DAMPING MODELLING AND IDENTIFICATION USING GENERALIZED PROPORTIONAL DAMPING

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ABSTRACT

In spite of a large amount of research, the understanding of damping forces in vibrating structures is not well developed. A major reason for this is that, by contrast with inertia and stiffness forces, the physics behind the damping forces is in general not clear. As a consequence, modelling of damping from the first principle is difficult, if not impossible, for real-life engineering structures. The common approach is to use the proportional damping model where it is assumed that the damping matrix is proportional to mass and stiffness matrices. The main limitation of the proportional damping approximation comes from the fact that the variation of damping factors with respect to vibration frequency cannot be modelled accurately by using this approach. Experimental results however suggest that damping factors can vary with frequency. In this paper a new generalized proportional damping model is proposed in order to capture the frequency-variation of the damping factors accurately. A simple identification method is proposed to obtain the damping matrix using the generalized proportional damping model. The proposed method requires only the measurements of natural frequencies and modal damping factors. Examples are provided to illustrate the proposed method.

NOMENCLATURE

α_1, α_2	proportional damping constants
$\mathbf{C}$	viscous damping matrix
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{q}(t)$	generalized coordinates
$\mathbf{T}$	a temporary matrix, $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$
$\mathbf{X}$	undamped modal matrix
ω_j	natural frequencies
$\widehat{f}(\bullet)$	fitted modal damping function
ζ_j	modal damping factors

1 INTRODUCTION

Modal analysis is the most popular and efficient method for solving engineering dynamic problems. The concept of modal analysis, as introduced by Lord Rayleigh [1], was originated from the linear dynamics of undamped systems. The undamped modes or classical normal modes satisfy an orthogonality relationship over the mass and stiffness matrices and uncouple the equations of motion, *i.e.*, if $\mathbf{X}$ is the modal matrix then $\mathbf{X}^T\mathbf{M}\mathbf{X}$ and $\mathbf{X}^T\mathbf{K}\mathbf{X}$ are both diagonal matrices. This significantly simplifies the dynamic analysis because complex multiple degree-of-freedom (MDOF) systems can be effectively treated as a collection of single degree-of-freedom oscillators.

Real-life systems are not undamped but possess some kind of energy dissipation mechanism or damping. In order to apply modal analysis of undamped systems to damped systems, it is common to assume the proportional damping, a special type of viscous damping. The proportional damping model expresses the damping matrix as a linear combination of the mass and stiffness matrices, that is

$$\mathbf{C} = \alpha_1 \mathbf{M} + \alpha_2 \mathbf{K} \tag{1}$$

where α_1, α_2 are real scalars. This damping model is also known as 'Rayleigh damping' or 'classical damping'. Modes of classically damped systems preserve the simplicity of the real normal modes as in the undamped case. Caughey and O'Kelly [2] have derived the condition which the system matrices must satisfy so that viscously damped linear systems possess classical normal modes. They have also proposed a series expression for the damping matrix in terms of the mass and stiffness matrices so that the system can be decoupled by the undamped modal matrix and have shown that the Rayleigh damping is a special case of this general expression. In this paper a more general expression of the damping matrix is proposed so that the system possess classical normal modes.

Complex engineering structures in general have non-proportional damping. For a non-proportionally damped system, the equations of motion in the modal coordinates are coupled through the off-diagonal terms of the modal damping matrix and consequently the system possess complex modes instead of real normal modes. Practical experience in modal testing also shows that most real-life structures possess complex modes. Complex modes can arise for various other reasons also [3], for example, due to the gyroscopic effects, aerodynamic effects, nonlinearity and experimental noise. Adhikari and Woodhouse [4, 5] have proposed few methods to identify damping from experimentally identified complex modes. In spite of a large amount of research, understanding and identification of complex modes is not well developed as real normal modes. The main reasons are:

- By contrast with real normal modes, the 'shapes' of complex modes are not in general clear. It appears that unlike the (real) scaling of real normal modes, the (complex) scaling or normalization of complex modes has a significant effect on their geometric appearance. This makes it particularly difficult to experimentally identify complex modes in a consistent manner [6].
- The imaginary parts of the complex modes are usually vary small compared to the real parts, especially when the damping is small. This makes it very difficult to reliably extract complex modes using numerical optimization methods in conjunction with experimentally obtained transfer function residues.
- The phase of complex modes are highly sensitive to experimental errors, ambient conditions and measurement noise and often not repeatable in a satisfactory manner.

