega) = m \int_0^L \mathbf{N}(x,\omega)\mathbf{N}^T(x,\omega)\mathrm{d}x \tag{39}$$

After some algebraic simplifications [34, 43] it can be shown that the dynamic stiffness matrix is given by the following closed-form expression

$$\mathbf{D}_e(\omega) = EI\Delta \times \begin{bmatrix} -\alpha\beta\left(cS\beta + Cs\alpha\right) & \beta\left(c\alpha C - \alpha - sS\beta\right) & \alpha\beta\left(S\beta + s\alpha\right) & -\left(C - c\right)\alpha\beta \\ \beta\left(c\alpha C - \alpha - sS\beta\right) & -sC\beta + c\alpha S & \left(C - c\right)\alpha\beta & -\alpha S + s\beta \\ \alpha\beta\left(S\beta + s\alpha\right) & \left(C - c\right)\alpha\beta & -\alpha\beta\left(cS\beta + Cs\alpha\right) & \alpha\left(s\alpha S - \beta + cC\beta\right) \\ -\left(C - c\right)\alpha\beta & -\alpha S + s\beta & \alpha\left(s\alpha S - \beta + cC\beta\right) & -sC\beta + c\alpha S \end{bmatrix} \tag{40}$$

where

$$\Delta = \frac{(\alpha^2 + \beta^2)}{sS(\alpha^2 - \beta^2) - 2\alpha\beta(1 - cC)} \tag{41}$$

with

$$C = \cosh(\beta L), \quad c = \cos(\alpha L), \quad S = \sinh(\beta L) \quad \text{and} \quad s = \sin(\alpha L) \tag{42}$$

These are frequency dependent complex quantities because α and β are functions of ω and damping factors.

In general the dynamic stiffness matrix in (40) is a 4×4 matrix with complex entries. The frequency response of the system at the nodal points can be obtained by simply solving Eq. (37) for all frequency values. The calculation only involves inverting a 4×4 complex matrix and the results are exact with only one element for any frequency value. This is a significant advantage of the proposed dynamic finite element approach compared to the conventional finite element approach discussed in the pervious subsection.

So far we did not explicitly consider any forces within the element. A distributed body force can be considered following the usual finite element approach [41] and replacing the static shape functions with the dynamic shape functions (30). Suppose $p_e(x, \omega), x \in [0, L]$ is the frequency depended distributed body force. The element nodal forcing vector can be obtained as

$$\mathbf{f}_e(\omega) = \int_0^L p_e(x, \omega)\mathbf{N}(x, \omega)\mathrm{d}x \tag{43}$$

As an example, if a point harmonic force of magnitude p is applied at length $b < L$ then, $p_e(x, \omega) = p\delta(x - b)$ where $\delta(\bullet)$ is the Dirac delta function. The element nodal force vector becomes

$$\mathbf{f}_e(\omega) = p \int_0^L \delta(x - b)\mathbf{N}(x, \omega)\mathrm{d}x \tag{44}$$

Next we illustrate the formulation derived in this section using an example.

4 Numerical results and discussions

A double-walled carbon nanotube (DWCNT) is considered to examine the bending vibration characteristics. An armchair (5, 5), (8, 8) DWCNT with Young's modulus $E = 1.0$ TPa, $L = 30$ nm, density $\rho = 2.3 \times 10^3$ kg/m^3 and thickness t = 0.35 nm is considered as in [44]. The inner and the outer diameters of the DWCNT are respectively 0.68nm and 1.1nm. The system considered here is shown in 2 . We consider pinned-pinned boundary condition for the DWCNT. Undamped nonlocal natural frequencies can be obtained [16] as

$$\lambda_j = \sqrt{\frac{EI}{m}} \frac{\beta_j^2}{\sqrt{1 + \beta_j^2 (e_0 a)^2}} \quad \text{where} \quad \beta_j = j\pi/L, \quad j = 1, 2, \cdots \tag{45}$$

EI is the bending rigidity and m is the mass per unit length of the DWCNT. For the finite element analysis the DWCNT is divided into 100 elements. The dimension of each of the system matrices become 200×200, that is $n = 200$. The global stiffness and mass matrices are obtained by assembling the element stiffness and mass matrix given by (27) and (28).

The natural frequencies obtained using the analytical expression (45) are compared with direct finite element simulation in 3. The frequency values are normalised with respect to the first local natural frequency. First 20 nonlocal natural frequencies are