In order to bypass these difficulties in experimental modal analysis often real normal modes are used. Ibrahim [7], Chen et al [8] and Balmès [9] have proposed methods to obtain the best real normal modes from identified complex modes. The damping identification method proposed in this paper assumes that the system is effectively proportionally damped so that the complex modes can be neglected. The outline of the paper is as follows. In section 2, a background of proportionally damped systems is provided. The concept of generalized proportional damping in introduced in section 3. The damping identification based on the generalized proportional damping is proposed in section 4. Numerical examples are provided to illustrate the proposed method.

2 BACKGROUND OF PROPORTIONALLY DAMPED SYSTEMS

The equations of motion of free vibration of a viscously damped system can be expressed by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}. \tag{2}$$

Caughey and O'Kelly [2] have proved that a damped linear system of the form (2) can possess classical normal modes if and only if the system matrices satisfy the relationship $\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K}$. This is an important result on modal analysis of viscously damped systems and is now well known. However, this result does not immediately generalize to systems with singular mass matrices [10]. This apparent restriction in Caughey and O'Kelly's result may be removed by considering the fact that all the three system matrices can be treated on equal basis and therefore can be interchanged. In view of this, when the system matrices are non-negative definite we have the following theorem:

Theorem 1 *A viscously damped linear system can possess classical normal modes if and only if at least one of the following conditions is satisfied:*
(a) $\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K}$, *(b)* $\mathbf{M}\mathbf{K}^{-1}\mathbf{C} = \mathbf{C}\mathbf{K}^{-1}\mathbf{M}$, *(c)* $\mathbf{M}\mathbf{C}^{-1}\mathbf{K} = \mathbf{K}\mathbf{C}^{-1}\mathbf{M}$.

This can be easily proved by following Caughey and O'Kelly's approach and interchanging **M**, **K** and **C** successively. If a system is $(\bullet)$-singular then the condition(s) involving $(\bullet)^{-1}$ have to be disregarded and remaining condition(s) have to be used. Thus, for a positive definite system, along with Caughey and O'Kelly's result (condition (a) of the theorem), there exist two other equivalent criterion to judge whether a damped system can possess classical normal modes. It is important to note that these three conditions are equivalent and simultaneously valid but in general *not* the same.

Example 1

Assume that a system's mass, stiffness and damping matrices are given by

$$\mathbf{M} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 2.0 \\ 1.0 & 2.0 & 3.0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1.2 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 15.25 & -9.8 & 3.4 \\ -9.8 & 6.48 & -1.84 \\ 3.4 & -1.84 & 2.22 \end{bmatrix}. \tag{3}$$

It may be verified that all the system matrices are positive definite. The mass-normalized undamped modal matrix is obtained as

$$\mathbf{X} = \begin{bmatrix} 0.4027 & -0.5221 & -1.2511 \\ 0.5845 & -0.4888 & 1.1914 \\ -0.1127 & 0.9036 & -0.4134 \end{bmatrix}. \tag{4}$$

Since Caughey and O'Kelly's condition

$$\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 125.45 & -80.92 & 28.61 \\ -80.92 & 52.272 & -18.176 \\ 28.61 & -18.176 & 7.908 \end{bmatrix}$$

is satisfied, the system possess classical normal modes and that **X** given in equation (4) is the modal matrix. Because the system is positive definite the other two conditions,

$$\mathbf{M}\mathbf{K}^{-1}\mathbf{C} = \mathbf{C}\mathbf{K}^{-1}\mathbf{M} = \begin{bmatrix} 2.0 & -1.0 & 0.5 \\ -1.0 & 1.2 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix}$$

and

$$\mathbf{M}\mathbf{C}^{-1}\mathbf{K} = \mathbf{K}\mathbf{C}^{-1}\mathbf{M} = \begin{bmatrix} 4.1 & 6.2 & 5.6 \\ 6.2 & 9.73 & 9.2 \\ 5.6 & 9.2 & 9.6 \end{bmatrix}$$

are also satisfied. Thus all three conditions described in Theorem 1 are simultaneously valid although none of them are the same. So, if any one of the three conditions proposed in Theorem 1 is satisfied, a viscously damped positive definite system possesses classical normal modes.

Example 2

Suppose for a system

$$\mathbf{M} = \begin{bmatrix} 7.0584 & 1.3139 \\ 1.3139 & 0.2446 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 4.0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix}. \tag{5}$$

It may be verified that the mass matrix is singular for this system. For this reason, Caughey and O'Kelly's criteria is not applicable. But, as the other two conditions in Theorem 1,

$$\mathbf{M}\mathbf{K}^{-1}\mathbf{C} = \mathbf{C}\mathbf{K}^{-1}\mathbf{M} = \begin{bmatrix} 1.6861 & 0.3139 \\ 0.3139 & 0.0584 \end{bmatrix}$$

and

$$\mathbf{M}\mathbf{C}^{-1}\mathbf{K} = \mathbf{K}\mathbf{C}^{-1}\mathbf{M} = \begin{bmatrix} 29.5475 & 5.5 \\ 5.5 & 1.0238 \end{bmatrix}$$

are satisfied, all three matrices can be diagonalized by a congruence transformation using the undamped modal matrix

$$\mathbf{X} = \begin{bmatrix} 0.9372 & -0.1830 \\ 0.3489 & 0.9831 \end{bmatrix}.$$

3 GENERALIZED PROPORTIONAL DAMPING

Obtaining a damping matrix from the 'first principles' like the mass and stiffness matrices is not possible for most systems. For this reason, assuming $\mathbf{M}$ and $\mathbf{K}$ are known, we often want to express $\mathbf{C}$ in terms of $\mathbf{M}$ and $\mathbf{K}$ such that the system still possesses classical normal modes. Of course, the earliest work along this line is the proportional damping shown in equation (1) by Rayleigh [1]. It may be verified that expressing $\mathbf{C}$ in such a way will always satisfy the conditions given by Theorem 1. Caughey [11] proposed that a *sufficient* condition for the existence of classical normal modes is: if $\mathbf{M}^{-1}\mathbf{C}$ can be expressed in a series involving powers of $\mathbf{M}^{-1}\mathbf{K}$. His result generalized Rayleigh's result, which turns out to be the first two terms of the series. Later, Caughey and O'Kelly [2] proved that the series representation of damping

$$\mathbf{C} = \mathbf{M}\sum_{j=0}^{N-1} \alpha_j \left(\mathbf{M}^{-1}\mathbf{K}\right)^j \tag{6}$$

is the *necessary and sufficient* condition for existence of classical normal modes for systems without any repeated roots. This series is now known as the 'Caughey series' and is possibly the most general form of damping under which the system will still possess classical normal modes.

Here, a further generalized and useful form of proportional damping will be proposed. We assume that the system is positive definite. Consider the conditions (a) and (b) of Theorem 1; premultiplying (a) by $\mathbf{M}^{-1}$ and (b) by $\mathbf{K}^{-1}$ one has

$$\begin{aligned} \left(\mathbf{M}^{-1}\mathbf{K}\right)\left(\mathbf{M}^{-1}\mathbf{C}\right) = \left(\mathbf{M}^{-1}\mathbf{C}\right)\left(\mathbf{M}^{-1}\mathbf{K}\right) \quad \text{or} \quad \mathbf{AB} = \mathbf{BA} \\ \left(\mathbf{K}^{-1}\mathbf{M}\right)\left(\mathbf{K}^{-1}\mathbf{C}\right) = \left(\mathbf{K}^{-1}\mathbf{C}\right)\left(\mathbf{K}^{-1}\mathbf{M}\right) \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{D} = \mathbf{D}\mathbf{A}^{-1} \end{aligned} \tag{7}$$