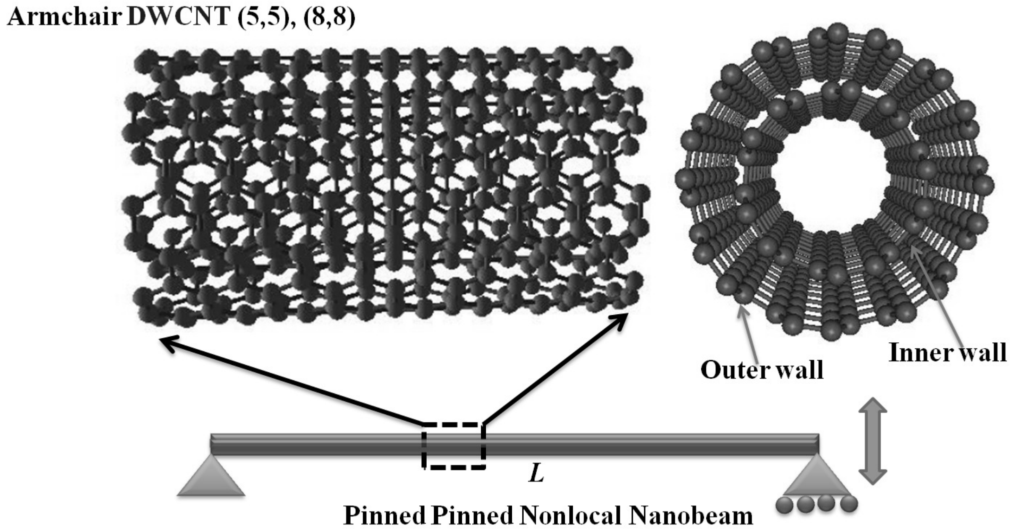


Figure 2: Bending vibration of an armchair (5, 5), (8, 8) double-walled carbon nanotube (DWCNT) with pinned-pinned boundary condition.

shown for four distinct values of e_0a, namely 0.5, 1.0, 1.5 and 2.0nm. Natural frequencies corresponding to the underlying local system is shown in 3. Local frequencies are qualitatively different from nonlocal frequencies as it increases quadratically with the number of modes. Nonlocal frequencies on the other hand increases approximately linearly with the number of modes.

In 4 the amplitude of the dynamic response function $v(\omega)$ at the right-end is shown for the four representative values of the nonlocal parameter. We consider the mass and stiffness proportional damping such that the damping factors $\zeta_2 = 0.05$ and $\zeta_1 = 10^{-4}$.

In the x-axis, excitation frequency normalised with respect to the first local frequency is considered. The frequency response is normalise by the static response d_{st} (this is the response when the excitation frequency is 0). The frequency response function of the underlying local model is also plotted to show the difference between the local and nonlocal response. For the nonlocal system, the frequency response is obtained by the direct finite element method and the dynamic stiffness method. For the finite element analysis we used 100 elements. This in turn, results in global mass and stiffness matrices of dimension 200 $\times$ 200. While for the dynamic stiffness method, only the inversion of a 2 $\times$ 2 matrix is necessary. The results demonstrates the computational efficiency and accuracy of the proposed dynamics stiffness method over the conventional finite element method.

5 Conclusions

A novel dynamic finite element approach for bending vibration of damped nonlocal beams is proposed. Strain rate dependent viscous damping and velocity dependent

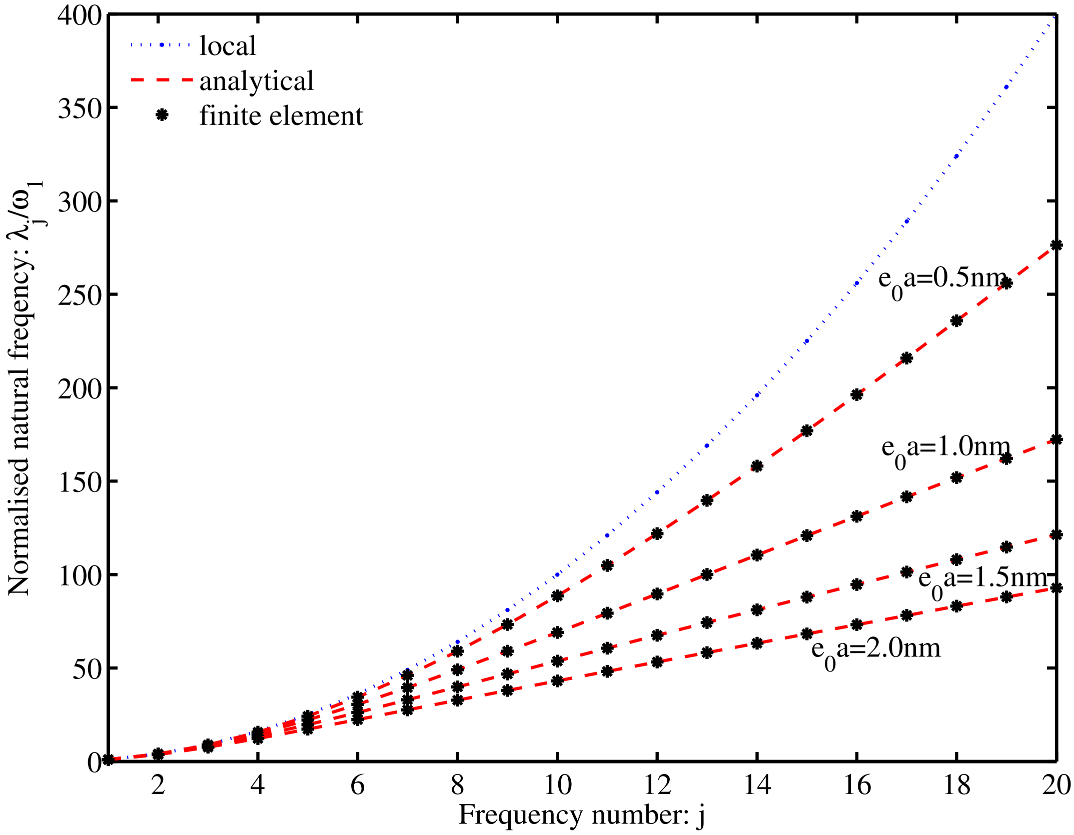
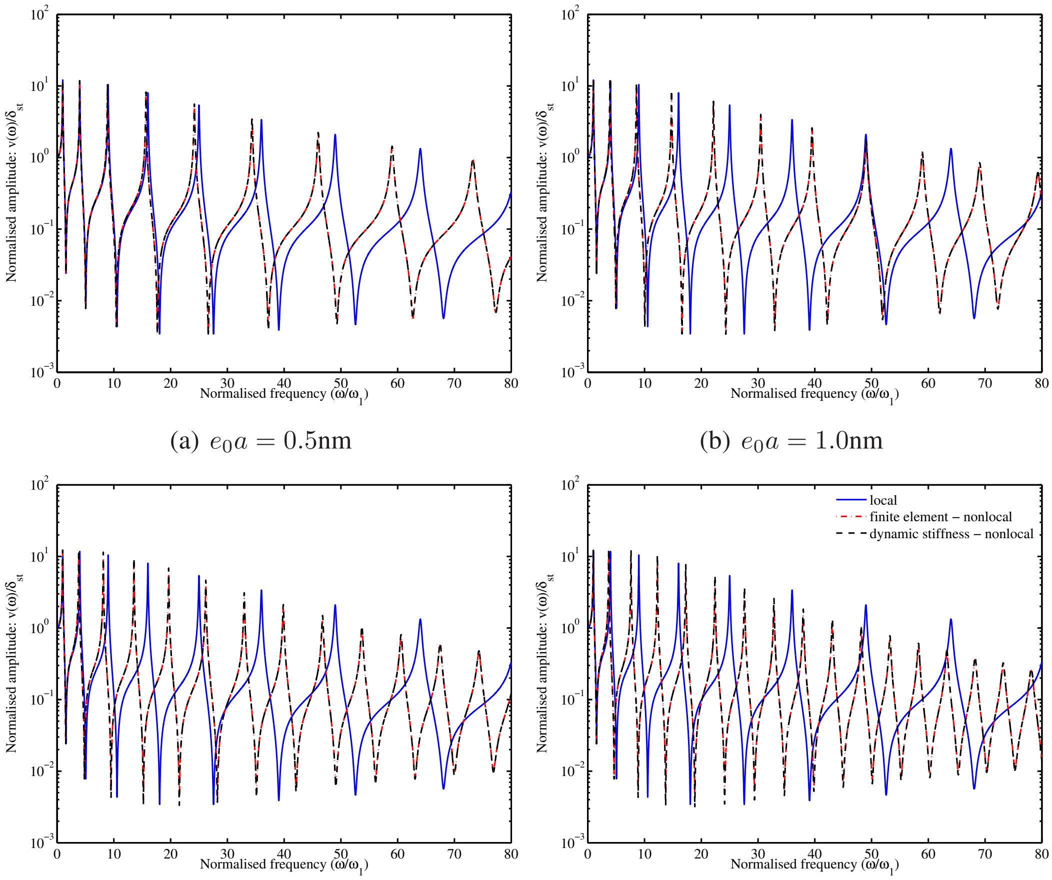


Figure 3: The variation of first 20 undamped natural frequencies for the bending vibration of DWCNT. Four representative values of e_0a (in nm) are considered.

viscous damping are considered. Damped and undamped dynamics are discussed. Frequency dependent complex-valued shape functions are used to obtain the dynamic stiffness matrix in closed-form. The dynamic response in the frequency domain can be obtained by inverting the dynamic stiffness matrix. The stiffness and mass matrices of the nonlocal beam was also obtained using the conventional finite element method. In the special case when the nonlocal parameter becomes zero, the expression of the mass matrix reduces to the classical case. The proposed method is numerically applied to the bending vibration of an armchair (5, 5), (8, 8) double-walled carbon nanotube with pinned-pinned boundary condition. The natural frequencies and the dynamic response obtained using the conventional finite element approach were compared with the results obtained using the dynamic finite method. Good agreement between conventional finite element with 100 elements and proposed dynamic finite element with only one element was found. This demonstrated the accuracy and computational efficiency of the proposed dynamic stiffness method.



(a) $e_0 a = 0.5$nm

(b) $e_0 a = 1.0$nm

(c) $e_0 a = 1.5$nm

(d) $e_0 a = 2.0$nm

Figure 4: Amplitude of the normalised frequency response of the DWCNT $v(\omega)$ at theright-end for different values of $e_0 a$. Exact finite element results are compared with the approximate analysis based on local eigensolutions.

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