where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$, $\mathbf{B} = \mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{D} = \mathbf{K}^{-1}\mathbf{C}$. Notice that we did not consider the condition (c) of Theorem 1. Premultiplying (c) by $\mathbf{C}^{-1}$, one would obtain a similar commutative condition but it would involve $\mathbf{C}$ terms in both the matrices, from which any meaningful expression of $\mathbf{C}$ in terms of $\mathbf{M}$ and $\mathbf{K}$ cannot be deduced. For this reason only the above two commutative relationships will be considered. It is well known that for any two matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A}$ commutes with $\mathbf{B}$, $f(\mathbf{A})$ also commutes with $\mathbf{B}$ where $f(z)$ is any *analytic function* of the variable z. Thus, in view of the commutative relationships represented by equation (7), one can use almost *all* well known functions to represent $\mathbf{M}^{-1}\mathbf{C}$ in terms of $\mathbf{M}^{-1}\mathbf{K}$ and also $\mathbf{K}^{-1}\mathbf{C}$ in terms of $\mathbf{K}^{-1}\mathbf{M}$, that is, representations like $\mathbf{C} = \mathbf{M}f(\mathbf{M}^{-1}\mathbf{K})$ and $\mathbf{C} = \mathbf{K}f(\mathbf{K}^{-1}\mathbf{M})$ are valid for any analytic $f(z)$. Adding these two quantities and also taking $\mathbf{A}$ and $\mathbf{A}^{-1}$ in the argument of the function as (trivially) $\mathbf{A}$ and $\mathbf{A}^{-1}$ always commute we can express the damping matrix in the form of

$$\mathbf{C} = \mathbf{M}f_1\left(\mathbf{M}^{-1}\mathbf{K}, \mathbf{K}^{-1}\mathbf{M}\right) + \mathbf{K}f_2\left(\mathbf{M}^{-1}\mathbf{K}, \mathbf{K}^{-1}\mathbf{M}\right) \tag{8}$$

such that the system possesses classical normal modes. Further, postmultiplying condition (a) of Theorem 1 by $\mathbf{M}^{-1}$ and (b) by $\mathbf{K}^{-1}$ one has

$$\begin{aligned} \left(\mathbf{K}\mathbf{M}^{-1}\right)\left(\mathbf{C}\mathbf{M}^{-1}\right) = \left(\mathbf{C}\mathbf{M}^{-1}\right)\left(\mathbf{K}\mathbf{M}^{-1}\right) \\ \left(\mathbf{M}\mathbf{K}^{-1}\right)\left(\mathbf{C}\mathbf{K}^{-1}\right) = \left(\mathbf{C}\mathbf{K}^{-1}\right)\left(\mathbf{M}\mathbf{K}^{-1}\right). \end{aligned} \tag{9}$$

Following a similar procedure we can express the damping matrix in the form

$$\mathbf{C} = f_3\left(\mathbf{K}\mathbf{M}^{-1}, \mathbf{M}\mathbf{K}^{-1}\right)\mathbf{M} + f_4\left(\mathbf{K}\mathbf{M}^{-1}, \mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K} \tag{10}$$

for which system (2) possesses classical normal modes. The functions f_i, $i = 1, \cdots, 4$ can have very general forms$-$ they may consist of an arbitrary number of multiplications, divisions, summations, subtractions or powers of any other functions or can even be functional compositions. Thus, any conceivable form of analytic functions that are valid for scalars can be used in equations (8) and (10). In a natural way, common restrictions applicable to scalar functions are also valid, for example logarithm of a negative number is not permitted. Although the functions f_i, $i = 1, \cdots, 4$ are general, the expression of $\mathbf{C}$ in (8) or (10) gets restricted because of the special nature of the *arguments* in the functions. As a consequence, $\mathbf{C}$ represented in (8) or (10) does not cover the whole $\mathbb{R}^{N\times N}$, which is well known that many damped systems do not possess classical normal modes.

Rayleigh's result (1) can be obtained directly from equation (8) or (10) as a very special $-$ one could almost say trivial $-$ case by choosing each matrix function f_i as real scalar times an identity matrix. The damping matrix expressed in equation (8) or (10) provides a new way of interpreting the 'Rayleigh damping' or 'proportional damping' where the identity matrices (always) associated in the right or left side of $\mathbf{M}$ and $\mathbf{K}$ are replaced by arbitrary matrix functions f_i with proper arguments. This kind of damping model will be called *generalized proportional damping*. We call the representation in equation (8) *right-functional form* and that in equation (10) *left-functional form*. Caughey series (6) is an example of right functional form. Note that if $\mathbf{M}$ or $\mathbf{K}$ is singular then the argument involving its corresponding inverse has to be removed from the functions.

All analytic functions have a power series form via Taylor expansion. It is also known that for any $\mathbf{A} \in \mathbb{R}^{N\times N}$, all $\mathbf{A}^k$, for integer $k > N$, can be expressed as a linear combination of $\mathbf{A}^j, j \leq (N-1)$ by a recursive relationship using the Cayley-Hamilton theorem [12]. For this reason the expression of $\mathbf{C}$ in (8) or (10) can in turn be expressed in the form of Caughey series (6). However, since all f_i can have very general forms, such a representation may not be always straight forward. For example, if $\mathbf{C} = \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^e$ the system possesses normal modes, but it is neither a direct member of the Caughey series (6) nor is it a member of the series involving rational fractional powers given by Caughey [11] as e is an irrational number. However, we know that $e = 1 + \frac{1}{1!} + \cdots + \frac{1}{r!} + \cdots\infty$, from which we can write $\mathbf{C} = \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})(\mathbf{M}^{-1}\mathbf{K})^{\frac{1}{1!}}\cdots(\mathbf{M}^{-1}\mathbf{K})^{\frac{1}{r!}}\cdots\infty$, which can in principle be represented by the Caughey series. It is easy to verify that, from a practical point of view, this representation is not simple and requires truncation of the series up to some finite number of terms. Hence, $\mathbf{C}$ expressed in the form of equation (8) or (10) is a more convenient representation of the Caughey series and we say that *viscously damped positive definite systems possess classical normal modes if and only if* $\mathbf{C}$ *can be represented by equation (8) or (10)*.

Example 3

It will be shown that the linear dynamic system satisfying the following equation of free vibration

$$\mathbf{M}\ddot{\mathbf{q}} + \left[\mathbf{M}e^{-\left(\mathbf{M}^{-1}\mathbf{K}\right)^2/2}\sinh(\mathbf{K}^{-1}\mathbf{M}\ln(\mathbf{M}^{-1}\mathbf{K})^{2/3}) \right. \\ \left. + \mathbf{K}\cos^2(\mathbf{K}^{-1}\mathbf{M})\sqrt[4]{\mathbf{K}^{-1}\mathbf{M}}\,\tan^{-1}\frac{\sqrt{\mathbf{M}^{-1}\mathbf{K}}}{\pi}\right]\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{11}$$

possesses classical normal modes and can be analyzed using modal analysis. Here $\mathbf{M}$ and $\mathbf{K}$ are the same as example 1.

Direct calculation shows

$$\mathbf{C} = -\begin{bmatrix} 67.9188 & 104.8208 & 95.9566 \\ 104.8208 & 161.1897 & 147.7378 \\ 95.9566 & 147.7378 & 135.2643 \end{bmatrix}. \tag{12}$$

Using the modal matrix calculated before in equation (4), we obtain

$$\mathbf{X}^T\mathbf{C}\mathbf{X} = \begin{bmatrix} -88.9682 & 0.0 & 0.0 \\ 0.0 & 0.0748 & 0.0 \\ 0.0 & 0.0 & 0.5293 \end{bmatrix},$$

a diagonal matrix. Analytically the modal damping factors can be obtained as

$$2\xi_j\omega_j = e^{-\omega_j^4/2}\sinh\left(\frac{1}{\omega_j^2}\ln\frac{4}{3}\omega_j\right) + \omega_j^2\cos^2\left(\frac{1}{\omega_j^2}\right)\frac{1}{\sqrt{\omega_j}}\tan^{-1}\frac{\omega_j}{\pi}. \tag{13}$$

This example shows that using the generalized proportional damping it is possible to model any variation of the damping factors with respect to the frequency. This is the basis of the damping identification method to be proposed here. Using Rayleigh's proportional damping in equation (1), the modal damping factors have a special form

$$\zeta_j = \frac{1}{2}\left(\frac{\alpha_1}{\omega_j} + \alpha_2\omega_j\right) \tag{14}$$

Clearly, not all form of variations of ζ_j with respect to ω_j can be captured using equation (14). The damping identification method proposed in the next section removes this restriction.

4 DAMPING IDENTIFICATION METHOD

The damping identification method relies on the expression of proportional damping given by equations (8) and (10). We consider only the left functional form in (8), but the right functional form can also be used in a similar way if required. To simplify the identification procedure, express the damping matrix by

$$\mathbf{C} = \mathbf{M}f\left(\mathbf{M}^{-1}\mathbf{K}\right) \tag{15}$$

The modal damping factors can be obtained as

$$2\zeta_j\omega_j = f\left(\omega_j^2\right) \tag{16}$$

$$\text{or}\quad \zeta_j = \frac{1}{2\omega_j}f\left(\omega_j^2\right) = \widehat{f}(\omega_j)\quad \text{(say)} \tag{17}$$

The function $\widehat{f}(\bullet)$ can be obtained by fitting a continuous function representing the variation of the measured modal damping factors with respect to the frequency. From equations (15) and (16) note that in the argument of $f(\bullet)$, the term ω_j can be replaced by $\sqrt{\mathbf{M}^{-1}\mathbf{K}}$ when obtaining the damping matrix. With the fitted function $\widehat{f}(\bullet)$, the damping matrix can be identified using equation (17) as

$$2\zeta_j\omega_j = 2\omega_j \widehat{f}(\omega_j) \tag{18}$$

$$\text{or} \quad \widehat{\mathbf{C}} = 2\mathbf{M}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\,\widehat{f}\left(\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right) \tag{19}$$

The following example will clarify the identification procedure.

Example 4

Suppose figure 1 shows modal damping factors as a function of frequency obtained by conducting simple vibration testing on a structure. The damping factors are such that, within the frequency range considered, they show very low values in the low frequency region, high values in the mid frequency region and again low values in the high frequency region. We want to identify

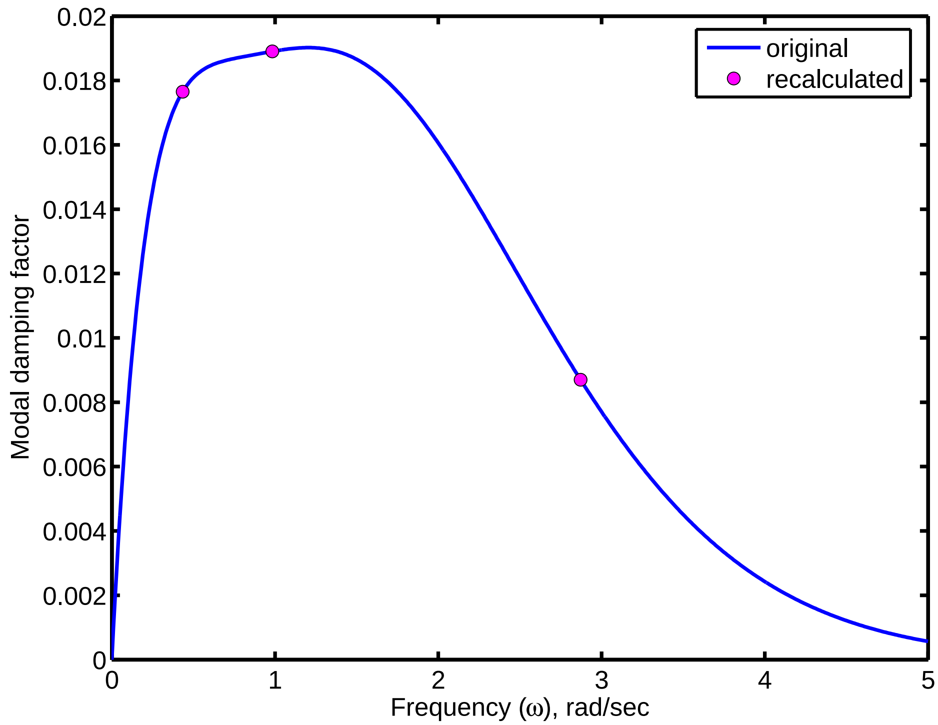

Figure 1: Curve of modal damping factors.

a damping model which shows this kind of behavior. The first step is to identify the function which produces this curve. Here this (continuous) curve was simulated using the equation

$$\widehat{f}(\omega) = \frac{1}{15}\left(e^{-2.0\omega} - e^{-3.5\omega}\right)\left(1 + 1.25\sin\frac{\omega}{7\pi}\right)\left(1 + 0.75\omega^3\right). \tag{20}$$

From the above equation, the modal damping factors in terms of the discrete natural frequencies, can be obtained by

$$2\xi_j\omega_j = \frac{2\omega_j}{15}\left(e^{-2.0\omega_j} - e^{-3.5\omega_j}\right)\left(1 + 1.25\sin\frac{\omega_j}{7\pi}\right)\left(1 + 0.75\omega_j^3\right). \tag{21}$$

To obtain the damping matrix, consider equation (21) as a function of ω_j^2 and replace ω_j^2 by $\mathbf{M}^{-1}\mathbf{K}$ and any constant terms by that constant times $\mathbf{I}$. Therefore, from equation (21) we have

$$\begin{aligned}\mathbf{C} =& \frac{2}{15}\mathbf{M}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\left[e^{-2.0\sqrt{\mathbf{M}^{-1}\mathbf{K}}} - e^{-3.5\sqrt{\mathbf{M}^{-1}\mathbf{K}}}\right] \\ &\times\left[\mathbf{I} + 1.25\sin\left(\frac{1}{7\pi}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right)\right]\left[\mathbf{I} + 0.75(\mathbf{M}^{-1}\mathbf{K})^{3/2}\right]\end{aligned} \tag{22}$$

as the identified damping matrix. Using the numerical values of **M** and **K** from example 1 we obtain

$$\mathbf{C} = \begin{bmatrix} 2.3323 & 0.9597 & 1.4255 \\ 0.9597 & 3.5926 & 3.7624 \\ 1.4255 & 3.7624 & 7.8394 \end{bmatrix} \times 10^{-2}. \quad (23)$$

If we recalculate the damping factors from the above constructed damping matrix, it will produce three points corresponding to the three natural frequencies which will exactly match with our initial curve as shown in figure 1.

The method outlined here can produce accurate estimate of the damping matrix if the modal damping factors are known. All polynomial fitting methods can be employed to approximate $\widehat{f}(\omega)$ and corresponding to the fitted function one can construct a damping matrix by the procedure outlined here. As an example, if $2\xi_j\omega_j$ can be represented in a Fourier series as

$$2\xi_j\omega_j = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi r\omega_j}{\Omega}\right) + b_r \sin\left(\frac{2\pi r\omega_j}{\Omega}\right)\right] \quad (24)$$

then the damping matrix can be expanded as

$$\mathbf{C} = \mathbf{M}\left(\frac{a_0}{2}\mathbf{I} + \sum_{r=1}^{\infty}\left[a_r \cos\left(2\pi r\Omega^{-1}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right) + b_r \sin\left(2\pi r\Omega^{-1}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right)\right]\right) \quad (25)$$

in a Fourier series. The damping identification procedure itself does not introduce significant errors as long as the modes are not significantly complex. From equation (19) it is obvious that the accuracy of the fitted damping matrix depends only on the accuracy the mass and stiffness matrix models. In summary, this identification procedure can be described by the following steps:

1. Measure a suitable transfer function $H_{ij}(\omega)$ by conducting a vibration testing.
2. Obtain the undamped natural frequencies as ω_j and modal damping factors ζ_j, for example, using the circle-fitting method.
3. Fit a function $\zeta = \widehat{f}(\omega)$ which represents the variation of ζ_j with respect to ω_j for the range of frequency considered in the study.
4. Calculate the matrix $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$
5. Obtain the damping matrix using $\widehat{\mathbf{C}} = 2\,\mathbf{M}\,\mathbf{T}\,\widehat{f}\,(\mathbf{T})$

Most of the currently available finite element based modal analysis packages only offer Rayleigh's proportional damping model. The proposed damping identification technique can be easily incorporated within the existing tools to enhance their damping modelling capabilities without using significant additional resources.

5 CONCLUSIONS

A method for identification of damping matrix using experimental modal analysis has been proposed. The method is based on generalized proportional damping. The generalized proportional damping expresses the damping matrix in terms of any non-linear function involving specially arranged mass and stiffness matrices so that the system still posses classical normal modes. This enables one to model practically any type of variations in the modal damping factors with respect to the frequency. Once a scalar function is fitted to model such variations, the damping matrix can be identified very easily using the proposed method. This implies that the problem of damping identification is effectively reduced to the problem of a scalar function fitting. The method is very simple and requires the measurement of damping factors and natural frequencies only (that is, the measurements of the mode shapes are not necessary). The proposed method is applicable to any linear structures as long as one have validated mass and stiffness matrix models which predict the natural frequencies accurately and modes are not significantly complex. In the case when a system is heavily damped and has significantly complex modes, the proposed identified damping matrix can be a good starting point for more sophisticated analyses.

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